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On isomorphisms of tetravalent Cayley digraphs over dihedral groups

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Abstract
Let m be a positive integer. A group G is said to be an m-DCI-group or an m-CI-group if G has the k-DCI property or k-CI property for all positive integers k at most m, respectively. Let G be a dihedral group of order $2n$ with $n \geq 3$. Qu and Yu proved that G is an m-DCI-group or m-CI-group, for every $m \in \{1, 2, 3\}$, if and only if n is odd. In this paper, it is shown that G is a 4-DCI-group if and only if n is odd and not divisible by 9, and G is a 4-CI-group if and only if n is odd.

Keywords Cayley digraph · m-DCI-group · m-CI-group · Dihedral group

Mathematics Subject Classification 20B25 · 05C25

1 Introduction

In this paper, a (finite) digraph Γ is an order pair $(V(\Gamma), Arc(\Gamma))$ of a finite set $V(\Gamma)$ and a set $Arc(\Gamma)$ consisting of ordered pairs of distinct elements from $V(\Gamma)$, while the elements in $V(\Gamma)$ and $Arc(\Gamma)$ are called vertices and arcs of Γ, respectively. Denote by $\Gamma^-(v)$ and $\Gamma^+(v)$, respectively, the sets of in-neighbors and out-neighbors of a vertex v in a digraph Γ, that is,

$$\Gamma^-(v) = \{u \in V(\Gamma) \mid (u, v) \in Arc(\Gamma)\},\ \Gamma^+(v) = \{u \in V(\Gamma) \mid (v, u) \in Arc(\Gamma)\}.$$

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We say a digraph Γ is k-regular, or of valency k, if $|\Gamma^-(v)| = k = |\Gamma^+(v)|$ for all $v \in V(\Gamma)$. For a digraph or group X, denote by $\mathrm{Aut}(X)$ its automorphism group. Graphs are also involved in this paper, and it is convenient to treat a graph as a digraph with each edge $\{u, v\}$ equated with two arcs (u, v) and (v, u). Thus, we define a graph as a digraph Γ in which (u, v) is an arc if and only if so does (v, u); in this case, $\Gamma^-(v) = \Gamma^+(v)$, written as $\Gamma(v)$.

Let G be a finite group and let $S \subseteq G \setminus \{1\}$, where 1 is the identity element of G. The *Cayley digraph* of G with respect to S, denoted by $\mathrm{Cay}(G, S)$, is the digraph having vertex set G such that (x, y) is an arc if and only if $yx^{-1} \in S$. Clearly, $\mathrm{Cay}(G, S)$ is $|S|$-regular. If $S = S^{-1}$, that is, S is closed under taking inverse, then (x, y) is an arc if and only if so does (y, x). In this case, (x, y) is an arc of $\mathrm{Cay}(G, S)$ if and only if so is (y, x), so call $\mathrm{Cay}(G, S)$ a *Cayley graph*.

For subsets $S, T \subseteq G \setminus \{1\}$, if $T = S^\sigma$ for some $\sigma \in \mathrm{Aut}(G)$, then σ induces an isomorphism from $\mathrm{Cay}(G, S)$ onto $\mathrm{Cay}(G, T)$, called a *Cayley isomorphism*. A subset S of $G \setminus \{1\}$ is called a *CI-subset* of G if for every $T \subseteq G$ with $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ there exists a Cayley isomorphism between $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$. In this case, $\mathrm{Cay}(G, S)$ is called a *CI-digraph*, or a *CI-graph* when $S = S^{-1}$. For a positive integer m, we say that G have the *m-DCI property* or *m-CI property* if every Cayley digraph or every Cayley graph on G of valency m is a CI-digraph or a CI-graph, respectively. The group G is said to be an *m-DCI-group* or an *m-CI-group* if G has the k-DCI property or k-CI property for every positive integer $k \leq m$, respectively; in particular, when $m = |G| - 1$, the group G is called a *DCI-group* or a *CI-group*, respectively. Clearly, if G has the m-DCI property, then G also has the m-CI property, and thus, a DCI-group is necessary a CI-group. We see from the definition that S is a CI-subset of G if and only if so does $G \setminus (S \cup \{1\})$. Then the m-DCI property yields the $(|G| - 1 - m)$-DCI property, and so G is a (D)CI-group if and only if G is an m-(D)CI-group with $m = \lfloor \frac{|G|-1}{2} \rfloor$.

Begun with a conjecture proposed by Ádám [1] in 1967, finite m-(D)CI-groups have been extensively studied for over fifty years, referred to surveys in [4, 29, 34]. In current terminology, Ádám's conjecture suggests that every cyclic group $\mathbb{Z}_n$ (with $n > 1$) is a DCI-group. This was disproved in 1970 by Elspas and Turner [13], who proved that $\mathbb{Z}_8$ has no the 3-DCI property and $\mathbb{Z}_{16}$ has no the 6-CI property. However, at that time, they also confirmed the conjecture for a prime n. In the following years, the conjecture was proved for certain classes integers n, see [3, 5, 15]. Finally, the finite cyclic DCI-groups were classified by Muzychuk [32, 33], that is, $\mathbb{Z}_n$ is a DCI-group if and only if $n = ab$ with $a \in \{1, 2\}$ and b a square-free integer, and $\mathbb{Z}_n$ is a CI-group but not a DCI-group if and only if $n \in \{8, 9, 18\}$. On the other hand, it has been proved that $\mathbb{Z}_n$ has the m-CI property for all positive integers $m \leq \min\{5, n\}$ (see [21, 37, 39]). In addition, Li [22] presented a necessary condition for cyclic groups with the m-DCI property and conjectured that the condition is also sufficient, which has been confirmed by Dobson [11].

The first class of nonablian CI-groups, namely the dihedral groups of order twice a prime, was given by Babai [5], who initiated the study on finite CI-groups other than cyclic groups. Later on, Babai and Frankl [6, 7] studied in depth the CI-groups of odd order and insoluble CI-groups. Li at al. extended the study of CI-groups and developed the theory of (D)CI-groups, see [8, 23–28]. It has been shown that a finite m-(D)CI-group has strict restrictions on the structure. In particular, the candidates of finite CI-groups and DCI-groups have been reduced to restricted lists, see [30, Theorem 1.2] and [24, Corollary 1.5]. Despite this, determining which finite groups are DCI-groups or CI-groups is still a highly challenging task. Besides the examples recorded in the references mentioned above, readers are referred [12, 14, 19, 20] for more DCI-groups or CI-groups. In this paper, we focus on the dihedral groups and make an attempt toward to determining the dihedral DCI-groups and CI-groups.

For an integer $n \geq 2$, the dihedral group D_{2n} is generated by two elements with the following presentation:

$$\mathrm{D}_{2n} = \langle a, b \mid a^n = b^2 = 1,\ abab = 1\rangle.$$

As mentioned above, if n is a prime, then D_{2n} is a DCI-group. Recently, the authors of [41] proved that if D_{2n} has the m-DCI property for some $1 \leq m \leq n-1$, then n is odd and not divisible by the square of any prime less than m, but in general it is unknown whether the converse is true. We also note that by [35], for $m \in \{1, 2, 3\}$, the dihedral group D_{2n} is an m-DCI-group if and only if D_{2n} is an m-CI-group if and only if n is odd. In this paper, we investigate the isomorphic problem of tetravalent Cayley digraphs on D_{2n} and consider the 4-DCI property and 4-CI property of D_{2n}. Our main result is stated in the following theorem.

Theorem 1.1 *Let $n \geq 3$ be an integer. Then*

(i) D_{2n} *has the* 4*-DCI property if and only if n is odd and not divisible by* 9*; and*
(ii) D_{2n} *has the* 4*-CI property if and only if n is odd.*

By [35], D_{2n} is a 3-DCI-group if and only if D_{2n} is a 3-CI-group if and only if n is odd. This together with Theorem 1.1 gives the following corollary.

Corollary 1.2 *Let $n \geq 3$ be an integer. Then*

(i) D_{2n} *is a* 4*-DCI-group if and only if n is odd and not divisible by* 9*; and*
(ii) D_{2n} *is a* 4*-CI-group if and only if n is odd.*

Following this introduction, we state in Sect. 2 some preliminary results which are used in the following sections. Section 3 proves that every connected tetravalent Cayley digraphs of D_{2n} with $n \geq 3$ odd is a CI-digraph, and Sect. 4 gives a proof of Theorem 1.1.

2 Preliminaries

This section collects some concepts and results, which are used in Sects. 3 and 4.

For a nonempty set Ω, denote by $\mathrm{Sym}(\Omega)$ the *symmetric group* on Ω. In particular, if $\Omega = \{1, \ldots, n\}$ for some positive integer n, we write $\mathrm{Sym}(\Omega)$ as S_n for convenience. Let Γ be a digraph. For $B, B_1, B_2 \subseteq V(\Gamma)$, denote $[B]$ the subdigraph of Γ induced by B, and denote $[B_1, B_2]$ the subdigraph of Γ obtained from $[B_1 \cup B_2]$ by deleting all arcs in $[B_1]$ and $[B_2]$. Denote by $\overrightarrow{\mathrm{K}}_{m,n}$ the orientation of the complete bipartite graph $\mathrm{K}_{m,n}$ with heads of all arcs lying in the part of size n.

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph of a group G, and let $A = \mathrm{Aut}(\Gamma)$. Each $g \in G$ induces $R(g)$ of A by the right multiplication on G. Write $R(G) = \{R(g) \mid g \in G\}$. Then $R(G)$ is a regular subgroup of A, and $g \mapsto R(g)$ gives an isomorphism from G to $R(G)$. Consider the normalizer of $R(G)$ in $\mathrm{Sym}(G)$ on G. We have $\mathrm{N}_{\mathrm{Sym}(G)}(R(G)) = R(G)\mathrm{Aut}(G)$, see [36, Lemma 7.16], for example. Then $\mathrm{N}_A(R(G)) = R(G)\mathrm{Aut}(G) \cap A = R(G)\mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\}$. In particular, $R(G)$ is a normal subgroup of A if and only if $A = R(G)\mathrm{Aut}(G, S)$; in this case, Γ is called a *normal* Cayley digraph of the group G. Thus, we have the following result, see also [42, Proposition 1.3 and Proposition 1.5].

Proposition 2.1 *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph of a group* G. *Then*

(i) $\mathrm{N}_{\mathrm{Aut}(\Gamma)}(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$*; and*
(ii) Γ *is normal with respect to* G *if and only if* $\mathrm{Aut}(\Gamma) = R(G)\mathrm{Aut}(G, S)$.

The result stated below is derived from the well-known Babai criterion [5] for determining whether a Cayley digraph is a CI-digraph, also refer to [29, Theorem 4.1].

Proposition 2.2 *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph of a group* G. *Then* Γ *is a CI-digraph if and only if every regular subgroup of* $\mathrm{Aut}(\Gamma)$ *isomorphic to* G *is conjugate to* $R(G)$ *in* $\mathrm{Aut}(\Gamma)$. *If further* Γ *is normal with respect to the* G, *then* Γ *is a CI-digraph if and only if* $R(G)$ *is the unique regular subgroup of* $\mathrm{Aut}(\Gamma)$ *isomorphic to* G.

Li [29] described the abelian Cayley digraphs which are CI-digraphs. By [29, Theorem 6.8] and [10, Theorem 2], we have the following proposition.

Proposition 2.3 *Let* G *be a cyclic group of order* $n > 1$, *and let* $\Gamma = \mathrm{Cay}(G, S)$ *be a connected Cayley digraphs of* G *with* $|S| = 4$.

(i) *If* n *is odd and indivisible by* 9, *then* Γ *is a CI-digraph.*
(ii) *If* $S = S^{-1}$, *then* Γ *is a CI-graph.*

A group G is called an *NDCI-group* or an *NCI-group* if each normal Cayley digraph or graph of G is a CI-digraph or a CI-graph, respectively. The following result, quoted from [40], characterizes the dihedral NDCI-groups and NCI-groups.

Proposition 2.4 *Let* $n \geq 2$ *be an integer. Then the dihedral group* D_{2n} *is an NDCI-group if and only if* D_{2n} *is an NCI-group if and only if either* $n \in \{2, 4\}$ *or* n *is odd.*

According to [41, Lemma 4.1 and Lemma 4.2], we have the following lemma.

Lemma 2.5 *Let* $n > 1$ *be an odd integer, and let* $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ *be a Cayley digraph.*

(i) *If X is a regular subgroup of* $\mathrm{Aut}(\Gamma)$ *that is isomorphic to* D_{2n}*, then X is conjugate to* $R(\mathrm{D}_{2n})$ *in* $\mathrm{Aut}(\Gamma)$ *if and only if the unique cyclic subgroup of order n of X is conjugate to the unique cyclic subgroup of order n of* $R(\mathrm{D}_{2n})$ *in* $\mathrm{Aut}(\Gamma)$.

(ii) *If the stabilizer of* 1 *in* $\mathrm{Aut}(\Gamma)$ *has order coprime to n, then Γ is a CI-digraph.*

(iii) *If Γ is connected and $|S|$ is less than the least prime divisor of n, then Γ is a CI-digraph.*

The statement below, derived from [41, Theorem 1.1], gives a necessary condition of dihedral groups with the m-DCI property.

Proposition 2.6 *Let $n \geq 3$ and m be integers with $1 \leq m \leq n-1$. If D_{2n} has the m-DCI property, then n is odd and indivisible by the square of any prime less than m.*

Given $\mathrm{D}_{2n} = \langle a, b \mid a^n = b^2 = 1,\ bab = a^{-1}\rangle$, it is shown that

$$\mathrm{Aut}(\mathrm{D}_{2n}) = \{\sigma_{r,s} \mid s, r \text{ are integers},\ (r, n) = 1\},$$

where $\sigma_{r,s}$ is defined by

$$(a^i)^{\sigma_{r,s}} = a^{ri},\ (a^j b)^{\sigma_{r,s}} = a^{rj+s} b.$$

Note that $\sigma_{r,s} = \sigma_{r',s'}$ if and only if $r \equiv r' \pmod n$ and $s \equiv s' \pmod n$. For an integer $w \geq 1$, we have

$$a^{\sigma_{r,s}^w} = a^{r^w},\ b^{\sigma_{r,s}^w} = a^{s(r^{w-1}+\cdots+r+1)} b.$$

Then

$$\sigma_{r,s}^w = 1 \text{ if and only if } r^w \equiv 1 \pmod n \text{ and } s(r^{w-1} + \cdots + r + 1) \equiv 0 \pmod n. \tag{1}$$

Clearly, if n is even, then $a^{\frac{n}{2}}$ has order 2 and lies in the center of D_{2n}; in this case, one cannot extend an isomorphism between $\langle a^{\frac{n}{2}}\rangle$ and $\langle b\rangle$ to some automorphism of D_{2n} unless $n = 2$. Recall that a group G is said to be *homogeneous* if every isomorphism between subgroups of G can be extended to an automorphism of G. By [35, Lemmas 1.6 and 1.9], the next lemma follows.

Lemma 2.7 *Let n be a positive integer. Then the cyclic group $\mathbb{Z}_n$ is homogeneous, and the dihedral group D_{2n} is homogeneous if and only if $n = 2$ or n is odd.*

We say two subsets S and T of a group G are equivalent, written as $S \equiv_G T$, if $T = S^\sigma$ for some $\sigma \in \mathrm{Aut}(G)$. Clearly,

$$S \equiv_G T \subseteq G \backslash \{1\} \Rightarrow \mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T).$$

In the special case where G is homogeneous, if $S, T \subseteq H \leq G$, then $S \equiv_H T$ if and only if $S \equiv_G T$. Thus, we write $S \equiv T$ in place of $S \equiv_H T$ (and $S \equiv_G T$) when G is homogeneous.

By [2, Theorem 1.1] and [31, Theorem 1], if $\mathrm{Cay}(\mathrm{D}_{2n}, S)$ is a connected arc-transitive graph of valency 3, then, up to isomorphism of graphs, S may be chosen as follows:

(i) $S = \{b, ab, a^2b\}$ with $n \in \{3, 4\}$, or $S = \{b, ab, a^3b\}$ with $n \in \{7, 8\}$; in this case, $\mathrm{Cay}(\mathrm{D}_{2n}, S)$ is not normal with respect to D_{2n}; or
(ii) $S = \{b, ab, a^{r+1}b\}$ with odd $n \geq 13$ and $r^2 + r + 1 \equiv 0 \pmod n$; in this case, $\mathrm{Cay}(\mathrm{D}_{2n}, S)$ is normal with respect to D_{2n}, and $\mathrm{Aut}(\mathrm{D}_{2n}, S) \cong \mathbb{Z}_3$.

For (ii), we may further require that $0 < r < n/2$. In fact, if r is a solution of $x^2 + x + 1 = 0$ in $\mathbb{Z}_n$, then

$$(r^2)^2 + r^2 + 1 = (r^2 + 1)^2 - r^2 = r^2 - r^2 = 0,$$

implying that r^2 is also a solution. Moreover, it is easy to see that $\mathrm{Cay}(\mathrm{D}_{2n}, S^{\sigma_{-r,0}}) \cong \mathrm{Cay}(\mathrm{D}_{2n}, S)$ and $S^{\sigma_{-r,0}} = \{b, ab, a^{r^2+1}b\}$. Therefore, if $r \geq n/2$, then we can choose r^2 in place of r, as $r^2 = -1 - r \in \{1, \ldots, (n-1)/2\}$.

In [35], it was proved that if n is odd, then D_{2n} is a 3-DCI-group. Thus, we have the following result.

Proposition 2.8 *Let $n \geq 3$ be an odd integer, and let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected arc-transitive Cayley graph with $|S| = 3$. Then*

(i) *$S \equiv S_0 := \{b, ab, a^{r+1}b\}$, where $0 < r < n/2$, $r^2 + r + 1 \equiv 0 \pmod n$, and either $n \geq 13$ or $n \in \{3, 7\}$;*
(ii) *$\mathrm{Aut}(\mathrm{D}_{2n}, S_0) \geq \langle \sigma_{r,1} \rangle \cong \mathbb{Z}_3$, where the equality holds if and only if $n \neq 3$;*
(iii) *Γ is normal with respect to D_{2n} if and only if $n \geq 13$; more precisely, if $n = 3$ then $\Gamma \cong \mathrm{K}_{3,3}$, and $\mathrm{Aut}(\Gamma) \cong (\mathrm{S}_3 \times \mathrm{S}_3){:}\mathrm{S}_2$; if $n = 7$ then Γ is isomorphic to the Heawood graph, and $\mathrm{Aut}(\Gamma) \cong \mathrm{PSL}_3(2).\mathbb{Z}_2$; if $n \geq 13$ then $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_n{:}\mathbb{Z}_6 \cong \mathrm{D}_{2n}{:}\mathbb{Z}_3$.*

Proof The conclusion (i) follows from the foregoing argument, and the conclusion (iii) is a corollary of [2, Theorem 1.1] and [31, Theorem 1]. Now it remains to show that (ii) holds. Since Γ is connected, $\mathrm{Aut}(\mathrm{D}_{2n}, S_0)$ acts faithfully on S_0, and so $\mathrm{Aut}(\mathrm{D}_{2n}, S_0) \lesssim \mathrm{S}_3$. It is easy to check that $\sigma_{r,1} \in \mathrm{Aut}(\mathrm{D}_{2n}, S_0)$ and $\sigma_{r,1}$ has order 3. Thus, if $n \geq 11$ then $\mathrm{Aut}(\mathrm{D}_{2n}, S_0) = \langle \sigma_{r,1} \rangle$ as Γ is normal with respect to D_{2n}. For $n = 3$, it is easy to check that $\mathrm{Aut}(\mathrm{D}_{2n}, S_0)$ contains $\sigma_{2,0}$, which has order 2, and so $\mathrm{Aut}(\mathrm{D}_{2n}, S_0) = \langle \sigma_{r,1}, \sigma_{2,0} \rangle \cong \mathrm{S}_3$.

Now let $n = 7$. Then $r = 2$. By (1), each element in $\mathrm{Aut}(\mathrm{D}_{2n})$ of order 2 has the form of $\sigma_{6,s}$, where $0 \leq s \leq 6$. Now

$$S_0^{\sigma_{6,s}} = \{a^s b, a^{6+s} b, a^{18+s} b\} = \{a^s b, a^{6+s} b, a^{4+s} b, \} \neq \{b, ab, a^3 b\} = S_0.$$

It follows that $\mathrm{Aut}(\mathrm{D}_{2n}, S_0)$ contains no element of order 2. Then $\mathrm{Aut}(\mathrm{D}_{2n}, S_0) = \langle \sigma_{r,1} \rangle$. This completes the proof. □

According to [17, Theorem 1.1] and [18, Theorem 5.1], we obtain the following proposition about the connected tetravalent arc-transitive Cayley graphs on D_{2n} with n odd.

Proposition 2.9 *Let $n \geq 3$ be an odd integer, and let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected arc-transitive Cayley graph with $|S| = 4$. If Γ is not normal with respect to D_{2n}, then either $n \in \{5, 7, 13, 15\}$ or $\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, \{a, a^{-1}, a^2b, b\})$.*

Let Γ be a digraph, and let $\mathcal{B}$ be a partition of $V(\Gamma)$. The *quotient digraph* $\Gamma_{\mathcal{B}}$ is defined as the digraph with vertex set $\mathcal{B}$ such that (B_1, B_2) is an arc if and only if (x, y) ia an arc of Γ for some $x \in B_1$ and $y \in B_2$. For a subgroup $G \leq \mathrm{Aut}(\Gamma)$, if $\mathcal{B}$ is G-invariant partition of $V(\Gamma)$, that is, $B^g \in \mathcal{B}$ for all $g \in G$ and $B \in \mathcal{B}$, then G induces an subgroup of $\mathrm{Aut}(\Gamma_{\mathcal{B}})$, say $G^{\mathcal{B}}$. The following fact is well known and easily proved.

Lemma 2.10 *Let Γ be a digraph and $G \leq \mathrm{Aut}(\Gamma)$. Assume that $\mathcal{B}$ is a G-invariant partition of $V(\Gamma)$, and let K be the kernel of G acting on $\mathcal{B}$. Then $G^{\mathcal{B}} \cong G/K$. Moreover, if G is transitively on $V(\Gamma)$, then $G^{\mathcal{B}}$ is transitive on $\mathcal{B}$, all $B \in \mathcal{B}$ have equal size, and the stabilizer G_B acts transitively on B.*

3 Connected Cayley digraphs on D_{2n} of valency 4

In this section, we prove the following result Theorem 3.1, which plays a key role in the proof of Theorem 1.1.

Theorem 3.1 *Let $n \geq 3$ be an odd integer. Then every connected Cayley digraph of valency 4 on D_{2n} is a CI-digraph.*

First, we construct in the following lemma certain non-normal connected CI-graph of valency 4 on the dihedral groups $\mathrm{D}_{2n} = \langle a, b \mid a^n = b^2 = 1,\ a^b = a^{-1}\rangle$.

Recall that the *lexicographic product* $X[Y]$ of two digraphs is the digraph with vertex set $V(X) \times V(Y)$ such that (x_1, y_1) is adjacent to (x_2, y_2) if and only if either $x_1 = x_2$ and $(y_1, y_2) \in Arc(Y)$, or $(x_1, x_2) \in Arc(X)$.

Lemma 3.2 *Let $n \geq 3$ be an odd integer. Then $\mathrm{Cay}(\mathrm{D}_{2n}, \{a, a^{-1}, a^2b, b\})$ is a connected CI-graph but not normal with respect to D_{2n}.*

Proof Let $S = \{a, a^{-1}, a^2b, b\}$ and $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$. It is easy to see that $S^{-1} = S$ and $\langle S\rangle = \mathrm{D}_{2n}$, and thus, Γ is a connected Cayley graph. Let $H = \langle ab\rangle$. Then

$$\mathrm{D}_{2n} = \sum_{i=0}^{n-1} Ha^i \text{ and } S = Ha \cup Ha^{n-1}.$$

Note that for every $i \in \{0, \dots, n-1\}$, we get

$$Ha^i = \{a^i, aba^i\} \text{ and } \Gamma(a^i) = \Gamma(aba^i) = Ha^{i-1} \cup Ha^{i+1}. \quad (2)$$

Let $\mathcal{B} = \{H, Ha, \dots, Ha^{n-1}\}$. Noting that Γ is a vertex-transitive graph, we can deduce from (2) that $\mathcal{B}$ is an $\mathrm{Aut}(\Gamma)$-invariant partition of $V(\Gamma)$. Moreover, we derive from $S = Ha \cup Ha^{n-1}$ that $[Ha^i, Ha^{i+1}]$ of Γ by $Ha^i \cup Ha^{i+1}$ is isomorphic to $\mathrm{K}_{2,2}$, which implies that $[Ha^j, Ha^\ell]$ is either a empty graph or isomorphic to $K_{2,2}$ for two arbitrary elements $j, \ell \in \{0, \ldots, n-1\}$. Thus, it is easily check that $\Gamma \cong \Gamma_{\mathcal{B}}[2K_1]$, where $2K_1$ is the empty graph on two vertices; in particular, $\Gamma_{\mathcal{B}} \cong C_n$, the cycle of length n.

Now we show that Γ is a CI-graph. Let $\tau_i = (a^i\, aba^i)$ for $0 \le i \le n-1$, and put $K = \langle \tau_i \mid 0 \le i \le n-1 \rangle$. Then $K \cong \mathbb{Z}_2^n$. Moreover, K is the kernel of $\mathrm{Aut}(\Gamma)$ acting on $\mathcal{B}$, and then, we derive from Lemma 2.10 that

$$\mathrm{Aut}(\Gamma)/K \cong \mathrm{Aut}(\Gamma)^{\mathcal{B}} \le \mathrm{Aut}(\Gamma_{\mathcal{B}}) \cong \mathrm{D}_{2n}.$$

In particular, $|\mathrm{Aut}(\Gamma)| \le 2n|K| = 2^{n+1}n$. Since n is odd and $R(\mathrm{D}_{2n}) \cong \mathrm{D}_{2n}$, we conclude that $R(\mathrm{D}_{2n})$ has no normal subgroup of order a power of 2. This forces that $K \cap R(\mathrm{D}_{2n}) = 1$, and so $|\mathrm{Aut}(\Gamma)| \ge |KR(\mathrm{D}_{2n})| = 2^n \cdot 2n$. It follows that $\mathrm{Aut}(\Gamma) = KR(\mathrm{D}_{2n})$; in particular, $\mathrm{Aut}(\Gamma)$ is soluble. Then $\langle R(a) \rangle$ is a Hall $2'$-subgroup of $\mathrm{Aut}(\Gamma)$, and every subgroup of $\mathrm{Aut}(\Gamma)$ with order n is conjugate to $\langle R(a) \rangle$. Now Lemma 2.5 (i) shows that every regular dihedral subgroup of $\mathrm{Aut}(\Gamma)$ is conjugate to $R(\mathrm{D}_{2n})$. Then Proposition 2.2 gives that Γ is a CI-graph, as required.

Next we show that Γ is not normal with respect to D_{2n}. Take $\tau = (1\, ab) \in \mathrm{Sym}(\mathrm{D}_{2n})$. We drive from (2) that $\Gamma(1) = S = \Gamma(ab)$, and so the transposition τ is an automorphism of Γ. Now, $(1, a)^{\tau^{-1}R(a)\tau} = (b, a^2)$. This implies that $\tau^{-1}R(a)\tau \notin R(\mathrm{D}_{2n})$, and so $R(\mathrm{D}_{2n})$ is not a normal subgroup of $\mathrm{Aut}(\Gamma)$, and the lemma follows. □

Recall that for an odd prime p, both D_{2p} and D_{6p} are CI-groups, refer to [5, 12]. Applying Lemma 3.2 and other facts, we have the following result, which says that Theorem 3.1 holds for arc-transitive digraphs.

Lemma 3.3 *Let $n \ge 3$ be an odd integer, and let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be of valency 4. If Γ is connected and arc-transitive, then Γ is a CI-digraph.*

Proof Assume that Γ is connected, i.e., $\mathrm{D}_{2n} = \langle S \rangle$. Then S contains at least one element u of order 2, and thus, both $(1, u)$ and $(u, 1)$ are arcs of Γ. Since Γ is arc-transitive, it follows that Γ is a graph. If Γ is normal with respect to D_{2n}, then Proposition 2.4 asserts that Γ is a CI-graph, as required. Thus, we suppose next that D_{2n} is not normal in $\mathrm{Aut}(\Gamma)$.

By Proposition 2.9, either $n \in \{5, 7, 13, 15\}$ or $\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, \{a, a^{-1}, a^2b, b\})$. The latter case implies that Γ is a CI-graph, see Lemma 3.2. Now let $n \in \{5, 7, 13, 15\}$; in particular, $2n$ has the form of $2p$ or $6p$ for some prime p. In this case, D_{2n} is a CI-group, refer to [5, 12]. Then Γ is a CI-graph. This completes the proof. □

In the following, we deal with those Cayley digraphs which are not arc-transitive.

Lemma 3.4 *Let $n \ge 3$ be an odd integer, and let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected Cayley digraph of valency 4. Let $A = \mathrm{Aut}(\Gamma)$ and A_1 be the stabilizer of 1. Suppose that A_1 has an orbit S_0 on S with $|S_0| = 3$ and $\langle S_0 \rangle$ dihedral. Then Γ is a CI-digraph.*

Proof In view of Proposition 2.4, we may assume that $R(\mathrm{D}_{2n})$ is not normal in A. Let $\Gamma_0 = \mathrm{Cay}(\mathrm{D}_{2n}, S_0)$. Then $A \le \mathrm{Aut}(\Gamma_0)$, and so $R(\mathrm{D}_{2n})$ is not normal in $\mathrm{Aut}(\Gamma_0)$. In addition, Γ_0 is arc-transitive. Since $\langle S_0\rangle$ is a dihedral group, S_0 contains at least one involution. Then Γ_0 is an arc-transitive graph.

Let $H = \langle S_0\rangle$. If Γ_0 is connected then, since $R(\mathrm{D}_{2n}) \ntrianglelefteq \mathrm{Aut}(\Gamma_0)$ and $n \geq 3$ is odd, Proposition 2.8 asserts that $n = 3$ or 7; in this case, D_{2n} is a DCI-group, and so Γ is a CI-digraph. Thus, we suppose next that $H \neq \mathrm{D}_{2n}$. Then $H \cong \mathrm{D}_{2k}$, where k is a proper divisor of n. We have $H = \langle a^m, b\rangle$, where $m = n/k \geq 3$.

Since n is odd, by Lemma 2.7, D_{2n} is homogeneous. Noting that $\mathrm{Cay}(H, S_0)$ is a connected arc-transitive cubic graph, it follows Proposition 2.8 that there exists $\alpha \in \mathrm{Aut}(\mathrm{D}_{2n})$ such that $S_0^\alpha = \{b, a^m b, a^{m(r_0+1)}b\}$, where $0 < r_0 < k/2$, $r_0^2 + r_0 + 1 \equiv 0 \pmod k$, and either $k \geq 13$ or $k \in \{3, 7\}$. Note that S is a CI-subset if and only if so is S^α. In the following, without loss of generality, we assume that

$$S_0 = \{b, a^m b, a^{m(r_0+1)}b\},\ S = \{b, a^m b, a^{m(r_0+1)}b, a^s\} \text{ or } \{b, a^m b, a^{m(r_0+1)}b, a^s b\}$$

for some integer $1 \leq s \leq n-1$. Since Γ is connected, we have $\langle S\rangle = \mathrm{D}_{2n}$, and then, $\langle a^m, a^s\rangle = \langle a\rangle$, which forces that $\gcd(m, s) = 1$. Recall that $k \geq 13$ or $k \in \{3, 7\}$.

Case 1:. $k = 3$.

In this case, $r_0 = 1$ and so $S_0 = \{b, a^m b, a^{2m}b\}$. Let $T \subseteq \mathrm{D}_{2n}$ with $\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, T)$. Then $\mathrm{Cay}(\mathrm{D}_{2n}, T)$ has an arc-transitive subgraph $\mathrm{Cay}(\mathrm{D}_{2n}, T_0)$, which is isomorphic to $\Gamma_0 = \mathrm{Cay}(\mathrm{D}_{2n}, S_0)$. Since D_{2n} is a 3-DCI-group (see [35]), there exists $\beta \in \mathrm{Aut}(\mathrm{D}_{2n})$ such that $T_0^\beta = S_0$ and

$$Y := T^\beta = \{b, a^m b, a^{2m}b, a^t\} \text{ or } \{b, a^m b, a^{2m}b, a^t b\},$$

where $1 \leq t \leq n-1$ with $\gcd(t, m) = 1$. Noting that $\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, Y)$, it follows that $\mathrm{Cay}(\mathrm{D}_{2n}, S\backslash S_0) \cong \mathrm{Cay}(\mathrm{D}_{2n}, Y\backslash S_0)$. This implies that $Y\backslash S_0 = \{a^t\}$ if $S\backslash S_0 = \{a^s\}$, and $Y\backslash S_0 = \{a^t b\}$ otherwise.

Suppose first that $S\backslash S_0 = \{a^s\}$. Then $Y\backslash S_0 = \{a^t\}$. Noting that $\mathrm{Cay}(\mathrm{D}_{2n}, \{a^s\}) \cong \mathrm{Cay}(\mathrm{D}_{2n}, \{a^t\})$, it follows that a^s and a^t has the same order, and so $\gcd(s, n) = \gcd(t, n)$. Recalling that $\gcd(s, m) = \gcd(t, m) = 1$ and $m = n/k = n/3$, we derive that

$$\text{either } \gcd(s, n) = 1 = \gcd(t, n), \text{ or } \gcd(s, n) = 3 = \gcd(t, n) \text{ and } \gcd(3, m) = 1.$$

Put $s = 3^i \cdot s_1$ and $t = 3^i \cdot t_1$ for $i \in \{0, 1\}$. Then $\gcd(s_1, n) = 1 = \gcd(t_1, n)$. This implies that $xs_1 \equiv t_1 \pmod n$ for some integer x with $\gcd(x, n) = 1$. It follows that $S_0^{\sigma_{x,0}} = \{b, a^{xm}b, a^{2xm}b\}$, and $(a^s)^{\sigma_{x,0}} = a^t$. Since $\gcd(3, x) = 1$, we have $x \equiv \pm 1 \pmod 3$. Then $S_0^{\sigma_{x,0}} = \{b, a^m b, a^{2m}b\} = S_0$, and $S^{\sigma_{x,0}} = Y$, which implies that $S \equiv Y \equiv T$. Therefore, S is a CI-subset, that is, Γ is a CI-digraph.

Now let $S \setminus S_0 = \{a^s b\}$. We have $Y\backslash S_0 = \{a^t b\}$. Note that either $\gcd(s, n) = 3^i = \gcd(t, n)$ for some $i \in \{0, 1\}$, or exactly one of s and t is coprime to n. For the former case, choosing x as above, we have $S^{\sigma_{x,0}} = Y$, and so $S \equiv Y \equiv T$. Thus, assume the latter case occurs, with out loss of generality, we let $\gcd(s, n) = 1$ and $\gcd(t, n) = 3$. Recalling that $\gcd(t, m) = 1$, we have $\gcd(3, m) = 1$. Since $\gcd(3, m) = 1 = \gcd(3, s)$, we have $s \equiv \pm 1 \pmod 3$ and $m \equiv \pm 1 \pmod 3$.

Choose $y \in \{-1, 1\}$ such that $s' := s + ym$ is divisible by 3. Then $\gcd(s', m) = 1$ and $\gcd(s', n) = 3 = \gcd(t, n)$. By a similar argument as above, there exists an integer x' such that $\{b, a^m b, a^{2m} b, a^{s'} b\}^{\sigma_{x',0}} = Y$. Note that

$$S^{\sigma_{1,ym}} = \{a^{my} b, a^{m(y+1)} b, a^{m(y+2)} b, a^{s'} b\} = \{b, a^m b, a^{2m} b, a^{s'} b\}.$$

It follows that $S \equiv \{b, a^m b, a^{2m} b, a^{s'} b\} \equiv Y \equiv T$, and so $S \equiv T$. Therefore, S is a CI-subset, that is, Γ is a CI-digraph.

Case 2: $k = 7$ or $k \geq 13$.

Note that $\Gamma_0 = \mathrm{Cay}(\mathrm{D}_{2n}, S_0)$ has exactly m connected components, say $\Lambda_i := [\langle a^m, b\rangle a^{is}]$, $0 \leq i \leq m-1$, of which each is isomorphic to the arc-transitive graph $\mathrm{Cay}(H, S_0)$. Moreover, A permutes these components. Put $\mathcal{B} = \{\langle a^m, b\rangle a^{is} \mid 0 \leq i \leq m-1\}$, and let K be the kernel of A acting on $\mathcal{B}$. It follows that $R(\langle a^m\rangle) \leq K$.

Claim I: The quotient graph $\Gamma_{\mathcal{B}}$ is a cycle of length m, $A/K \cong \mathrm{D}_{2m}$, $A = K \rtimes R(\langle a^s, b\rangle)$, and K is edge-transitive but not vertex-transitive on each Λ_i.

Note the Λ_i is a connected bipartite graph with the bipartition $(\langle a^m\rangle a^{is}, \langle a^m\rangle b a^{is})$. Assume first that $S \setminus S_0 = \{a^s b\}$. We have

$$a^s b \langle a^m\rangle a^{is} = \langle a^m\rangle b a^{is-s},\ a^s b \langle a^m\rangle b a^{is} = \langle a^m\rangle a^{is+s}, \forall i \in \{0, \ldots, m-1\}.$$

It follows that $\{\langle a^m, b\rangle a^{is}, \langle a^m, b\rangle a^{js}\}$ is an edge of $\Gamma_{\mathcal{B}}$ if and only if $j - i \equiv \pm 1 \pmod{m}$, and $[\langle a^m, b\rangle a^{is}, \langle a^m, b\rangle a^{(i+1)s}]$ is the union of a perfect matching $[\langle a^m\rangle b a^{is}, \langle a^m\rangle a^{(i+1)s}]$ ($\cong k\mathrm{K}_{1,1}$) and an independent set $\langle a^m\rangle a^{is} \cup \langle a^m\rangle b a^{(i+1)s}$. Thus, $\Gamma_{\mathcal{B}} \cong \mathrm{C}_m$, the cycle of length m, and K fixes each of $\langle a^m\rangle b a^{is}$ and $\langle a^m\rangle a^{is}$ setwise.

Next assume that $S \setminus S_0 = \{a^s\}$. We have

$$a^s \langle a^m\rangle a^{is} = \langle a^m\rangle a^{is+s},\ a^s \langle a^m\rangle b a^{is} = \langle a^m\rangle b a^{is-s}, \forall i \in \{0, \ldots, m-1\}.$$

It follows that $Arc([\langle a^m, b\rangle a^{is}, \langle a^m, b\rangle a^{js}]) \neq \emptyset$ if and only if $j - i \equiv \pm 1 \pmod{m}$, and the subdigraph $[\langle a^m, b\rangle a^{is}, \langle a^m, b\rangle a^{(i+1)s}]$ is the union of two directed matching $[\langle a^m\rangle a^{is}, \langle a^m\rangle a^{(i+1)s}]$ ($\cong k\overrightarrow{\mathrm{K}}_{1,1}$) and $[\langle a^m\rangle b a^{(i+1)s}, \langle a^m\rangle b a^{is}]$ ($\cong k\overrightarrow{\mathrm{K}}_{1,1}$), both of which do not contain any isolated vertex. This implies that $\Gamma_{\mathcal{B}}$ is also isomorphic to C_m, and K fixes each of $\langle a^m\rangle a^{is}$ and $\langle a^m\rangle b a^{is}$ setwise.

The argument above shows that $A/K \lesssim \mathrm{Aut}(\mathrm{C}_m) \cong \mathrm{D}_{2m}$, and K preserves the bipartition of each Λ_i. Further, it is easily shown that $R(\langle a^s, b\rangle)$ acts transitively but not regularly on the vertex set of $\Gamma_{\mathcal{B}}$. Then

$$A/K \cong \mathrm{D}_{2m} \text{ and } A = K \rtimes R(\langle a^s, b\rangle),$$

desired as in the claim.

For each i, since $|A : A_{\langle a^m, b\rangle a^{is}}| = m$, we conclude that K has index 2 in $A_{\langle a^m, b\rangle a^{is}}$. Since Λ_i is a connected component of Γ_0 and Γ_0 is arc-transitive, $A_{\langle a^m, b\rangle a^{is}}$ acts transitively on $Arc(\Lambda_i)$, and hence, K acts transitively on the edge set of Λ_i. Suppose that K is unfaithful on one of $\langle a^m\rangle a^{is}$ and $\langle a^m\rangle b a^{is}$. Noting that Λ_i has valency 3, it follows that Λ_i is the complete bipartite graph of order 6. This implies that a^m has order 3, and so $k = 3$, which is not the case. Then the claim follows.

Claim II: If $K \cong \mathbb{Z}_k{:}\mathbb{Z}_3$, then $R(\mathrm{D}_{2n}) \trianglelefteq A$.

Assume that $K \cong \mathbb{Z}_k{:}\mathbb{Z}_3$. Then $R(\langle a^m \rangle) \trianglelefteq K$ and $|A| = 2m|K| = 3|R(\mathrm{D}_{2n})|$. In particular, $A = R(\mathrm{D}_{2n})K_1$. It further follows from $R(\langle a^m \rangle) \trianglelefteq R(\mathrm{D}_{2n})$ that $R(\langle a^m \rangle) \trianglelefteq A$. Let C be the centralizer of $R(\langle a^m \rangle)$ in A. Clearly, $R(\langle a \rangle) \leq C \trianglelefteq A$, and so

$$A = R(\mathrm{D}_{2n})K_1 = C(K\langle R(b)\rangle) = (CK_1)\langle R(b)\rangle.$$

Noting that K is not abelian, we have $K \not\leq C$; in particular, $K_1 \not\leq C$, and so $C \cap K_1 = 1$. Then $CK_1/C \cong K_1 \cong \mathbb{Z}_3$. In addition, $R(b) \notin C$, and so $C\langle R(b)\rangle/C \cong \mathbb{Z}_2$. Considering the action of A on $R(\langle a^m \rangle)$ by conjugation, we have

$$A/C \lesssim \mathrm{Aut}(R(\langle a^m \rangle)) \cong \mathrm{Aut}(\mathbb{Z}_k).$$

Then A/C is abelian, and so

$$A/C = (CK_1/C)(C\langle R(b)\rangle/C) \cong \mathbb{Z}_3 \times \mathbb{Z}_2.$$

This implies that $C = R(\langle a \rangle)$, and $R(\mathrm{D}_{2n})/C \trianglelefteq A/C$. Then $R(\mathrm{D}_{2n}) \trianglelefteq A$, as claimed.

Now we are ready to show that S is a CI-subset. Note that $k \geq 13$ or $k \in \{3, 7\}$. For $k \geq 13$, by Claim I and (iii) of Proposition 2.8, $K \cong \mathbb{Z}_k{:}\mathbb{Z}_3$; in this case, by Claim II, $R(\mathrm{D}_{2n}) \trianglelefteq A$, and so S is a CI-subset by Proposition 2.4. Thus, we let $k = 7$. If $K \cong \mathbb{Z}_7{:}\mathbb{Z}_3$ then S is a CI-subset by Claim II and Proposition 2.4. Thus, assume next that $K \not\cong \mathbb{Z}_7{:}\mathbb{Z}_3$.

By (iii) of Proposition 2.8, $\mathrm{Aut}(\Lambda_0) \cong \mathrm{PSL}_3(2).\mathbb{Z}_2$. Recall that K is an edge-transitive but not vertex-transitive subgroup of $\mathrm{Aut}(\Lambda_0)$, see Claim I. We have $K \lesssim \mathrm{PSL}_3(2)$. Checking the subgroups of $\mathrm{PSL}_3(2)$ in the Atlas [9], since $|K|$ is divisible by 7, we conclude that $K \cong \mathrm{PSL}_3(2)$. Let C be the centralizer of K in A. Then

$$A/KC \cong (A/C)/(KC/C) \lesssim \mathrm{Aut}(K)/\mathrm{Inn}(K) \cong \mathbb{Z}_2.$$

In particular, $|A : (KC)| \leq 2$. Since $R(a)$ has odd order n, we conclude that $R(a) \in KC$. In addition, since $|(A/K) : (KC/K)| = |A : (KC)| \leq 2$ and $A/K \cong \mathrm{D}_{2m}$, we have

$$KC/K \cong \mathbb{Z}_m \text{ or } \mathrm{D}_{2m}.$$

Since K is a nonabelian simple group, we have $K \cap C = 1$, and so

$$KC = K \times C, \text{and } C \cong KC/K \cong \mathbb{Z}_m \text{ or } \mathrm{D}_{2m}.$$

It follows from $|A : (KC)| \leq 2$ and $K \cong \mathrm{PSL}_3(2)$ that either $\mathrm{PSL}_3(2) \times \mathbb{Z}_m$ or $\mathrm{PSL}_3(2) \times \mathrm{D}_{2m}$ contains a cyclic subgroup of order $7m$. The only possibility is that $(7, m) = 1$.

Now let R be an arbitrary cyclic subgroup of order n in A. Then, since $n = 7m$ and $(7, m) = 1$, we have $R = C_1 \times C_2$ with $C_1 \cong \mathbb{Z}_7$ and $C_2 \cong \mathbb{Z}_m$. Note that $|KC_1 : K| = |C_1|/|K \cap C_1| = 1$ or 7. Since $KC_1 \leq A$, we have

$$2m = |A : K| = |A : (KC_1)||KC_1 : K|.$$

This implies that $|KC_1 : K| = 1$, and so $C_1 \leq K$. Then we can obtain

$$R \leq \mathbf{C}_{KC}(C_1) = \mathbf{C}_K(C_1) \times \mathbf{C}_C(C_1) = C_1 \times C,$$

which implies that $C_2 \leq C$. Clearly, C has a unique cyclic subgroup of order m, and all cyclic subgroups of order 7 in K are conjugate under K. It follows that all cyclic subgroups of order $7m$ in KC are conjugate under KC. Then we derive from $|A : (KC)| \leq 2$ that all cyclic subgroups of order n in A are contained and conjugate in KC. Then, by Proposition 2.2 and (i) of Lemma 2.5, S is a CI-subset, and hence, Γ is a CI-digraph. This completes the proof. □

In the following, we can deduce from Lemma 3.4 and other facts that Theorem 3.1 holds for non-arc-transitive digraphs.

Lemma 3.5 Let $n \geq 3$ be an odd integer, and let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected Cayley digraph with $|S| = 4$. If Γ is not arc-transitive, then Γ is a CI-digraph.

Proof Let $A = \mathrm{Aut}(\Gamma)$, and so A does not acts transitively on $Arc(\Gamma)$. Let A_1 be the stabilizer of 1 in A. If $\gcd(|A_1|, n) = 1$ then, by (ii) of Lemma 2.5, Γ is a CI-digraph. Thus, we may assume that $\gcd(|A_1|, n) \neq 1$. Then 3 divides $|A_1|$ as n is odd. Let α be an element of order 3 in A_1. By [16, Lemma 2.6.1], Γ is strongly connected, and hence, there exists an m-arc $(1 = u_0, u_1, \ldots, u_m)$ for some $m \geq 1$ such that α fixes u_i for every $0 \leq i \leq m-1$ and do not fix u_m (that is, α acting on $\Gamma^+(u_{m-1})$ has a 3-orbit). Since Γ is not arc-transitive, $A_{u_{m-1}}$ has two orbits on $\Gamma^+(u_{m-1})$ with length 1 and 3, and since A is vertex-transitive on Γ, A_1 has exactly two orbits on S with length 1 and 3.

Let S_0 be the A_1-orbit of length 3. Then A acts transitively on $Arc(\mathrm{Cay}(\mathrm{D}_{2n}, S_0))$. If $\langle S_0 \rangle$ is dihedral, then Γ is a CI-digraph by Lemma 3.4. Thus, we may assume that $\langle S_0 \rangle$ is not dihedral, and since Γ is connected, we may write

$$S_0 = \{a^s, a^t, a^r\} \text{ and } S = \{a^s, a^t, a^r, a^i b\}.$$

Let $T \subseteq \mathrm{D}_{2n}$ with $\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, T)$. Since A_1 acting on S has an orbit S_0 of length 3 such that $\langle S_0 \rangle \leq \langle a \rangle$, it is easy to check that $|T \cap \langle a \rangle| = 3$. Similarly, we have

$$T_0 := \{a^{s'}, a^{t'}, a^{r'}\} \text{ and } T = \{a^{s'}, a^{t'}, a^{r'}, a^j b\},$$

where $\mathrm{Cay}(\mathrm{D}_{2n}, S_0) \cong \mathrm{Cay}(\mathrm{D}_{2n}, T_0)$. Since D_{2n} is a 3-DCI-group, D_{2n} has an automorphism β such that $S_0^\beta = T_0$. It follows that $S^\beta = \{a^{s'}, a^{t'}, a^{r'}, a^k b\}$, and since $\sigma_{1,j-k}$ fixes a and maps $a^k b$ to $a^j b$, we have $S \equiv T$ and hence Γ is a CI-digraph, as required. This completes the proof. □

Finally, Theorem 3.1 follows from Lemmas 3.3 and 3.5.

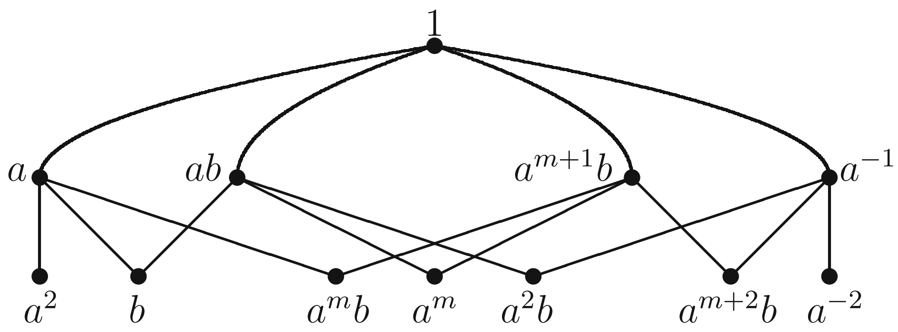

Fig. 1 Induced subgraph $[\Gamma_0(1) \cup \Gamma_1(1) \cup \Gamma_2(1)]$ in Γ

4 Proof of Theorem 1.1

For positive integer k and a vertex in a graph Γ, denote by $\Gamma_k(u)$ the set of vertices at distance k from u. Note that $\Gamma_1(u) = \Gamma(u)$, called the *neighborhood* of u in Γ.

Lemma 4.1 *Let $n \geq 4$ be an even integer. Let $S = \{a, a^{-1}, ab, a^{\frac{n}{2}+1}b\} \subseteq \mathrm{D}_{2n}$, and $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$. Then S is not a CI-subset, and if $n > 4$ then $R(\mathrm{D}_{2n}) \trianglelefteq \mathrm{Aut}(\Gamma)$.*

Proof Note that $S^{-1} = S$ and $\langle S \rangle = \mathrm{D}_{2n}$. Then Γ is a connected Cayley graph. Let $A = \mathrm{Aut}(\Gamma)$. Assume that $n = 4$. Then Γ is isomorphic to the complete bipartite graph of order 8, and so $A \cong (\mathrm{S}_4 \times \mathrm{S}_4){:}\mathrm{S}_2$. Computation with GAP [38] shows that A has two regular subgroups which are isomorphic to D_8 but not conjugate in A. Then S is not a CI-subset by Proposition 2.2, and the lemma holds for $n = 4$.

Assume that $n = 2m > 4$ in the following. Let A_1 be the stabilizer 1 in A. Then

$$\mathbb{Z}_2^2 \cong \langle \sigma_{1,m}, \sigma_{-1,2} \rangle \leq \mathrm{Aut}(\mathrm{D}_{2n}, S) \leq A_1. \tag{3}$$

Clearly, $\Gamma_2(1) = \{a^2, b, a^m b, a^m, a^2 b, a^{m+2} b, a^{-2}\}$. Since $n = 2m > 4$, we have $|\Gamma_2(1)| = 7$. The induced subgraph $[\Gamma_0(1) \cup \Gamma_1(1) \cup \Gamma_2(1)]$ is depicted in Fig. 1.

First, we prove that $R(\mathrm{D}_{2n}) \trianglelefteq \mathrm{Aut}(\Gamma)$. Let A_1^* be the kernel of A_1 acting on $\Gamma(1) = S$. For $\Gamma_2(1)$, according to Fig. 1, A_1^* fixes $\{a^2, a^{-2}\}$ pointwise, and A_1^* fixes

$$\{b, a^m b\}, \{b, a^m, a^2 b\}, \{a^m b, a^m, a^{m+2} b\} \text{ and } \{a^2 b, a^{m+2} b\}$$

setwise, respectively. Of course, A_1^* fixes those sets obtained from the last four sets by set operations: $\cup$, $\cap$ and $\setminus$. It follows that A_1^* fixes $\Gamma_2(1)$ pointwise. By the transitivity of A on $V(\Gamma)$, A_w^* fixes $\Gamma_2(w)$ pointwise for every $w \in V(\Gamma)$. Since Γ is connected, for each positive integer k, an easy inductive argument on k gives rise to that A_1^* fixes $\Gamma_k(1)$ pointwise. Thus, A_1^* fixes each vertex of Γ, and so $A_1^* = 1$. Then A_1 acts faithfully on S. Again by Fig. 1, A_1 fixes $\{a, a^{-1}\}$ and $\{ab, a^{m+1}b\}$ setwise, respectively. This implies that $|A_1| \leq 4$. It follows from (3) that $A_1 = \langle \sigma_{1,m}, \sigma_{-1,2} \rangle = \mathrm{Aut}(\mathrm{D}_{2n}, S)$. Then, by Proposition 2.1, $R(\mathrm{D}_{2n}) \trianglelefteq \mathrm{Aut}(\Gamma)$.

Next, we prove that S is not a CI-subset. According to Proposition 2.2, it suffices to shown that A has a regular dihedral subgroup, which is not $R(\mathrm{D}_{2n})$. Write $\beta = \sigma_{-1,2}$.

Then $\beta R(a^2b)\beta = R(b)$ and $\beta^2 = 1$. We get $(R(a^2b)\beta)^2 = R(a^2)$, which has order m. Thus either $R(a^2b)\beta$ has order m and m is odd, or $R(a^2b)\beta$ has order n. For the former case, we have

$$\begin{aligned}1 &= (R(a^2b)\beta)^m = ((R(a^2b)\beta)^2)^{\frac{m-1}{2}}(R(a^2b)\beta) = R(a^2)^{\frac{m-1}{2}}(R(a^2b)\beta)\\ &= R(a^{m+1}b)\beta \neq 1,\end{aligned}$$

a contradiction. Therefore, $R(a^2b)\beta$ has order n. Noting that $\beta R(ab)\beta = R(ab)$, we have

$$R(ab)R(a^2b)\beta R(ab) = R(ab)R(a^2b)R(ab)\beta = R(b)\beta.$$

Calculation shows that $\beta R(a)\beta = R(a^{-1})$, and so

$$\begin{aligned}(R(a^2b)\beta)^{-1} &= \beta R(a^2b) = \beta R(a)R(ab)\beta^2 = \beta R(a)\beta R(ab)\beta\\ &= R(a^{-1})R(ab)\beta = R(b)\beta.\end{aligned}$$

It follows that $R(ab)(R(a^2b)\beta)R(ab) = (R(a^2b)\beta)^{-1}$. Since $R(ab)$ has order 2, we have

$$L := \langle R(a^2b)\beta, R(ab)\rangle \cong \mathrm{D}_{2n}.$$

If $L = R(\mathrm{D}_{2n})$, then $\beta \in R(\mathrm{D}_{2n})$, a contradiction. This implies that $L \neq R(\mathrm{D}_{2n})$. Noting that $\langle a^2, ab\rangle$ is a dihedral group of order $n = 2m$, we have $|R(\mathrm{D}_{2n}) : \langle R(a^2), R(ab)\rangle| = 2$. Since $R(a^2) = (R(a^2b)\beta)^2 \in L$ and $R(ab) \in L$, we have $L \cap R(\mathrm{D}_{2n}) = \langle R(a^2), R(ab)\rangle$. Clearly, $\langle R(a^2), R(ab)\rangle$ has two orbits on D_{2n}, that is, $\langle a^2, ab\rangle$ and $b\langle a^2, ab\rangle$. Noting that $1^{R(a^2b)\beta} = (a^2b)^\beta = b \in b\langle a^2, ab\rangle$, it follows that L acts transitively on $V(\Gamma)$, and so L is a regular subgroup of A. This completes the proof. □

For a positive integer m, a group G is said to be a connected m-(D)CI-group if every connected Cayley (di)graph of G with valency at most m is a CI-(di)graph. If n is odd then, by [35, Theorem 3.5], D_{2n} is a 3-DCI-group, and so D_{2n} is a connected 4-DCI-group by Theorem 3.1. If $n \geq 4$ is even, then D_{2n} is not a connected 4-CI-group by Lemma 4.1. Thus, we have the following lemma.

Lemma 4.2 *Let $n \geq 3$ be an integer. Then D_{2n} is a connected 4-DCI-group if and only if D_{2n} is a connected 4-CI-group if and only if n is odd.*

Now we are ready to prove Theorem 1.1.

***Proof of Theorem* 1.1.** First we prove part (i). If D_{2n} has the 4-DCI property, then Proposition 2.6 asserts that n is odd and is indivisible by 9. Conversely, suppose that n is odd and is indivisible by 9. Let $S, T \subseteq \mathrm{D}_{2n}\setminus\{1\}$ with $|S| = 4 = |T|$ and $\mathrm{Cay}(\mathrm{D}_{2n}, S) \cong \mathrm{Cay}(\mathrm{D}_{2n}, T)$. Then we need to prove that $S \equiv T$. Note that $\mathrm{Cay}(\mathrm{D}_{2n}, S)$ and $\mathrm{Cay}(\mathrm{D}_{2n}, T)$ have connected components $\mathrm{Cay}(\langle S\rangle, S)$ and

$\mathrm{Cay}(\langle T\rangle, T)$, respectively. It follows that $\mathrm{Cay}(\langle S\rangle, S) \cong \mathrm{Cay}(\langle T\rangle, T)$. In particular, $|\langle S\rangle| = |\langle T\rangle|$. If $|\langle S\rangle|$ is odd then $\langle S\rangle$ and $\langle T\rangle$ are both cyclic and contained in $\langle a\rangle$, and thus, $\langle S\rangle = \langle T\rangle$; in this case, we have $S \equiv T$ by (i) of Proposition 2.3 and Lemma 2.7, where the condition $9 \nmid n$ applies (this is the only place to use the condition, and under this condition, the digraph is not a graph). Suppose that $|\langle S\rangle|$ is even. Then we obtain that $\langle S\rangle = \langle a^s, a^t b\rangle$ and $\langle T\rangle = \langle a^s, a^r b\rangle$ for some integers s, t and r. Let $\alpha = \sigma_{1,r-t} \in \mathrm{Aut}(\mathrm{D}_{2n})$. Then $\langle S^\alpha\rangle = \langle T\rangle$. Note that

$$\mathrm{Cay}(\langle T\rangle, S^\alpha) = \mathrm{Cay}(\langle S^\alpha\rangle, S^\alpha) \cong \mathrm{Cay}(\langle S\rangle, S) \cong \mathrm{Cay}(\langle T\rangle, T).$$

By Lemmas 4.2 and 2.7, we have $S^\alpha \equiv T$, and so $S \equiv T$. Therefore, Theorem 1.1 (i) holds.

For Theorem 1.1 (ii), the necessity follows from Lemma 4.1. Applying Proposition 2.3(ii), Lemmas 4.2, and 2.7, we derive the sufficiency using a similar argument to the above proof of the sufficiency of Theorem 1.1 (i). □

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