Rainbow directed version of Dirac's theorem

Hao Li[1], Luyi Li[1,2], Ping Li[3], Xueliang Li[2]

[1]Laboratoire Interdisciplinaire des Sciences du Numérique

CNRS-Université Paris-Saclay, Orsay 91405, France

[2]Center for Combinatorics and LPMC

Nankai University, Tianjin 300071, China

[3]School of Mathematics and Statistics

Shaanxi Normal University, Xi'an, Shaanxi 710062, China

Emails: li@lri.fr, liluyi@mail.nankai.edu.cn, lp-math@snnu.edu.cn, lxl@nankai.edu.cn

Abstract

Let $\mathcal{G} = \{G_i : i \in [s]\}$ be a collection of not necessarily distinct graphs on the same vertex set V. A graph H is called *rainbow* in $\mathcal{G}$ if any two edges of H belong to different graphs of $\mathcal{G}$. In 2020, Joos and Kim proved a rainbow version of Dirac's theorem. In this paper, we prove a rainbow directed version of Dirac's theorem asymptotically: For each $0 < \varepsilon < 1$, there exists an integer N such that when $n \geq N$ the following holds. Let $\mathcal{D} = \{D_i : i \in [n]\}$ be a collection of n-vertex digraphs on the same vertex set V. If both the out-degree and the in-degree of v are at least $\left(\frac{1}{2} + \varepsilon\right) n$ for each vertex $v \in V$ and each integer $i \in [n]$, then $\mathcal{D}$ contains a rainbow Hamiltonian cycle. Furthermore, we provide a sufficient condition for the existence of arbitrary rainbow tournaments in a collection of n-vertex digraphs, where a *tournament* is an oriented graph of a complete graph.

Keywords: rainbow; directed Hamiltonian cycle; tournament

AMS subject classification 2020: 05C15, 05C38, 05C07.

1 Introduction

Hamiltonicity of graphs is a historic subject in graph theory which has been researched extensively. A well-known result by Dirac [12] asserts that n-vertex graphs satisfying $\delta(G) \geq n/2$ are Hamiltonian, where $\delta(G)$ denotes the minimum degree of G. In 2020, Joos and Kim [16] introduced the concept of transversal in a collection of graphs on the same vertex set.

For a collection $\mathcal{G} = \{G_i : i \in [t]\}$ of not necessarily distinct graphs with common vertex set V, a simple graph H is a *partial transversal* of $\mathcal{G}$ if $V(H) \subseteq V$, $|E(H)| \leq t$ and there exists an injection $\theta : E(H) \to [t]$ such that $e \in E(G_{\theta(e)})$ for every $e \in E(H)$, where $[t] = \{1, 2, \ldots, t\}$. In particular, H is a *transversal* of $\mathcal{G}$ if H is a partial transversal of $\mathcal{G}$ with $|E(H)| = t$. Since all edges of H come from different graphs of $\mathcal{G}$, we also call H a *rainbow* subgraph of $\mathcal{G}$. Joos and Kim [16] proved a result which can be seen as a generalization of Dirac's theorem.

Theorem 1.1. *[16] Suppose that $\mathcal{G} = \{G_i : i \in [n]\}$ is a collection of not necessarily distinct n-vertex graphs with the same vertex set and $\delta(G_i) \geq \frac{n}{2}$ for $i \in [n]$. Then there exists a rainbow Hamiltonian cycle.*

Since then, many scholars focused on generalizing results in extremal graph theory to the setting of graph transversals, including cycles [2, 6, 7, 11, 14, 18, 19, 21], matchings [1, 3, 13], trees [8, 18] and factors [10, 20]. Recently, some results in digraph theory are investigated in a collection of digraphs. In 2023, Chakraborti, Kim, Lee and Seo [9] considered the existence of rainbow Hamiltonian paths in a collection of tournaments, where a *tournament* is an oriented graph of a complete graph.

Theorem 1.2. *[9] Let $\mathcal{T} = \{T_i : i \in [n-1]\}$ be a collection of tournaments with the same vertex set $V(\mathcal{T}) = n$. If n is sufficiently large, then $\mathcal{T}$ contains a rainbow Hamiltonian path.*

In fact, there are many different sufficient conditions to guarantee Hamiltonicity of digraphs, please refer to a survey [17]. In 1960, Ghouila-Houri [15] proved that if the degree (the sum of in-degree and out-degree) of every vertex in a strong n-vertex digraph D is at least n, then D is Hamiltonian. It is natural that we can deduce a corollary with respect to the minimum semi-degree $\delta^0(D)$, where $\delta^0(D)$ denotes the minimum value of both in-degrees and out-degrees of all vertices in D.

Theorem 1.3. *[15] Let D be a digraph of order $n \geq 2$. If $\delta^0(D) \geq \frac{n}{2}$, then D is Hamiltonian.*

Inspired by the above results, we consider the rainbow Hamiltonicity of a collection of n-vertex digraphs. The main results are as follows.

Theorem 1.4. *Let $0 < \varepsilon < 1$. Then there exists an integer N such that when $n \geq N$ the following holds. Given a collection $\mathcal{D} = \{D_i : i \in [n]\}$ of n-vertex digraphs on the same vertex set V such that $\delta^0(D_i) \geq (\frac{1}{2} + \varepsilon)n$ for all $i \in [n]$, then $\mathcal{D}$ contains a rainbow Hamiltonian cycle.*

Theorem 1.5. *Suppose $\mathcal{D} = \{D_i : i \in [\binom{s}{2}]\}$ is a collection of n-vertex digraphs with the same vertex set V. If $\delta^0(D_i) \geq (1 - \frac{1}{s-1})n$ for $i \in [\binom{s}{2} - 1]$ and $\delta^0(D_{\binom{s}{2}}) > (1 - \frac{1}{s-1})n$, then $\mathcal{D}$ contains a rainbow copy of arbitrary tournaments T_s.*

Remark 1.1. A directed graph D is called a *biorientation* of a graph G if D is obtained from G by replacing each edge $\{x, y\}$ of G by either xy or yx or the pair xy and yx (loops are not allowed). For a graph G, the *complete biorientation* of G (denoted by $\overrightarrow{G}$) is a biorientation D of G such that $xy \in A(D)$ implies $yx \in A(D)$. Let $T_{n,s-1}$ be a balanced complete $(s-1)$-partite graph and $\overrightarrow{T}_{n,s-1}$ be a complete biorientation of $T_{n,s-1}$. Note that if $(s-1)|n$ and $\mathcal{D} = \{D_1, \dots, D_{\binom{s}{2}}\}$ consists of $\binom{s}{2}$ copies of $\overrightarrow{T}_{n,s-1}$, then $\delta^0(D_i) = (1 - \frac{1}{s-1})n$ for each $i \in [\binom{s}{2}]$ and $\mathcal{D}$ does not contain a rainbow tournament $\overrightarrow{T}_{n,s-1}$. This implies that the bound of Theorem 1.5 is sharp.

Before we proceed further, let us introduce some additional terminology and notation. A *directed graph* (or just *digraph*) D consists of a vertex set $V(D)$ and an arc set $A(D)$ of ordered pairs of distinct members of $V(D)$. We use uv to denote the arc from u to v. Given a digraph $D = (V, A)$ and a vertex $v \in V$, set $N_D^+(v) = \{u \in V : vu \in A\}$ and $N_D^-(v) = \{u \in V : uv \in A\}$ as *out-neighbourhood* and *in-neighbourhood* of v, respectively. We use $d_D^+(v)$ and $d_D^-(v)$ to denote the *out-degree* and *in-degree* of v in D. Set $\delta^+(D) = \min\{d_D^+(v) : v \in V\}$ and $\delta^-(D) = \min\{d_D^-(v) : v \in V\}$. Clearly, $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. For two positive integers $a < b$ we use $[a, b]$ to denote the set $\{a, a+1, \dots, b-1, b\}$. For terminology and notation not defined here, we refer the reader to [4, 5].

2 Proofs of Theorems 1.4 and 1.5

We first prove Theorems 1.4 which started with several lemmas.

Lemma 2.1. *Let $\mathcal{D} = \{D_i : i \in [n+1]\}$ be a collection of n-vertex digraphs on the same vertex set V such that $\delta^0(D_i) \geq \frac{n}{2}$ for all $i \in [n+1]$. Then $\mathcal{D}$ contains a rainbow Hamiltonian cycle.*

Proof. Suppose that $C = x_1x_2 \cdots x_tx_1$ is a maximum-length rainbow cycle with $x_{t+1} = x_1$ and $x_ix_{i+1} \in D_i$ for all $i \in [t]$ in $\mathcal{D}$. Let $\mathcal{D}' = \{D'_i = D_i - V(C) : i \in [t+1, n+1]\}$ and $P = y_1y_2 \cdots y_s$ be a maximum-length rainbow path with $y_iy_{i+1} \in D'_{t+i}$ for all $i \in [s-1]$ in $\mathcal{D}'$. We claim that $t \geq \frac{n}{2} + 1$. Clearly, if we choose a maximum directed rainbow path $L = v_1v_2 \dots v_z$ (without loss of generality, assume that $v_iv_{i+1} \in A(D_i)$), then $N_{D_n}(v_z) \subseteq V(P)$; otherwise we can obtain a longer directed rainbow path than L, a contradiction. Since $d^+_{D_n}(v) \geq \frac{n}{2}$, we can choose the minimum integer $j \in [z]$ with $v_j \in N_{D_n}(v_z)$, then $v_zv_jLv_z$ is a directed cycle of length at least $\frac{n}{2} + 1$.

It follows from $V(C) \cap V(P) = \emptyset$ that $t + s \leq n$. Note that neither C nor P contains the edges of D_n and D_{n+1}. We define the following two sets:

$$A = \{i \in [t] : x_{i-1}y_1 \in A(D_n)\},$$

and

$$B = \{i \in [t] : y_s x_i \in A(D_{n+1})\}.$$

If $A \cap B \neq \emptyset$, choose an integer $i \in A \cap B$, then $x_i C x_{i-1} y_1 P y_s x_i$ is a rainbow cycle of length $s + t$ in $\mathcal{D}$. This contradicts the hypothesis that C is a maximum rainbow cycle in $\mathcal{D}$. Therefore, $A \cap B = \emptyset$. The maximality of P implies that $N^-_{D_n}(y_1) \subseteq V(P) \cup V(C)$, and hence $d^-_{D_n}(y_1) = d^-_{D_n[V(P)]}(y_1) + |A|$. Since $d^-_{D_n}(y_1) \geq \frac{n}{2}$ and $d^-_{D_n[V(P)]}(y_1) \leq s - 1$, we have

$$|A| \geq \frac{n}{2} - s + 1.$$

A similar argument yields

$$|B| \geq \frac{n}{2} - s + 1.$$

Recall that $t + s \leq n$. It follows from $t \geq \frac{n}{2} + 1$ that A and B are nonempty. Then

$$|A| + |B| \geq n - 2s + 2 \geq t - s + 2. \tag{1}$$

Since A and B are disjoint, we have that

$$|A \cup B| \geq t - s + 2. \tag{2}$$

Assume that $Q = x_{\ell_1} x_{\ell_1+1} \cdots x_{\ell_2}$ is a subpath of C such that $i \notin A \cup B$ for each $i \in [\ell_1, \ell_2]$. Then

$$|V(Q)| \leq |V(C)| - |A \cup B| \leq s - 2.$$

Since A and B are nonempty sets, there are two positive integers i and $k \in [s-1]$ such that $i \in A$, $i + k \in B$ and

$$i + j \notin A \cup B \text{ for all } 1 \leq j < k, \tag{3}$$

where addition is taken modulo t. Recall that $s+t \leq n$ and $t \geq \frac{n}{2}+1$, we have $s \leq \frac{n}{2}-1 \leq t-2$. Since $k \leq s - 1$, it follows that $x_{i+k} \neq x_{i-1}$. Thus $x_{i+k} C x_{i-1} y_1 P y_s x_{i+k}$ is a rainbow cycle of length $t + s - k$, which contradicts the hypothesis that C is a maximum rainbow cycle in $\mathcal{D}$. □

Note that if $n + 1$ digraphs are allowed in $\mathcal{D}$, then the degree condition $\delta^0(D_i) \geq \frac{n}{2}$ for each $i \in [n+1]$ is sufficient for the existence of rainbow Hamiltonian cycles in a collection of digraphs. Theorems 1.4 indicates that if only n digraphs are allowed in $\mathcal{D}$, then we can elevate the semi-degree condition by $o(n)$ to ensure the existence of rainbow Hamiltonian cycles in a collection of digraphs. The following corollary is necessary for the proof of Theorems 1.4.

Corollary 2.1. Let $\mathcal{D} = \{D_i : i \in [n]\}$ be a collection of n-vertex digraphs on the same vertex set V such that $\delta^0(D_i) \geq \frac{n}{2}$ for all $i \in [n]$. Then $\mathcal{D}$ contains a rainbow Hamiltonian path.

To prove the rainbow Hamiltonicity of a collection of n-vertex digraph, we introduce some definitions.

Definition 1. Let $\mathcal{D} = \{D_i : i \in [n]\}$ be a collection of n-vertex digraphs on the same vertex set V. For any two vertices $x_1, x_2 \in V$ and four integers $s, i, j, k \in [n]$, a rainbow path $P = z_1z_2z_3z_4$ is an *absorbing path* for (x_1, x_2) if the following conditions hold:

(1) $z_1z_2 \in A(D_i)$, $z_2z_3 \in A(D_j)$ and $z_3z_4 \in A(D_k)$;

(2) $V(P) \cap \{x_1, x_2\} = \emptyset$;

(3) $z_2x_1 \in A(D_s)$ and $x_2z_3 \in A(D_j)$.

In addition, we use $\mathcal{L}_{s,i,j,k}(x_1, x_2)$ to denote the set of absorbing paths P for (x_1, x_2) with respect to an ordered quadruple (s, i, j, k). Specifically, $\mathcal{L}_{s,i,j,k}(x_1, x_2) = \emptyset$ if $s \in \{i, j, k\}$.

Lemma 2.2. *For any $0 < \varepsilon < 1$, there exists an integer N such that the following holds for any integers $n \geq N$. Given a collection $\mathcal{D} = \{D_i : i \in [n]\}$ of n-vertex digraphs on the same vertex set V such that $\delta^0(D_i) \geq (\frac{1}{2} + \varepsilon)n$ for all $i \in [n]$, then $|\mathcal{L}_{s,i,j,k}(x_1, x_2)| > \frac{\varepsilon n^4}{8}$ for any two vertices $x_1, x_2 \in V$ and four integers $s, i, j, k \in [n]$.*

Proof. Fix two vertices $x_1, x_2 \in V$ and four integers $s, i, j, k \in [n]$. First, choose a vertex $z_2 \in V - \{x_1, x_2\}$ such that $z_2x_1 \in A(D_s)$ and a vertex $z_1 \in V - \{x_1, x_2, z_2\}$ such that $z_1z_2 \in A(D_i)$. It is clear that there are at least $(\frac{n}{2} + \varepsilon n - 1)(\frac{n}{2} + \varepsilon n - 2)$ choices for z_1 and z_2. Choose a vertex $z_3 \in V - \{x_1, x_2, z_1, z_2\}$ such that $z_2z_3, x_2z_3 \in A(D_j)$. Then there are at least

$$2\left(\frac{n}{2} + \varepsilon n - 3\right) - (n - 4) = 2\varepsilon n - 2$$

choices for z_3. By a similar argument, there are at least $\frac{n}{2} + \varepsilon n - 5$ choices for $z_4 \in V - \{x_1, x_2, z_1, z_2, z_3\}$ such that $z_3z_4 \in A(D_k)$. Hence, there are at least

$$\left(\frac{n}{2} + \varepsilon n - 1\right)\left(\frac{n}{2} + \varepsilon n - 2\right)(2\varepsilon n - 2)\left(\frac{n}{2} + \varepsilon n - 5\right) > \frac{\varepsilon n^4}{8}$$

choices of absorbing paths for (x_1, x_2) with respect to (s, i, j, k) for n large enough. □

Lemma 2.3. *For n, μ, ε with $\frac{1}{n} \ll \mu \ll \varepsilon < 1$ suppose that $\mathcal{D} = \{D_i : i \in [n]\}$ is a collection of n-vertex digraphs on the same vertex set V such that $\delta^0(D_i) \geq (\frac{1}{2} + \varepsilon)n$ for all $i \in [n]$. Then there is a family $\mathcal{F}'$ of pairwise vertex-disjoint rainbow directed 4-paths such that the following statements hold.*

1. *$|\mathcal{F}'| \leq \mu n$.*
2. *Any two elements of $\mathcal{F}'$ contain no common color.*
3. *For any two vertices $x_1, x_2 \in V$ and an integer $s \in [n]$, there are three distinct integers $i, j, k \in [n]$ such that $\mathcal{L}_{s,i,j,k}(x_1, x_2) \cap \mathcal{F}' \neq \emptyset$.*

Proof. Without loss of generality, we assume that ℓ is an integer and $3\ell = \mu n$. Let $\mathcal{P}_i$ be the set of all 4-paths $z_1z_2z_3z_4$ such that $z_1z_2 \in A(D_{3i-2})$, $z_2z_3 \in A(D_{3i-1})$ and $z_3z_4 \in A(D_{3i})$ for all $i \in [\ell]$. Now we construct a random set $\mathcal{F}$ of size ℓ as follows: For all $i \in [\ell]$, let us take an element from $\mathcal{P}_i$ and put it into $\mathcal{F}$ uniformly and independently.

For $s \in [n]$ and a pair of vertices (x_1, x_2) of V, let

$$\mathcal{L}_s(x_1, x_2) = \bigcup_{i=1}^{\ell} (\mathcal{L}_{s,3i-2,3i-1,3i}(x_1, x_2) \cap \mathcal{F})$$

and X_i be an indicator random variable as follows:

$$X_i = \begin{cases} 1, & \mathcal{L}_{s,3i-2,3i-1,3i}(x_1, x_2) \cap \mathcal{F} \neq \emptyset; \\ 0, & \text{otherwise,} \end{cases}$$

and let $X = |\mathcal{L}_s(x_1, x_2)| = \sum_{i=1}^{\ell} X_i$. It is clear that all X_is are independent. The definition of $\mathcal{L}_{s,3i-2,3i-1,3i}(x_1, x_2)$ implies that $\mathcal{L}_{s,3i-2,3i-1,3i}(x_1, x_2) \subseteq \mathcal{P}_i$. If $s \notin \{3i-2, 3i-1, 3i\}$, then from Lemma 2.2,

$$P(X_i = 1) = \frac{|\mathcal{L}_{s,3i-2,3i-1,3i}(x_1, x_2)|}{|\mathcal{P}_i|} \geq \frac{\frac{\varepsilon n^4}{8}}{n^4} \geq \frac{\varepsilon}{8};$$

if $s \in \{3i-2, 3i-1, 3i\}$, then $P(X_i = 1) = 0$. Hence,

$$E(X) = \sum_{i=1}^{\ell} E(X_i) \geq \frac{\varepsilon(\ell - 1)}{8} = \frac{\varepsilon \mu n}{25}.$$

Since X obeys the binomial distribution, Using Chernoff's bound, we have

$$P\left(X \leq \frac{E(X)}{2}\right) \leq P\left(|X - E(X)| \geq \frac{E(X)}{2}\right) \leq 2e^{-\frac{E(X)}{12}}.$$

Hence, we have

$$P\left(X \leq \frac{\varepsilon \mu n}{50}\right) \leq P\left(|X - E(X)| \geq \frac{\varepsilon \mu n}{50}\right) \leq 2e^{-\frac{\varepsilon \mu n}{300}}.$$

Let us come back to considering the set $\mathcal{F}$. Note that $\mathcal{F}$ is a set of rainbow 4-paths and two elements of $\mathcal{F}$ may intersect. Since any two elements of $\mathcal{F}$ are chosen from different sets of $\{\mathcal{P}_i : i \in [\ell]\}$ independently, let $S_{i,j}$ be an indicator random variable as follows:

$$S_{i,j} = \begin{cases} 1, & \text{the path chosen from } \mathcal{P}_i \text{ intersects with the path chosen from } \mathcal{P}_j; \\ 0, & \text{otherwise.} \end{cases}$$

Define S as the number of pairs of rainbow paths in $\mathcal{F}$ that intersect in at least one vertex. Then $S = \sum_{i,j \in [\ell]} S_{i,j}$. For $i, j \in [\ell]$, let T be the set of unordered pair (P_i, P_j) with $P_i \in \mathcal{P}_i$, $P_j \in \mathcal{P}_j$ and $V(P_i) \cap V(P_j) \neq \emptyset$. There are at most $n(n-1)(n-2)(n-3)$ choices for

P_i. Fixing P_i, there are at most $16(n-1)(n-2)(n-3)$ choices for P_j, which implies that $|T| < 16n^7$. Note that

$$|\mathcal{P}_i| \geq n\left(\frac{n}{2}+\varepsilon n\right)\left(\frac{n}{2}+\varepsilon n-1\right)\left(\frac{n}{2}+\varepsilon n-2\right) > \frac{n^4}{8}.$$

Similarly, we have $|\mathcal{P}_j| > \frac{n^4}{8}$. Fix two integers $i, j \in [\ell]$. The probability choosing each pair (P_i, P_j) with $V(P_i) \cap V(P_j) \neq \emptyset$ is $\frac{1}{|\mathcal{P}_i||\mathcal{P}_j|} \leq \frac{1}{(\frac{n^4}{8})^2} = \frac{64}{n^8}$ when n is sufficiently large. Consequently, we have

$$E(S) = E\left(\sum_{i,j\in[\ell]} S_{i,j}\right) \leq \binom{\ell}{2} \times 16n^7 \times \frac{64}{n^8} \leq 57\mu^2 n \leq \frac{\varepsilon\mu n}{200}.$$

Using Markov's Inequality, we can deduce

$$P\left(S \geq \frac{\varepsilon\mu n}{100}\right) \leq P\left(S \geq 2E(S)\right) \leq \frac{1}{2}.$$

Recall that $P\left(X \leq \frac{\varepsilon\mu n}{50}\right) \leq 2e^{-\frac{\varepsilon\mu n}{300}}$ for any color $s \in [n]$ and any two vertices $v_1, v_2 \in V$. Since there are at most $n \times (n-1) \times n \leq n^3$ choices of s, x_1, x_2, by the union bound we can choose a sufficiently large n such that

$$2e^{-\frac{\varepsilon\mu n}{300}} \times n^3 < \frac{1}{2}.$$

Combining $P(S \geq \frac{\varepsilon\mu n}{100}) \leq \frac{1}{2}$, we conclude that there exists a choice of $\mathcal{F}$ when n is sufficiently large such that the following statements hold.

- for any pair (x_1, x_2) and integer $s \in [n]$, there are at least $\frac{\varepsilon\mu n}{50}$ choices of $i \in [\ell]$ such that $\mathcal{L}_{s,i-2,i-1,i}(x_1, x_2) \cap \mathcal{F} \neq \emptyset$;
- $S \leq \frac{\varepsilon\mu n}{100}$.

Assume that $\mathcal{F}'$ is the set of remaining rainbow 4-paths in $\mathcal{F}$ obtained by deleting one rainbow 4-path in each intersecting pair. Then any two elements of $\mathcal{F}'$ are vertex-disjoint, and for each integer $s \in [n]$ and any pair (x_1, x_2) of vertices, we have

$$|\cup_{i=1}^{\ell}\left(\mathcal{L}_{s,3i-2,3i-1,3i}(x,y) \cap \mathcal{F}'\right)| \geq \frac{\varepsilon\mu n}{50} - \frac{\varepsilon\mu n}{100} = \frac{\varepsilon\mu n}{100}.$$

Consequently, $\mathcal{F}'$ is as desired. $\square$

Proof of Theorem 1.4. Given a constant μ with $\frac{1}{n} \ll \mu \ll \varepsilon < 1$, from Lemma 2.3, there is a family $\mathcal{F}'$ of vertex-disjoint rainbow directed 4-paths such that the following statements hold.

1. $|\mathcal{F}'| \leq \mu n$.

2. Any two elements of $\mathcal{F}'$ contain no common color.

3. For any two vertices $x_1, x_2 \in V$ and an integer $s \in [n]$, there are three distinct integers $i, j, k \in [n]$ such that $\mathcal{L}_{s,i,j,k}(x_1, x_2) \cap \mathcal{F}' \neq \emptyset$.

Without loss of generality, suppose that $|\mathcal{F}'| = a$ and $\mathcal{F}' = \{Q_i = v_i y_i z_i w_i : i \in [a]\}$ with $v_i y_i \in D_{3i-2}$, $y_i z_i \in D_{3i-1}$ and $z_i w_i \in D_{3i}$. Set $\mathcal{D}_1 = \{D_i : i \in [3a+1, n]\}$ and $V_1 = V - \cup_{i=1}^{a} V(Q_i)$. Clearly, $a \leq \mu n$ and $|\cup_{i=1}^{a} V(Q_i)| \leq 4\mu n$. For Q_1 and Q_2, note that

$$|N^+_{D_{3a+1}}(w_1)| + |N^-_{D_{3a+2}}(v_2)| - n \geq 2\varepsilon n \geq 10\mu n.$$

Then it is easily to find a vertex $u_1 \in V_1$ such that $Q_1 u_1 Q_2 = v_1 y_1 z_1 w_1 u_1 v_2 y_2 z_2 w_2$ is a rainbow path with $w_1 u_1 \in A(D_{3a+1})$ and $u_1 v_2 \in A(D_{3a+2})$. Repeating this argument, we can find a distinct vertices $\{u_i : i \in [a]\} \subseteq V_1$ such that $Q_i u_i Q_{i+1}$ is a rainbow path with $w_i u_i \in A(D_{3a+2i-1})$ and $u_i v_{i+1} \in A(D_{3a+2i})$ for all $i \in [a]$, where $Q_{a+1} = Q_1$. It follows that $C = Q_1 u_1 Q_2 u_2 \cdots Q_a u_a Q_1$ is a rainbow cycle and the set of colors appearing on C is $[5a]$.

Set $V_2 = V - V(C)$ and $\mathcal{D}_2 = \{D_i[V_2] : i \in [5a+1, n]\}$. Recall that $a \leq \mu n$ and $\delta^0(D_i) \geq (\frac{1}{2} + \varepsilon)n$ for all $i \in [n]$, we have

$$\delta^0(D_i[V_2]) \geq \left(\frac{1}{2} + \varepsilon\right) n - 5a \geq \left(\frac{1}{2} + \varepsilon\right) n - 5\mu n \geq \frac{n}{2} > \frac{|V_2|}{2}.$$

Using Corollary 2.1, there is a rainbow Hamiltonian path R in $\mathcal{D}_2$. Without loss of generality, suppose that R begins with x_1 and ends with x_2, and does not use the arc of $D_n[V_2]$.

By the definition of $\mathcal{F}'$, we have $\mathcal{L}_{n,i,j,k}(x_1, x_2) \cap \mathcal{F}' \neq \emptyset$, which implies that there exists some $Q_i = v_i y_i z_i w_i \in \mathcal{F}'$ such that $y_i x_1 \in A(D_n)$ and $x_2 z_i \in A(D_{3i-1})$. Then $z_i C y_i x_1 R x_2 z_i$ is a rainbow Hamiltonian cycle in $\mathcal{D}$. □

Now we ready to prove Theorem 1.5.

Proof of Theorem 1.5. For $2 \leq j \leq s$, let $\mathcal{D}_j = \{D_i : i \in [\binom{j}{2}]\}$. The proof proceeds by induction on s. It is obvious that $\mathcal{D}_2$ contains a rainbow T_2. Since the minimum semi-degree of each digraph in $\mathcal{D}_{s-1}$ is larger than $(1 - \frac{1}{s-2})n$, it follows that $\mathcal{D}_{s-1}$ contains a rainbow copy of an arbitrary tournament on $s-1$ vertices. For an arbitrary tournament T_s of s vertices, assume that $V(T_s) = \{v_1, \cdots, v_{s-1}, v_s\}$ and $T_{s-1} = T - \{v_s\}$. Then $\mathcal{D}_{s-1}$ contains a rainbow tournament T_{s-1}. Without loss of generality, we assume that the in-neighborhood of v_s in T_s is $\{v_1, v_2, ..., v_\ell\}$ and the out-neighborhood of v_s in T_s is $\{v_{\ell+1}, v_{\ell+3}, \ldots, v_{s-1}\}$, respectively. We prove that $\mathcal{D}_s$ contains a rainbow tournament T_s. First we construct an auxiliary digraph D with $V(D) = V$ and $A(D)$ as follows:

- $v_i u \in A(D)$ if $v_i u$ is an arc of $D_{\binom{s-1}{2}+i}$ for all $i \in [\ell]$;

- $uv_i \in A(D)$ if uv_i is an arc of $D_{\binom{s-1}{2}+i}$ for all $i \in [\ell+1, s-1]$.

Now we verify that there is a vertex $w \in V - V(T_{s-1})$ with $d_D^+(w) + d_D^-(w) \geq s-1$. Note that

$$\sum_{v \in V(D)-V(T_{s-1})} \left[d_D^+(v) + d_D^-(v)\right] > (s-1)\left[(1-\frac{1}{s-1})n - (s-2)\right] = (s-1)(n-s+2) - n.$$

Hence, there is a vertex $w \in V(D) - V(T_{s-1})$ such that

$$d_D^+(w) + d_D^-(w) > \frac{\sum_{v \in V(D)-V(T_{s-1})}[d_D^+(v) + d_D^-(v)]}{n-s+1} = s-2.$$

This implies that $d_D^+(w) + d_D^-(w) \geq s-1$. Note that $D_{\binom{s-1}{2}+i}$ contributes one to $d_D^-(w)$ for each $i \in [\ell]$ and $D_{\binom{s-1}{2}+i}$ contributes one to $d_D^+(w)$ for each $i \in [\ell+1, s-1]$. Then $d_D^+(w) + d_D^-(w) \leq s-1$, which implies $d_D^+(w) + d_D^-(w) = s-1$. Consequently, we have $N_D^-(w) = \{v_1, v_2 \ldots, v_\ell\}$ and $N_D^+(w) = \{v_{\ell+1}, v_{\ell+2}, \ldots, v_{s-1}\}$. Then $\mathcal{D}_s$ contains a rainbow tournament T_s. □

3 Acknowledgment

The authors would like to thank anonymous referees for their detailed comments and helpful advice to improve the presentation of this paper. The work was supported by the National Natural Science Foundation of China (Nos. 12131013, 11871034 and 12201375).

References

[1] R. Aharoni, E. Berger, Rainbow matchings in r-partite r-graphs, Electron. J. Combin. 16, 2009.

[2] R. Aharoni, M. DeVos, S. González Hermosillo de la Maza, A. Montejano, R. Šámal. A rainbow version of Mantel's theorem, Adv. Comb. 2(2020), 12 pp.

[3] R. Aharoni, D. Howard, A rainbow r-partite version of the Erdös-Ko-Rado theorem, Combin. Probab. Comput. 26(2017), 321–337.

[4] J. Bang-Jensen, G. Gutin, (eds.), Classes of Directed Graphs, SMM. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-71840-8

[5] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM No.244, Springer, Berlin, 2008.

[6] P. Bradshaw, Transversals and bipancyclicity in bipartite graph families, Electron. J. Comb. 28(4)(2021), 4–25.

[7] P. Bradshaw, K. Halasz, L. Stacho, From one to many rainbow Hamiltonian cycles, Graphs Comb. 38(6)(2022), 1–21.

[8] P. Bradshaw, Rainbow spanning trees in random edge-colored graphs, arXiv:2102.12012.

[9] D. Chakraborti, J. Kim, H. Lee, J. Seo, Hamilton transversals in tournaments, Combinatorica 44 (2024), 1381–1400. https://doi.org/10.1007/s00493-024-00123-1.

[10] Y. Cheng, J. Han, B. Wang G. Wang, Rainbow spanning structure in graph hypergraph systems, Forum of Mathematics, Sigma, 2023, 11(e95): 1–20.

[11] Y. Cheng, G. Wang, Y. Zhao, Rainbow pancyclicity in graph systems, Electron. J. Combin. **28** (2021).

[12] G. A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. **2** (1952), 69–81.

[13] A. Ferber, J. Han, D. Mao, Dirac-type problem of rainbow matchings and Hamilton cycles in random graphs, arXiv: 2211.05477.

[14] J. Hu, L. Li, X. Li, N. Xu, Vertex-bipancyclicity in a bipartite graph collection, Discrete Mathematics, 347 (7) (2024), 113980.

[15] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit hamiltonien. CR Acad. Sci. Paris 25 (1960) 495-497.

[16] F. Joos, J. Kim, On a rainbow version of Dirac's theorem, Bull. London Math. Soc. 52(3)(2020), 498–504.

[17] D. Kühn, D. Osthus, A survey on Hamilton cycles in directed graphs, European Journal of Combinatorics, 33(5)(2012), 750–766.

[18] L. Li, P. Li, X. Li, Rainbow structures in a collection of graphs with degree conditions, J. Graph. Theory. (2023) 104:341–359. https://doi.org/10.1002/jgt.22966.

[19] L. Li, P. Li, X. Li, Rainbow pancyclicity in a collection of graphs under the Dirac-type condition, Acta Mathematicae Applicatae Sinica, English Series, 40(2)(2024) 269–274.

[20] R. Montgomery , A. Müyesser, Y. Pehova, Transversal factors and spanning trees, Adv. Comb. 3 (2022), 25pp.

[21] V. Rödl, A. Ruciński, E. Szemerédi, Dirac-type conditions for Hamiltonian paths and cycles in 3-uniform hypergraphs, Adv. Math. 227(3)(2011), 1225–1299.