Spectral radius of graphs of given size with a forbidden fan graph F_6

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Abstract

Let $F_k = K_1 \vee P_{k-1}$ be the fan graph on k vertices. A graph is said to be F_k-free if it does not contain a copy of F_k as a subgraph. Yu, Li and Peng [Discrete Math. 348 (2025) 114391] conjectured that for $k \geq 2$ and m sufficiently large, if G is an F_{2k+1}-free or F_{2k+2}-free graph, then $\lambda(G) \leq \frac{k-1+\sqrt{4m-k^2+1}}{2}$ and the equality holds if and only if $G \cong K_k \vee \left(\frac{m}{k} - \frac{k-1}{2}\right) K_1$. Recently, Li, Zhao and Zou [arXiv:2409.15918] showed that the conjecture above holds for $k \geq 3$. The only left case is for $k = 2$, which corresponds to F_5 or F_6. Since the case of F_5 was solved by Zhang and Wang in [Discrete Math. 347 (2024) 114171] and Yu, Li and Peng in [Discrete Math. 348 (2025) 114391]. So, one needs only to deal with the case of F_6. In this paper, we solve the only left case by determining the maximum spectral radius of F_6-free graphs with size $m \geq 88$, and the corresponding extremal graph is characterized.

Keywords: Spectral radius; $\mathcal{F}$-free graph; Fan graph; Extremal graph
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1 Introduction

Throughout this paper, we consider only simple and finite undirected graphs. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We use $|G|$ and $e(G)$ to denote the order and the size of G, respectively. Let $A(G)$ be the adjacency matrix of a graph G. Since $A(G)$ is real symmetric, its eigenvalues are real. Hence they can be ordered as $\lambda_1(G) \geq \cdots \geq \lambda_{|G|}(G)$ where $\lambda_1(G)$ is called the spectral radius of G and also denoted by $\lambda(G)$. The neighborhood of a vertex $u \in V(G)$ is denoted by $N_G(u)$. Let

$N_G[u] = N_G(u) \cup \{u\}$. The degree of a vertex u in G is denoted by $d_G(u)$. All the subscripts defined here will be omitted if it is clear from the context. As usual, $\Delta(G)$ stands for the maximum degree of G. For two subsets $X, Y \subseteq V(G)$, we use $e(X, Y)$ to denote the number of all edges of G with one end vertex in X and the other in Y. Particularly, $e(X, X)$ is simplified by $e(X)$. Denote $G[X]$ the subgraph of G induced by X.

Let K_n, C_n, P_n and $K_{1,n-1}$ be the complete graph, cycle, path and star on n vertices, respectively. Let $K_{1,n-1} + e$ be the graph obtained from $K_{1,n-1}$ by adding one edge within its independent set. An (a, b)-double star, denoted by $D_{a,b}$, is the graph obtained by taking an edge and joining one of its end vertices with a vertices and the other end vertex with b vertices which are different from the a vertices. The join of two disjoint graphs G and H, denoted by $G \vee H$, is obtained from $G \cup H$ by adding all possible edges between G and H. For graph notation and concept undefined here, readers are referred to [1].

A graph is said to be F-free if it does not contain a subgraph isomorphic to F. For a graph F and an integer m, let $\mathcal{G}(m, F)$ be the set of F-free graphs of size m without isolated vertices. An interesting spectral Turán type problem asks what is the maximum spectral radius of an F-free graph with given size m, which is also known as Brualdi-Hoffman-Turán type problem [2]. These extremal spectral graph problems have attracted wide attention recently, see [5, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 21, 22].

Let $F_k = K_1 \vee P_{k-1}$ be the fan graph on k vertices. Note that F_3 is a triangle and F_4 is a book on 4 vertices. Nosal [16] showed that $\lambda(G) \leq \sqrt{m}$ for any F_3-free graph G with size m. In 2021, Nikiforov [15] showed that if G is a graph with m edges and $\lambda(G) \geq \sqrt{m}$, then the maximum number of triangles with a common edge in G is greater than $\frac{1}{12}\sqrt[4]{m}$, unless G is a complete bipartite graph with possibly some isolated vertices. From this, we obtain that the complete bipartite graphs attain the maximum spectral radius when the graphs are F_4-free. For $k = 5$, Zhang and Wang [23] and Yu et al. [20] respectively considered the extremal problem on spectral radius for F_5-free graphs with odd size m. Later, Chen and Yuan [3] addressed Brualdi-Hoffman-Turán type problem on F_5-free graphs for both odd m and even m. In [20], Yu et al. proposed the following conjecture on spectral radius for F_k-free graphs with given size m.

Conjecture 1.1 *Let $k \geq 2$ be fixed and m be sufficiently large. If G is an F_{2k+1}-free or F_{2k+2}-free graph with m edges, then*

$$\lambda(G) \leq \frac{k - 1 + \sqrt{4m - k^2 + 1}}{2},$$

and equality holds if and only if $G \cong K_k \vee \left(\frac{m}{k} - \frac{k-1}{2}\right) K_1$.

Recently, Li et al. [6] gave a unified approach to resolve Conjecture 1.1 for $k \geq 3$. In order to resolve Conjecture 1.1 completely, one only need to consider F_6-free graphs. Motivated by it, we will show that Conjecture 1.1 holds for F_6-free graphs.

Theorem 1.2 *Let G be an F_6-free graph with $m \geq 88$ edges. Then $\lambda(G) \leq \frac{1+\sqrt{4m-3}}{2}$ and equality holds if and only if $G \cong K_2 \vee \frac{m-1}{2}K_1$.*

2 Preliminaries

In this section, we present some preliminary results, which play an important role in the subsequent sections.

Lemma 2.1 *[12, 16] If G is a K_3-free graph with size m, then $\lambda(G) \leq \sqrt{m}$ and equality holds if and only if G is a complete bipartite graph.*

For a connected graph G, by Perron-Frobenius theorem [4], we know that there exists a positive unit eigenvector corresponding to $\lambda(G)$, which is called the Perron vector of G.

Lemma 2.2 *[19] Let u and v be two vertices of the connected graph G. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_G(v))$ are some vertices of $N_G(v) \setminus N_G(u)$ and $\mathbf{x}$ is the Perron vector of G with x_w corresponding to the vertex $w \in V(G)$. Let $G' = G - \{vv_i | 1 \leq i \leq s\} + \{uv_i | 1 \leq i \leq s\}$. If $x_u \geq x_v$, then $\lambda(G') > \lambda(G)$.*

A cut vertex of a graph is a vertex whose deletion increases the number of components. A graph is called 2-connected, if it is a connected graph without cut vertices. Let $\mathbf{x}$ be the Perron vector of G with coordinate x_u corresponding to the vertex $u \in V(G)$ and u^* be a vertex satisfying $x_{u^*} = \max\{x_u | u \in V(G)\}$, which is said to be an extremal vertex.

Lemma 2.3 *[21] Let G be a graph in $\mathcal{G}(m, F)$ with the maximum spectral radius. If F is a 2-connected graph and u^* is an extremal vertex of G, then G is connected and $d(u) \geq 2$ for any $u \in V(G) \setminus N[u^*]$.*

The following result is mentioned in [6] which is easy to get.

Lemma 2.4 *[6] Let G be a graph in $\mathcal{G}(m, F_k)$. Then for all $u \in V(G)$, the graph $G[N(u)]$ is P_{k-1}-free.*

3 Proof of Theorem 1.2

Let G^* be a graph in $\mathcal{G}(m, F_6)$ with the maximum spectral radius. By Lemma 2.3, we have G^* is connected. Let $\lambda = \lambda(G^*)$ and $\mathbf{x}$ be the Perron vector of G^* with coordinate x_v corresponding to the vertex $v \in V(G^*)$. Assume that u^* is an extremal vertex of G^*. Set $U = N_{G^*}(u^*)$ and $W = V(G^*) \setminus N_{G^*}[u^*]$. Let U_0 be the isolated vertices of the induced subgraph $G^*[U]$, and $U_+ = U \setminus U_0$ be the vertices of U with degree at least one in $G^*[U]$. Let $W_H = N_W(V(H))$ for any subset H of $G^*[U]$ and $W_0 = \{w \in W | d_W(w) = 0\}$.

Note that $\lambda\left(K_2 \vee \frac{m-1}{2}K_1\right) = \frac{1+\sqrt{4m-3}}{2}$ and $K_2 \vee \frac{m-1}{2}K_1$ is F_6-free, we have

$$\lambda(G^*) \geq \lambda\left(K_2 \vee \frac{m-1}{2}K_1\right) = \frac{1+\sqrt{4m-3}}{2}.$$

Hence $\lambda^2 - \lambda \geq m-1$. Furthermore, since $m \geq 88$, we can get $\lambda = \lambda(G^*) > \frac{49}{5}$. Since $\lambda(G^*)\mathbf{x} = A(G^*)\mathbf{x}$, we have

$$\lambda x_{u^*} = \sum_{u\in U_+} x_u + \sum_{u\in U_0} x_u.$$

Furthermore, $\mathbf{x}$ is also an eigenvector of $A^2(G^*)$ corresponding to $\lambda^2(G^*)$. It follows that

$$\lambda^2 x_{u^*} = |U|x_{u^*} + \sum_{u\in U_+} d_U(u)x_u + \sum_{w\in W} d_U(w)x_w.$$

Therefore,

$$(\lambda^2 - \lambda)x_{u^*} = |U|x_{u^*} + \sum_{u\in U_+}(d_U(u)-1)x_u + \sum_{w\in W} d_U(w)x_w - \sum_{u\in U_0} x_u.$$

Since $\lambda^2 - \lambda \geq m-1 = |U| + e(U_+) + e(U,W) + e(W) - 1$, we have

$$\sum_{u\in U_+}(d_U(u)-1)x_u + \sum_{w\in W} d_U(w)x_w \geq \left(e(U_+) + e(U,W) + e(W) + \sum_{u\in U_0}\frac{x_u}{x_{u^*}} - 1\right)x_{u^*}.$$

That is,

$$\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \geq e(U_+) + e(U,W) + e(W) + \sum_{u\in U_0}\frac{x_u}{x_{u^*}} - 1. \quad (1)$$

Lemma 3.1 $e(U) \geq 4$.

Proof. Since $\lambda(G^*) \geq \frac{1+\sqrt{4m-3}}{2} > \sqrt{m+3}$ when $m \geq 88$, we have

$$m+3 < \lambda^2 = |U| + \sum_{u\in U_+} d_U(u)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}}$$

$$\leq |U| + \sum_{u\in U_+} d_U(u) + \sum_{w\in W} d_U(w)$$
$$= |U| + 2e(U) + e(U,W).$$

Note that $m = |U| + e(U) + e(U,W) + e(W)$. It follows that $e(U) > e(W) + 3 \geq 3$. Hence $e(U) \geq 4$. $\square$

By Lemma 3.1, there exists at least one non-trivial component in $G^*[U]$. Let $\mathcal{H}$ be the set of all non-trivial components in $G^*[U]$. For each non-trivial component H of $\mathcal{H}$, we denote $\gamma(H) := \sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} - e(H)$. Clearly, by (1) we have

$$\begin{aligned} e(W) &\leq \sum_{H\in\mathcal{H}} \gamma(H) + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} - e(U,W) - \sum_{u\in U_0} \frac{x_u}{x_{u^*}} + 1 \\ &\leq \sum_{H\in\mathcal{H}} \gamma(H) - \sum_{u\in U_0} \frac{x_u}{x_{u^*}} + 1 \end{aligned} \tag{2}$$

with equality if and only if $\lambda^2 - \lambda = m-1$ and $x_w = x_{u^*}$ for any $w \in W$ with $d_U(w) \geq 1$.

Since G^* is F_6-free, by Lemma 2.4, we know that $G^*[U]$ does not contain a path with 5 vertices. Similarly to the proof of Lemma 4.5 in [9], we have the following result.

Lemma 3.2 *Let G^* be an F_6-free graph with $u \in V(G^*)$ and H be a component of $G^*[N(u)]$. Then H is one of the following cases:*

(i) a star $K_{1,r}$ for $r \geq 0$, where $K_{1,0}$ is an isolated vertex;

(ii) a double star $D_{a,b}$ for $a, b \geq 1$;

(iii) $K_{1,r} + e$ where $r \geq 2$ and $K_{1,2} + e$ is a triangle;

(iv) $C_4, K_4 - e$ or K_4.

Next we provide some upper bounds on $\gamma(H)$ where H is a non-trivial component of $G^*[U]$.

Lemma 3.3 *Let H be a non-trivial component of $G^*[U]$, then*

$$\gamma(H) \leq \begin{cases} -1, & \text{if } H \cong K_{1,r} \text{ or } D_{a,b} \text{ where } r,a,b \geq 1, \\ 0, & \text{if } H \cong K_{1,r} + e \text{ where } r \geq 2 \text{ or } C_4, \\ 1, & \text{if } H \cong K_4 - e, \\ 2, & \text{if } H \cong K_4. \end{cases}$$

Proof. Since $\frac{x_u}{x_{u^*}} \le 1$ and $d_H(u) \ge 1$ for any $u \in V(H)$, we have

$$\gamma(H) = \sum_{u\in V(H)} (d_H(u)-1)\frac{x_u}{x_{u^*}} - e(H) \le \sum_{u\in V(H)} (d_H(u)-1) - e(H) = e(H) - |H|.$$

If $H \cong K_{1,r}$ or $D_{a,b}$, then $e(H) - |H| = -1$. When $H \cong K_{1,r} + e$ or C_4, we have $e(H) - |H| = 0$. For $H \cong K_4 - e$, we get $e(H) - |H| = 1$. If $H \cong K_4$, it follows that $e(H) - |H| = 2$. Hence the lemma holds. □

In fact, we can give a tighter upper bound about $\gamma(H)$.

Lemma 3.4 *For every non-trivial component H in $G^*[U]$, we have $\gamma(H) \le 0$.*

Proof. Suppose to the contrary that there exists at least one component $H \in \mathcal{H}$ satisfying $\gamma(H) > 0$. Let $\widetilde{\mathcal{H}} = \{H \in \mathcal{H} \colon \gamma(H) > 0\}$. By Lemma 3.3, we know that H is K_4 or $K_4 - e$ for any component $H \in \widetilde{\mathcal{H}}$. We have the following claims.

Claim 3.5 $\sum_{u\in V(H)} x_u > \frac{5}{2}x_{u^*}$ *for any $H \in \widetilde{\mathcal{H}}$.*

Proof. Suppose that $\sum_{u\in V(H_0)} x_u \le \frac{5}{2}x_{u^*}$ for some $H_0 \in \widetilde{\mathcal{H}}$. Then

$$\begin{aligned}
\gamma(H_0) &= \sum_{u\in V(H_0)} (d_{H_0}(u)-1)\frac{x_u}{x_{u^*}} - e(H_0)\\
&\le (\Delta(H_0)-1)\sum_{u\in V(H_0)} \frac{x_u}{x_{u^*}} - e(H_0)\\
&\le 2\sum_{u\in V(H_0)} \frac{x_u}{x_{u^*}} - 5\\
&\le 0,
\end{aligned}$$

which contradicts $H_0 \in \widetilde{\mathcal{H}}$. Therefore, $\sum_{u\in V(H)} x_u > \frac{5}{2}x_{u^*}$ for any $H \in \widetilde{\mathcal{H}}$. □

Claim 3.6 *The number $|\widetilde{\mathcal{H}}|$ of members in $\widetilde{\mathcal{H}}$ satisfying $|\widetilde{\mathcal{H}}| < \frac{2}{5}\lambda + \frac{7}{10}$.*

Proof. Suppose $|\widetilde{\mathcal{H}}| \ge \frac{2}{5}\lambda + \frac{7}{10}$. For every $H \in \widetilde{\mathcal{H}}$, we have

$$\begin{aligned}
\lambda\sum_{u\in V(H)} x_u &= \sum_{u\in V(H)}\left(x_{u^*} + \sum_{v\in N_H(u)} x_v + \sum_{w\in N_W(u)} x_w\right)\\
&= 4x_{u^*} + \sum_{u\in V(H)}\sum_{v\in N_H(u)} x_v + \sum_{u\in V(H)}\sum_{w\in N_W(u)} x_w\\
&\le 4x_{u^*} + \sum_{u\in V(H)} d_H(u)x_u + e(H,W)x_{u^*}
\end{aligned}$$

$$= 4x_{u^*} + \gamma(H)x_{u^*} + e(H)x_{u^*} + \sum_{u \in V(H)} x_u + e(H,W)x_{u^*}.$$

Note that H is K_4 or $K_4 - e$, by Lemma 3.3, we have $\gamma(H) \leq 2$. Furthermore, $e(H) \leq 6$. Therefore,

$$\begin{aligned}(\lambda - 1)\sum_{u \in V(H)} x_u &\leq 4x_{u^*} + \gamma(H)x_{u^*} + e(H)x_{u^*} + e(H,W)x_{u^*} \\ &\leq 4x_{u^*} + 2x_{u^*} + 6x_{u^*} + e(H,W)x_{u^*} \\ &= 12x_{u^*} + e(H,W)x_{u^*}.\end{aligned}$$

By Claim 3.5, we obtain

$$\begin{aligned}e(H,W)x_{u^*} &\geq (\lambda - 1)\sum_{u \in V(H)} x_u - 12x_{u^*} \\ &> \frac{5(\lambda-1)}{2}x_{u^*} - 12x_{u^*} \\ &= \frac{5\lambda - 29}{2}x_{u^*}.\end{aligned}$$

It follows that $e(H,W) > \frac{5\lambda-29}{2}$. Since

$$\begin{aligned}m &\geq |U| + \sum_{H \in \widetilde{\mathcal{H}}} e(H) + \sum_{H \in \widetilde{\mathcal{H}}} e(H,W) \\ &> 4|\widetilde{\mathcal{H}}| + 5|\widetilde{\mathcal{H}}| + \left(\frac{5\lambda - 29}{2}\right)|\widetilde{\mathcal{H}}| \\ &= \left(\frac{5\lambda - 11}{2}\right)|\widetilde{\mathcal{H}}| \\ &\geq \left(\frac{5\lambda - 11}{2}\right)\left(\frac{2}{5}\lambda + \frac{7}{10}\right) \\ &= \lambda^2 - \frac{9}{20}\lambda - \frac{77}{20}.\end{aligned}$$

Since $\lambda > \frac{49}{5}$, we have $m > \lambda^2 - \frac{9}{20}\lambda - \frac{77}{20} > \lambda^2 - \lambda + 1$, which contradicts $\lambda \geq \frac{1+\sqrt{4m-3}}{2}$. Therefore, $|\widetilde{\mathcal{H}}| < \frac{2}{5}\lambda + \frac{7}{10}$. $\square$

Claim 3.7 *If $W_0 \neq \emptyset$, then $x_w \leq \frac{2\lambda-3}{2\lambda}x_{u^*}$ for any $w \in W_0$.*

Proof. Suppose to the contrary that there exists $w_0 \in W_0$ such that $x_{w_0} > \frac{2\lambda-3}{2\lambda}x_{u^*}$. For every $H \in \widetilde{\mathcal{H}}$, if $w_0 \notin N_{G^*}(u)$ for every $u \in V(H)$, then $\lambda x_{w_0} \leq \lambda x_{u^*} - \sum_{u \in V(H)} x_u$. It follows that $\sum_{u \in V(H)} x_u \leq \lambda x_{u^*} - \lambda x_{w_0} < \lambda x_{u^*} - \frac{2\lambda-3}{2}x_{u^*} = \frac{3}{2}x_{u^*}$. Hence $\gamma(H) \leq 2\sum_{u \in V(H)} \frac{x_u}{x_{u^*}} - e(H) < 3 - 5 < 0$, contradicting $H \in \widetilde{\mathcal{H}}$. Next, we consider that there exists $H' \in \widetilde{\mathcal{H}}$ satisfying $w_0 \in N_{G^*}(u)$ for some $u \in V(H')$. Let $V(H') = \{u_1, u_2, u_3, u_4\}$ and

$u_1u_2u_3u_4u_1$ is a cycle in H'. Without loss of generality, assume $w_0 \in N_W(u_1)$. If $H' \cong K_4$, then $N_U(w_0) \cap V(H') = \{u_1\}$. Otherwise, $G^*[u^*, u_1, u_2, u_3, u_4, w_0]$ contains a copy of F_6. Therefore, $\lambda x_{w_0} \leq \lambda x_{u^*} - \sum_{i=2}^{4} x_{u_i}$. It follows that $\sum_{i=2}^{4} x_{u_i} \leq \lambda x_{u^*} - \lambda x_{w_0} < \frac{3}{2}x_{u^*}$. Thus $\gamma(H') \leq 2\sum_{u\in V(H')} \frac{x_u}{x_{u^*}} - e(H') \leq 2+2\sum_{i=2}^{4} \frac{x_{u_i}}{x_{u^*}} - 6 < 2+3-6 < 0$, contradicting $H' \in \widetilde{\mathcal{H}}$. When $H' \cong K_4 - e$, if $d_{H'}(u_1) = 2$, then $u_2, u_4 \notin N_{G^*}(w_0)$. Otherwise, G^* contains a copy of F_6. So $\lambda x_{w_0} \leq \lambda x_{u^*} - (x_{u_2} + x_{u_4})$. That is, $x_{u_2} + x_{u_4} \leq \lambda x_{u^*} - \lambda x_{w_0} < \frac{3}{2}x_{u^*}$. Hence $\gamma(H') = \frac{x_{u_1}}{x_{u^*}} + 2\cdot\frac{x_{u_2}}{x_{u^*}} + \frac{x_{u_3}}{x_{u^*}} + 2\cdot\frac{x_{u_4}}{x_{u^*}} - 5 < 2 + 2\cdot\frac{3}{2} - 5 = 0$, a contradiction. If $d_{H'}(u_1) = 3$, then $u_2, u_3, u_4 \notin N_{G^*}(w_0)$. By $\lambda x_{w_0} \leq \lambda x_{u^*} - (x_{u_2} + x_{u_3} + x_{u_4})$, we obtain $x_{u_2} + x_{u_3} + x_{u_4} \leq \lambda x_{u^*} - \lambda x_{w_0} < \frac{3}{2}x_{u^*}$. Hence $\gamma(H') = 2\cdot\frac{x_{u_1}}{x_{u^*}} + \frac{x_{u_2}}{x_{u^*}} + 2\cdot\frac{x_{u_3}}{x_{u^*}} + \frac{x_{u_4}}{x_{u^*}} - 5 < 3 + \frac{3}{2} - 5 < 0$, a contradiction. Therefore, $x_w \leq \frac{2\lambda-3}{2\lambda}x_{u^*}$ for any $w \in W_0$. □

Now, we come back to prove Lemma 3.4. Choose a component $H \in \widetilde{\mathcal{H}}$. Let $V(H) = \{u_1, u_2, u_3, u_4\}$. We proceed by distinguishing the following two cases.

Case 1. $e(W) = 0$.

If $W_H = \emptyset$, then $d_{G^*}(u) \leq 4$ for every $u \in V(H)$. Therefore, $\lambda x_u = \sum_{v\in N(u)} x_v \leq 4x_{u^*}$. That is, $x_u \leq \frac{4}{\lambda}x_{u^*}$. By $d_H(u) \leq 3$ for every $u \in V(H)$ and $e(H) \geq 5$, we have $\gamma(H) \leq 2\sum_{u\in V(H)} \frac{x_u}{x_{u^*}} - 5 \leq 2\cdot 4\cdot\frac{4}{\lambda} - 5 = \frac{32}{\lambda} - 5$. Note that $\lambda > \frac{49}{5}$, then $\gamma(H) < 0$, which contradicts $H \in \widetilde{\mathcal{H}}$. Now we consider $W_H \neq \emptyset$. If $H \cong K_4$, we have $|N_H(w)\cap V(H)| = 1$ for any $w \in W_H$. Otherwise, G^* contains a copy of F_6, a contradiction. Let $x_{u_1} = \max\{x_u | u \in V(H)\}$. It follows that $N_H(w)\cap V(H) = \{u_1\}$. Otherwise, suppose $N_H(w_0)\cap V(H) = \{u_2\}$ for some $w_0 \in W_H$. We get $G' = G^* - w_0u_2 + w_0u_1$ is F_6-free. According to $x_{u_1} \geq x_{u_2}$ and Lemma 2.2, we get $\lambda(G') > \lambda(G^*)$, which contradicts the maximality of G^*. Thus $N_W(u) = \emptyset$ for any $u \in V(H) \setminus \{u_1\}$. That is, $d_{G^*}(u) = 4$ for every $u \in V(H) \setminus \{u_1\}$. So we obtain $\lambda x_u \leq 4x_{u^*}$ for every $u \in V(H) \setminus \{u_1\}$. Therefore, by $\lambda > \frac{49}{5}$, we have $\gamma(H) \leq 2 + 6\cdot\frac{4}{\lambda} - 6 < 0$, a contradiction. If $H \cong K_4 - e$, suppose $d_H(u_1) = d_H(u_2) = 3$ and $x_{u_1} \geq x_{u_2}$. If $N_W(u_1) \cup N_W(u_2) = \emptyset$, then $d_{G^*}(u_1) = d_{G^*}(u_2) = 4$. It follows that $\lambda x_{u_i} \leq 4x_{u^*}$ for $i = 1, 2$. Then $\gamma(H) \leq 2\sum_{i=1}^{2} \frac{x_{u_i}}{x_{u^*}} + \sum_{u\in V(H)\setminus\{u_1,u_2\}} \frac{x_{u_i}}{x_{u^*}} - 5 \leq 4\cdot\frac{4}{\lambda} + 2 - 5 < 0$, a contradiction. If $N_W(u_1)\cup N_W(u_2) \neq \emptyset$, since G^* is F_6-free, we have $|N_G(w)\cap\{u_1, u_2\}| = 1$ and $|N_G(w)\cap(V(H)\setminus\{u_1, u_2\})| = 0$ for every $w \in N_W(u_1)\cup N_W(u_2)$. Recall that $x_{u_1} \geq x_{u_2}$. It follows that $N_G(w) \cap \{u_1, u_2\} = \{u_1\}$. Otherwise, there is a vertex $w_0 \in N_W(u_2)$. We have $G' = G^* - w_0u_2 + w_0u_1$ is F_6-free. According to Lemma 2.2, we get $\lambda(G') > \lambda(G^*)$, which contradicts the maximality of G^*. Thus $N_W(u_2) = \emptyset$. That is, $d_{G^*}(u_2) = 4$. So $\lambda x_{u_2} \leq 4x_{u^*}$. Therefore, by $\lambda > \frac{49}{5}$, we get $\gamma(H) \leq 2\sum_{i=1}^{2} \frac{x_{u_i}}{x_{u^*}} + \sum_{u\in V(H)\setminus\{u_1,u_2\}} \frac{x_{u_i}}{x_{u^*}} - 5 \leq 2 + 2\cdot\frac{4}{\lambda} + 2 - 5 < 0$, a contradiction.

Case 2. $1 \leq e(W) \leq 3$.

If $W_H = \emptyset$, similarly to Case 1. Now we consider $W_H \neq \emptyset$. If $H \cong K_4$, we have $|N_H(w)\cap V(H)| = 1$ for any $w \in W_H$. Otherwise, G^* contains a copy of F_6, a contradiction.

Let $x_{u_1} = \max\{x_u | u \in V(H)\}$. It follows that $N_H(w) \cap V(H) = \{u_1\}$ for any $w \in W_H \cap W_0$. Otherwise, suppose $N_H(w_0) \cap V(H) = \{u_i\}$ for some $w_0 \in W_H \cap W_0$ and $i \in \{2, 3, 4\}$. We can get that $G' = G^* - w_0 u_i + w_0 u_1$ is F_6-free. According to $x_{u_1} \geq x_{u_i}$ and Lemma 2.2, we get $\lambda(G') > \lambda(G^*)$, which contradicts the maximality of G^*. By $e(W) \leq 3$, we have $|W_H \setminus W_0| \leq 6$. It follows that $\sum_{i=2}^{4} \sum_{w \in N_{W_H}(u_i)} x_w \leq 6x_{u^*}$. By

$$\begin{cases} \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_H}(u_2)} x_w, \\ \lambda x_{u_3} = x_{u_1} + x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_H}(u_3)} x_w, \\ \lambda x_{u_4} = x_{u_1} + x_{u_2} + x_{u_3} + x_{u^*} + \sum_{w \in N_{W_H}(u_4)} x_w, \end{cases}$$

we obtain

$$(\lambda - 2)(x_{u_2} + x_{u_3} + x_{u_4}) \leq 6x_{u^*} + \sum_{i=2}^{4} \sum_{w \in N_{W_H}(u_i)} x_w \leq 12x_{u^*}.$$

Therefore, by $\lambda > \frac{49}{5}$, we have $\sum_{u \in V(H)} (d_H(u) - 1) \frac{x_u}{x_{u^*}} = 2 \sum_{i=1}^{4} \frac{x_{u_i}}{x_{u^*}} \leq 2 + 2 \cdot \frac{12}{\lambda - 2} < 6 = e(H)$.

If $H \cong K_4 - e$, let $d_H(u_2) = d_H(u_4) = 3$. Then $N_W(u_1) \cap N_W(u_2) \cap N_W(u_4) = \emptyset$. Hence $\sum_{i \in \{1,2,4\}} \sum_{w \in N_{W_H}(u_i)} x_w = \sum_{w \in W_0 \cap (\cup_{i \in \{1,2,4\}} N_W(u_i))} x_w + \sum_{w \in (\cup_{i \in \{1,2,4\}} N_W(u_i)) \setminus W_0} x_w$. By Claim 3.7, we have $\sum_{w \in W_0 \cap (\cup_{i \in \{1,2,4\}} N_W(u_i))} x_w \leq \frac{(2\lambda - 3)e(H, W_0)}{2\lambda} x_{u^*}$. Since $d_W(w) \geq 1$ for $w \in (\cup_{i \in \{1,2,4\}} N_W(u_i)) \setminus W_0$, we have $\sum_{w \in (\cup_{i \in \{1,2,4\}} N_W(u_i)) \setminus W_0} x_w \leq \sum_{w \in W_H \setminus W_0} d_W(w) x_w \leq 2e(W) x_{u^*} \leq 6x_{u^*}$. Since

$$\begin{cases} \lambda x_{u_1} = x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_H}(u_1)} x_w, \\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_H}(u_2)} x_w, \\ \lambda x_{u_4} = x_{u_1} + x_{u_2} + x_{u_3} + x_{u^*} + \sum_{w \in N_{W_H}(u_4)} x_w, \end{cases}$$

we have

$$\lambda(x_{u_1} + x_{u_2} + x_{u_4}) = 2(x_{u_1} + x_{u_2} + x_{u_4}) + 2x_{u_3} + 3x_{u^*} + \sum_{i \in \{1,2,4\}} \sum_{w \in N_{W_H}(u_i)} x_w.$$

That is,

$$\begin{aligned} x_{u_1} + x_{u_2} + x_{u_4} &\leq \frac{5x_{u^*} + \sum_{i \in \{1,2,4\}} \sum_{w \in N_{W_H}(u_i)} x_w}{\lambda - 2} \\ &\leq \frac{11}{\lambda - 2} x_{u^*} + \frac{(2\lambda - 3)e(H, W_0)}{2\lambda(\lambda - 2)} x_{u^*}. \end{aligned}$$

Thus

$$\sum_{u \in V(H)} (d_H(u) - 1) \frac{x_u}{x_{u^*}} = \frac{x_{u_1} + 2x_{u_2} + x_{u_3} + 2x_{u_4}}{x_{u^*}}$$

$$
\begin{aligned}
&= \frac{x_{u_2}+x_{u_3}+x_{u_4}}{x_{u^*}} + \frac{x_{u_1}+x_{u_2}+x_{u_4}}{x_{u^*}} \\
&\le 3 + \frac{11}{\lambda-2} + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}.
\end{aligned}
$$

Since $\lambda > \frac{49}{5}$, we have $3+\frac{11}{\lambda-2} < 5$. Hence

$$
\begin{aligned}
\sum_{u\in V(H)} (d_H(u)-1)\frac{x_u}{x_{u^*}} &< 5 + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)} \\
&= e(H) + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}.
\end{aligned}
$$

Therefore, $\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} < e(H) + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}$ for any $H \in \widetilde{\mathcal{H}}$. For any $H \in \mathcal{H}\setminus\widetilde{\mathcal{H}}$, since $\gamma(H)\le 0$, we have $\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} \le e(H)$. By $\lambda > \frac{49}{5}$, we obtain

$$
\begin{aligned}
&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \\
&= \sum_{H\in\widetilde{\mathcal{H}}}\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} + \sum_{H\in\mathcal{H}\setminus\widetilde{\mathcal{H}}}\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W_0} d_U(w)\frac{x_w}{x_{u^*}} + \sum_{w\in W\setminus W_0} d_U(w)\frac{x_w}{x_{u^*}} \\
&< \sum_{H\in\widetilde{\mathcal{H}}}\left(e(H) + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}\right) + \sum_{H\in\mathcal{H}\setminus\widetilde{\mathcal{H}}} e(H) + \sum_{w\in W_0} d_U(w)\cdot\frac{2\lambda-3}{2\lambda} + \sum_{w\in W\setminus W_0} d_U(w) \\
&\le e(U_+) + \frac{(2\lambda-3)e(U,W_0)}{2\lambda(\lambda-2)} + \frac{(2\lambda-3)e(U,W_0)}{2\lambda} + e(U, W\setminus W_0) \\
&= e(U_+) + \left(\frac{2\lambda-3}{2\lambda(\lambda-2)} + \frac{2\lambda-3}{2\lambda}\right)e(U,W_0) + e(U,W\setminus W_0) \\
&< e(U_+) + e(U,W_0) + e(U,W\setminus W_0) \\
&\le e(U_+) + e(U,W) + e(W) - 1,
\end{aligned}
$$

a contradiction.

Case 3. $e(W)\ge 4$.

If $H\cong K_4$, then $N_W(u_1)\cap N_W(u_2)\cap N_W(u_3)\cap N_W(u_4)=\emptyset$. Otherwise, G^* contains a copy of F_6. Therefore, $\sum_{i=1}^4\sum_{w\in N_{W_H}(u_i)} x_w = \sum_{w\in W_0\cap W_H} x_w + \sum_{w\in W_H\setminus W_0} x_w$. By Claim 3.7, we have $\sum_{w\in W_0\cap W_H} x_w \le \frac{(2\lambda-3)e(H,W_0)}{2\lambda}x_{u^*}$. Since $d_W(w)\ge 1$ for $w\in W_H\setminus W_0$, we have $\sum_{w\in W_H\setminus W_0} x_w \le \sum_{w\in W_H\setminus W_0} d_W(w)x_w \le 2e(W)x_{u^*}$. Since

$$
\begin{cases}
\lambda x_{u_1} = x_{u_2}+x_{u_3}+x_{u_4}+x_{u^*}+\sum_{w\in N_{W_H}(u_1)} x_w, \\
\lambda x_{u_2} = x_{u_1}+x_{u_3}+x_{u_4}+x_{u^*}+\sum_{w\in N_{W_H}(u_2)} x_w, \\
\lambda x_{u_3} = x_{u_1}+x_{u_2}+x_{u_4}+x_{u^*}+\sum_{w\in N_{W_H}(u_3)} x_w, \\
\lambda x_{u_4} = x_{u_1}+x_{u_2}+x_{u_3}+x_{u^*}+\sum_{w\in N_{W_H}(u_4)} x_w,
\end{cases}
$$

we have

$$(\lambda-3)(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4}) = 4x_{u^*} + \sum_{i=1}^{4}\sum_{w\in N_{W_H}(u_i)} x_w.$$

That is,

$$\begin{aligned} x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4} &= \frac{4x_{u^*}+\sum_{i=1}^{4}\sum_{w\in N_{W_H}(u_i)} x_w}{\lambda-3} \\ &\le \frac{4}{\lambda-3}x_{u^*} + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-3)}x_{u^*} + \frac{2e(W)}{\lambda-3}x_{u^*}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} &= \frac{2(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4})}{x_{u^*}} \\ &= \frac{x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4}}{x_{u^*}} + \frac{x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4}}{x_{u^*}} \\ &\le 4 + \frac{4}{\lambda-3} + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-3)} + \frac{2e(W)}{\lambda-3} \\ &= \left(4+\frac{4}{\lambda-3}+\frac{\frac{2\lambda-5}{2\lambda-4}e(W)}{\lambda-3}\right) + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-3)} + \frac{\frac{2\lambda-3}{2\lambda-4}e(W)}{\lambda-3}. \end{aligned}$$

By (2), Lemma 3.3 and Claim 3.6, we have $e(W)\le 2|\widetilde{\mathcal{H}}|+1<\frac{4}{5}\lambda+\frac{12}{5}$. This implies that $\frac{4}{\lambda-3}+\frac{\frac{2\lambda-5}{2\lambda-4}e(W)}{\lambda-3}<\frac{8\lambda^2+44\lambda-140}{5(\lambda-3)(2\lambda-4)}$. Since $\lambda>\frac{49}{5}$, we have $\frac{4}{\lambda-3}+\frac{\frac{2\lambda-5}{2\lambda-4}e(W)}{\lambda-3}<2$. Hence

$$\begin{aligned} \sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} &< 6 + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-3)} + \frac{\frac{2\lambda-3}{2\lambda-4}e(W)}{\lambda-3} \\ &= e(H) + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-3)} + \frac{\frac{2\lambda-3}{2\lambda-4}e(W)}{\lambda-3}. \end{aligned}$$

If $H\cong K_4-e$, similarly to Case 2, we have

$$\begin{aligned} x_{u_1}+x_{u_2}+x_{u_4} &\le \frac{5x_{u^*}+\sum_{i\in\{1,2,4\}}\sum_{w\in N_{W_H}(u_i)} x_w}{\lambda-2} \\ &\le \frac{5}{\lambda-2}x_{u^*} + \frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}x_{u^*} + \frac{2e(W)}{\lambda-2}x_{u^*}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} &= \frac{x_{u_1}+2x_{u_2}+x_{u_3}+2x_{u_4}}{x_{u^*}} \\ &= \frac{x_{u_2}+x_{u_3}+x_{u_4}}{x_{u^*}} + \frac{x_{u_1}+x_{u_2}+x_{u_4}}{x_{u^*}} \end{aligned}$$

$$\begin{aligned}
&\le 3+\frac{5}{\lambda-2}+\frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}+\frac{2e(W)}{\lambda-2}\\
&=\left(3+\frac{5}{\lambda-2}+\frac{e(W)}{\lambda-2}\right)+\frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}+\frac{e(W)}{\lambda-2}.
\end{aligned}$$

By (2), Lemma 3.3 and Claim 3.6, we have $e(W)\le 2|\widetilde{\mathcal{H}}|+1<\frac{4}{5}\lambda+\frac{12}{5}$. This implies that $\frac{5}{\lambda-2}+\frac{e(W)}{\lambda-2}<\frac{4\lambda+37}{5(\lambda-2)}$. Since $\lambda>\frac{49}{5}$, we have $\frac{5}{\lambda-2}+\frac{e(W)}{\lambda-2}<2$. Hence

$$\begin{aligned}
\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}&<5+\frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}+\frac{e(W)}{\lambda-2}\\
&=e(H)+\frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-2)}+\frac{e(W)}{\lambda-2}.
\end{aligned}$$

By $\lambda>\frac{49}{5}$, we get $\frac{1}{\lambda-2}\le\frac{1}{\lambda-3}\cdot\frac{2\lambda-3}{2\lambda-4}$. Therefore, $\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}<e(H)+\frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-3)}+\frac{\frac{2\lambda-3}{2\lambda-4}e(W)}{\lambda-3}$ for any $H\in\widetilde{\mathcal{H}}$. For any $H\in\mathcal{H}\setminus\widetilde{\mathcal{H}}$, since $\gamma(H)\le 0$, we have $\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}\le e(H)$. By $\lambda>\frac{49}{5}$, we obtain

$$\begin{aligned}
&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W}d_U(w)\frac{x_w}{x_{u^*}}\\
&=\sum_{H\in\widetilde{\mathcal{H}}}\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}+\sum_{H\in\mathcal{H}\setminus\widetilde{\mathcal{H}}}\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W_0}d_U(w)\frac{x_w}{x_{u^*}}+\sum_{w\in W\setminus W_0}d_U(w)\frac{x_w}{x_{u^*}}\\
&<\sum_{H\in\widetilde{\mathcal{H}}}\left(e(H)+\frac{(2\lambda-3)e(H,W_0)}{2\lambda(\lambda-3)}+\frac{\frac{2\lambda-3}{2\lambda-4}e(W)}{\lambda-3}\right)+\sum_{H\in\mathcal{H}\setminus\widetilde{\mathcal{H}}}e(H)+\sum_{w\in W_0}d_U(w)\cdot\frac{2\lambda-3}{2\lambda}\\
&+\sum_{w\in W\setminus W_0}d_U(w)\\
&<e(U_+)+\frac{(2\lambda-3)e(U,W_0)}{2\lambda(\lambda-3)}+\frac{\frac{2\lambda-3}{2\lambda-4}e(W)}{\lambda-3}\cdot\left(\frac{2}{5}\lambda+\frac{7}{10}\right)+\frac{(2\lambda-3)e(U,W_0)}{2\lambda}+e(U,W\setminus W_0)\\
&=e(U_+)+\left(\frac{2\lambda-3}{2\lambda(\lambda-3)}+\frac{2\lambda-3}{2\lambda}\right)e(U,W_0)+e(U,W\setminus W_0)+\frac{1}{\lambda-3}\cdot\frac{2\lambda-3}{2\lambda-4}\cdot\left(\frac{2}{5}\lambda+\frac{7}{10}\right)e(W)\\
&<e(U_+)+e(U,W_0)+e(U,W\setminus W_0)+\frac{3}{4}e(W)\\
&\le e(U_+)+e(U,W)+e(W)-1,
\end{aligned}$$

which contradicts (1). This completes the proof of Lemma 3.4. □

By (2) and Lemma 3.4, we know that $e(W)\le 1$.

Lemma 3.8 *For any $H\in\mathcal{H}$, we have $H\ncong K_4$.*

Proof. Suppose to the contrary that there exists a component $H_0\in\mathcal{H}$ satisfying $H_0\cong K_4$. Let $V(H_0)=\{u_1,u_2,u_3,u_4\}$. We distinguish the following two cases to lead a contradiction, respectively.

Case 1. $W_{H_0} = \emptyset$.

In this case, it is easy to get $x_{u_1} = x_{u_2} = x_{u_3} = x_{u_4}$. By $\lambda x_{u_1} = x_{u_2} + x_{u_3} + x_{u_4} + x_{u^*}$, we obtain $x_{u_1} = \frac{x_{u^*}}{\lambda-3}$. By (2) and Lemma 3.4, we have

$$\begin{aligned} e(W) &\leq \sum_{H\in\mathcal{H}\setminus\{H_0\}} \gamma(H) + \gamma(H_0) + 1 \\ &\leq 0 + \frac{2(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4})}{x_{u^*}} - 6 + 1 \\ &\leq \frac{8}{\lambda - 3} - 5. \end{aligned}$$

Since $\lambda > \frac{49}{5}$, we get $e(W) < 0$, a contradiction.

Case 2. $W_{H_0} \neq \emptyset$.

Since G^* does not contain a subgraph isomorphic to F_6, we have $|N_U(w) \cap V(H_0)| = 1$ for every $w \in W_{H_0}$. Suppose $x_{u_1} \geq x_{u_2} \geq x_{u_3} \geq x_{u_4}$. Then $N_U(w) \cap V(H_0) = \{u_1\}$ for every $w \in W_{H_0} \cap W_0$. Otherwise, assume $N_U(w_0) \cap V(H_0) = \{u_i\}$ for some $w_0 \in W_{H_0} \cap W_0$ and $i \in \{2, 3, 4\}$. We can verify that $G' = G^* - w_0u_i + w_0u_1$ is F_6-free. By Lemma 2.2, we have $\lambda(G') > \lambda(G^*)$, which contradicts the maximality of G^*. Combining with $e(W) \leq 1$, we have $|W_{H_0} \setminus W_0| \leq 2$. Hence $\sum_{i=2}^{4} \sum_{w\in N_{W_H}(u_i)} x_w \leq 2x_{u^*}$. By $\lambda x_{u_i} \leq d_H(u_i)x_{u^*} + x_{u^*} + \sum_{w\in N_{W_H}(u_i)} x_w$ for $i \in \{2, 3, 4\}$, we obtain $\sum_{i=2}^{4} \lambda x_{u_i} \leq \sum_{i=2}^{4} d_H(u_i)x_{u^*} + 3x_{u^*} + \sum_{i=2}^{4} \sum_{w\in N_{W_H}(u_i)} x_w \leq 14x_{u^*}$. Hence

$$\begin{aligned} e(W) &\leq \sum_{H\in\mathcal{H}\setminus\{H_0\}} \gamma(H) + \gamma(H_0) + 1 \\ &\leq 0 + \frac{2(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4})}{x_{u^*}} - 6 + 1 \\ &\leq 2\left(1 + \frac{14}{\lambda}\right) - 5 \\ &\leq \frac{28}{\lambda} - 3 < 0. \end{aligned}$$

a contradiction. This completes the proof. □

Lemma 3.9 *For any $H \in \mathcal{H}$, we have $H \ncong K_4 - e$.*

Proof. Suppose to the contrary that H_1 is a component in $\mathcal{H}$ such that $H_1 \cong K_4 - e$. Let $V(H_1) = \{u_1, u_2, u_3, u_4\}$ with $d_{H_1}(u_2) = d_{H_1}(u_4) = 3$. We discuss the following two cases to lead a contradiction, respectively.

Case 1. $N_W(u_2) \cup N_W(u_4) = \emptyset$.

In this case, it is easy to get $x_{u_2} = x_{u_4}$. By $\lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u_4} + x_{u^*}$, we obtain $x_{u_2} \leq \frac{3x_{u^*}}{\lambda-1}$. By (2) and Lemma 3.4, we have

$$\begin{aligned} e(W) &\leq \sum_{H \in \mathcal{H}\setminus\{H_1\}} \gamma(H) + \gamma(H_1) + 1 \\ &\leq 0 + \frac{x_{u_1} + 2x_{u_2} + x_{u_3} + 2x_{u_4}}{x_{u^*}} - 5 + 1 \\ &\leq 2 + 4 \cdot \frac{3}{\lambda - 1} - 4 \\ &\leq \frac{12}{\lambda - 1} - 2. \end{aligned}$$

Note that $\lambda > \frac{49}{5}$, we have $e(W) < 0$, a contradiction.

Case 2. $N_W(u_2) \cup N_W(u_4) \neq \emptyset$.

If $\gamma(H_1) < -1$, then by (2) and Lemma 3.4, we know that $e(W) < 0$, a contradiction. So $-1 \leq \gamma(H_1) \leq 0$. Since G^* does not contain a copy of F_6, we have $|N_U(w) \cap \{u_2, u_4\}| = 1$ and $|N_U(w) \cap \{u_1, u_3\}| = 0$ for any $w \in N_W(u_2) \cup N_W(u_4)$. Suppose $x_{u_2} \geq x_{u_4}$. Then $N_U(w) \cap V(H_1) = \{u_2\}$ for any $w \in (N_W(u_2) \cup N_W(u_4)) \cap W_0$. Otherwise, assume $N_U(w_0) \cap V(H_1) = \{u_4\}$ for some $w_0 \in (N_W(u_2) \cup N_W(u_4)) \cap W_0$. It is easy to find that $G' = G^* - w_0u_4 + w_0u_2$ is F_6-free. By Lemma 2.2, we have $\lambda(G') > \lambda(G^*)$, contradicting the maximality of G^*. Hence $N_W(u_4) \subseteq W_{H_1} \setminus W_0$. Note that $e(W) \leq 1$, we have $|N_W(u_4)| \leq 2$. By $\lambda x_{u_4} = x_{u_1} + x_{u_2} + x_{u_3} + x_{u^*} + \sum_{w \in N_{W_H}(u_4)} x_w \leq 6x_{u^*}$, we have $x_{u_4} \leq \frac{6x_{u^*}}{\lambda}$.

We first show that if $W_0 \neq \emptyset$, then $x_w \leq \frac{\lambda-1}{\lambda} x_{u^*}$ for any $w \in W_0$. Suppose that there exists a vertex $w_0 \in W_0$ such that $x_{w_0} > \frac{\lambda-1}{\lambda} x_{u^*}$. If $w_0 \in N_W(u_2)$, then $u_1, u_3, u_4 \notin N_U(w_0)$. Therefore, $\lambda x_{w_0} \leq \lambda x_{u^*} - x_{u_1} - x_{u_3} - x_{u_4}$. That is, $x_{u_1} + x_{u_3} + x_{u_4} \leq \lambda x_{u^*} - \lambda x_{w_0} < x_{u^*}$. Hence $\gamma(H_1) = \frac{x_{u_1}+2x_{u_2}+x_{u_3}+2x_{u_4}}{x_{u^*}} - 5 < 1 + 3 - 5 = -1$, a contradiction. If $w_0 \notin N_W(u_2)$, then by $N_W(u_4) \subseteq W_{H_1} \setminus W_0$, we have $u_2, u_4 \notin N_U(w_0)$. So $\lambda x_{w_0} \leq \lambda x_{u^*} - x_{u_2} - x_{u_4}$. Thus $x_{u_2} + x_{u_4} < x_{u^*}$. It follows that $\gamma(H_1) = \frac{x_{u_1}+2x_{u_2}+x_{u_3}+2x_{u_4}}{x_{u^*}} - 5 < 2 + 2 - 5 = -1$, a contradiction. Therefore, $x_w \leq \frac{\lambda-1}{\lambda} x_{u^*}$ for any $w \in W_0$.

Since G^* is F_6-free, we have $N_W(u_1) \cap N_W(u_2) = \emptyset$. Then $\sum_{i=1}^{2} \sum_{w \in N_{W_{H_1}}(u_i)} x_w = \sum_{w \in W_0 \cap \left(\cup_{i\in\{1,2\}} N_W(u_i)\right)} x_w + \sum_{w \in \left(\cup_{i\in\{1,2\}} N_W(u_i)\right)\setminus W_0} x_w$. Since $x_w \leq \frac{\lambda-1}{\lambda} x_{u^*}$ for any $w \in W_0$, we have $\sum_{w \in W_0 \cap \left(\cup_{i\in\{1,2\}} N_W(u_i)\right)} x_w \leq \frac{(\lambda-1)e(H_1,W_0)}{\lambda} x_{u^*}$. Since $d_W(w) \geq 1$ for $w \in (\cup_{i\in\{1,2\}} N_W(u_i)) \setminus W_0$, we have $\sum_{w \in \left(\cup_{i\in\{1,2\}} N_W(u_i)\right)\setminus W_0} x_w \leq \sum_{w \in W_{H_1} \setminus W_0} d_W(w) x_w \leq 2e(W) x_{u^*}$. By

$$\begin{cases} \lambda x_{u_1} = x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_{H_1}}(u_1)} x_w, \\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_{H_1}}(u_2)} x_w, \end{cases}$$

we obtain $\lambda(x_{u_1}+x_{u_2}) = x_{u_1}+x_{u_2}+x_{u_3}+2x_{u_4}+2x_{u^*}+\sum_{i=1}^{2}\sum_{w\in N_{W_{H_1}}(u_i)} x_w$. That is,

$$\begin{aligned}
x_{u_1}+x_{u_2} &= \frac{x_{u_3}+2x_{u_4}+2x_{u^*}+\sum_{i=1}^{2}\sum_{w\in N_{W_{H_1}}(u_i)} x_w}{\lambda-1} \\
&\le \frac{x_{u^*}+2\cdot\frac{6x_{u^*}}{\lambda}+2x_{u^*}+\frac{(\lambda-1)e(H_1,W_0)}{\lambda}x_{u^*}+2e(W)x_{u^*}}{\lambda-1} \\
&= \frac{(3+\frac{12}{\lambda})x_{u^*}+\frac{(\lambda-1)e(H_1,W_0)}{\lambda}x_{u^*}+2e(W)x_{u^*}}{\lambda-1}.
\end{aligned}$$

Furthermore, by Lemma 3.4, we know that $\gamma(H)=\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}-e(H)\le 0$ for any $H\in\mathcal{H}$. That is, $\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}\le e(H)$ for any $H\in\mathcal{H}\setminus\{H_1\}$. Thus

$$\begin{aligned}
&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \\
&= \sum_{H\in\mathcal{H}\setminus\{H_1\}}\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}+\sum_{u\in V(H_1)}(d_{H_1}(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W_0} d_U(w)\frac{x_w}{x_{u^*}}+\sum_{w\in W\setminus W_0} d_U(w)\frac{x_w}{x_{u^*}} \\
&\le \sum_{H\in\mathcal{H}\setminus\{H_1\}} e(H)+\frac{3+\frac{12}{\lambda}+\frac{(\lambda-1)e(H_1,W_0)}{\lambda}+2e(W)}{\lambda-1}+2+2\cdot\frac{6}{\lambda}+\frac{(\lambda-1)e(U,W_0)}{\lambda}+e(U,W\setminus W_0) \\
&= \sum_{H\in\mathcal{H}\setminus\{H_1\}} e(H)+\frac{3+\frac{12}{\lambda}}{\lambda-1}+2+2\cdot\frac{6}{\lambda}+\frac{e(H_1,W_0)}{\lambda}+\frac{(\lambda-1)e(U,W_0)}{\lambda}+\frac{2e(W)}{\lambda-1}+e(U,W\setminus W_0).
\end{aligned}$$

Note that $\lambda>\frac{49}{5}$, it can find that $\frac{3+\frac{12}{\lambda}}{\lambda-1}+2+2\cdot\frac{6}{\lambda}=\frac{15}{\lambda-1}+2<4=e(H_1)-1$. Hence

$$\begin{aligned}
&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \\
&< \sum_{H\in\mathcal{H}\setminus\{H_1\}} e(H)+e(H_1)-1+e(U,W_0)+e(W)+e(U,W\setminus W_0) \\
&= e(U_+)-1+e(U,W)+e(W),
\end{aligned}$$

which contradicts (1). This completes the proof. □

Lemma 3.10 $e(W)=0$.

Proof. Suppose that $e(W)\neq 0$. Recall that $e(W)\le 1$. Then $e(W)=1$. Combining Lemma 3.4 and Inequality (2), we obtain $0\ge\sum_{H\in\mathcal{H}}\gamma(H)\ge\sum_{u\in U_0}\frac{x_u}{x_{u^*}}$. Since $x_u>0$ for any $u\in V(G^*)$, we get $U_0=\emptyset$ and $\gamma(H)=0$ for any $H\in\mathcal{H}$. Moreover, $x_w=x_{u^*}$ for $w\in W$ and $d_U(w)\ge 1$. Let $E(W)=\{w_1w_2\}$. By Lemma 2.3, we know that $d_U(w_1)\ge 1$ and $d_U(w_2)\ge 1$. So $x_{w_1}=x_{w_2}=x_{u^*}$. Note that $\mathcal{H}$ is not empty. Choose a component $H\in\mathcal{H}$. By Lemmas 3.2, 3.3, 3.8 and 3.9, we have $H\cong K_{1,r}+e$ or C_4. If $H\cong K_{1,r}+e$ with $r\ge 2$, then H contains a copy of triangle C_3. Let $V(C_3)=\{u_1,u_2,u_3\}$. It follows that w_i is adjacent to at least two vertices of C_3 for $i \in \{1,2\}$. Otherwise, assume that w_1 is adjacent to at most one vertex, without loss of generality, suppose u_1. Then $\lambda x_{w_1} \leq x_{w_2} + \lambda x_{u^*} - x_{u_2} - x_{u_3}$. Since $\gamma(H) = 0$, we know that $x_{u_i} = x_{u^*}$ for $i \in \{1,2,3\}$. So $\lambda x_{w_1} \leq (\lambda - 1)x_{u^*}$, which contradicts $x_{w_1} = x_{u^*}$. Therefore, w_i is adjacent to at least two vertices of C_3 for $i \in \{1,2\}$. One can find that G^* contains a subgraph isomorphic to F_6, a contradiction. If $H \cong C_4$, similarly, w_i is adjacent to at least three vertices of C_4 for $i \in \{1,2\}$. One can verify that G^* contains a copy of F_6, a contradiction. Hence $e(W) = 0$. $\square$

Lemma 3.11 *For any $H \in \mathcal{H}$, we have $H \ncong C_4$.*

Proof. Suppose that $G^*[U]$ contains a component $H_2 \in \mathcal{H}$ satisfying $H_2 \cong C_4$. If $\gamma(H_2) < -1$, combining Lemma 3.4 and (2), we have $e(W) < 0$, a contradiction. So $-1 \leq \gamma(H_2) \leq 0$. We proceed by distinguishing the following two cases.

Case 1. $|W_{H_2}| \leq 1$.

For any $u \in V(H_2)$, we have $d_{G^*}(u) \leq 4$. Then $\lambda x_u = \sum_{v \in N(u)} x_v \leq 4x_{u^*}$. That is, $x_u \leq \frac{4x_{u^*}}{\lambda}$. Therefore, $\gamma(H_2) = \sum_{u \in V(H_2)} \frac{x_u}{x_{u^*}} - 4 \leq \frac{16}{\lambda} - 4 < -1$, a contradiction.

Case 2. $|W_{H_2}| \geq 2$.

Let $H_2 = u_1u_2u_3u_4u_1$. We first show the following two claims.

Claim 3.12 *There is at most one vertex w in W_{H_2} such that $d_{H_2}(w) \geq 3$.*

Proof. Suppose to the contrary that there are two vertices w_1 and w_2 in W_{H_2} satisfying $d_{H_2}(w_1) \geq 3$ and $d_{H_2}(w_2) \geq 3$. Then $|N_{H_2}(w_1) \cap N_{H_2}(w_2)| \geq 2$. If $|N_{H_2}(w_1) \cap N_{H_2}(w_2)| \geq 3$, without loss of generality, assume $\{u_1, u_2, u_3\} \subseteq N_{H_2}(w_1) \cap N_{H_2}(w_2)$, then $\{u_2\} \vee \{w_1u_1u^*u_3w_2\}$ is an F_6, a contradiction. If $|N_{H_2}(w_1) \cap N_{H_2}(w_2)| = 2$, let $N_{H_2}(w_1) \cap N_{H_2}(w_2) = \{u_i, u_j\}$ with $i \neq j \in \{1,2,3,4\}$. If u_i and u_j are adjacent, without loss of generality, assume $\{u_i, u_j\} = \{u_1, u_2\}$. Since $d_{H_2}(w_1) \geq 3$, we get that u_3 or u_4 is a neighbor of w_1. Suppose that u_3 is a neighbor of w_1. Because $d_{H_2}(w_2) \geq 3$ and $|N_{H_2}(w_1) \cap N_{H_2}(w_2)| = 2$, we know that u_4 is a neighbor of w_2. We can observe that $\{u_1\} \vee \{w_1u_2u^*u_4w_2\}$ is an F_6, a contradiction. If u_i and u_j are not adjacent, without loss of generality, assume $\{u_i, u_j\} = \{u_1, u_3\}$. Since $d_{H_2}(w_i) \geq 3$ for $i \in \{1,2\}$ and $|N_{H_2}(w_1) \cap N_{H_2}(w_2)| = 2$, suppose that u_2 is a neighbor of w_1 and u_4 is a neighbor of w_2. We can find that $\{u_1\} \vee \{w_1u_2u^*u_4w_2\}$ is an F_6, a contradiction. The proof is complete. $\square$

Claim 3.13 *For any $w \in W$ satisfying $d_{H_2}(w) \leq 2$, we have $x_w \leq \frac{\lambda-1}{\lambda}x_{u^*}$.*

Proof. Suppose that there exists a $w_0 \in W$ such that $d_{H_2}(w_0) \leq 2$ and $x_{w_0} > \frac{\lambda-1}{\lambda}x_{u^*}$. Since $d_{H_2}(w_0) \leq 2$, there are at least two vertices, denoted by u_i and u_j with $i \neq j \in$

$\{1,2,3,4\}$, in $V(H_2)$ that are not neighbors of w_0. Then $\lambda x_{w_0} \le \lambda x_{u^*} - x_{u_i} - x_{u_j}$. That is, $x_{u_i} + x_{u_j} \le \lambda x_{u^*} - \lambda x_{w_0} < x_{u^*}$. Therefore, $\gamma(H_2) = \sum_{u\in V(H_2)} \frac{x_u}{x_{u^*}} - 4 < 2+1-4 = -1$, which contradicts $-1 \le \gamma(H_2) \le 0$. The proof is complete. $\square$

Now we come back to show Lemma 3.11. By

$$\begin{cases} \lambda x_{u_1} = x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w\in N_{W_{H_2}}(u_1)} x_w, \\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*} + \sum_{w\in N_{W_{H_2}}(u_2)} x_w, \\ \lambda x_{u_3} = x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w\in N_{W_{H_2}}(u_3)} x_w, \\ \lambda x_{u_4} = x_{u_1} + x_{u_3} + x_{u^*} + \sum_{w\in N_{W_{H_2}}(u_4)} x_w, \end{cases}$$

we have $(\lambda-1)(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4}) \le 8x_{u^*} + \sum_{i=1}^4 \sum_{w\in N_{W_{H_2}}(u_i)} x_w$. That is, $(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4}) \le \frac{8x_{u^*}+\sum_{i=1}^4 \sum_{w\in N_{W_{H_2}}(u_i)} x_w}{\lambda-1}$. By Claims 3.12 and 3.13, we obtain $\sum_{i=1}^4 \sum_{w\in N_{W_{H_2}}(u_i)} x_w \le 4x_{u^*} + (e(H_2,W)-3)\cdot\frac{\lambda-1}{\lambda} x_{u^*}$ and $\sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \le 4 + (e(H_2,W)-3)\cdot\frac{\lambda-1}{\lambda} + e(U\setminus H_2, W)$. Recall that $\gamma(H)\le 0$. It follows that $\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} \le e(H)$ for any $H\in\mathcal{H}\setminus\{H_2\}$. By $\lambda > \frac{49}{5}$, we obtain

$$\begin{aligned}
&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \\
&= \sum_{H\in\mathcal{H}\setminus\{H_2\}} \sum_{u\in V(H)} (d_H(u)-1)\frac{x_u}{x_{u^*}} + \sum_{u\in V(H_2)} \frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \\
&\le \sum_{H\in\mathcal{H}\setminus\{H_2\}} e(H) + \frac{8x_{u^*} + \sum_{i=1}^4 \sum_{w\in N_{W_{H_2}}(u_i)} x_w}{(\lambda-1)x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \\
&\le \sum_{H\in\mathcal{H}\setminus\{H_2\}} e(H) + \frac{8}{\lambda-1} + \frac{4+(e(H_2,W)-3)\cdot\frac{\lambda-1}{\lambda}}{\lambda-1} + 4 + (e(H_2,W)-3)\cdot\frac{\lambda-1}{\lambda} + e(U\setminus H_2, W) \\
&= e(U_+) - 4 + \frac{8}{\lambda-1} + \frac{4}{\lambda-1} + \frac{e(H_2,W)}{\lambda} - \frac{3}{\lambda} + 4 + \frac{(\lambda-1)e(H_2,W)}{\lambda} - \frac{3(\lambda-1)}{\lambda} + e(U\setminus H_2, W) \\
&= e(U_+) + \frac{12}{\lambda-1} + e(H_2,W) - 3 + e(U\setminus H_2,W) \\
&< e(U_+) + e(U,W) + e(W) - 1,
\end{aligned}$$

which contradicts (1). This completes the proof. $\square$

Lemma 3.14 *For any $H\in\mathcal{H}$, we have $H \ncong K_{1,r}+e$ where $r\ge 2$.*

Proof. Suppose to the contrary that $G^*[U]$ contains a component $H_3 \cong K_{1,r}+e$ for some $r\ge 2$. Then H_3 contains a copy of triangle C_3. Let $V(C_3) = \{u_1,u_2,u_3\}$ with $d_{H_3}(u_1) = d_{H_3}(u_2) = 2$. We proceed by considering the following two possible cases.

Case 1. $|W_{H_3}| \le 1$.

By

$$\begin{cases} \lambda x_{u_1} \le x_{u_2} + x_{u_3} + x_{u^*} + x_{u^*}, \\ \lambda x_{u_2} \le x_{u_1} + x_{u_3} + x_{u^*} + x_{u^*}, \end{cases}$$

we get $x_{u_1} \le \frac{4x_{u^*}}{\lambda}$ and $x_{u_2} \le \frac{4x_{u^*}}{\lambda}$. Therefore, $\gamma(H_3) = \frac{x_{u_1}}{x_{u^*}} + \frac{x_{u_2}}{x_{u^*}} + (r-1)\frac{x_{u_3}}{x_{u^*}} - (r+1) \le \frac{8}{\lambda} - 2$. Note that $\lambda > \frac{49}{5}$. It is easy to get $\gamma(H_3) < -1$ and $e(W) < 0$, a contradiction.

Case 2. $|W_{H_3}| \ge 2$.

We first show that $d_{C_3}(w) \le 2$ for any $w \in W$. Otherwise, there exists a vertex $w_0 \in W$ such that $d_{C_3}(w_0) = 3$. If $r \ge 3$, then G^* contains a copy of F_6, a contradiction. So $r = 2$. That is, $H_3 \cong C_3$. If there exists another vertex w_1 satisfying $d_{C_3}(w_1) \ge 2$, without loss of generality, assume $u_1, u_2 \in N_{C_3}(w_1)$. Then $\{u_1\} \vee \{w_0 u_3 u^* u_2 w_1\}$ is an F_6, a contradiction. Thus $d_{C_3}(w) \le 1$ for any $w \in W \setminus \{w_0\}$. Suppose $x_{u_1} \ge x_{u_2} \ge x_{u_3}$. We have $N_{C_3}(w) = \{u_1\}$ or $\emptyset$ for any $w \in W \setminus \{w_0\}$. Otherwise, assume $N_{C_3}(w) = \{u_i\}$ where $i \in \{2,3\}$ for some $w_1 \in W \setminus \{w_0\}$, then $G' = G^* - wu_i + wu_1$ is F_6-free. By Lemma 2.2, we know that $\lambda(G') > \lambda(G^*)$, which contradicts the maximality of G^*. Hence $N_W(u_2) = N_W(u_3) = \{w_0\}$. It follows that $\lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*} + x_{w_0} \le 4x_{u^*}$ and $\lambda x_{u_3} = x_{u_1} + x_{u_2} + x_{u^*} + x_{w_0} \le 4x_{u^*}$. Equivalently, $x_{u_2} \le \frac{4}{\lambda}x_{u^*}$ and $x_{u_3} \le \frac{4}{\lambda}x_{u^*}$. By $\lambda > \frac{49}{5}$, we obtain $\gamma(H_3) = \sum_{i=1}^{3} \frac{x_{u_i}}{x_{u^*}} - 3 \le 1 + \frac{8}{\lambda} - 3 < -1$ and $e(W) < 0$, a contradiction. Therefore, $d_{C_3}(w) \le 2$ for any $w \in W$.

If there exists another non-trivial component H', then $H' \cong K_{1,r'} + e$ or $K_{1,p}$ or $D_{a,b}$ where $r' \ge 2$ and $p, a, b \ge 1$. If $H' \cong K_{1,p}$ or $D_{a,b}$, by Lemma 3.3, we know $\gamma(H') \le -1$. Since $d_{C_3}(w) \le 2$ for any $w \in W$ and Lemma 3.10, we have $N_{G^*}(w) \subset N_{G^*}(u^*)$. Furthermore, $\lambda x_w = \sum_{u \in N_{G^*}(w)} x_u < \sum_{u \in N_{G^*}(u^*)} x_u = \lambda x_{u^*}$. That is, $x_w < x_{u^*}$ for any $w \in W$. Therefore, $\sum_{w \in W} d_U(w) \frac{x_w}{x_{u^*}} < e(U, W)$. By the first inequality of (2), we have $e(W) < \sum_{H \in \mathcal{H}} \gamma(H) - \sum_{u \in U_0} \frac{x_u}{x_{u^*}} + 1 \le -1 - \sum_{u \in U_0} \frac{x_u}{x_{u^*}} + 1 \le 0$, a contradiction. Next we consider $H' \cong K_{1,r'} + e$ with $r' \ge 2$. Let C_3' be a triangle in $K_{1,r'} + e$ with $V(C_3') = \{v_1, v_2, v_3\}$ and $d_{H'}(v_1) = d_{H'}(v_2) = 2$. Similarly, $d_{C_3'}(w) \le 2$ for any $w \in W$. Now we have $x_w \le \frac{\lambda - 1}{\lambda} x_{u^*}$ for any $w \in W$. Otherwise, there is a vertex $w_0 \in W$ such that $x_{w_0} > \frac{\lambda-1}{\lambda} x_{u^*}$. Since $d_{C_3}(w_0) \le 2$ and $d_{C_3'}(w_0) \le 2$, It follows that there are $u_i \in V(C_3)$ and $v_j \in V(C_3')$ which are not neighbors of w_0 for some $i, j \in \{1,2,3\}$. Because $e(W) = 0$, we obtain $\lambda x_{w_0} \le \lambda x_{u^*} - x_{u_i} - x_{v_j}$. Equivalently, $x_{u_i} + x_{v_j} \le \lambda x_{u^*} - \lambda x_{w_0} < x_{u^*}$. Thus

$$\begin{aligned} \gamma(H_3) + \gamma(H') = & \sum_{u \in V(H_3)} (d_{H_3}(u) - 1)\frac{x_u}{x_{u^*}} - \frac{x_{u_i}}{x_{u^*}} + \frac{x_{u_i}}{x_{u^*}} - e(H_3) \\ & + \sum_{u \in V(H')} (d_{H'}(u) - 1)\frac{x_u}{x_{u^*}} - \frac{x_{v_j}}{x_{u^*}} + \frac{x_{v_j}}{x_{u^*}} - e(H') \end{aligned}$$

$$
\begin{aligned}
&= \frac{x_{u_1}}{x_{u^*}} + \frac{x_{u_2}}{x_{u^*}} + (r-1)\frac{x_{u_3}}{x_{u^*}} - \frac{x_{u_i}}{x_{u^*}} + \frac{x_{u_i}}{x_{u^*}} - (r+1)\\
&+ \frac{x_{v_1}}{x_{u^*}} + \frac{x_{v_2}}{x_{u^*}} + (r'-1)\frac{x_{v_3}}{x_{u^*}} - \frac{x_{v_j}}{x_{u^*}} + \frac{x_{v_j}}{x_{u^*}} - (r'+1)\\
&\le r + \frac{x_{u_i}}{x_{u^*}} - (r+1) + r' + \frac{x_{v_j}}{x_{u^*}} - (r'+1)\\
&< r - (r+1) + r' - (r'+1) + 1 = -1.
\end{aligned}
$$

By (2) and Lemma 3.4, we find that $e(W) \le \sum_{H\in\mathcal{H}} \gamma(H) - \sum_{u\in U_0} \frac{x_u}{x_{u^*}} + 1 < -1+0+1 = 0$, a contradiction. Therefore, $x_w \le \frac{\lambda-1}{\lambda} x_{u^*}$ for any $w \in W$. Furthermore, $\sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \le \frac{\lambda-1}{\lambda} e(U,W)$. Since

$$
\begin{cases}
\lambda x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + \sum_{w\in N_W(u_1)} x_w,\\
\lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*} + \sum_{w\in N_W(u_2)} x_w,
\end{cases}
$$

we have

$$
\begin{aligned}
x_{u_1} + x_{u_2} &\le \frac{4x_{u^*} + \sum_{i=1}^{2}\sum_{w\in N_W(u_i)} x_w}{\lambda-1}\\
&\le \frac{4x_{u^*} + \frac{\lambda-1}{\lambda} e(H_3,W)x_{u^*}}{\lambda-1}\\
&= \frac{4x_{u^*}}{\lambda-1} + \frac{e(H_3,W)x_{u^*}}{\lambda}.
\end{aligned}
$$

By $\lambda > \frac{49}{5}$, we have

$$
\begin{aligned}
&\sum_{u\in U_+} (d_U(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}}\\
&= \sum_{H\in\mathcal{H}\setminus\{H_3\}} \sum_{u\in V(H)} (d_H(u)-1)\frac{x_u}{x_{u^*}} + \sum_{u\in V(H_3)} (d_{H_3}(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}}\\
&\le \sum_{H\in\mathcal{H}\setminus\{H_3\}} e(H) + \frac{4}{\lambda-1} + \frac{e(H_3,W)}{\lambda} + (r-1) + \frac{\lambda-1}{\lambda} e(U,W)\\
&\le e(U_+) - (r+1) + \frac{4}{\lambda-1} + e(U,W) + (r-1)\\
&= e(U_+) + e(U,W) + \frac{4}{\lambda-1} - 2\\
&< e(U_+) + e(U,W) + e(W) - 1,
\end{aligned}
$$

which contradicts (1). Thus there is exactly one non-trivial component H_3. By Lemma 3.1, we know that $r \ge 3$.

By (2), we have $0 = e(W) \le \gamma(H_3) - \sum_{u\in U_0} \frac{x_u}{x_{u^*}} + 1$. By Lemma 3.4, we get $0 \le 0 - \sum_{u\in U_0} \frac{x_u}{x_{u^*}} + 1$. That is, $\sum_{u\in U_0} \frac{x_u}{x_{u^*}} \le 1$. Since $\lambda x_{u^*} = x_{u_1} + x_{u_2} + x_{u_3} + \sum_{v\in N_{H_3}(u_3)\setminus\{u_1,u_2\}} x_v +$

$\sum_{u\in U_0} x_u$, we have

$$\begin{aligned}\sum_{v\in N_{H_3}(u_3)\setminus\{u_1,u_2\}} x_v &= \lambda x_{u^*} - x_{u_1} - x_{u_2} - x_{u_3} - \sum_{u\in U_0} x_u\\ &\geq (\lambda-1)x_{u^*} - x_{u_1} - x_{u_2} - x_{u_3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda x_{u_3} &= x_{u_1} + x_{u_2} + x_{u^*} + \sum_{v\in N_{H_3}(u_3)\setminus\{u_1,u_2\}} x_v + \sum_{w\in N_W(u_3)} x_w\\ &\geq x_{u_1} + x_{u_2} + x_{u^*} + (\lambda-1)x_{u^*} - x_{u_1} - x_{u_2} - x_{u_3} + \sum_{w\in N_W(u_3)} x_w\\ &= \lambda x_{u^*} - x_{u_3} + \sum_{w\in N_W(u_3)} x_w.\end{aligned}$$

It follows that $x_{u_3} \geq \frac{\lambda}{\lambda+1} x_{u^*}$ and $\sum_{w\in N_W(u_3)} x_w \leq (\lambda+1)x_{u_3} - \lambda x_{u^*} \leq x_{u^*}$. For any $w \in W$, if $w \in N_W(u_3)$, then $d_{H_3}(w) = 1$. Otherwise, recall that $r \geq 3$, there is an F_6 in G^*. So $\lambda x_w \leq x_{u_3} + \sum_{u\in U_0} x_u \leq 2x_{u^*}$. If $w \notin N_W(u_3)$, then $\lambda x_w \leq \lambda x_{u^*} - x_{u_3} \leq \lambda x_{u^*} - \frac{\lambda}{\lambda+1}x_{u^*} = \frac{\lambda^2}{\lambda+1}x_{u^*}$. That is, $x_w \leq \frac{\lambda}{\lambda+1}x_{u^*}$. Note that $\lambda > \frac{49}{5}$, we have $x_w \leq \frac{\lambda}{\lambda+1}x_{u^*}$ for any $w \in W$. By

$$\begin{cases}\lambda x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + \sum_{w\in N_W(u_1)} x_w,\\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*} + \sum_{w\in N_W(u_2)} x_w,\end{cases}$$

we have

$$\begin{aligned}\lambda(x_{u_1} + x_{u_2}) &\leq 6x_{u^*} + \sum_{i=1}^{2}\sum_{w\in N_W(u_i)} x_w\\ &\leq 6x_{u^*} + e(H_3, W)\cdot\frac{\lambda}{\lambda+1}x_{u^*}.\end{aligned}$$

Equivalently, $x_{u_1} + x_{u_2} \leq \frac{6}{\lambda}x_{u^*} + \frac{e(H_3,W)}{\lambda+1}x_{u^*}$. By $\lambda > \frac{49}{5}$, we have

$$\begin{aligned}&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}}\\ &= \sum_{u\in V(H_3)}(d_{H_3}(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}}\\ &\leq \frac{6}{\lambda} + \frac{e(H_3,W)}{\lambda+1} + (r-1) + \frac{\lambda}{\lambda+1}e(U,W)\\ &= \frac{6}{\lambda} + e(H_3,W) + \frac{\lambda}{\lambda+1}e(U\setminus H_3, W) + (r-1)\\ &\leq \frac{6}{\lambda} + e(U,W) + (r-1)\end{aligned}$$

$$< r + e(U,W)$$
$$= e(U_+) + e(U,W) - 1,$$

which contradicts (1). This completes the proof. $\square$

Proof of Theorem 1.2. By Lemmas 3.2, 3.8, 3.9, 3.11, and 3.14, we have that each non-trivial component of $G^*[U]$ is $K_{1,r}$ or $D_{a,b}$ where $r, a, b \geq 1$. By (2) and Lemma 3.3, we obtain that the non-trivial component number of $G^*[U]$ is at most 1. If the non-trivial component number of $G^*[U]$ is 0, then G^* is bipartite. By Lemma 2.1, $\lambda \leq \sqrt{m} < \frac{1+\sqrt{4m-3}}{2}$, a contradiction. Hence the non-trivial component number of $G^*[U]$ is 1. Let H be the unique non-trivial component of $G^*[U]$. Then $H \cong K_{1,r}$ or $D_{a,b}$. Since $\gamma(H) \leq -1$ and $0 = e(W) \leq \gamma(H) - \sum_{u \in U_0} \frac{x_u}{x_{u^*}} + 1$, we have $\gamma(H) = -1$ and $U_0 = \emptyset$. By (2), we also have $x_w = x_{u^*}$ for any $w \in W$ and $d_U(w) \geq 1$.

If $H \cong D_{a,b}$, let $V(D_{a,b}) = \{u_1, u_2, u_{11}, \cdots, u_{1a}, u_{21}, \cdots, u_{2b}\}$ with $N_H(u_1) = \{u_2, u_{11}, \cdots, u_{1a}\}$ and $N_H(u_2) = \{u_1, u_{21}, \cdots, u_{2b}\}$. According to the definition of $\gamma(H)$ and $\gamma(H) = -1$, we have $x_{u_1} = x_{u_2} = x_{u^*}$. If $W_H = \emptyset$, then $\lambda x_{u^*} = x_{u_1} + x_{u_2} + \sum_{i=1}^{a} x_{u_{1i}} + \sum_{j=1}^{b} x_{u_{2j}} = 2x_{u^*} + \sum_{i=1}^{a} x_{u_{1i}} + \sum_{j=1}^{b} x_{u_{2j}}$ and $\lambda x_{u_1} = x_{u_2} + x_{u^*} + \sum_{i=1}^{a} x_{u_{1i}} = 2x_{u^*} + \sum_{i=1}^{a} x_{u_{1i}}$. Recall that $x_{u_1} = x_{u^*}$. It follows that $\sum_{j=1}^{b} x_{u_{2j}} = 0$. which contradicts $x_u > 0$ for any $u \in V(G^*)$ and $b \geq 1$. If $|W_H| = 1$, let $W_H = \{w\}$. By Lemma 2.3, we know $d_U(w) \geq 2$. Therefore, $x_w = x_{u^*}$. Furthermore, $\lambda x_w = \sum_{u \in N(w)} x_u$ and $\lambda x_{u^*} = \sum_{u \in N(u^*)} x_u$. By $N(w) \subseteq N(u^*)$, we obtain $N(w) = N(u^*)$. If a or $b \geq 2$, suppose $a \geq 2$. It is easy to find $\{u_1\} \vee \{u_{11} w u_2 u^* u_{12}\}$ is an F_6, a contradiction. Thus $a = b = 1$ and $m = 11$, which contradicts $m \geq 88$. If $|W_H| \geq 2$, let $w_1, w_2 \in W_H$. By Lemma 2.3, we have $d_U(w_1) \geq 2$ and $d_U(w_2) \geq 2$. Therefore, $x_{w_1} = x_{w_2} = x_{u^*}$. Furthermore, $N(w_1) = N(w_2) = N(u^*)$. Observed that $\{u_1\} \vee \{w_1 u_{11} u^* u_2 w_2\}$ is an F_6, a contradiction. Hence $H \ncong D_{a,b}$. That is, $H \cong K_{1,r}$.

Let $V(H) = \{u_0, u_1, \cdots, u_r\}$ with the central vertex u_0. By the definition of $\gamma(H)$ and $\gamma(H) = -1$, we know $x_{u_0} = x_{u^*}$. Since $\lambda x_{u^*} = x_{u_0} + \sum_{i=1}^{r} x_{u_i} = x_{u^*} + \sum_{i=1}^{r} x_{u_i}$ and $\lambda x_{u_0} = x_{u^*} + \sum_{i=1}^{r} x_{u_i} + \sum_{w \in N_W(u_0)} x_w$, we have $\sum_{w \in N_W(u_0)} x_w = 0$. Hence $N_W(u_0) = \emptyset$. If $W \neq \emptyset$, then $d_U(w) \geq 2$ for any $w \in W$. So $x_w = x_{u^*}$. However, $\lambda x_w = \sum_{u \in N(w)} x_u \leq \sum_{u \in N(u^*)} x_u - x_{u_0} < \lambda x_{u^*}$. It is a contradiction. Therefore, $W = \emptyset$. Thus $G^* \cong K_1 \vee K_{1,r}$ with $2r + 1 = m$. Equivalently, $G^* \cong K_2 \vee \frac{m-1}{2} K_1$. This completes the proof. $\square$

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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