The maximum spectral radius of graphs without a theta subgraph

Jing Gao[1], Xueliang Li[1,2]

[1]Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China

[2]School of Mathematical Sciences, Xinjiang Normal University, Urumchi, Xinjiang 830017, China

gjing1270@163.com, lxl@nankai.edu.cn

Abstract

A theta graph $\theta_{r,p,q}$ is the graph obtained by connecting two distinct vertices with three internally disjoint paths of length r, p, q, where $q \geq p \geq r \geq 1$ and $p \geq 2$. A graph is $\theta_{r,p,q}$-free if it does not contain $\theta_{r,p,q}$ as a subgraph. The maximum spectral radius of $\theta_{1,p,q}$-free graphs with given size has been determined for any $q \geq p \geq 2$. Zhai, Lin and Shu [Spectral extrema of graphs with fixed size: cycles and complete bipartite graphs, European J. Combin. 95 (2021) 103322] characterized the extremal graph with the maximum spectral radius of $\theta_{2,2,2}$-free graphs having m edges. In this paper, we determine the maximum spectral radius of $\theta_{2,2,3}$-free graphs with size m and characterize the extremal graph.

Keywords: Spectral radius; F-free graphs; Theta graph; Brualdi-Hoffman-Turán problem

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1 Introduction

Let G be an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$, where $n := |G| = |V(G)|$ and $m := e(G) = |E(G)|$ are the order and the size of G, respectively. The adjacency matrix of a connected graph G is defined as $A(G) = (a_{u,v})_{n\times n}$ where $a_{u,v} = 1$ if $uv \in E(G)$ and $a_{u,v} = 0$ otherwise. The spectral radius $\lambda(G)$ of G is the largest eigenvalue of $A(G)$. Given two vertex-disjoint graphs G and H, denote $G\cup H = (V(G)\cup V(H), E(G)\cup E(H))$. Let $G \vee H$ be the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. As usual, let P_n, C_n, $K_{1,n-1}$ and K_n be the path, the cycle, the star and the complete graph on n vertices, respectively. Let $K_{1,n-1}+e$ be the graph obtained from $K_{1,n-1}$ by adding one edge within its independent set and K_n-e be a graph obtained from K_n by deleting any one edge.

Given a graph F, a graph G is said to be F-free if it does not contain F as a subgraph. Let $\mathcal{G}(m,F)$ denote the set of F-free graphs with m edges and without isolated vertices. The Brualdi-Hoffman-Turán type problem [3] is to determine the maximum spectral radius of F-free graphs with given size. This problem has attracted wide attention recently, see [6, 7, 10, 11, 12, 16, 17].

A theta graph, say $\theta_{r,p,q}$, is the graph obtained by connecting two distinct vertices with three internally disjoint paths of length r,p,q, where $q\geq p\geq r\geq 1$ and $p\geq 2$. About $\theta_{r,p,q}$-free graphs, the Brualdi-Hoffman-Turán type problem has been determined completely for $r=1$. First, Sun et al. [14] confirmed the graphs having the largest spectral radius among all $\theta_{1,2,3}$-free and $\theta_{1,2,4}$-free graphs with odd size, respectively. Fang and You [4] characterized the extremal graph with maximum spectral radius of $\theta_{1,2,3}$-free graphs with even size. Liu and Wang [8] characterized the extremal graph with maximum spectral radius of $\theta_{1,2,4}$-free graphs with even size. Later, Lu et al. [9] characterized the extremal graph with the largest spectral radius of $\theta_{1,2,5}$-free graphs. For $q\geq 5$, Li et al. [7] determined the largest spectral radius of $\theta_{1,2,q}$-free graphs. Recently, Gao and Li [5] gave the largest spectral radius of $\theta_{1,3,3}$-free graphs. For $q\geq p\geq 3$ and $p+q\geq 7$, Li et al. [6] obtained the largest spectral radius of $\theta_{1,p,q}$-free graphs. In the same paper, they proposed a problem about $\theta_{r,p,q}$-free graphs where $q\geq p\geq r\geq 2$.

Problem 1.1 *[6] How can we characterize the graphs among $\mathcal{G}(m,\theta_{r,p,q})$ having the largest spectral radius for $q\geq p\geq r\geq 2$?*

For $r=p=q=2$, we have $\theta_{2,2,2}\cong K_{2,3}$. Zhai et al. [17] determined the extremal graph for $K_{2,r}$-free graphs with $r\geq 3$.

Theorem 1.2 *[17] If $G\in\mathcal{G}(m,K_{2,r+1})$ with $r\geq 2$ and $m\geq 16r^2$, then $\lambda(G)\leq\sqrt{m}$, and equality holds if and only if G is a star.*

In this paper, we give an upper bound of the spectral radius of $\theta_{2,2,3}$-free graphs and characterize the unique graph with the maximum spectral radius among $\mathcal{G}(m,\theta_{2,2,3})$.

Theorem 1.3 *Let $G\in\mathcal{G}(m,\theta_{2,2,3})$ with $m\geq 57$. Then $\lambda(G)\leq\frac{1+\sqrt{4m-3}}{2}$ and equality holds if and only if $G\cong K_2\vee\frac{m-1}{2}K_1$.*

Note that C_5 is a subgraph of $\theta_{2,2,3}$. It is easy to have $\mathcal{G}(m,C_5)\subseteq\mathcal{G}(m,\theta_{2,2,3})$. Therefore, when m is large, by Theorem 1.3, we can imply the following result which was obtained by Zhai et al. [17].

Theorem 1.4 *[17] If $G \in \mathcal{G}(m, C_5)$ with $m \geq 8$. Then $\lambda(G) \leq \frac{1+\sqrt{4m-3}}{2}$ and equality holds if and only if $G \cong K_2 \vee \frac{m-1}{2}K_1$.*

2 Preliminaries

At the beginning of this section, we give some notations and terminology. Readers are referred to [1] and [2]. For any vertex $v \in V(G)$, we denote by $N(v)$ or $N_G(v)$ the neighborhood set of v in G and $N[v] = N(v) \cup \{v\}$. Let $d(v)$ or $d_G(v)$ be the degree of a vertex v in G. For any two subsets $X, Y \subseteq V(G)$, we denote $N_X(Y) = \cup_{v \in Y} N(v) \cap X$. Let $e(X, Y)$ denote the number of all edges of G with one end vertex in X and the other in Y. Particularly, let $e(X) := e(X, X)$. Denote by $G[X]$ the subgraph of G induced by X.

For a matrix (or vector) A, $A > 0 (\geq 0)$ means that all its entries are positive (nonnegative). Here, we state the famous Perron-Frobenius theorem.

Lemma 2.1 (Perron-Frobenius Theorem) *[2] Let $A \geq 0$ be an irreducible symmetric matrix. Then the largest eigenvalue $\lambda(A)$ of A is a real number, and the entries of eigenvector corresponding to $\lambda(A)$ are all positive.*

Note that $A(G)$ is irreducible and nonnegative for a connected graph G. By Lemma 2.1, there exists a unique positive unit eigenvector $\mathbf{x}$ corresponding to $\lambda(G)$, which is called Perron vector of G. Let $\mathbf{x}$ be the Perron vector of G with coordinate x_v corresponding to the vertex $v \in V(G)$. A vertex u^* is said to be an extremal vertex if $x_{u^*} = \max\{x_u | u \in V(G)\}$.

A cut vertex of a graph is a vertex whose deletion increases the number of components. A graph is called 2-connected, if it is a connected graph without cut vertices.

Lemma 2.2 *[17] Let G be a graph in $\mathcal{G}(m, F)$ with the maximum spectral radius. If F is a 2-connected graph and u^* is an extremal vertex of G, then G is connected and $d(u) \geq 2$ for any $u \in V(G) \setminus N[u^*]$.*

The following result is about the largest spectral radius of triangle-free graphs which will be used in the subsequent section.

Lemma 2.3 *[10, 13] Let G be a graph with m edges. If G is triangle-free, then $\lambda(G) \leq \sqrt{m}$. Equality holds if and only if G is a complete bipartite graph.*

3 Proof of Theorem 1.3

Let G^* be the extremal graph with the maximum spectral radius among all graphs in $\mathcal{G}(m, \theta_{2,2,3})$. For convenience, denote $\lambda = \lambda(G^*)$. By Lemma 2.2, we know that G^* is connected. In the view of Lemma 2.1, there is the Perron vector $\mathbf{x}$ in G^*. Let u^* be the extremal vertex of G^*. Note that $K_2 \vee \frac{m-1}{2}K_1$ is $\theta_{2,2,3}$-free, we have

$$\lambda \geq \lambda\left(K_2 \vee \frac{m-1}{2}K_1\right) = \frac{1+\sqrt{4m-3}}{2}.$$

Denote $U = N_{G^*}(u^*)$ and $W = V(G^*) \setminus N_{G^*}[u^*]$. Let U_0 be the set of all isolated vertices in the induced subgraph $G^*[U]$ and $U_+ = U \setminus U_0$ be the set of all vertices with degree at least one in $G^*[U]$. Let $W_H = N_W(V(H))$ for any subgraph H of $G^*[U]$. Since $\lambda(G^*)\mathbf{x} = A(G^*)\mathbf{x}$, we have

$$\lambda x_{u^*} = \sum_{u\in U} x_u = \sum_{u\in U_+} x_u + \sum_{u\in U_0} x_u.$$

Furthermore, we also have $\lambda^2(G^*)\mathbf{x} = A^2(G^*)\mathbf{x}$. It follows that

$$\lambda^2 x_{u^*} = |U|x_{u^*} + \sum_{u\in U_+} d_U(u)x_u + \sum_{w\in W} d_U(w)x_w.$$

Therefore,

$$(\lambda^2 - \lambda)x_{u^*} = |U|x_{u^*} + \sum_{u\in U_+}(d_U(u)-1)x_u + \sum_{w\in W} d_U(w)x_w - \sum_{u\in U_0} x_u.$$

Recall that $\lambda \geq \frac{1+\sqrt{4m-3}}{2}$. It is easy to get that $\lambda^2 - \lambda \geq m-1$. Then

$$|U|x_{u^*} + \sum_{u\in U_+}(d_U(u)-1)x_u + \sum_{w\in W} d_U(w)x_w - \sum_{u\in U_0} x_u \geq (m-1)x_{u^*}.$$

Since $m = |U| + e(U_+) + e(U,W) + e(W)$, we have

$$\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \geq e(U_+) + e(U,W) + e(W) + \sum_{u\in U_0}\frac{x_u}{x_{u^*}} - 1. \quad (1)$$

Let $\mathcal{H}$ be the set of all non-trivial components in $G^*[U]$. Note that G^* is $\theta_{2,2,3}$-free. This implies that $G^*[U]$ contains no double star $S_{1,2}$, which is a tree with a central edge uv, 1 leaf connected to u and 2 leaves connected to v. It follows that every element H in $\mathcal{H}$ is $K_{1,r}$ where $r \geq 1$, $K_{1,3}+e$, $K_4 - e$, K_4, P_k where $k \geq 4$ or C_l where $l \geq 3$.

Lemma 3.1 *Let H be a component of $G^*[U]$ which contains a cycle of length at least four. If $W_H \neq \emptyset$, then $d_U(w) \leq 2$ for any $w \in W_H$.*

Proof. Assume that $d_U(w_0) \geq 3$ for some $w_0 \in W_H$. Let C_l be the cycle of H where $l \geq 4$. We have $V(C_l) = V(H)$. Since $w_0 \in W_H$, without loss of generality, suppose $w_0 \in N_W(u_1)$ where $u_1 \in V(C_l)$. Note that $d_U(w_0) \geq 3$. Suppose $u_2, u_3 \in N_U(w_0)$. Since $l \geq 4$, there is at least a vertex $u_i \in \{u_1, u_2, u_3\}$ such that u_i has a neighbor $u_4 \in V(C_l)$ different from

$\{u_1, u_2, u_3\}$. Hence $G^*[\{u^*, u_1, u_2, u_3, u_4, w_0\}]$ contains a $\theta_{2,2,3}$, which is a contradiction. We complete the proof. $\square$

Let $W_0 = \{w \in W | d_W(w) = 0\}$. By Lemma 3.1, if $w \in W_0 \cup N_W(C_l)$ where $l \geq 4$, then $d(w) \leq 2$.

Lemma 3.2 $G^*[U]$ *contains no any cycle of length at least four.*

Proof. Suppose that $G^*[U]$ contains C_l where $l \geq 4$. Let $\mathcal{H}'$ be the family of components of $G^*[U]$ each of which contains cycle of length at least four as a subgraph, then $\mathcal{H} \setminus \mathcal{H}'$ is the family of other components of $G^*[U]$ each of which is $K_{1,r}$ where $r \geq 1$, $K_{1,3} + e$, P_k where $k \geq 4$, C_3. Therefore, for each $H \in \mathcal{H} \setminus \mathcal{H}'$, we have $e(H) \leq |H|$. It is clear that

$$\sum_{u \in V(H)} (d_H(u) - 1)x_u \leq (2e(H) - |H|)x_{u^*} \leq e(H)x_{u^*}.$$

For any $H \in \mathcal{H}'$, H is $K_4 - e$, K_4 or C_l where $l \geq 4$. In the following, we consider the two cases.

Case 1. $W_H = \emptyset$.

If $H \cong C_l = u_1 u_2 \cdots u_l u_1$ where $l \geq 4$, we have

$$\begin{cases} \lambda x_{u_1} = x_{u_l} + x_{u_2} + x_{u^*}, \\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*}, \\ \vdots \\ \lambda x_{u_l} = x_{u_{l-1}} + x_{u_1} + x_{u^*}. \end{cases}$$

Then $\lambda(x_{u_1} + x_{u_2} + \cdots + x_{u_l}) = 2(x_{u_1} + x_{u_2} + \cdots + x_{u_l}) + l x_{u^*}$. Therefore,

$$\sum_{u \in V(H)} (d_H(u) - 1)x_u = \sum_{i=1}^{l} x_{u_i} = \frac{l}{\lambda - 2} x_{u^*}.$$

Since $m \geq 57$, we have $\lambda \geq 8$. Hence

$$\sum_{u \in V(H)} (d_H(u) - 1)\frac{x_u}{x_{u^*}} < (e(H) - 1).$$

If $H \cong K_4$ or $K_4 - e$, suppose $V(H) = \{u_1, u_2, u_3, u_4\}$. For any vertex $u_i \in \{u_1, u_2, u_3, u_4\}$, we have $d_H(u_i) \leq 3$. Therefore, $\lambda x_{u_i} \leq x_{u^*} + 3x_{u^*} = 4x_{u^*}$. It follows that $x_{u_i} \leq \frac{4}{\lambda} x_{u^*}$ for any $i \in \{1, 2, 3, 4\}$. Hence, according to $\lambda \geq 8$, we have

$$\sum_{u \in V(H)} (d_H(u) - 1)x_u = 2\sum_{i=1}^{4} x_{u_i} \leq \frac{32}{\lambda} x_{u^*} \leq 4x_{u^*} < (e(H) - 1)x_{u^*}$$

for $H \cong K_4$, and

$$\sum_{u\in V(H)}(d_H(u)-1)x_u < 2\sum_{i=1}^{4}x_{u_i} \le \frac{32}{\lambda}x_{u^*} \le 4x_{u^*} = (e(H)-1)x_{u^*}$$

for $H \cong K_4 - e$. So $\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} < (e(H)-1)$ when $W_H = \emptyset$.

Case 2. $W_H \neq \emptyset$.

If $H \cong C_l = u_1u_2\cdots u_lu_1$ where $l \ge 4$, then

$$\begin{cases}\lambda x_{u_1} = x_{u_l} + x_{u_2} + x_{u^*} + \sum_{w\in N_W(u_1)} x_w,\\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*} + \sum_{w\in N_W(u_2)} x_w,\\ \vdots\\ \lambda x_{u_l} = x_{u_{l-1}} + x_{u_1} + x_{u^*} + \sum_{w\in N_W(u_l)} x_w,\end{cases}$$

we have

$$\lambda(x_{u_1}+x_{u_2}+\cdots+x_{u_l}) = lx_{u^*} + 2(x_{u_1}+x_{u_2}+\cdots+x_{u_l}) + \sum_{i=1}^{l}\sum_{w\in N_W(u_i)} x_w.$$

That is,

$$x_{u_1}+x_{u_2}+\cdots+x_{u_l} = \frac{l}{\lambda-2}x_{u^*} + \frac{1}{\lambda-2}\sum_{u\in V(H)}\sum_{w\in N_W(u)} x_w.$$

Thus, by $\lambda \ge 8$, we obtain

$$\begin{aligned}\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} &= \frac{x_{u_1}+x_{u_2}+\cdots+x_{u_l}}{x_{u^*}}\\ &= \frac{l}{\lambda-2} + \frac{1}{\lambda-2}\sum_{u\in V(H)}\sum_{w\in N_W(u)}\frac{x_w}{x_{u^*}}\\ &< e(H)-1+\frac{1}{\lambda-2}\sum_{u\in V(H)}\sum_{w\in N_W(u)}\frac{x_w}{x_{u^*}}.\end{aligned}$$

If $H \cong K_4 - e$, then let $V(H) = \{u_1, u_2, u_3, u_4\}$. Without loss of generality, we suppose that $d_H(u_2) = d_H(u_4) = 3$. Then

$$\begin{cases}\lambda x_{u_1} = x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w\in N_W(u_1)} x_w,\\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u_4} + x_{u^*} + \sum_{w\in N_W(u_2)} x_w,\\ \lambda x_{u_3} = x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w\in N_W(u_3)} x_w,\\ \lambda x_{u_4} = x_{u_1} + x_{u_2} + x_{u_3} + x_{u^*} + \sum_{w\in N_W(u_4)} x_w,\end{cases}$$

we have

$$\lambda(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4}) = 4x_{u^*}+2(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4})+x_{u_2}+x_{u_4}+\sum_{i=1}^{4}\sum_{w\in N_W(u_i)}x_w$$
$$\leq 6x_{u^*}+2(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4})+\sum_{i=1}^{4}\sum_{w\in N_W(u_i)}x_w.$$

That is,

$$x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4}\leq \frac{6}{\lambda-2}x_{u^*}+\frac{1}{\lambda-2}\sum_{u\in V(H)}\sum_{w\in N_W(u)}x_w.$$

Note that $\lambda\geq 8$. It follows that

$$\begin{aligned}\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} &= \frac{x_{u_1}+2x_{u_2}+x_{u_3}+2x_{u_4}}{x_{u^*}}\\ &\leq 2+\frac{6}{\lambda-2}+\frac{1}{\lambda-2}\sum_{u\in V(H)}\sum_{w\in N_W(u)}\frac{x_w}{x_{u^*}}\\ &< e(H)-1+\frac{1}{\lambda-2}\sum_{u\in V(H)}\sum_{w\in N_W(u)}\frac{x_w}{x_{u^*}}.\end{aligned}$$

If $H\cong K_4$ with $V(H)=\{u_1,u_2,u_3,u_4\}$, we have $d_H(u_i)=3$ for any $i\in\{1,2,3,4\}$. Therefore,

$$\begin{cases}\lambda x_{u_1}\leq 3x_{u^*}+x_{u^*}+\sum_{w\in N_W(u_1)}x_w,\\ \lambda x_{u_2}\leq 3x_{u^*}+x_{u^*}+\sum_{w\in N_W(u_2)}x_w,\\ \lambda x_{u_3}\leq 3x_{u^*}+x_{u^*}+\sum_{w\in N_W(u_3)}x_w,\\ \lambda x_{u_4}\leq 3x_{u^*}+x_{u^*}+\sum_{w\in N_W(u_4)}x_w,\end{cases}$$

we have

$$\lambda(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4})\leq 16x_{u^*}+\sum_{i=1}^{4}\sum_{w\in N_W(u_i)}x_w.$$

Thus,

$$\begin{aligned}\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} &= \frac{2(x_{u_1}+x_{u_2}+x_{u_3}+x_{u_4})}{x_{u^*}}\\ &\leq \frac{32}{\lambda}+\frac{2}{\lambda}\sum_{u\in V(H)}\sum_{w\in N_W(u)}\frac{x_w}{x_{u^*}}.\end{aligned}$$

Since $\lambda \geq 8$, we obtain

$$\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} < e(H)-1+\frac{2}{\lambda}\sum_{u\in V(H)}\sum_{w\in N_W(u)}\frac{x_w}{x_{u^*}}.$$

It is easy to get $\frac{2}{\lambda} \geq \frac{1}{\lambda-2}$ for $\lambda \geq 8$. Then $\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}} < e(H)-1+\frac{2}{\lambda}\sum_{u\in V(H)}\sum_{w\in N_W(u)}\frac{x_w}{x_{u^*}}$ for each $H \in \mathcal{H}'$. Thus,

$$\begin{aligned}\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}} &= \sum_{H\in\mathcal{H}\setminus\mathcal{H}'}\left(\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}\right)+\sum_{H\in\mathcal{H}'}\left(\sum_{u\in V(H)}(d_H(u)-1)\frac{x_u}{x_{u^*}}\right)\\ &< \sum_{H\in\mathcal{H}\setminus\mathcal{H}'}e(H)+\sum_{H\in\mathcal{H}'}(e(H)-1)+\sum_{H\in\mathcal{H}'}\frac{2}{\lambda}\sum_{u\in V(H)}\sum_{w\in N_W(u)}\frac{x_w}{x_{u^*}}\\ &\leq e(U_+)-|\mathcal{H}'|+\frac{2}{\lambda}\sum_{w\in\cup_{H\in\mathcal{H}'}W_H}d_U(w)\frac{x_w}{x_{u^*}}.\end{aligned}$$

Let $W_1 = \cup_{H\in\mathcal{H}'}W_H \cap W_0$. By Lemma 3.1, we have $d_U(w) \leq 2$ for any $w \in W_H$ where $H \in \mathcal{H}'$ and $d(w) \leq 2$ for any $w \in W_1$. Therefore, for $w \in W_1$, we get $\lambda x_w \leq 2x_{u^*}$. That is, $x_w \leq \frac{2}{\lambda}x_{u^*}$ for any $w \in W_1$. This implies that

$$\begin{aligned}&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W}d_U(w)\frac{x_w}{x_{u^*}}\\ &< e(U_+)-|\mathcal{H}'|+\frac{2}{\lambda}\cdot 2\sum_{w\in\bigcup_{H\in\mathcal{H}'}W_H}\frac{x_w}{x_{u^*}}+2\sum_{w\in W_1}\frac{x_w}{x_{u^*}}+\sum_{w\in W\setminus W_1}d_U(w)\frac{x_w}{x_{u^*}}\\ &= e(U_+)-|\mathcal{H}'|+\frac{4}{\lambda}\sum_{w\in W_1}\frac{x_w}{x_{u^*}}+\frac{4}{\lambda}\sum_{w\in\cup_{H\in\mathcal{H}'}W_H\setminus W_1}\frac{x_w}{x_{u^*}}+2\sum_{w\in W_1}\frac{x_w}{x_{u^*}}+\sum_{w\in W\setminus W_1}d_U(w)\frac{x_w}{x_{u^*}}\\ &\leq e(U_+)-|\mathcal{H}'|+\left(\frac{4}{\lambda}+2\right)\cdot\frac{2}{\lambda}e(U,W_1)+\frac{4}{\lambda}\sum_{w\in\cup_{H\in\mathcal{H}'}W_H\setminus W_1}d_W(w)\frac{x_w}{x_{u^*}}+e(U,W\setminus W_1)\\ &\leq e(U_+)-|\mathcal{H}'|+\frac{8+4\lambda}{\lambda^2}e(U,W_1)+\frac{8}{\lambda}e(W)+e(U,W\setminus W_1).\end{aligned}$$

Since $\lambda \geq 8$, we have $\frac{8+4\lambda}{\lambda^2} < 1$ and $\frac{8}{\lambda} \leq 1$. Then

$$\begin{aligned}\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W}d_U(w)\frac{x_w}{x_{u^*}} &< e(U_+)-1+e(U,W_1)+e(W)+e(U,W\setminus W_1)\\ &= e(U_+)-1+e(U,W)+e(W),\end{aligned}$$

which contradicts with (1). This completes the proof. □

By Lemma 3.2, we obtain that every non-trivial component of $G^*[U]$ is $K_{1,r}$ where $r \geq 1$, $K_{1,3}+e$, P_k where $k \geq 4$ or C_3.

Lemma 3.3 $e(W) = 0$.

Proof. If $W = \emptyset$, then $e(W) = 0$, as desired. So we consider $W \neq \emptyset$ in the following. Suppose to the contrary that $e(W) \geq 1$. Since every non-trivial component of $G^*[U]$ is a tree or a unicyclic graph, we have $e(U_+) \leq |U_+|$. By inequality (1), we get

$$\begin{aligned} e(W) &\leq \sum_{u \in U_+} (d_U(u) - 1)\frac{x_u}{x_{u^*}} + \sum_{w \in W} d_U(w)\frac{x_w}{x_{u^*}} - e(U_+) - e(U, W) - \sum_{u \in U_0} \frac{x_u}{x_{u^*}} + 1 \\ &\leq 2e(U_+) - |U_+| + e(U, W) - e(U_+) - e(U, W) + 1 \\ &\leq 1. \end{aligned}$$

So $e(W) = 1$ and $x_w = x_{u^*}$ for any $w \in W$ satisfying $d_U(w) \geq 1$. Let $E(W) = \{w_1w_2\}$. By Lemma 2.2, we know that $d(w_1) \geq 2$ and $d(w_2) \geq 2$. This implies that $d_U(w_1) \geq 1$ and $d_U(w_2) \geq 1$. It follows that $x_{w_1} = x_{w_2} = x_{u^*}$. Since G^* is $\theta_{2,2,3}$-free, we have $d_U(w_1) + d_U(w_2) \leq 4$. Otherwise, there is a vertex $w_i \in \{w_1, w_2\}$ such that $d_U(w_i) \geq 3$. Without loss of generality, suppose $w_i = w_1$. Let $u_1, u_2, u_3 \in N_U(w_1)$. Note that $d_U(w_2) \geq 1$. Let $u_4 \in N_U(w_2)$. Then there are at least two vertices of $\{u_1, u_2, u_3\}$ which are different from u_4. Assume that u_1 and u_2 are different from u_4. We can find that $G^*[\{u^*, u_1, u_2, u_4, w_1, w_2\}]$ contains a $\theta_{2,2,3}$, a contradiction. Therefore,

$$\begin{aligned} 2\lambda x_{u^*} &= \lambda x_{w_1} + \lambda x_{w_2} \\ &= x_{w_2} + \sum_{u \in N_U(w_1)} x_u + x_{w_1} + \sum_{u \in N_U(w_2)} x_u \\ &\leq x_{u^*} + 4x_{u^*} + x_{u^*} \\ &= 6x_{u^*}. \end{aligned}$$

It yields that $\lambda \leq 3$, a contradiction. This completes the proof. $\square$

From now on, if $W_H = \emptyset$, we denote $\sum_{w \in N_W(u)} x_w = 0$ for any $u \in V(H)$.

Lemma 3.4 *For any $H \in \mathcal{H}$, we have $H \ncong K_{1,3} + e$.*

Proof. Suppose that there is a component $H \in \mathcal{H}$ such that $H \cong K_{1,3} + e$. Let $V(H) = \{u_1, u_2, u_3, u_4\}$ with $d_H(u_1) = 3$ and $d_H(u_4) = 1$. We first prove that $d_U(w) \leq 2$ for any vertex $w \in W_H$ if $W_H \neq \emptyset$. Otherwise, there is a vertex $w_0 \in W_H$ such that $d_U(w_0) \geq 3$. Since $w_0 \in W_H$, we can see that $w_0 \in N_W(u_i)$ for some $i \in \{1, 2, 3, 4\}$. Note that $d_U(w_0) \geq 3$. Suppose $v_1, v_2 \in N_U(w_0) \setminus \{u_i\}$. If $i = 1$, then at least one vertex $u_j \in \{u_2, u_3, u_4\}$ is different from v_1 and v_2. Therefore, $u^*v_1w_0, u^*v_2w_0, u^*u_ju_iw_0$ are three internally disjoint paths of length 2,2,3 between u^* and w_0, a contradiction. So $w_0 \notin N_W(u_1)$. Then $u^*v_1w_0, u^*v_2w_0, u^*u_1u_iw_0$ are three internally disjoint paths of length

2,2,3 between u^* and w_0, a contradiction. Thus, $d_U(w) \leq 2$ for any vertex $w \in W_H$. By Lemmas 2.2 and 3.3, we have $d(w) = 2$ for any vertex $w \in W_H$. Therefore, $\lambda x_w \leq 2x_{u^*}$. That is, $x_w \leq \frac{2}{\lambda} x_{u^*}$ for any vertex $w \in W_H$. According to $\lambda \mathbf{x} = A(G^*)\mathbf{x}$, we obtain

$$\begin{cases} \lambda x_{u_1} = x_{u_2} + x_{u_3} + x_{u_4} + x_{u^*} + \sum_{w \in N_W(u_1)} x_w, \\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*} + \sum_{w \in N_W(u_2)} x_w, \\ \lambda x_{u_3} = x_{u_1} + x_{u_2} + x_{u^*} + \sum_{w \in N_W(u_3)} x_w. \end{cases}$$

Thus,

$$(\lambda - 2)(x_{u_1} + x_{u_2} + x_{u_3}) = 3x_{u^*} + x_{u_4} + \sum_{i=1}^{3} \sum_{w \in N_W(u_i)} x_w.$$

By $\lambda \geq 8$, we have

$$\begin{aligned} x_{u_1} + x_{u_2} + x_{u_3} &\leq \frac{4x_{u^*}}{\lambda - 2} + \frac{1}{\lambda - 2} \sum_{i=1}^{3} \sum_{w \in N_W(u_i)} x_w \\ &< (e(H) - 2)x_{u^*} + \frac{1}{\lambda - 2} \sum_{i=1}^{3} \sum_{w \in N_W(u_i)} x_w. \end{aligned}$$

Recall that $x_w \leq \frac{2}{\lambda} x_{u^*}$ for any vertex $w \in W_H$. We conclude that

$$\begin{aligned} \sum_{u \in V(H)} (d_H(u) - 1) \frac{x_u}{x_{u^*}} &= \frac{2x_{u_1} + x_{u_2} + x_{u_3}}{x_{u^*}} \\ &< 1 + e(H) - 2 + \frac{1}{\lambda - 2} \cdot \frac{2}{\lambda} e(H, W_H) \\ &= e(H) - 1 + \frac{2}{\lambda(\lambda - 2)} e(H, W_H). \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{u \in U_+} (d_U(u) - 1) \frac{x_u}{x_{u^*}} + \sum_{w \in W} d_U(w) \frac{x_w}{x_{u^*}} \\ &= \sum_{u \in U_+ \setminus V(H)} (d_U(u) - 1) \frac{x_u}{x_{u^*}} + \sum_{u \in V(H)} (d_U(u) - 1) \frac{x_u}{x_{u^*}} + \sum_{w \in W_H} d_U(w) \frac{x_w}{x_{u^*}} + \sum_{w \in W \setminus W_H} d_U(w) \frac{x_w}{x_{u^*}} \\ &< e(U_+ \setminus V(H)) + e(H) - 1 + \frac{2}{\lambda(\lambda - 2)} e(H, W_H) + \frac{2}{\lambda} e(U, W_H) + e(U, W \setminus W_H) \\ &\leq e(U_+) - 1 + \frac{2\lambda - 2}{\lambda(\lambda - 2)} e(U, W_H) + e(U, W \setminus W_H) \\ &\leq e(U_+) - 1 + e(U, W). \end{aligned}$$

This is a contradiction. We complete the proof. $\square$

Lemma 3.5 *For any $H \in \mathcal{H}$, we have $H \ncong C_3$.*

Proof. Suppose to the contrary that there is a component $H \in \mathcal{H}$ such that $H \cong C_3$. Let $V(H) = \{u_1, u_2, u_3\}$. If $W_H \neq \emptyset$, we have $d_U(w) \leq 3$ for any $w \in W_H$. Otherwise, there is a vertex $w_0 \in W_H$ satisfying $d_U(w_0) \geq 4$. Without loss of generality, assume $w_0 \in N_W(u_1)$. Note that $|V(H)| = 3$. Suppose that $v \in N_U(w_0)$ is different from u_1, u_2, u_3. Since $d_U(w_0) \geq 4$, there is another vertex $v' \in N_U(w_0) \setminus \{u_1, v\}$. Then at least one of u_2 and u_3 is different from v'. Suppose that u_2 is different from v'. It is easy to find that $u^*vw_0, u^*v'w_0, u^*u_2u_1w_0$ are three internally disjoint paths of length 2,2,3, a contradiction. By Lemma 3.3, we obtain $\lambda x_w \leq 3x_{u^*}$ for any vertex $w \in W_H$. Since

$$\begin{cases} \lambda x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + \sum_{w \in N_W(u_1)} x_w, \\ \lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*} + \sum_{w \in N_W(u_2)} x_w, \\ \lambda x_{u_3} = x_{u_1} + x_{u_2} + x_{u^*} + \sum_{w \in N_W(u_3)} x_w, \end{cases}$$

we get

$$(\lambda - 2)(x_{u_1} + x_{u_2} + x_{u_3}) = 3x_{u^*} + \sum_{i=1}^{3} \sum_{w \in N_W(u_i)} x_w.$$

Recall that $\lambda \geq 8$. It follows that

$$\begin{aligned} x_{u_1} + x_{u_2} + x_{u_3} &= \frac{3x_{u^*}}{\lambda - 2} + \frac{1}{\lambda - 2} \sum_{i=1}^{3} \sum_{w \in N_W(u_i)} x_w \\ &< (e(H) - 1)x_{u^*} + \frac{1}{\lambda - 2} \sum_{i=1}^{3} \sum_{w \in N_W(u_i)} x_w. \end{aligned}$$

Since $x_w \leq \frac{3}{\lambda} x_{u^*}$ for any vertex $w \in W_H$, we obtain

$$\begin{aligned} \sum_{u \in V(H)} (d_H(u) - 1) \frac{x_u}{x_{u^*}} &= \frac{x_{u_1} + x_{u_2} + x_{u_3}}{x_{u^*}} \\ &< e(H) - 1 + \frac{1}{\lambda - 2} \cdot \frac{3}{\lambda} e(H, W_H) \\ &= e(H) - 1 + \frac{3}{\lambda(\lambda - 2)} e(H, W_H). \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{u \in U_+} (d_U(u) - 1) \frac{x_u}{x_{u^*}} + \sum_{w \in W} d_U(w) \frac{x_w}{x_{u^*}} \\ &= \sum_{u \in U_+ \setminus V(H)} (d_U(u) - 1) \frac{x_u}{x_{u^*}} + \sum_{u \in V(H)} (d_U(u) - 1) \frac{x_u}{x_{u^*}} + \sum_{w \in W_H} d_U(w) \frac{x_w}{x_{u^*}} + \sum_{w \in W \setminus W_H} d_U(w) \frac{x_w}{x_{u^*}} \end{aligned}$$

$$
\begin{aligned}
&< e(U_+ \setminus V(H)) + e(H) - 1 + \frac{3}{\lambda(\lambda-2)} e(H, W_H) + \frac{3}{\lambda} e(U, W_H) + e(U, W \setminus W_H) \\
&\le e(U_+) - 1 + \frac{3\lambda - 3}{\lambda(\lambda-2)} e(U, W_H) + e(U, W \setminus W_H) \\
&\le e(U_+) - 1 + e(U, W).
\end{aligned}
$$

This is a contradiction. We complete the proof. $\square$

Lemma 3.6 *For any $H \in \mathcal{H}$, we have $H \not\cong P_k$ where $k \geq 4$.*

Proof. Let $P_k = u_1 u_2 \cdots u_k$ where $k \geq 4$. If $W_H \neq \emptyset$, we show that $d_U(w) \leq 2$ for any $w \in W_H$. Suppose that there exists a vertex $w_0 \in W_H$ satisfying $d_U(w_0) \geq 3$. Assume $v_1, v_2, v_3 \in N_U(w_0)$. Since $k \geq 4$, there is a vertex $u_i \in V(P_k)$ such that $u_{i-1} \in N_U(w_0)$ or $u_{i+1} \in N_U(w_0)$ and $u_i \notin \{v_1, v_2, v_3\}$. Without loss of generality, suppose $u_{i-1} \in N_U(w_0)$. Then at least two vertices of v_1, v_2, v_3 are different from u_{i-1}. Suppose the two vertices are v_1, v_2. It follows that $u^* v_1 w_0, u^* v_2 w_0, u^* u_i u_{i-1} w_0$ are three internally disjoint paths of length 2,2,3, a contradiction. Therefore, $d_U(w) \leq 2$ for any $w \in W_H$. By Lemma 3.3, we have $d(w) \leq 2$ for any $w \in W_H$. This implies that $\lambda x_w \leq 2 x_{u^*}$ for any $w \in W_H$. By

$$
\begin{cases}
\lambda x_{u_2} = x_{u_1} + x_{u_3} + x_{u^*} + \sum_{w \in N_W(u_2)} x_w, \\
\lambda x_{u_3} = x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w \in N_W(u_3)} x_w, \\
\vdots \\
\lambda x_{u_{k-1}} = x_{u_{k-2}} + x_{u_k} + x_{u^*} + \sum_{w \in N_W(u_{k-1})} x_w,
\end{cases}
$$

we have

$$
\begin{aligned}
&\lambda(x_{u_2} + x_{u_3} + \cdots + x_{u_{k-1}}) \\
&= (k-2) x_{u^*} + x_{u_1} + x_{u_2} + 2(x_{u_3} + \cdots + x_{u_{k-2}}) + x_{u_{k-1}} + x_{u_k} + \sum_{i=2}^{k-1} \sum_{w \in N_W(u_i)} x_w \\
&\le 2(k-2) x_{u^*} + x_{u_2} + x_{u_3} + \cdots + x_{u_{k-1}} + \sum_{i=2}^{k-1} \sum_{w \in N_W(u_i)} x_w.
\end{aligned}
$$

Note that $\lambda x_w \leq 2 x_{u^*}$ for any $w \in W_H$. We obtain

$$
\begin{aligned}
x_{u_2} + x_{u_3} + \cdots + x_{u_{k-1}} &\le \frac{2(k-2)}{\lambda - 1} x_{u^*} + \frac{1}{\lambda - 1} \cdot \frac{2}{\lambda} e(H, W_H) x_{u^*} \\
&= \frac{2(k-2)}{\lambda - 1} x_{u^*} + \frac{2}{\lambda(\lambda-1)} e(H, W_H) x_{u^*}.
\end{aligned}
$$

By $\lambda \geq 8$, we have

$$
\sum_{u \in V(H)} (d_H(u) - 1) \frac{x_u}{x_{u^*}} = \frac{x_{u_2} + x_{u_3} + \cdots + x_{u_{k-1}}}{x_{u^*}}
$$

$$
\begin{aligned}
&\le \frac{2(k-2)}{\lambda-1}+\frac{2}{\lambda(\lambda-1)}e(H,W_H)\\
&< e(H)-1+\frac{2}{\lambda(\lambda-1)}e(H,W_H).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W}d_U(w)\frac{x_w}{x_{u^*}}\\
&=\sum_{u\in U_+\setminus V(H)}(d_U(u)-1)\frac{x_u}{x_{u^*}}+\sum_{u\in V(H)}(d_U(u)-1)\frac{x_u}{x_{u^*}}+\sum_{w\in W_H}d_U(w)\frac{x_w}{x_{u^*}}+\sum_{w\in W\setminus W_H}d_U(w)\frac{x_w}{x_{u^*}}\\
&< e(U_+\setminus V(H))+e(H)-1+\frac{2}{\lambda(\lambda-1)}e(H,W_H)+\frac{2}{\lambda}e(U,W_H)+e(U,W\setminus W_H)\\
&\le e(U_+)-1+\frac{2}{\lambda-1}e(U,W_H)+e(U,W\setminus W_H)\\
&\le e(U_+)-1+e(U,W),
\end{aligned}
$$

a contradiction. This completes the proof. $\square$

According to Lemmas 3.2, 3.4, 3.5, 3.6, we get that every element H in $\mathcal{H}$ is $K_{1,r}$ where $r\ge 1$. Then $e(H)=|H|-1$. This implies that

$$
\begin{aligned}
\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}} &\le \sum_{H\in\mathcal{H}}(2e(H)-|H|)\\
&=\sum_{H\in\mathcal{H}}(e(H)-1)\\
&=e(U_+)-|\mathcal{H}|.
\end{aligned}
$$

By (1) and $\sum_{w\in W}d_U(w)\frac{x_w}{x_{u^*}}\le e(U,W)$, we have

$$
\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}}\ge e(U_+)+\sum_{u\in U_0}\frac{x_u}{x_{u^*}}-1.
$$

Combining with the two inequalities, we have $|\mathcal{H}|+\sum_{u\in U_0}\frac{x_u}{x_{u^*}}\le 1$. Next we finish the proof of Theorem 1.3.

Proof of Theorem 1.3. If $|\mathcal{H}|=0$, then G^* is bipartite. By Lemma 2.3, we have $\lambda\le\sqrt{m}<\frac{1+\sqrt{4m-3}}{2}$. This contradicts with $\lambda\ge\frac{1+\sqrt{4m-3}}{2}$. So $|\mathcal{H}|=1$. It follows that $U_0=\emptyset$ due to $x_u>0$ for any $u\in V(G^*)$ and $x_u=x_{u^*}$ for any $u\in U$ satisfying $d_U(u)\ge 2$. That is, $G^*[U]\cong K_{1,r}$. If $r=1$, then $G^*[U]$ contains an edge u_0u_1. We have $\lambda x_{u^*}=x_{u_0}+x_{u_1}$ and $\lambda x_{u_0}=x_{u^*}+x_{u_1}+\sum_{w\in N_w(u_0)}x_w$. It yields that $\sum_{w\in N_w(u_0)}x_w=(\lambda+1)(x_{u_0}-x_{u^*})\le 0$. Since $\sum_{w\in N_w(u_0)}x_w\ge 0$, we obtain $\sum_{w\in N_w(u_0)}x_w=0$. That is, $N_w(u_0)=\emptyset$. By Lemmas 2.2 and 3.3, we have $W=\emptyset$. Then $m=3$, a contradiction. So $r\ge 2$. Let $U=\{u_0,u_1,\dots,u_r\}$ with the center u_0. Because $d_U(u_0)\ge 2$, we have $x_{u_0}=x_{u^*}$. Since

$$
\lambda x_{u^*}=x_{u_0}+x_{u_1}+\cdots+x_{u_r}
$$

and

$$\lambda x_{u_0} = x_{u^*} + x_{u_1} + \cdots + x_{u_r} + \sum_{w \in N_w(u_0)} x_w,$$

we get $\sum_{w\in N_w(u_0)} x_w = 0$. Thus, $N_w(u_0) = \emptyset$. If $r = 2$, then we have $N(w) = \{u_1, u_2\}$ for any $w \in W$ by Lemma 2.2. It follows that $|W| \leq 1$. Otherwise there is a $\theta_{2,2,3}$, contradiction. Therefore, $m = e(G^*) = 7$, a contradiction. This implies that $r \geq 3$. If $W \neq \emptyset$, we have $d(w) \leq 2$ for any $w \in W$. Otherwise, suppose that u_1, u_2, u_3 are three neighbors of $w_0 \in W$. Then $u^*u_1w_0, u^*u_2w_0, u^*u_0u_3w_0$ are three internally disjoint paths of length 2,2,3, a contradiction. Therefore, $\lambda x_w \leq 2x_{u^*}$ for any $w \in W$. It follows that

$$\begin{aligned}
&\sum_{u\in U_+}(d_U(u)-1)\frac{x_u}{x_{u^*}} + \sum_{w\in W} d_U(w)\frac{x_w}{x_{u^*}} \\
&\leq (e(U_+)-1) + \frac{2}{\lambda}e(U,W) \\
&< e(U_+) - 1 + e(U,W),
\end{aligned}$$

a contradiction. Hence, $W = \emptyset$. This implies that $G^* \cong K_1 \vee K_{1,r}$ with $2r+1 = m$. That is, $G^* \cong K_2 \vee \frac{m-1}{2}K_1$. We complete the proof. □

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