The degree and codegree threshold for generalized triangle and some trees covering

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Abstract

Given two k-uniform hypergraphs F and G, we say that G has an F-covering if for every vertex in G there is a copy of F cover it. For $1 \leq i \leq k-1$, the minimum i-degree $\delta_i(G)$ of G is the minimum integer such that every i vertices are contained in at least $\delta_i(G)$ edges. Let $c_i(n, F)$ be the largest minimum i-degree among all n-vertex k-uniform hypergraphs that have no F-covering. In this paper, we mainly consider the F-covering problem in 3-uniform hypergraphs. When F is a generalized triangle T, we give the exact value of $c_2(n, T)$ and asymptotically determine $c_1(n, T)$. Moreover, when F is a linear k-path P_k or a star S_k, we provide bounds of $c_i(n, P_k)$ and $c_i(n, S_k)$ for $k \geq 3$, where $i = 1, 2$.

Keywords: 3-graphs; Covering; Extremal;

AMS Subject Classification (2020): 68R10, 05C35, 05C22

1 Introduction

Let k be an integer with $k \geq 2$. We say a k-uniform hypergraph, or a k-graph, is a pair $G = (V(G), E(G))$, where $V(G)$ is a set of vertices and $E(G)$ is a collection of k-subsets of V. When $k = 2$, the k-graph is the simple graph. We simply denote 2-graph by graph. Let $G = (V(G), E(G))$ be a k-graph. For any $S \subset V(G)$, let the neighborhood $N_G(S)$ of S be $\{T \subset V(G) \setminus S : T \cup S \in E(G)\}$ and the degree d_G of S be $|N_G(S)|$. For $1 \le i \le k-1$, we denote the *minimum i-degree* of G by $\delta_i(G)$, which is the minimum of $d_G(S)$ over all $S \in \binom{V(G)}{i}$. We call $\delta_1(G)$ the *minimum degree* of G and $\delta_{k-1}(G)$ the *minimum codegree* of G. When $|S| = k-1$, we also call the vertex in $N_G(S)$ the co-neighbor of S. For a vertex x in V, we define the link graph G_x to be a $(k-1)$-graph with the vertex set $V(G) \setminus \{x\}$ and the edge set $N_G(\{x\})$.

Given a k-graph F, we say a k-graph G has an F-covering if for any vertex of G, we can find a copy of F containing it. For $1 \le i \le k-1$, define

$$c_i(n, F) = \max\{\delta_i(G) : G \text{ is a } k\text{-graph on } n \text{ vertices with no } F\text{-covering}\}.$$

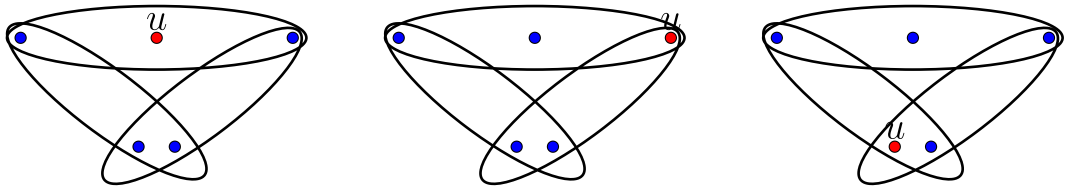
and call $c_1(n, F)$ the F-covering degree-threshold and $c_{k-1}(n, F)$ the F-covering codegree-threshold.

For graphs F, the F-covering problem was solved asymptotically in [7] and showed that $c_1(n, F) = (\frac{\chi(F)-2}{\chi(F)-1} + o(1))n$ where $\chi(F)$ is the chromatic number of F. Falgas-Ravry and Zhao [1] initiated the study of the F-covering problem in 3-graphs. For $n \ge k$, let K_n^k denote the complete k-graph on n vertices and K_n^{k-} denote the k-graph by removing one edge from K_n^k. In [1], Falgas-Ravry and Zhao determined the exact value of $c_2(n, K_4^3)$ for $n > 98$ and gave bounds of $c_2(n, F)$ when F is K_4^{3-}, K_5^3 or the tight cycle C_5^3 on 5 vertices. Yu, Hou, Ma and Liu [3] gave the exact value of $c_2(n, K_4^{3-})$, $c_2(n, K_5^{3-})$ and showed that $c_2(n, K_4^{3-}) = \lfloor \frac{n}{3} \rfloor$, $c_2(n, K_5^{3-}) = \lfloor \frac{2n-2}{3} \rfloor$. Soon after that, Falgas-Ravry, Markström, and Zhao [2] gave near optimal bounds of $c_1(n, K_4^3)$ and asymptotically determined $c_1(n, K_4^{3-})$. Recently, Tang, Ma and Hou [4] determined the exact value of $c_2(n, C_6^3)$ and an asymptotic optimal value of $c_1(n, C_6^3)$. There are some other related results in literature, for example in [5],[6].

In this paper, we also focus on 3-graphs. Let the generalized triangle T be a 3-graph with the vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and the edge set $\{\{v_1v_2v_3\}, \{v_1v_2v_4\}, \{v_3 v_4v_5\}\}$. We determine the exact value of $c_2(n, T)$ and give the bounds of $c_1(n, T)$. What's more, let G be a graph and fix a vertex u in $V(G)$. If u is covered by a generalized triangle, then there are three possible positions for u to have, see Figure 1. We denote these three ways by T^1, T^2 and T^3. We give the upper bounds of $\delta_1(G)$ guaranteeing that every vertex in $V(G)$ is contained in T^1, T^2 and T^3. The main results on generalized triangle are as follows.

Theorem 1. *For $n \ge 5$, we have:*

$$c_2(n, T) = \begin{cases} 1, \textit{when } n \in [5, 10] \\ 2, \textit{when } n \ge 11 \textit{ and } n-1 \equiv 0 \pmod 3 \\ 1, \textit{when } n \ge 11 \textit{ and } n-1 \equiv 1, 2 \pmod 3 \end{cases}.$$



(a) u is contained in T^1 (b) u is contained in T^2 (c) u is contained in T^3

Figure 1: Different positions of u in a generalized triangle

Theorem 2. *For $n \geq 5$, we have:*

(i) $\frac{n^2}{9} \leq c_1(n, T) \leq \frac{n^2}{6} + \frac{5}{6}n - 3$.

(ii) If G is an n-vertex 3-graph satisfying that $\delta_1(G) > \frac{n^2}{6} + \frac{5}{6}n - 3$, then for any vertex u in G, there is a generalized triangle T^1 or T^2 covering u.

(iii) If G is an n-vertex 3-graph satisfying that $\delta_1(G) > \frac{n^2}{4} + \frac{1}{4}n - 2$, then for any vertex u in G, there are generalized triangles T^1 and T^2 covering u.

(iv) If G is an n-vertex 3-graph satisfying that $\delta_1(G) > \frac{\sqrt{5}-1}{4}n^2 + O(n)$, then for any vertex u in G, there are generalized triangles T^1, T^2 and T^3 covering u.

Now we pay attention to some trees covering problems. For $k \geq 2$, let S_k be the 3-graph k-star with the vertex set $\{v_0, v_1, v_2..., v_{2k-1}, v_{2k}\}$ and the edge set $\{\{v_0, v_1, v_2\}, \{v_0, v_3, v_4\}, ..., \{v_0, v_{2k-1}, v_{2k}\}\}$. Let v_0 be the center of S_k. For $k \geq 2$, let the 3-graph P_k be the the linear k-path with the vertex set $\{v_1, v_2..., v_{2k}, v_{2k+1}\}$ and the edge set $\{\{v_1, v_2, v_3\} \{v_3, v_4, v_5\}, ..., \{v_{2k-1}, v_{2k}, v_{2k+1}\}\}$. In this paper, we consider the F-covering problem when F is the k-star S_k or the linear k-path P_k.

When $k = 2$, the 2-star S_2 is the linear 2-path P_2. Figure 2 is an example of the linear 2-path P_2. We determine the exact values of $c_2(n, P_2)$ and $c_1(n, P_2)$. The results on the linear 2-path covering or the 2-star covering are as follows.

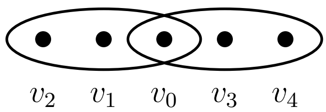


Figure 2: Linear 2-path P_2

Theorem 3. *For $n \geq 5$, we have $c_2(n, P_2) = 0$.*

Theorem 4. *For $n \geq 8$, we have $c_1(n, P_2) = 3$.*

In addition, we determine the codegree threshold for the property of a 3-graph G that for any vertex $u \in V(G)$ we can find a linear 2-path P_2 with the center u.

Theorem 5. *If G is an n-vertex 3-graph satisfying that $n \geq 5$ and $\delta_2(G) \geq 2$, then for any vertex $u \in V(G)$, we can find 4 vertices p, q, s, t where $\{u, p, q\}$ and $\{u, s, t\}$ form a linear 2-path P_2 covering u.*

Through exploring the structure of graphs without some specific matchings, we obtain the following result on the 3-star S_3-covering.

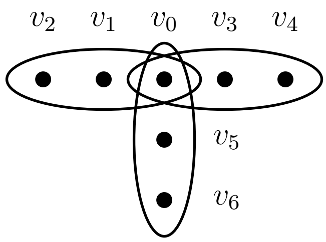


Figure 3: The 3-star S_3

Theorem 6. *If H is an n-vertex 3-graph satisfying $n \geq 7$ and $\delta_2(H) \geq 2$, then for any vertex $u \in V(H)$, there is a 3-star S_3 covering u.*

By Theorem 6, we can directly get the following corollary.

Corollary 1. *For $n \geq 7$, $c_2(n, S_3) \leq 1$.*

As well, we determine the codegree threshold for the property of a 3-graph G that for any vertex $u \in V(G)$ we can find a 3-star S_3 with the center u.

Theorem 7. *If H is an n-vertex 3-graph with $n \geq 7$ and $\delta_2(H) \geq 3$, then for any vertex $u \in V(H)$ we can find a S_3 with the center u.*

Using the similar technique in the proof of Theorem 6 we also give bounds of $c_2(n, S_k)$ and $c_1(n, S_k)$ for $k \geq 3$.

Proposition 1. *Let k be an integer with $k \geq 3$. Let H be an n-vertex 3-graph with $n \geq 2k + 1$. We have:*

(i) $c_2(n, S_k) \leq \max\{\frac{4k^2-6k+2}{n-1}, k - 2 - \frac{k^2-nk}{n-1}\}$.

(ii) $c_1(n, S_k) \leq \max\{\binom{2k-1}{2}, \binom{n-1}{2} - \binom{n-k}{2}\}$.

Figure 4 is an example of the linear 3-path P_3. We determine the exact value of $c_2(n, P_3)$ and asymptotically determine $c_1(n, P_3)$ as follows.

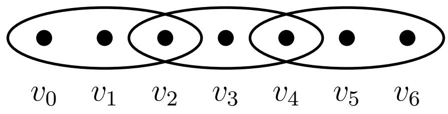


Figure 4: Linear 3-path P_3

Theorem 8. *For $n \geq 8$, we have $c_2(n, P_3) = 1$.*

Theorem 9. *For $n \geq 8$, we have $n - 2 \leq c_1(n, P_3) \leq n + 4$.*

Moreover, we determine the codegree threshold for the property of a 3-graph G that for any vertex $u \in V(G)$ we can find a linear 3-path with the vertex set $\{u, v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set $\{\{v_1v_2u\}, \{uv_3v_4\}, \{v_4v_5v_6\}\}$ covering u.

Theorem 10. *If H is an n-vertex 3-graph with $n \geq 8$ and $\delta_2(H) \geq 3$, then for any vertex $u \in V(H)$ we can find a P_3 with the vertex set $\{u, v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set $\{\{uv_1v_2\}, \{uv_3v_4\}, \{v_4v_5v_6\}\}$ covering u.*

We also give the bounds of $c_2(n, P_k)$ and $c_1(n, P_k)$ for $k \geq 4$ as follows.

Proposition 2. *Let k be an integer with $k \geq 4$. We have:*

(i) For $n \geq 2k + 1$, $k - 3 \leq c_2(n, P_k) \leq 2k - 2$.

(ii) For $n \geq 4k$, $\max\{n - 2, \binom{2k-1}{2}\} \leq c_1(n, P_k) \leq \binom{n-1}{2} - \binom{n-2k+1}{2}$.

The rest of the paper is arranged as follows. In Section 2, we give some extremal constructions and proofs of theorems for generalized triangle covering. And in Sections 3 we give some extremal constructions and proofs of theorems for some trees covering.

2 Results on generalized triangle covering

2.1 Construction

We introduce some constructions involving our results. For two families of sets $\mathcal{A}$ and $\mathcal{B}$, define $A \vee B = \{A \cup B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$.

Construction 1. *Let V_1 be a vertex set. Fix $u \in V_1$, let $V' = V_1 \backslash \{u\}$, $E_1 = \{u\} \vee \binom{V'}{2}$ which means E_1 is a 3-set family and every 3-set from E_1 contains u and two other vertices form V'. Let $G_1 = (V_1, E_1)$ be a 3-graph.*

The following observation can be checked directly.

Observation 1. *G_1 is a 3-graph with $\delta_2(G_1) = 1$ and there is no generalized triangle T covering u.*

Construction 2. *Let k be an integer with $k \geq 4$. Let $G_2 = (V_2, E_2)$ be a 3-graph with $V_2 = \{u\} \cup \sum_{i=1}^{k} C_i$ where C_i is a 3-vertex set for $i \in [1, k]$. E_2 consists of two types of edges. For the first type, edges induced in the vertex set $\{u\} \cup C_i$ form a K_4^3 for any $i \in [1, k]$. For the second type, let C_a, C_b and C_c be any three elements in $\{C_i : i \in [1, k]\}$. Without loss of generality, we assume C_a is $\{v_1, v_2, v_3\}$, C_b is $\{v_4, v_5, v_6\}$ and C_c is $\{v_7, v_8, v_9\}$. The edges induced in C_a, C_b and C_c are:*

$$\left\{ \begin{array}{l} \{v_1, v_4, v_7\}, \{v_2, v_4, v_8\}, \{v_3, v_4, v_9\}; \\ \{v_1, v_5, v_8\}, \{v_2, v_5, v_9\}, \{v_3, v_5, v_7\}; \\ \{v_1, v_6, v_9\}, \{v_2, v_6, v_7\}, \{v_3, v_6, v_8\}; \end{array} \right\}$$

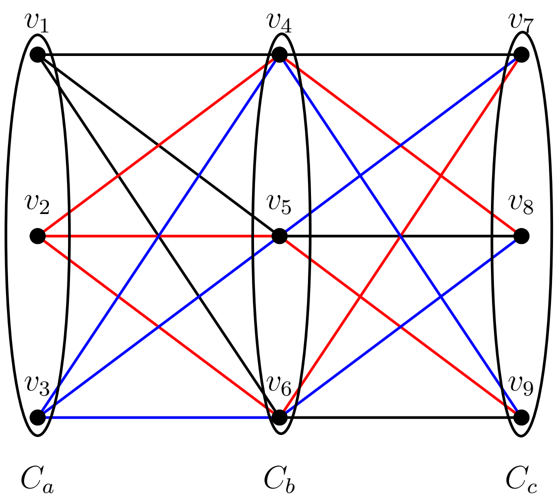


Figure 5: Edges induced in C_a, C_b and C_c.

In Construction 2, the subgraph induced in every three elements of $\{C_i\}$ is isomorphic to the 3-graph in Figure 5. And we get the following observation for the Construction 2.

Observation 2. *G_2 is a 3-graph with $\delta_2(G_2) = 2$ and there is no generalized triangle T covering u.*

Proof. We first check that G_2 has no generalized triangle T covering u. If u is covered as the first case in Figure 1, then there is an edge $e_0 = \{u, v_1, v_2\}$ such that

$v_1, v_2 \in C_i$ for $i \in [1,k]$. By the definition of G_2, we can not find two vertices v_3, v_4 from $V_2 \setminus \{u, v_1, v_2\}$ making $\{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}$ being edges in E_2, a contradiction with the fact that there is a T^1 covering u. If u is covered as the second case in Figure 1, then there are two edges $e_1 = \{u, v_5, v_6\}$ and $e_2 = \{u, v_7, v_8\}$ such that $v_5, v_6 \in C_i$ and $v_3, v_4 \in C_j$ for $i \neq j$ and $i, j \in [1,k]$. However, there is no edge induced in any two $C_i's$ in G_2 , which means we can not find an edge together with e_1 and e_2 to form a T^2 covering u, a contradiction. If u is covered as the third case in Figure 1, then there are two edges $e_3 = \{u, v_9, v_{10}\}$ and $e_4 = \{u, v_9, v_{11}\}$ such that $v_9, v_{10}, v_{11} \in C_i$ for some $i \in [1,k]$. Actually, there is no vertex v_{12} making $\{v_{10}, v_{11}, v_{12}\}$ being an edge in G_2, a contradiction with the fact that there is a T^3 covering u.

Next we prove that $\delta_2(G_2) = 2$. Let s,t be any two vertices in $V(G_2)$. We have:

- If the two vertices s, t belong to different $C_i's$, then $d_G(\{s,t\}) \geq 2$.
- If the two vertices s, t belong to any C_i, then $d_G(\{s,t\}) = 2$.
- If the vertex s is u and the vertex t belongs to any C_i, then $d_G(\{s,t\}) = 2$.

In conclusion, we have $\delta_2(G_2) = 2$ and there is no generalized triangle T covering u.

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Construction 3. *Let $G_3 = (V_3, E_3)$ be an n-vertex 3-graph with V_3=$\{u\} \cup A_1 \cup A_2 \cup B$ and $E_3 = \left(\{u\} \vee \binom{A_1}{1} \vee \binom{A_2}{1}\right) \cup \left(\binom{A_1}{1} \vee \binom{A_2}{1} \vee \binom{B}{1}\right) \cup \binom{B}{3}$, where $|A_1| = |A_2| = \lceil \frac{n}{3} \rceil$.*

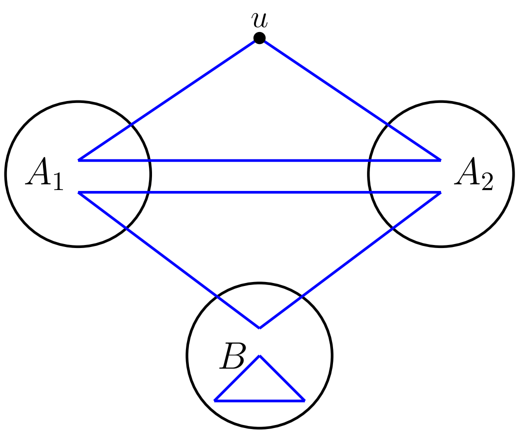

Figure 6: Construction 3

Observation 3. *G_3 is a 3-graph with $\delta_1(G_3) \geq \frac{n^2}{9}$ and there is no generalized triangle T covering u.*

Proof. We check that G_3 has no generalized triangle T covering u. If u is covered as the first case in Figure 1, then there is an edge $e_0 = \{u, v_1, v_2\}$ such that v_1, v_2 are in different A_i for $i = 1, 2$. By the definition of G_3, we can not find two vertices v_3, v_4 from $V_3 \setminus \{u, v_1, v_2\}$ such that $\{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}$ are edges in E_3, a contradiction with the fact that there is a T^1 covering u. If u is covered as the second case in Figure 1, then there are two edges $e_1 = \{u, v_5, v_6\}$ and $e_2 = \{u, v_7, v_8\}$ such that both v_5, v_6 and v_7, v_8 are in different A_i for $i = 1, 2$. However, there is no edge induced in A_1 and A_2 in G_3, which means we can not find an edge together with e_1 and e_2 to form a T^2 covering u, a contradiction. If u is covered as the third case in Figure 1, then there are two edges $e_3 = \{u, v_9, v_{10}\}$ and $e_4 = \{u, v_9, v_{11}\}$ such that v_{10}, v_{11} are in one of A_i and v_9 in another one for $i = 1, 2$. Actually, there is no vertex v_{12} making $\{v_{10}, v_{11}, v_{12}\}$ being an edge in G_3, a contradiction with the fact that there is a T^3 covering u.

Next we prove that $\delta_1(G_3) \geq \frac{n^2}{9}$. Let v be a vertex from $V(G_3)$.

If $v = u$, then

$$d_{G_3}(\{v\}) = \lceil \frac{n}{3} \rceil \cdot \lceil \frac{n}{3} \rceil \geq \frac{n^2}{9}.$$

If $v \in A_1 \cup A_2$, then

$$d_{G_3}(\{v\}) = \lceil \frac{n}{3} \rceil + \lceil \frac{n}{3} \rceil \cdot (n - 1 - 2\lceil \frac{n}{3} \rceil) \geq \frac{n^2}{9}.$$

If $v \in B$, then

$$d_{G_3}(\{v\}) = \lceil \frac{n}{3} \rceil \cdot \lceil \frac{n}{3} \rceil + \binom{n - 1 - 2\lceil \frac{n}{3} \rceil}{2} > \lceil \frac{n}{3} \rceil \cdot \lceil \frac{n}{3} \rceil \geq \frac{n^2}{9}.$$

Therefore, we have $\delta_1(G_3) \geq \frac{n^2}{9}$.

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2.2 The proof of Theorem 1

We divide the proof of Theorem 1 into two parts according to the value of n.

2.2.1 When $n \in [5, 10]$

When $n \in [5, 10]$, the lower bound of $c_2(n, T)$ can be directly gotten from Observation 1. Therefore, we only need to prove $c_2(n, T) \leq 1$ when $n \in [5, 10]$. We assume to the contrary that there is a 3-graph G with $\delta_2(G) \geq 2$ and a vertex $u \in V(G)$ that is not covered by T.

Let $y \in V(G)$ be a vertex different from u. As $d_G(\{u, y\}) \geq 2$, $N_G(\{u, y\})$ has at least two vertices. Considering any two vertices p, q from $N_G(\{u, y\})$, we have $N_G(\{p, q\}) \subseteq \{u, y\}$. Otherwise, if there is a vertex $t \in N_G(\{p, q\})$ different from u and y, then $\{\{p, q, t\}, \{u, y, p\}, \{u, y, q\}\}$ is a T covering u, a contradiction. On the other hand, as $\delta_2(G) \geq 2$, we have $N_G(\{p, q\}) = \{u, y\}$. A direct corollary is that $G[\{u, y, p, q\}] = K_4^3$. Moreover, any two vertices from $\{u, y, p, q\}$ have codegree 2 in G. Otherwise, we can find a T covering u, a contradiction.

Now consider the link graphs of vertices y, p, q, we denote them by G_y, G_p and G_q, respectively. Let G_a be the 3-graph obtained by deleting the vertices u, p, q (and related edges) from G_y, G_b be the 3-graph obtained by deleting the vertices u, y, q (and related edges) from G_p, and G_c be the 3-graph obtained by deleting the vertices u, y, p (and related edges) from G_q. As $\delta_2(G) \geq 2$ and any two vertices in $\{u, y, p, q\}$ have no co-neighbor out of $\{u, y, p, q\}$, we have $\delta_1(G_a) \geq 2$, $\delta_1(G_b) \geq 2$ and $\delta_1(G_c) \geq 2$. Actually, G_a, G_b and G_c are simple graphs defined on the same vertex set. As $n \in [5, 10]$, we have:

$$e(G_a) + e(G_b) + e(G_c) \geq \frac{2(n-4)}{2} 3 = 3(n-4) > \binom{n-4}{2}.$$

The inequality above implies that there must be at least one edge contained in at least two graphs of G_a, G_b and G_c. Without loss of generality, let $\{s, t\}$ be the common edge in G_a and G_b. As a result, we can find a $T = \{\{s, t, y\}, \{s, t, p\}, \{u, y, p\}\}$ covering u, a contradiction.

2.2.2 When $n \geq 11$

We first consider the case for $n \geq 11$ and $n - 1 \equiv 0 \pmod 3$. By Observation 2, we have $c_2(n, T) \geq 2$ for $n \geq 11$ and $n - 1 \equiv 0 \pmod 3$. Therefore, we only need to prove $c_2(n, T) \leq 2$ for $n \geq 11$ and $n - 1 \equiv 0 \pmod 3$. We suppose to the contrary that there is an n-vertex 3-graph G with $\delta_2(G) \geq 3$ for $n \geq 11$ and $n - 1 \equiv 0 \pmod 3$ and a vertex $u \in V(G)$ that is not covered by T.

Let y be any other vertex different from u in G. As $\delta_2(G) \geq 3$, we have $d_G(\{u, y\}) \geq 3$. Therefore, we can find two edges containing $\{u, y\}$. Let the two edges be $\{u, y, p\}$ and $\{u, y, q\}$. Considering $d_G(\{p, q\}) \geq 3$, we can find a vertex o different from u and y, such that $\{o, p, q\}$ forms an edge in G. Therefore, we find a generalized triangle T with the edge set $\{\{u, y, p\}, \{u, y, q\}, \{o, p, q\}\}$ covering u, a contradiction.

Next, we consider the case for $n \geq 11$ and $n - 1 \equiv 1, 2 \pmod 3$. By Observation 1, we have $c_2(n, T) \geq 1$ for $n \geq 11$ and $n - 1 \equiv 1, 2 \pmod 3$. Therefore, we only need to prove $c_2(n,T) \leq 1$ for $n \geq 11$ and $n-1 \equiv 1,2 \pmod 3$. We suppose to the contrary that there is an n-vertex 3-graph G with $\delta_2(G) \geq 2$ for $n \geq 11$ and $n-1 \equiv 1,2 \pmod 3$ and a vertex $u \in V(G)$ that is not covered by T.

Let $v_1 \in V(G)$ be a vertex in $V(G)$ different from u. As $\delta_2(G) \geq 2$, we have $d_G(\{u,v_1\}) \geq 2$ and $N_G(\{u,v_1\})$ has at least two vertices. Considering any two vertices v_2, v_3 in $N_G(\{u,v_1\})$, we have $N_G(\{v_2,v_3\}) \subseteq \{u,v_1\}$. Otherwise, if there is a vertex $h \in N_G(\{v_2,v_3\})$ different from u and v_1, then there is a generalized triangle T with the edge set $\{\{v_2,v_3,h\},\{u,v_1,v_2\},\{u,v_1,v_3\}\}$ covering u, a contradiction. On the other hand, as $\delta_2(G) \geq 2$, we have $N_G(\{v_2,v_3\}) = \{u,v_1\}$. Actually, we find $G[\{u,v_1,v_2,v_3\}]$ is a complete 3-graph on 4 vertices. Besides, any two vertices in $\{u,v_1,v_2,v_3\}$ have codegree 2, which means any two vertices in $\{u,v_1,v_2,v_3\}$ have no co-neighbor out of $\{u,v_1,v_2,v_3\}$. Otherwise, we can find a T covering u, a contradiction.

Let v_4 be a vertex in $V(G)$ different from v_1, v_2, v_3 and u. As $\delta_2(G) \geq 2$, we have $d_G(\{u,v_4\}) \geq 2$ and $N_G(\{u,v_4\})$ has at least two vertices. Let v_5 and v_6 be any two vertices in $N_G(\{u,v_4\})$. The same as the above analysis, we have $N_G(\{v_5,v_6\}) = \{u,v_4\}$ and $G[\{u,v_4,v_5,v_6\}]$ is a complete 3-graph on 4 vertices. Continue to consider other vertices in this way. There must exist a lot of 3-vertex sets: $T_1 = \{v_1,v_2,v_3\}$, $T_2 = \{v_4,v_5,v_6\}$,...,$T_l = \{v_{3l-2},v_{3l-1},v_{3l}\}$, such that edges induced in the vertex set $T_i \cup \{u\}$ form a complete 3-graph on 4 vertices for $i \in [1,l]$. Apart from these $3l+1$ vertices, there are one or two vertices left since $n-1 \equiv 1,2 \pmod 3$.

- If there is exactly one vertex left, let it be a. As $\delta_2(G) \geq 2$, we have $d_G(\{u,a\}) \geq 2$ and $N_G(\{u,a\})$ has at least two vertices. For the vertex in $N_G(\{u,a\})$, it must be a vertex in a T_i for $i \in [1,l]$. Without loss of generality, let v_{3i} in T_i be a vertex from $N_G(\{u,a\})$, which means $\{v_{3i},u,a\}$ is an edge in G. Then we find a generalized triangle T with the edge set $\{\{v_{3i},u,a\},\{v_{3i-2},v_{3i-1},v_{3i}\},\{v_{3i-2}, v_{3i-1},u\}\}$ covering u, a contradiction.

- If there are two vertices left, let them be b and c. As $\delta_2(G) \geq 2$, we have $d_G(\{u,b\}) \geq 2$ and $N_G(\{u,b\})$ has at least two vertices. For the vertex in $N_G(\{u,b\}) \setminus \{c\}$, it must be a vertex in a T_i for $i \in [1,l]$. Then through the same analysis as the case before for one vertex left, we can find a T covering u, a contradiction.

Therefore, we have $c_2(n,T) = 1$ for $n \geq 11$ *and* $n-1 \equiv 1,2 \pmod 3$.

2.3 The proof of Theorem 2

2.3.1 The proof of (i)

We can directly get the lower bound of $c_1(n, T)$ from Observation 3. Therefore, it is sufficient to show that every 3-graph G on n vertices with $\delta_1(G) > \frac{n^2}{6} + \frac{5}{6}n - 3$ has a T-covering. Suppose to the contrary that there is an n-vertex 3-graph G with $\delta_1(G) > \frac{n^2}{6} + \frac{5}{6}n - 3$ and a vertex $u \in V(G)$ that is not covered by T.

Consider an edge $e = \{u, x, y\}$ containing u in G. Let G_x, G_y and G_u be the link graphs of x, y and u, respectively. Let $G_x^{'}$ be the 3-graph obtained by deleting the vertices u, y (and related edges) from G_x, $G_y^{'}$ be the 3-graph obtained by deleting the vertices u, x (and related edges) from G_y and $G_u^{'}$ be the 3-graph obtained by deleting the vertices x, y (and related edges) from G_u. Then $G_x^{'}$, $G_y^{'}$ and $G_u^{'}$ are simple graphs defined on the same $n-3$ vertices. As

$$\delta_1(G) > \frac{n^2}{6} + \frac{5}{6}n - 3,$$

we have

$$e(G_x^{'}) + e(G_y^{'}) + e(G_u^{'}) > 3 \cdot (\frac{n^2}{6} + \frac{5}{6}n - 3 - 2(n-3) - 1) > \binom{n-3}{2},$$

which means that there must be a common edge in at least two of $G_x^{'}$, $G_y^{'}$ and $G_u^{'}$. Without loss of generality, let $\{s, t\}$ be the common edge in both $G_x^{'}$ and $G_y^{'}$. As a result, we can find a $T = \{\{s, t, x\}, \{s, t, y\}, \{u, x, y\}\}$ covering u, a contradiction.

2.3.2 The proof of (ii)

Let G be a 3-graph with $\delta_1(G) > \frac{n^2}{6} + \frac{5}{6}n - 3$. We will prove for any vertex u, there is a generalized triangle T^1 or T^2 covering u.

For any vertex $u \in V(G)$, as $d_G(\{u\}) > 1$, there is an edge $\{u, x, y\}$ containing u. Let G_x, G_y and G_u be the link graphs of x, y and u, respectively. Let $G_x^{'}$ be the 3-graph obtained by deleting the vertices u, y (and related edges) from G_x, $G_y^{'}$ be the 3-graph obtained by deleting the vertices u, x (and related edges) from G_y, and $G_u^{'}$ be the 3-graph obtained by deleting the vertices x, y (and related edges) from G_u. Then $G_x^{'}$, $G_y^{'}$ and $G_u^{'}$ are simple graphs defined on the same $n-3$ vertices. As

$$\delta_1(G) > \frac{n^2}{6} + \frac{5}{6}n - 3,$$

we have

$$e(G_x^{'}) + e(G_y^{'}) + e(G_u^{'}) > 3 \cdot (\frac{n^2}{6} + \frac{5}{6}n - 3 - 2(n-3) - 1) > \binom{n-3}{2},$$

which means that there must be a common edge in at least two of $G_x^{'}$, $G_y^{'}$ and $G_u^{'}$. If there is an edge $\{s,t\}$ in both $G_x^{'}$ and $G_y^{'}$, we can find a $T^1 = \{\{s,t,x\},\{s,t,y\},\{u,x,y\}\}$ covering u. If there is an edge $\{p,q\}$ in both $G_x^{'}$ and $G_u^{'}$, we can find a $T^2 = \{\{p,q,x\},\{p,q,u\},\{x,y,u\}\}$ covering u. If there is an edge $\{g,h\}$ in both $G_y^{'}$ and $G_u^{'}$, we can find a $T^2 = \{\{g,h,y\},\{g,h,u\},\{u,x,y\}\}$ covering u.

In conclusion, if G is a 3-graph satisfying that $\delta_1(G) > \frac{n^2}{6} + \frac{5}{6}n - 3$, then for any vertex $u \in V(G)$, we can find a generalized triangle T^1 or T^2 covering u.

2.3.3 The proof of (iii)

Let G be a 3-graph with $\delta_1(G) > \frac{n^2}{4} + \frac{1}{4}n - 2$. We will prove for any vertex $u \in V(G)$, there are generalized triangles T^1 and T^2 covering u.

Since $d_G(\{u\}) > 1$, there is an edge $\{u,x,y\}$ containing u. Let G_x, G_y and G_u be the link graphs of x, y and u, respectively. Let $G_x^{'}$ be the 3-graph obtained by deleting the vertices u,y (and related edges) from G_x, $G_y^{'}$ be the 3-graph obtained by deleting the vertices u,x (and related edges) from G_y, and $G_u^{'}$ be the 3-graph obtained by deleting the vertices x,y (and related edges) from G_u. Then $G_x^{'}$, $G_y^{'}$ and $G_u^{'}$ are simple graphs defined on the same $n-3$ vertices. As

$$\delta_1(G) > \frac{n^2}{4} + \frac{1}{4}n - 2,$$

we have

$$e(G_x^{'}) + e(G_y^{'}) > 2 \cdot (\frac{n^2}{4} + \frac{1}{4}n - 2 - 2(n-3) - 1) > \binom{n-3}{2};$$

$$e(G_x^{'}) + e(G_u^{'}) > 2 \cdot (\frac{n^2}{4} + \frac{1}{4}n - 2 - 2(n-3) - 1) > \binom{n-3}{2};$$

which means that there must be a common edge in both $G_x^{'}$ and $G_y^{'}$ and a common edge in both $G_x^{'}$ and $G_u^{'}$. Without loss of generality, let $\{s,t\}$ be the common edge in $G_x^{'}$ and $G_y^{'}$ and $\{p,q\}$ be the common edge in $G_x^{'}$ and $G_u^{'}$. As a result, we can find a $T^1 = \{\{s,t,x\},\{s,t,y\},\{u,x,y\}\}$ covering u and a $T^2 = \{\{p,q,x\},\{p,q,u\},\{x,y,u\}\}$ covering u.

In conclusion, if G is a 3-graph satisfying that $\delta_1(G) > \frac{n^2}{4} + \frac{1}{4}n - 2$, then for any vertex $u \in V(G)$, there are generalized triangles T^1 and T^2 covering u.

2.3.4 The proof of (iv)

Let G be a 3-graph with $\delta_1(G) > \frac{\sqrt{5}-1}{4}n^2 + O(n)$.

We first prove that for any vertex u in $V(G)$, there is a generalized triangle T^3 covering u. Suppose to the contrary that there is an n-vertex 3-graph G with $\delta_1(G) > \frac{\sqrt{5}-1}{4}n^2 + O(n)$ and a vertex $u \in V(G)$ that is not covered by a T^3.

Let G_u be the link graph of u. For any vertex $x \in V(G_u)$, let $N = N_{G_u}(\{x\})$ be the neighbor set of x in G_u. Considering any vertex $x \in V(G_u)$ with $d_{G_u}(\{x\}) \geq 2$, let y, z be any two vertices in N. As there is no T^3 covering u, for any vertex $f \in V(G_u) - \{x, y, z\}$, we have $\{y, z, f\}$ is not an edge in G. We call these triples like $\{y, z, f\}$ as the non-edges. Considering the fact that every vertex $x \in V(G_u)$ with $d_{G_u}(\{x\}) \geq 2$ is contained in no T^3, we collect the triples of these non-edges for every vertex $x \in V(G_u)$. And we denote the multiset of the non-edges by M. Actually, every non-edge in M can be repeated at most $3 \cdot \Delta_{G_u}$ times, where Δ_{G_u} denotes the maximum degree of G_u. As a result, there are at least m non-edges in G, in which $m \geq \frac{|M|}{3 \cdot \Delta_{G_u}}$. Counting the size of M, we have

$$\begin{aligned}
|M| &= \sum_{x \in V(G_u), d_{G_u}(\{x\}) \geq 2} (n-4)\binom{d_{G_u}(\{x\})}{2} \\
&= \frac{n-4}{2} \cdot \sum_{x \in V(G_u), d_{G_u}(\{x\}) \geq 2} (d_{G_u}^2(\{x\}) - d_{G_u}(\{x\})) \\
&= \frac{n-4}{2} \cdot \sum_{x \in V(G_u), d_{G_u}(\{x\}) \geq 2} ((d_{G_u}(\{x\}) - \frac{1}{2})^2 - \frac{1}{4}) \\
&\geq \frac{n-4}{2} \cdot \sum_{x \in V(G_u), d_{G_u}(\{x\}) \geq 2} (d_{G_u}(\{x\}) - \frac{1}{2})^2 - \frac{n \cdot (n-4)}{8}.
\end{aligned}$$

By handshaking theorem, we have:

$$\begin{aligned}
\sum_{v \in V(G_u), d_{G_u}(\{x\}) \geq 2} d_{G_u}(\{x\}) &\geq 2 \cdot e(G_u) - n \\
&\geq 2 \cdot \delta_1(G) - n.
\end{aligned}$$

And by Hölder inequality, we have:

$$\begin{aligned}
\sum_{x \in V(G_u), d_{G_u}(\{x\}) \geq 2} (d_{G_u}(\{x\}) - \frac{1}{2})^2 &\geq \frac{(\sum_{x \in V(G_u), d_{G_u}(\{x\}) \geq 2} (d_{G_u}(\{x\}) - \frac{1}{2}))^2}{\sum_{x \in V(G_u), d_{G_u}(\{x\}) \geq 2} 1} \\
&\geq \frac{(2 \cdot \delta_1(G) - n - \frac{1}{2} \cdot n)^2}{n} \\
&= \frac{4 \cdot \delta_1(G)^2 - 6 \cdot n \cdot \delta_1(G) + \frac{9}{4} \cdot n^2}{n}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
m &\geq \frac{\frac{n-4}{2}\cdot \sum\limits_{x\in V(G_u), d_{G_u}(\{x\})\geq 2} (d_{G_u}(\{x\})-\frac{1}{2})^2 - \frac{n\cdot(n-4)}{8}}{3n}\\
&\geq \frac{n-4}{2\cdot 3n}\cdot\frac{4\cdot\delta_1(G)^2-6\cdot n\cdot\delta_1(G)+\frac{9}{4}\cdot n^2}{n}-\frac{n\cdot(n-4)}{8\cdot 3n}\\
&\geq \frac{2(n-4)}{3n^2}\delta_1^2-\frac{n-4}{n}\delta_1-\frac{n-4}{3}. \qquad (1)
\end{aligned}
$$

On the other hand, as $e(G_u)=d_1(\{u\})\geq\delta_1(G)$, there are at most m non-edges in G, in which

$$m\leq\binom{n}{3}-\frac{\delta_1(G)\cdot n}{3}. \qquad (2)$$

As $\delta_1(G)>\frac{\sqrt{5}-1}{4}n^2+O(n)$, we have a contradiction between (1) and (2), which means that every vertex in G can be covered by a T^3 if $\delta_1(G)>\frac{\sqrt{5}-1}{4}n^2+O(n)$. As

$$\frac{\sqrt{5}-1}{4}n^2+O(n)>\frac{n^2}{4}+\frac{1}{4}n-2,$$

combining with the proof of (iii) we have for any vertex in G, we can find generalized triangles T^1 and T^2 covering it if $\delta_1(G)>\frac{\sqrt{5}-1}{4}n^2+O(n)$.

In conclusion, if G is a 3-graph satisfying that $\delta_1(G)>\frac{\sqrt{5}-1}{4}n^2+O(n)$, then for any vertex $u\in G$, there are generalized triangles T^1, T^2 and T^3 covering u.

3 Results on some trees covering

3.1 Star covering

3.1.1 Construction

Construction 4. *Let V_4 be an n-vertex set with $n\geq 8$ and E_4 be a 3-element set. Let A be a 4-vertex subset of V_4 and B be the remain vertex set $V\setminus A$. Let $E_4=\binom{A}{3}\cup\binom{B}{3}$. Let $G_4=(V_4,E_4)$ be a 3-graph. Actually, we have $G_4=K_4^3\cup K_{n-4}^3$.*

Observation 4. *G_4 is a 3-graph with $\delta_1(G_4)=3$ and there is no P_2 in G_4 covering vertices in A.*

Proof. As $|A|=4$, there is no P_2 in the induced graph $G[A]$. And A and B are disconnected, so there is no P_2 covering the vertices in A.

Now we prove that $\delta_1(G_4)=3$. Let v be a vertex in $V(G_4)$.

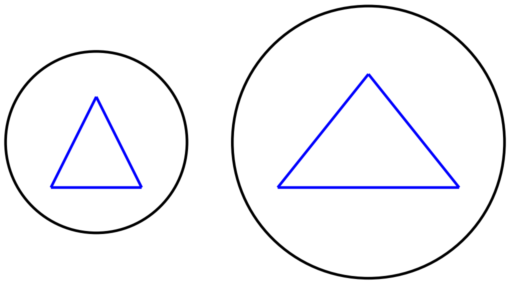

Figure 7: Construction 4

If $v \in A$, then

$$d_{G_4}(\{v\}) = \binom{4-1}{2} = 3.$$

If $v \in B$, as $n \geq 8$ we have

$$d_{G_4}(\{v\}) = \binom{n-5}{2} \geq 3.$$

Therefore, we have $\delta_1(G_4) = 3$.

∎

Construction 5. *Let V_5 be a vertex set. Fix two vertices $u, v_0 \in V_5$, let $V_5' = V_5 \setminus \{u, v_0\}$ and $E_5 = \{u, v_0\} \vee \binom{V_5'}{1} \cup \{v_0\} \vee \binom{V_5'}{2}$. Let $G_5 = (V_5, E_5)$ be a 3-graph.*

Observation 5. *G_5 is a 3-graph with $\delta_2(G_5) = 1$. For the vertex u, we can not find 4 vertices p, q, s, t such that $\{u, p, q\}$ and $\{u, s, t\}$ form a linear 2-path P_2 covering u.*

Proof. Considering any two vertices $v_1, v_2 \in V(G_5)$, we have:

- If $v_1 = u$ and $v_2 = v_0$, we have $d_{G_5}(\{u, v_0\}) \geq 1$.
- If $v_1 = u$ and $v_2 \in V_5'$, we have $d_{G_5}(\{u, v_2\}) = 1$.
- If $v_1 = v_0$ and $v_2 \in V_5'$, we have $d_{G_5}(\{u, v_0\}) \geq 1$.
- If $v_1, v_2 \in V_5'$, we have $d_{G_5}(\{u, v_0\}) = 1$.

Hence, we have $\delta_2(G_5) = 1$.

Let G_u be the link graph of u. As G_u is a star, we can not find 4 vertices p, q, s, t where $\{u, p, q\}$ and $\{u, s, t\}$ form a linear 2-path P_2 covering u. ∎

Construction 6. *Let V_6 be a vertex set. Fix three vertices u, a, b in V_6, let V_6' be $V_6 \backslash \{u, a, b\}$. Let $E_6 = \{u, a, b\} \cup \{u, a\} \vee \binom{V_6'}{1} \cup \{u, b\} \vee \binom{V_6'}{1} \cup \binom{V_6 \backslash \{u\}}{3}$. Let $G_6 = (V_6, E_6)$ be a 3-graph.*

Observation 6. *G_6 is a 3-graph with $\delta_2(G_6) = 2$ and there is no S_3 with the center u covering u.*

Proof. Considering any two vertices $v_1, v_2 \in V_6$, we have:

- If $v_1 = u$ and $v_2 = a$, we have $d_{G_6}(\{u, a\}) \geq 2$.
- If $v_1 = u$ and $v_2 = b$, we have $d_{G_6}(\{u, b\}) \geq 2$.
- If $v_1 = u$ and $v_2 \in V_6'$, we have $d_{G_6}(\{u, v_2\}) = 2$.
- If $v_1 \in V_6 \setminus \{u\}$ and $v_2 \in V_6 \setminus \{u\}$, we have $d_{G_6}(\{v_1, v_2\}) \geq 2$.

Hence, we have $\delta_2(G_6) = 2$. Now we only need to prove there is no S_3 with the center u.

Let G_u be the link graph of u. Then we find G_u is the book graph which has no 3-matching. Hence there is no S_3 with the center u. ∎

3.1.2 The proof of Theorem 3

As $c_2(n, P_2) \geq 0$, we only need to prove $c_2(n, P_2)$ could not be larger than 0. Suppose to the contrary that there is a 3-graph G with $\delta_2(G) \geq 1$ and a vertex $u \in V(G)$ that is not covered by P_2.

Let v_1 be a vertex different from u in G. As $d(\{u, v_1\}) \geq 1$, there is a vertex v_2 making $\{u, v_1, v_2\}$ being an edge in G. Consider another vertex v_3 in G, as $d(\{u, v_3\}) \geq 1$ and there is no P_2 containing u, we have $N_G(\{u, v_3\}) \subseteq \{v_1, v_2\}$.

If $d(\{u, v_3\}) > 1$, we have $N_G(\{u, v_3\}) = \{v_1, v_2\}$. Let v_4 be a vertex in $V(G) \setminus \{u, v_1, v_2, v_3\}$. As $d(\{v_1, v_4\}) \geq 1$ and there is no P_2 containing u, we have $N_G(\{v_1, v_4\}) \subseteq \{u, v_2, v_3\}$. If $u \in N_G(\{v_1, v_4\})$, then we find a P_2 with the edge set $\{\{u, v_2, v_3\}, \{u, v_1, v_4\}\}$ containing u, a contradiction. If $v_2 \in N_G(\{v_1, v_4\})$, then we find a P_2 with the edge set $\{\{u, v_2, v_3\}, \{v_1, v_2, v_4\}\}$ containing u, a contradiction. If $v_3 \in N_G(\{v_1, v_4\})$, then we find a P_2 with the edge set $\{\{u, v_2, v_3\}, \{v_1, v_3, v_4\}\}$ containing u, a contradiction. As $N_G(\{v_1, v_4\}) \neq \emptyset$, we have a contradiction with $N_G(\{v_1, v_4\}) \subseteq \{u, v_2, v_3\}$.

If $d(\{u, v_3\}) = 1$, without loss of generality, let $N_G(\{u, v_3\}) = \{v_1\}$. Let v_5 be a vertex in $V(G) \setminus \{u, v_1, v_2, v_3\}$. As $d(\{v_3, v_5\}) \geq 1$ and there is no P_2 containing u, we have $N_G(\{v_3, v_5\}) \subseteq \{u, v_1, v_2\}$. If $u \in N_G(\{v_3, v_5\})$, then we find a P_2 with the edge set $\{\{u, v_3, v_5\}, \{u, v_1, v_2\}\}$ containing u, a contradiction. If $v_1 \in N_G(\{v_3, v_5\})$, then we find a P_2 with the edge set $\{\{u, v_1, v_2\}, \{v_1, v_3, v_5\}\}$ containing u, a contradiction. If $v_2 \in N_G(\{v_3, v_5\})$, then we find a P_2 with the edge set $\{\{u, v_1, v_3\}, \{v_2, v_3, v_5\}\}$ containing u, a contradiction. As $N_G(\{v_3, v_5\}) \neq \emptyset$, we have a contradiction with $N_G(\{v_3, v_5\}) \subseteq \{u, v_1, v_2\}$.

In conclusion, we have $c_2(n, P_2) = 0$.

3.1.3 The proof of Theorem 4

The lower bound of $c_2(n, P_2)$ can be directly get from Observation 4. Therefore, we only need to prove $c_2(n, T) \leq 3$ when $n \geq 8$. Suppose to the contrary that there is a 3-graph G with $\delta_1(G) \geq 4$ and a vertex $u \in V(G)$ that is not covered by P_2. Let G_u be the link graph of u.

As there is no P_2 covering u, there is no subgraph G' with the vertex set $\{a, b, c, d\}$ and the edge set $\{\{a, b\}, \{c, d\}\}$ in G_u. Otherwise, we can find a P_2 with the edge set $\{\{a, b, u\}, \{c, d, u\}\}$ covering u. Cause $\delta_1(G) \geq 4$, we have there are at least 4 edges in G_u. Therefore, there must be a star S with the vertex set $\{v_1, v_2, v_3\}$ and the edge set $\{\{v_1, v_2\}, \{v_2, v_3\}\}$ in G_u. We claim that $N_G(\{v_1\}) \subseteq \{uv_2, uv_3, v_2v_3\}$. Otherwise, if there are two vertices x, y in $V(G) \setminus \{v_1, v_2, v_3\}$ making $\{xy\} \in N_G(\{v_1\})$, then we find a P_2 with the edge set $\{\{u, v_1, v_2\}, \{v_1, x, y\}\}$ covering u, a contradiction. If there is a vertex s in $V(G) \setminus \{v_1, v_2, v_3\}$ making $\{su\} \in N_G(\{v_1\})$, then we find a P_2 with the edge set $\{\{u, v_1, s\}, \{u, v_2, v_3\}\}$ covering u, a contradiction. If there is a vertex t in $V(G) \setminus \{v_1, v_2, v_3\}$ making $\{tv_2\} \in N_G(\{v_1\})$, then we find a P_2 with the edge set $\{\{t, v_1, v_2\}, \{u, v_2, v_3\}\}$ covering u, a contradiction. If there is a vertex q in $V(G) \setminus \{v_1, v_2, v_3\}$ making $\{qv_3\} \in N_G(\{v_1\})$, then we find a P_2 with the edge set $\{\{q, v_1, v_3\}, \{u, v_1, v_2\}\}$ covering u, a contradiction. Therefore, we have $N_G(\{v_1\}) \subseteq \{uv_2, uv_3, v_2v_3\}$. However, that contradicts $\delta_1(G) \geq 4$.

In conclusion, we have $c_1(n, P_2) = 3$ when $n \geq 8$.

3.1.4 The proof of Theorem 5

Suppose to the contrary that there is a 3-graph G with $\delta_2(G) \geq 2$ and a vertex u, such that there is no 4 vertices p, q, s, t making $\{u, p, q\}$ and $\{u, s, t\}$ forming a linear 2-path P_2 covering u. Let G_u be the link graph of u.

As $\delta_2(G) \geq 2$, we have $\delta(G_u) \geq 2$. Since there is no 4 vertices p, q, s, t such that $\{u, p, q\}$ and $\{u, s, t\}$ form a linear 2-path P_2 covering u, G_u has no 2-matching. We claim that G_u has only one component. Otherwise, as $\delta(G_u) \geq 2$, there is a

2-matching in G_u. As $|V(G_u)| = n - 1$ and $n \geq 5$, we have G_u must be a star. However, the leaves in G_u only have degree 1, a contradiction with $\delta(G_u) \geq 2$.

Hence, if G is an n-vertex 3-graph satisfying that $n \geq 5$ and $\delta_2(G) \geq 2$, then for any vertex $u \in V(G)$, we can find 4 vertices p, q, s, t such that $\{u, p, q\}$ and $\{u, s, t\}$ form a linear 2-path P_2 covering u.

Furthermore, by Observation 5, we have the inequality in Theorem 5 is sharp.

3.1.5 The proof of Theorem 6

For a positive integer t, the book graph B_t is the graph obtained by the amalgamation of t triangles along the same edge. Let B_t^- be the graph obtained by deleting the common edge from the book graph B_t. Before we prove Theorem 6, we explore the structure of graphs without some specific matchings and obtain the following result.

Theorem 11. *Let G be an n-vertex simple graph with $n \geq 7$. If G has no 3-matching and $\delta(G) \geq 2$, then G be the book graph B_{n-2} or the graph B_{n-2}^-.*

Proof of Theorem 11. Consider the components of G.

If G has more than 2 components, as $\delta(G) \geq 2$, every component has no isolated vertices. Hence we can choose one edge in each component and then we can find at least three vertex-disjoint edges, a contradiction with G has no 3-matching.

If G has 2 components, let them be G_1, G_2. As $\delta(G) \geq 2$, every component has no isolated vertices. We can choose one edge in G_1. As there is no 3-matching in G, there is no 2-matching in G_2. Hence G_2 must be a 3-cycle or a star. If G_2 is a star, then the leaves of G_2 only have degree 1, a contradiction with $\delta(G) \geq 2$. If G_2 be a 3-cycle, as $n \geq 7$, G_1 has at least 4 vertices. And G_1 must be a star because G_1 also can not have 2-matching. Hence the leaves of G_1 only have degree 1, a contradiction with $\delta(G) \geq 2$.

Now we only need to consider the case that G is a connected graph. We claim that G must have a cycle. Otherwise, let P be the longest path in G with endpoints u, v. As $\delta(G) \geq 2$, u must have at least two neighbors. Since P is the longest path, the neighbors of u must in $V(P)$, which makes a cycle in G, a contradiction.

Now let C be the longest cycle in G. Consider the length of C.

- If the length of C is more than 5, then there exists a 3-matching in G, a contradiction with G has no 3-matching.

- If the longest cycle C is a 5-cycle, let the vertex set $V(C)$ be $\{v_1, v_2, v_3, v_4, v_5\}$ and the edge set $E(C)$ be $\{\{v_1v_2\}, \{v_2v_3\}, \{v_3v_4\}, \{v_4v_5\}, \{v_5v_1\}\}$. As G is a connected n-vertex graph and $n \geq 7$, there is at least one vertex in $V(C)$ sending at least one edge to $V(G) \setminus V(C)$. Without loss of generality, let v_1 send one edge to $v_6 \in V(G) \setminus V(C)$. Then we find a 3-matching $\{\{v_1v_6\}, \{v_2v_3\}, \{v_4v_5\}\}$, a contradiction.

- If the longest cycle C is a 3-cycle, let the vertex set $V(C)$ be $\{v_1, v_2, v_3\}$ and the edge set $E(C)$ be $\{\{v_1v_2\}, \{v_2v_3\}, \{v_3v_1\}\}$. As G is connected, there must be some vertices in $\{v_1, v_2, v_3\}$ sending edges into $V(G) \setminus \{v_1, v_2, v_3\}$. Without loss of generality, let v_1 adjacent to $v_4 \in V(G) \setminus \{v_1, v_2, v_3\}$. As $\delta(G) \geq 2$, we have v_4 has at least 2 neighbors. If v_4 is adjacent to v_2 or v_3, then there is a 4-cycle in G, a contradiction with the longest cycle is a 3-cycle. Hence v_4 has a neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4\}$, let it be v_5. Since $d(v_5) \geq 2$, v_5 has at least one neighbor other than v_4. If v_5 has a neighbor in $\{v_2, v_3\}$, then there is a 4-cycle in G, a contradiction with the longest cycle is 3-cycle. If v_5 has a neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, let it be v_6, then there is a 3-matching $\{\{v_2v_3\}, \{v_1v_4\}, \{v_5v_6\}\}$ in G, a contradiction with G has no 3-matching. If v_1 is a neighbor of v_5, then there must be some vertices in $\{v_1, v_2, v_3, v_4\}$ sending edges into $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$ since G is connected. If v_1 is adjacent to $v_7 \in V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then there is a 3-matching $\{v_1v_7, v_2v_3, v_4v_5\}$, a contradiction. If there is a vertex in $\{v_2, v_3, v_4\}$ adjacent to $v_8 \in V(G)\setminus\{v_1, v_2, v_3, v_4, v_5\}$, let v_2 be adjacent to v_8, then there is a 3-matching $\{\{v_2v_8\}, \{v_1v_3\}, \{v_4v_5\}\}$, a contradiction.

Hence, the longest cycle C must be a 4-cycle. Let the vertex set $V(C)$ be $\{v_1, v_2, v_3, v_4\}$ and the edge set $E(C)$ be $\{\{v_1v_2\}, \{v_2v_3\}, \{v_3v_4\}, \{v_4v_1\}\}$.

As G is connected, there must be some vertices in $\{v_1, v_2, v_3, v_4\}$ sending edges into $V(G) \setminus \{v_1, v_2, v_3, v_4\}$. Without loss of generality, let v_5 be such a vertex that is adjacent to v_1. Since $\delta(G) \geq 2$, v_5 has at at least one neighbor other than v_1. If v_5 has a neighbor in $\{v_2, v_4\}$, then there is a 5 cycle, a contradiction with the longest cycle is a 4-cycle. If v_5 has a neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4\}$, let v_5 be adjacent to $v_6 \in V(G)\setminus\{v_1, v_2, v_3, v_4\}$. Then there is a 3-matching $\{\{v_1v_2\}, \{v_3v_4\}, \{v_5v_6\}\}$ in G, a contradiction. Hence $N_G(\{v_5\}) \subseteq \{v_1, v_3\}$. As $d(\{v_5\}) \geq 2$, we have $N_G(\{v_5\}) = \{v_1, v_3\}$. As G is connected, there must be some vertices in $\{v_1, v_2, v_3, v_4, v_5\}$ sending edges into $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. If there are some vertices in $\{v_2, v_4, v_5\}$ sending edges into $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, without loss of generality, let the vertex v_2 be adjacent to $v_6 \in V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. Then we find a 3-matching $\{\{v_1v_4\}, \{v_3v_5\}, \{v_2v_6\}\}$ in

G, a contradiction. Hence v_1 or v_3 must send edges into $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$.

Let M be the vertex set satisfying that every vertex in M is adjacent to v_1 or v_3. We claim that for every vertex $a \in M$ we have $N_G(\{a\}) \subseteq \{v_1, v_3\}$. Otherwise, if a is adjacent to v_1 and there is a vertex $b \in V(G) \setminus \{v_1, v_2, v_3, v_4, v_5, a\}$ such that $\{ab\}$ is an edge in G, then we will find a 3-matching $\{\{ab\}, \{v_1v_2\}, \{v_3v_4\}\}$ in G, a contradiction. As $\delta(G) \geq 2$, we have $N_G(\{a\}) = \{v_1, v_3\}$. Actually, we have $M = V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. Otherwise there is a contradiction with the fact that G is connected.

Now consider the adjacency of v_2 and v_4. If $\{v_2v_4\}$ is an edge in G, then we will find a 5-cycle with the edge set $\{\{v_2v_1\}, \{v_1v_5\}, \{v_5v_3\}, \{v_3v_4\}, \{v_4v_2\}\}$ in G, a contradiction with the fact that the longest cycle is a 4-cycle in G. Hence v_2 and v_4 are not adjacent. Considering the adjacency of v_1 and v_3, if $\{v_1v_3\}$ is an edge in G, then G is the book graph B_{n-2}. If $\{v_1v_3\}$ is not an edge in G, then G is the graph B^-_{n-2}.

In conclusion, if G has no 3-matching and $\delta(G) \geq 2$, then G is the book graph B_{n-2} or the graph B^-_{n-2}. ∎

Now we prove Theorem 6.

Proof of Theorem 6. Suppose to the contrary that there is an n-vertex 3-graph H with $\delta_2(H) \geq 2$ and a vertex u that is not covered by a S_3. Let G_u be the link graph of u. As $\delta_2(H) \geq 2$ and there is no S_3 covering u, then $\delta(G_u) \geq 2$ and there is no 3-matching in G_u. By Theorem 11, we have G_u is the book graph B_{n-3} or the graph B^-_{n-3}. Let A be the set of vertices with degree 2 in $V(B^-_{n-3})$. And let b_1 and b_2 be the remained two vertices in $V(B^-_{n-3}) \setminus A$.

As $\delta_2(H) \geq 2$, we have $d_H(\{b_1, b_2\}) \geq 2$ and $\{b_1, b_2\}$ has at least two co-neighbors. Even if $\{b_1b_2u\}$ is an edge in H, there still must be at least one co-neighbor of $\{b_1b_2\}$ in A. Without loss of generality, let $a_0 \in A$ and $\{b_1b_2a_0\}$ be an edge in H. Consider the vertex set $A \setminus \{a_0\}$. Let a_1, a_2 and a_3 be three different vertices in $A \setminus \{a_0\}$. If $\{a_1a_2\}$ has a co-neighbor b_1 or b_2, then we will find a S_3 with the edge set $\{\{b_1b_2a_0\}, \{b_1a_1a_2\}, \{b_1ua_3\}\}$ or $\{\{b_1b_2a_0\}, \{b_2a_1a_2\}, \{b_2ua_3\}\}$ covering u, a contradiction. Hence $N_H(\{a_1, a_2\}) \subseteq A$. Consider the induced graph $H[A \setminus \{a_0\}]$, as $\delta_2(H) \geq 2$ we have $\delta_2(H[A \setminus \{a_0\}]) \geq 1$. By Theorem 3, $H[A \setminus \{a_0\}]$ has a P_2 covering. Then there must be a P_2 in $H[A \setminus \{a_0\}]$. Without loss of generality, let P_2 with the vertex set $\{a_1, a_2, a_3, a_3, a_5\}$ and the edge set $\{\{a_1a_2a_3\}, \{a_1a_4a_5\}\}$ be a linear 2-path in $H[A \setminus \{a_0\}]$. As a result, we find a S_3 with the edge set $\{\{a_1a_2a_3\}, \{a_1a_4a_5\}, \{a_1b_1u\}\}$ covering u, a contradiction.

In conclusion, if H is an n-vertex 3-graph satisfying that $n \geq 7$ and $\delta_2(H) \geq 2$, then for any vertex $u \in V(H)$ there is a 3-star S_3 covering u. ∎

3.1.6 The proof of Theorem 7

Before we prove Theorem 7, we first prove a useful theorem as follows.

Theorem 12. *Let G be a simple graph and δ be the minimum degree of G with $\delta \geq 2$. Then G contains a cycle of length at least $\delta + 1$.*

Proof. Let P be the longest path in G with endpoints x and y. Then $N_G(x) \subseteq V(P)$. As $d(x) \geq \delta$, there is a cycle of length $\delta + 1$. ∎

Proof of Theorem 7. Suppose to the contrary that there is an n-vertex 3-graph H with $\delta_2(H) \geq 3$ and a vertex $u \in V(H)$ such that there is no S_3 with center u covering it. Let H_u be the link graph of u. Then we have $\delta(H_u) \geq 3$ and there is no 3-matching in H_u.

We claim that H_u must be a connected graph. Otherwise, if H_u has more than two components, then as $\delta(H_u) \geq 3$ we have there is no isolated vertices. Then selecting one edge in every component generates a 3-matching in H_u, a contradiction. If G_u has two components, then we claim there must be at least one component containing a 2-matching. Otherwise, as $\delta(H_u) \geq 3$ we have the two components can not be 3-cycles. As the two components have no 2-matching, the two components must be two stars, a contradiction with $\delta(H_u) \geq 3$. Hence selecting a 2-matching in such a component and an edge in another component will generate a 3-matching in H_u, a contradiction. Therefore, H_u must be a connected graph.

Also, we have the following claim.

Claim 1. *The longest cycle in H_u is a 4-cycle.*

Proof. As $\delta(H_u) \geq 3$, there is a cycle with length at least 4 in H_u by Theorem 12. We only need to prove there is no cycle with length more than 4 in H_u.

Firstly, there is no cycle with length more than 5. Otherwise, such cycle will generate a 3-matching in H_u, a contradiction. Secondly, there is no 5-cycle in H_u. Suppose to the contrary that there is a 5-cycle in H_u with the vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and the edge set $\{\{v_1v_2\}, \{v_2v_3\}, \{v_3v_4\}, \{v_4v_5\}, \{v_5v_1\}\}$. As H_u is connected, then there must be a vertex in $\{v_1, v_2, v_3, v_4, v_5\}$ send one edge to $V(H_u)\setminus\{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality, let v_1 be adjacent to $v_6 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. As a result, there is a 3-matching with the edge set $\{\{v_1v_6\}, \{v_2v_3\}, \{v_4v_5\}\}$ in H_u, a contradiction.

Therefore, the longest cycle in H_u is a 4-cycle.

∎

Let C_4 be a 4-cycle in H_u with the vertex set $\{v_1, v_2, v_3, v_4\}$ and the edge set $\{\{v_1v_2\}, \{v_2v_3\}, \{v_3v_4\}, \{v_4v_1\}\}$. As H_u is connected, then there must be a vertex in $\{v_1, v_2, v_3, v_4\}$ sending one edge to $V(H_u) \setminus \{v_1, v_2, v_3, v_4\}$. Without loss of generality, let v_1 be adjacent to $v_5 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4\}$. We find v_5 can not have a neighbor in $V(H_u) \setminus \{v_1, v_2, v_3, v_4\}$. Otherwise, let v_6 be a neighbor in $V(H_u) \setminus \{v_1, v_2, v_3, v_4\}$. Then we will find a 3-matching with the edge set $\{\{v_1v_2\}, \{v_3v_4\}, \{v_5v_6\}\}$ in H_u, a contradiction. As $\delta(H_u) \geq 3$, we have $N_{H_u}(v_5) \subset \{v_1, v_2, v_3, v_4\}$.

If $\delta(H_u) = 4$, then we have $N_{H_u}(v_5) = \{v_1, v_2, v_3, v_4\}$. As H_u is connected, there must be a vertex in $\{v_1, v_2, v_3, v_4, v_5\}$ sending edges to $V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality, let v_1 be adjacent to $v_6 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. Then we find there is a 3-matching with the edge set $\{\{v_1v_6\}, \{v_2v_3\}, \{v_4v_5\}\}$ in H_u, a contradiction.

If $\delta(H_u) = 3$, then we have v_5 has two neighbors in $\{v_2, v_3, v_4\}$. We first consider $N_{H_u}(v_5) = \{v_1, v_2, v_3\}$. As H_u is connected, there must be a vertex in $\{v_1, v_2, v_3, v_4\}$ sending edges to $V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. If v_1 is adjacent to $v_6 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then we will find a 3-matching with the edge set $\{\{v_1v_6\}, \{v_2v_5\}, \{v_3 v_4\}\}$ in H_u, a contradiction. If v_2 is adjacent to $v_7 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then we will find a 3-matching with the edge set $\{\{v_1v_5\}, \{v_2v_7\}, \{v_3v_4\}\}$ in H_u, a contradiction. If v_3 is adjacent to $v_8 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then we will find a 3-matching with the edge set $\{\{v_1v_4\}, \{v_2v_5\}, \{v_3v_8\}\}$ in H_u, a contradiction. If v_4 is adjacent to $v_9 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then we will find a 3-matching with the edge set $\{\{v_1v_5\}, \{v_2v_3\}, \{v_4v_9\}\}$ in H_u, a contradiction. Next we consider the case $N_{H_u}(v_5) = \{v_1, v_2, v_4\}$ and $N_{H_u}(v_5) = \{v_1, v_3, v_4\}$. By symmetry, we only need to consider $N_{H_u}(v_5) = \{v_1, v_2, v_4\}$. As H_u is connected, there must be a vertex in $\{v_1, v_2, v_3, v_4\}$ sending one edge to $V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$. If v_1 is adjacent to $v_6 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then we will find a 3-matching with the edge set $\{\{v_1v_6\}, \{v_2v_5\}, \{v_3v_4\}\}$ in H_u, a contradiction. If v_2 is adjacent to $v_7 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then we will find a 3-matching with the edge set $\{\{v_1v_5\}, \{v_2v_7\}, \{v_3v_4\}\}$ in H_u, a contradiction. If v_3 is adjacent to $v_8 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then we will find a 3-matching with the edge set $\{\{v_1v_4\}, \{v_2v_5\}, \{v_3 v_8\}\}$ in H_u, a contradiction. If v_4 is adjacent to $v_9 \in V(H_u) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, then we will find a 3-matching with the edge set $\{\{v_1v_5\}, \{v_2v_3\}, \{v_4v_9\}\}$ in H_u, a contradiction.

In conclusion, if H is an n-vertex 3-graph with $n \geq 7$ and $\delta_2(H) \geq 3$, then for any vertex $u \in V(H)$ we can find a S_3 with the center u.

Besides, by observation 6, we have the bound in Theorem 7 is sharp.

∎

3.1.7 The proof of Proposition 1

First we introduce a result about k-matching in extremal graph theory due to Erdős and Gallai [8] as follows.

Theorem 13. *[8] Let G be an n-vertex graph. If G has no k-matching, then* $e(G) \leq \max\{\binom{2k-1}{2}, \binom{n}{2} - \binom{n-k+1}{2}\}$.

Let k be an integer with $k \geq 3$. Then we prove Proposition 1.

Proof of (i) in Proposition 1. Let H be an n-vertex 3-graph satisfying that $n \geq 2k+1$ and $\delta_2(H) > \max\{\frac{4k^2-6k+2}{n-1}, k-2-\frac{k^2-nk}{n-1}\}$. Let u be a vertex in $V(H)$ and G_u be the link graph of u. As $\delta_2(H) > \max\{\frac{4k^2-6k+2}{n-1}, k-2-\frac{k^2-nk}{n-1}\}$, we have $\delta(G_u) > \max\{\frac{4k^2-6k+2}{n-1}, k-2-\frac{k^2-nk}{n-1}\}$. By handshaking theorem, we have:

$$\begin{aligned} e(G_u) &= \frac{1}{2} \sum_{x \in V(G_u)} d_{G_u}(\{x\}) \\ &\geq \frac{1}{2}(n-1) \cdot \delta(G_u) \\ &> \max\{\binom{2k-1}{2}, \binom{n}{2} - \binom{n-k+1}{2}\} \end{aligned}$$

Then by Theorem 13, we have there is a k-matching in G_u, which means there is a k-star S_k with the center u in H.

Hence, if H is an n-vertex 3-graph satisfying that $n \geq 2k+1$ and $\delta_2(H) > \max\{\frac{4k^2-6k+2}{n-1}, k-2-\frac{k^2-nk}{n-1}\}$, then for any vertex $u \in V(H)$ there is a 3-star S_3 covering u, where the center of S_3 is u. As a direct corollary, we have $c_2(n, S_k) \leq \max\{\frac{4k^2-6k+2}{n-1}, k-2-\frac{k^2-nk}{n-1}\}$, which ends our proof.

∎

Proof of (ii) in Proposition 1 . Let H be an n-vertex 3-graph satisfying that $n \geq 2k+1$ and $\delta_1(H) > \max\{\binom{2k-1}{2}, \binom{n-1}{2} - \binom{n-k}{2}\}$. Let u be a vertex in $V(H)$ and G_u be the link graph of u. As $\delta_1(H) > \max\{\binom{2k-1}{2}, \binom{n-1}{2} - \binom{n-k}{2}\}$, we have $e(H_u) > \max\{\binom{2k-1}{2}, \binom{n-1}{2} - \binom{n-k}{2}\}$. By Theorem 13, we have there is a k-matching in H_u, which means there is a S_k covering u. As a direct corollary, we have $c_1(n, S_k) \leq \max\{\binom{2k-1}{2}, \binom{n-1}{2} - \binom{n-k}{2}\}$, which ends our proof. ∎

3.2 Path covering

3.2.1 Construction

Construction 7. *Let V_7 be a vertex set. Fix $u \in V_7$, let $V' = V_7 \setminus \{u\}$ and $E_7 = \{u\} \vee \binom{V'}{2}$. Let $G_7 = (V_7, E_7)$ be a 3-graph.*

Observation 7. *G_7 is a 3-graph with $\delta_2(G_7) = 1$ and $\delta_1(G_7) = n - 2$. There is no P_3 covering u.*

Proof. Considering any two vertices $v_1, v_2 \in V_7$, we have:

- If $v_1 = u$ and $v_2 \in V'$, then $d_{G_7}(\{u, v_2\}) \geq 1$.
- If $v_1 \in V'$ and $v_2 \in V'$, then $d_{G_7}(\{v_1, v_2\}) = 1$.

Considering any vertex $v_0 \in V_7$, we have:

- If $v_0 = u$, then $d_{G_7}(\{u\}) = \binom{n-1}{2}$.
- If $v_0 \in V'$, then $d_{G_7}(\{v_0\}) = n - 2$.

Hence $\delta_2(G_7) = 1$ and $\delta_1(G_7) = n - 2$. Meanwhile, as all edges in G_7 intersect in u, it follows that there is no linear 3-path P_3 covering u.

∎

Construction 8. *For $k \geq 4$, let V_8 be a vertex set with size more than $2k + 1$. Let A be a $(k - 2)$-subset of V_8 and B be the remain vertex set. Let G_8 be the complete bipartite 3-graph with vertex set V_8 and edge set $E_8 = (\binom{A}{2} \vee \binom{B}{1}) \cup (\binom{A}{1} \vee \binom{B}{2})$.*

Observation 8. *For $k \geq 4$, we have $\delta_2(G_8) = k - 3$ and G_8 has no P_k covering.*

Proof. Considering any two vertices $v_1, v_2 \in V_8$, we have:

- If $v_1 \in A$ and $v_2 \in B$, then $d_{G_8}(\{v_1, v_2\}) = n - 2$. As $n \geq 2k + 1$, we have $d_{G_8}(\{v_1, v_2\}) \geq 2k - 1$.
- If v_1 and v_2 are vertices in A, then $d_{G_8}(\{v_1, v_2\}) = n - k + 2$. As $n \geq 2k + 1$, we have $d_{G_8}(\{v_1, v_2\}) \geq k + 3$.
- If v_1 and v_2 are vertices in B, then $d_{G_8}(\{v_1, v_2\}) = k - 3$.

Hence we have $\delta_2(G_8) = k-3$. Next we prove that G_8 has no P_k-covering.

Let P_l the longest linear path in G_8. If $l = k$, then let the vertex set of P_l be $\{v_0, v_1, v_2, ..., v_{2l-1}, v_{2l}\}$ and the edge set of P_l be $\{\{v_0v_1v_2\}, \{v_2v_3v_4\}, ..., \{v_{2l-2}v_{2l-1}v_{2l}\}\}$. We denote the vertex set $\{v_2, v_4, ..., v_{2l-2}\}$ by A' and the vertex set $\{v_0, v_1, ..., v_{2l-1}, v_{2l}\}$ by B'. Then $P_l = (A', B')$ is a bipartite 3-graph. As $l = k$, we have $|A| < |A'|$ and $|A| < |B'|$. Therefore, P_l cannot be a subgraph of G_8, a contradiction with the hypothesis. Hence $l < k$ and G_8 has no P_k covering. ∎

Construction 9. *Let k be an integer with $k \geq 3$. Let V_9 be an n-vertex set with $n \geq 4k$ and E_9 be a 3-element set. Let A be a $2k$-vertex subset of V_9 and B be the remain vertex set $V \setminus A$. Let $E_9 = \binom{A}{3} \cup \binom{B}{3}$. Let $G_9 = (V_9, E_9)$ be a 3-graph. Actually, we have $G_9 = K^3_{2k} \cup K^3_{n-2k}$.*

Observation 9. *G_9 is a 3-graph with $\delta_1(G_9) = \binom{2k-1}{2}$ and G_9 has no P_k-covering.*

Proof. As $|A| = 2k$, there is no P_k in the induced graph $G[A]$. And A and B are disconnected, so there is no P_k covering the vertices in A.

Now we prove that $\delta_1(G_9) = \binom{2k-1}{2}$. Let v be a vertex in $V(G_9)$.

If $v \in A$, then

$$d_{G_9}(\{v\}) = \binom{2k-1}{2}.$$

If $v \in B$, as $n \geq 4k$ we have

$$d_{G_9}(\{v\}) = \binom{n-2k}{2} \geq \binom{2k-1}{2}.$$

Therefore, we have $\delta_1(G_9) = \binom{2k-1}{2}$.

∎

3.2.2 The proof of Theorem 8

The lower bound of $c_2(n, P_3)$ is a direct corollary of Observation 7. Therefore, it is sufficient to show that every 3-graph H on n vertices with $\delta_2(H) \geq 2$ has a P_3-covering.

Suppose to the contrary that there is a 3-graph H on n vertices with $\delta_2(H) \geq 2$ and a vertex $u \in V(H)$ that is not contained in a copy of P_3. As $\delta_2(H) \geq 2$, by Theorem 5 we have for every vertex $v_0 \in V(H)$ there is a P_2 with the center v_0 covering it. Let P_2 be such a linear 2-path in H with the vertex set $\{u, v_1, v_2, v_3, v_4\}$ and the edge set $\{\{uv_1v_2\}, \{uv_3v_4\}\}$. We denote $V(H) \setminus \{u, v_1, v_2, v_3, v_4\}$ by A. Let v_5 and v_6

be any two vertices in A. Then any vertex in $\{v_1, v_2, v_3, v_4\}$ can not be the co-neighbor of $\{v_5, v_6\}$. Otherwise, without loss of generality let v_4 be a co-neighbor of $\{v_5, v_6\}$. We find there is a linear 3-path P_3 with the edge set $\{\{v_1v_2u\}, \{uv_3v_4\}, \{v_4v_5v_6\}\}$ covering u, a contradiction. Hence we have $N_H(\{v_5v_6\}) \subseteq A \cup \{u\}$. As $\delta_2(H) \geq 2$, there is at least one co-neighbor in A. Let $v_7 \in A$ be a co-neighbor of $\{v_5v_6\}$.

Consider the co-neighbors of $\{v_4v_5\}$. As v_4 is not the co-neighbor of $\{v_5, v_6\}$, we have $N_H(\{v_4v_5\}) \subseteq \{v_1, v_2, v_3, u\}$. If v_1 or v_2 is a co-neighbor of $\{v_4v_5\}$, then we assume without loss of generality that v_1 is a co-neighbor of $\{v_4v_5\}$. Then we find a linear 3-path P_3 with the edge set $\{\{uv_1v_2\}, \{v_1v_4v_5\}, \{v_5v_6v_7\}\}$ covering u, a contradiction. If neither v_1 nor v_2 is a co-neighbor of $\{v_4v_5\}$, then we have $N_H(\{v_4v_5\}) \subseteq \{v_3, u\}$. By Theorem 5 we have $\{v_1, v_2, v_4\}$ must be an edge in H. Otherwise, there is no P_2 with the center u. Hence we find a linear 3-path P_3 with the edge set $\{\{v_1v_2v_4\}, \{v_4uv_5\}, \{v_5v_6v_7\}\}$ covering u, a contradiction.

Therefore, we have $c_2(n, P_3) = 1$ for $n \geq 8$.

3.2.3 The proof of Theorem 9

We first prove a lemma before we prove Theorem 9.

Lemma 1. *If G is an n-vertex 3-graph satisfying that $n \geq 5$ and $\delta_1(G) \geq n-1$, then for any vertex $u \in V(G)$, we can find 4 vertices p, q, s, t where $\{u, p, q\}$ and $\{u, s, t\}$ form a linear 2-path P_2 covering u.*

Proof of the Lemma 1. Let u be any vertex in $V(G)$. Let G_u be the link graph of u. As $\delta_1(G) \geq n-1$, we have:

$$e(G_x) \geq n - 1 > \max\{\binom{4-1}{2}, \binom{n-1}{2} - \binom{n-2}{2}\}.$$

By Theorem 13, there is a 2 matching in G_u. We assume without loss of generality that $\{p, q\}$ and $\{s, t\}$ form a 2-matching in G_u. Then we find a P_2 with the edge set $\{\{u, p, q\}, \{u, s, t\}\}$ covering u, which ends our proof.

∎

Now we begin to prove Theorem 9.

Proof of Theorem 9. The lower bound of $c_1(n, P_3)$ is a direct corollary of Observation 7. Therefore, it is sufficient to show that every 3-graph H on n vertices with $n \geq 9$ and $\delta_1(H) \geq n + 5$ has a P_3-covering.

Suppose to the contrary that there is a 3-graph H on n vertices with $n \geq 9$ and $\delta_1(H) \geq n+5$ and a vertex $u \in V(H)$ that is not contained in a copy of P_3. As $\delta_1(H) \geq n-1$, by Lemma 1 we have there is a P_2 with the center u covering it. Let P_2 be such a linear 2-path in H with the vertex set $\{u, v_1, v_2, v_3, v_4\}$ and the edge set $\{\{uv_1v_2\}, \{uv_3v_4\}\}$. We denote the vertex set $V(H) \setminus \{u, v_1, v_2, v_3, v_4\}$ by A. Then any two vertices in A has no co-neighbor in $\{v_1, v_2, v_3, v_4\}$. Otherwise, without loss of generality we assume v_1 is a co-neighbor of $\{v_5v_6\}$ with $v_5, v_6 \in A$. Then there is a P_3 with the edge set $\{\{v_5v_6v_1\}, \{v_1v_2u\}, \{uv_3v_4\}\}$ covering u, a contradiction.

If there is a vertex v in $\{v_1, v_2, v_3, v_4\}$ such that $\{uv\}$ has a co-neighbor in A, then we assume without loss of generality that $v_5 \in A$ is the co-neighbor of $\{uv_1\}$. Let H_{v_5} be the link graph of v_5. As $\delta_1(H) \geq n+5$, we have $e(H_{v_5}) \geq n+5$. In H_{v_5}, there are at most $n-6$ edges between A and $\{u, v_1, v_2, v_3, v_4\}$. And $\{u, v_1, v_2, v_3, v_4\}$ can span at most $\binom{5}{2}$ edges. Hence there is at least one edge induced in A. Let it be $\{v_6v_7\}$. Then we find a P_3 with the edge set $\{\{uv_1v_5\}, \{uv_3v_4\}, \{v_5v_6v_7\}\}$ covering u, a contradiction.

If there is no vertex v in $\{v_1, v_2, v_3, v_4\}$ such that $\{uv\}$ has a co-neighbor in A, then there must exist $v_8, v_9 \in A$ such that $\{uv_8v_9\}$ is an edge in H as $\delta_1(H) \geq n+5 > \binom{4}{2}$. Let H_{v_1} be the link graph of v_1.

If there is a vertex a in A such that $\{a, v_3\}$ or $\{a, v_4\}$ is an edge in H_{v_1}, then we assume without loss of generality that $\{a, v_4\}$ is an edge in H_{v_1}. Considering the position of a in A, we have:

- If a is a different vertex form v_8 and v_9 in A, then we find a P_3 with the edge set $\{\{av_1v_4\}, \{v_4v_3u\}, \{uv_8v_9\}\}$ covering u, a contradiction.
- If a is v_8 or v_9, then we can assume a is v_8. Let H_{v_8} be the link graph of v_8. As $\delta_1(H) \geq n+5$, we have $e(H_{v_8}) \geq n+5$. In H_{v_8}, there are at most $n-6$ edges between A and $\{u, v_1, v_2, v_3, v_4\}$. And $\{u, v_1, v_2, v_3, v_4\}$ can span at most $\binom{5}{2}$ edges. Hence, in H_{v_8}, there is at least one edge induced in A. Let it be $\{v_{10}v_{11}\}$. Then we find a P_3 with the edge set $\{\{uv_1v_2\}, \{uv_8v_9\}, \{v_8v_{10}v_{11}\}\}$ covering u, a contradiction.

If there is no vertex a in A such that $\{a, v_3\}$ or $\{a, v_4\}$ is an edge in H_{v_1}, then $\{v_1v_3v_4\}$ must be an edge in H as there must be at least one P_2 with the center v_1. Then we find a P_3 with the edge set $\{\{v_8v_9u\}, \{uv_2v_1\}, \{v_1v_3v_4\}\}$ covering u, a contradiction.

In conclusion, we have $n-2 \leq c_2(n, P_3) \leq n+4$ for $n \geq 9$.

∎

3.2.4 The proof of Theorem 10

Let H be an n-vertex 3-graph with $n \geq 8$ and $\delta_2(H) \geq 3$. Let u be any vertex in $V(H)$. By Theorem 7, we have there is a S_3 with the center u covering u. Let $V(S_3)$ be $\{u, v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(S_3)$ be $\{\{uv_1v_2\}, \{uv_3v_4\}, \{uv_5v_6\}\}$. Let v_7 be a vertex in $V(H) \setminus \{u, v_1, v_2, v_3, v_4, v_5, v_6\}$. As $\delta_2(H) \geq 3$, we have $d_H(\{v_6v_7\}) \geq 3$. Hence there is at least one co-neighbor of $\{v_6v_7\}$ in $V(H) \setminus \{u, v_5, v_6, v_7\}$. If there is a co-neighbor of $\{v_6v_7\}$ in $V(H) \setminus \{u, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, let $v_8 \in V(H) \setminus \{u, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a co-neighbor of $\{v_6v_7\}$. Then we find a P_3 with the edge set $\{\{uv_1v_2\}, \{uv_5v_6\}, \{v_6v_7v_8\}\}$ covering u. If there is a co-neighbor of $\{v_6v_7\}$ in $\{v_1, v_2, v_3, v_4\}$, without loss of generality let v_1 be a co-neighbor of $\{v_6v_7\}$. Then we find a P_3 with the edge set $\{\{uv_3v_4\}, \{uv_5v_6\}, \{v_6v_7v_1\}\}$ covering u.

Hence, If H is an n-vertex 3-graph with $n \geq 8$ and $\delta_2(H) \geq 3$, then for any vertex $u \in V(H)$ we can find a P_3 with the vertex set $\{u, v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set $\{\{uv_1v_2\}, \{uv_3v_4\}, \{v_4v_5v_6\}\}$ covering u.

3.2.5 The proof of Proposition 2

We use the same method of Lemma 2.1 in [1] to prove (i) in Proposition 2.

Proof of (i) in Proposition 2. The lower bound of $c_2(n, P_k)$ is a direct corollary of Observation 8. Therefore, it is sufficient to show that $c_2(n, P_k) \leq 2k - 2$. We prove that if H is an n-vertex 3-graph with $n \geq 2k + 1$ and $\delta_2(H) \geq 2k - 1$, then for any vertex we can find a linear k-path P_k covering it.

For $k \geq 4$, we order the vertices of P_k as $x_0, x_1, x_2, ..., x_{2k-1}, x_{2k}$ such that the edge set of P_k is $\{\{x_0x_1x_2\}, \{x_3x_4x_5\}, ..., \{x_{2k}x_{2k-1}x_{2k}\}\}$. Let H be an n-vertex 3-graph with $n \geq 2k + 1$ and $\delta_2(H) \geq 2k - 1$. Fix a vertex v_0 in $V(H)$. We can find a copy of P_k by mapping x_0 to v_0, x_1 to any other vertex v_1 in $V(H)$, and x_2 to any $v_2 \in N_H(\{v_0v_1\})$. Suppose that $x_0, ..., x_{i-1}$ has been embedded to $v_0, ..., v_{i-1}$. Considering the embedding of x_i for $i \leq 2k$, if i is odd, then we embedded x_i to any vertex $v_i \in V(H) \setminus \{v_0, v_1, ..., v_{i-1}\}$. If i is even, as $\delta_2(H) \geq 2k - 1$, $\{v_{i-2}v_{i-1}\}$ has at least $2k - 1$ co-neighbors. Hence $\{v_{i-2}v_{i-1}\}$ has at least one co-neighbor in $V(H) \setminus \{v_0, v_1, ..., v_{i-1}\}$, let it be v_i. Then we embed x_i to v_i. Continuing this process, we obtain a copy of P_k when we embed the $2k + 1$ vertices.

Hence, if H is an n-vertex 3-graph with $n \geq 2k + 1$ and $\delta_2(H) \geq 2k - 1$, then for any vertex we can find a linear k-path P_k covering it. As a direct corollary, we have $c_2(n, P_k) \leq 2k - 2$ for $n \geq 2k + 1$ and $k \geq 4$.

∎

Proof of (ii) in Proposition 2. The lower bound of $c_1(n, P_k)$ is a direct corollary of Observation 7 and Observation 9. Therefore, it is sufficient to show that $c_1(n, P_k) \leq \binom{n-1}{2} - \binom{n-2k+1}{2}$. We prove that if H is an n-vertex 3-graph with $n \geq 4k$ and $\delta_1(H) \geq \binom{n-1}{2} - \binom{n-2k+1}{2} + 1$, then for any vertex we can find a linear k-path P_k covering it.

For any vertex u, suppose we have found a linear i-path P_i with $1 \leq i \leq k-1$ containing u. Now we prove that we can extend this P_i to P_{i+1}. We assume that the vertex set of P_i is $\{v_1, ..., v_{2i+1}\}$ and the edge set of P_i is $\{\{v_1v_2v_3\}, \{v_3v_4v_5\}, ..., \{v_{2i-1}v_{2i}v_{2i+1}\}\}$. As $\delta_1(H) \geq \binom{n-1}{2} - \binom{n-2k+1}{2} + 1$, we have $d_H(\{v_{2i+1}\}) \geq \binom{n-1}{2} - \binom{n-2k+1}{2} + 1$. Hence there must be two vertices v_{2i+2} and v_{2i+3} in $V(H) \setminus \{v_1, ..., v_{2i+1}\}$ such that $\{v_{2i+1}v_{2i+2}v_{2i+3}\}$ is an edge in H. Then we find a linear $(i+1)$-path P_{i+1} by adding $\{v_{2i+1}v_{2i+2}v_{2i+3}\}$ to P_i. When $i = k-1$, we can get a linear k-path P_k covering u.

Hence we have for $n \geq 4k$ and $k \geq 3$, $\max\{n-2, \binom{2k-1}{2}\} \leq c_1(n, P_k) \leq \binom{n-1}{2} - \binom{n-2k+1}{2}$.

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References

[1] V. Falgas-Ravry and Y. Zhao, Codegree thresholds for covering 3-uniform hypergraphs, SIAM Journal on Discrete Math., 30(4)(2016),1899-1917.

[2] V. Falgas-Ravry, K. Markström and Y. Zhao, Triangle-degrees in graphs and tetrahedron coverings in 3-graphs. Combinatorics, Probability and Computing, 30(2)(2021), 175-199.

[3] L. Yu, X. Hou, Y. Ma and B. Liu, Exact minimum codegree thresholds for K_4^--covering and K_5^--covering, The Electronic Journal of Combinatorics, 27(3)(2020), P3.22.

[4] Y. Tang, Y. Ma and X. Hou, The degree and codegree threshold for linear tringle covering in 3-graphs, arxiv:2212.03718.

[5] A. Freschi and A. T. Dirac-type results for tilings and coverings in ordered graphs. Forum of Mathematics, Sigma, Vol.10, Issue. 2022.

[6] Z. Füredi and Y. Zhao, Shadows of 3-Uniform Hypergraphs under a Minimum Degree Condition. SIAM Journal on Discrete Mathematics, 36(4)(2022), 2453-3057.

[7] C. Zhang, Matching and tilings in hypergraphs, PhD thesis, Georgia State University, 2016.

[8] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Mathematica Hungarica, 10(3)(1959), 337-356.