LINKED PARTITION IDEALS AND TWO-COLOR PARTITIONS

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Abstract. Two-color partitions are partitions whose parts can be two colors, such as red and green. Let $\mathcal{L}_d$ denote the set of two-color partitions into numerically distinct parts with the added condition that red parts are at least d larger than the next largest part and green parts are at least $d+1$ larger than the next largest part, and with no green part 1_g or $(d-1)_g$. Recently, Andrews established three partition theorems related to $\mathcal{L}_d$ for $d = 1, 2, 3$. Subsequently, Fu found the refinements of these three theorems by providing the bijective proofs. In this paper, in view of linked partition ideals, we give the analytic proofs of three refinements of the theorems given by Andrews. Meanwhile, we find an analogue of Euler pentagonal number theorem related to $\mathcal{L}_d$ for $d \geqslant 3$. Furthermore, the combinatorial proofs of the main theorems are presented.

1. Introduction

A partition λ of n is a weakly decreasing positive integer sequence $\lambda_1, \lambda_2, \ldots, \lambda_\ell$, such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. Usually, for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, each λ_i is called the part of λ, the number of parts ℓ is called the length of λ, denoted by $\sharp(\lambda)$, and $|\lambda| = n$ is referred to as the weight of λ.

Two-color partitions are partitions whose parts can be two colors, such as red and green. For example, there are ten two-color partitions of 3, namely $3_r,\ 3_g,\ 2_r + 1_r,\ 2_r + 1_g,\ 2_g + 1_r,\ 2_g + 1_g,\ 1_r + 1_r + 1_r,\ 1_r + 1_r + 1_g,\ 1_r + 1_g + 1_g,\ 1_g + 1_g + 1_g$. Given a two-color partition, we say that two parts are distinct if they are of different colors or different numerical values or both. We say two parts are numerically distinct if they have different numerical values. For instance, 1_r and 1_g are distinct, but not numerically distinct.

Define $\mathcal{L}_d$ to be the set of two-color partitions into numerically distinct parts with the condition that red parts are at least d larger than the next largest part and green parts are at least $d + 1$ larger than the next largest part, and with no green part 1_g or $(d-1)_g$. Let $L_d(n)$ denote the number of partitions of n in $\mathcal{L}_d$. Recently, Andrews [7] found the following three theorems.

Theorem 1.1. *[7, Theorem 1.3] $L_1(n)$ equals the number of partitions of n in which no part is divisible by 4.*

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Theorem 1.2. *[7, Theorem 1.4] $L_2(n)$ equals the number of basis partitions of n.*

Theorem 1.3. *[7, Theorem 1.5] $L_3(n)$ equals the number of partitions of n into distinct parts.*

Basis partitions mentioned here were first introduced by Gupta [20] in 1978 when considering rank vectors. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$, its Ferrers graph is an array of ℓ rows of dots, where the ith row has λ_i dots and rows are left justified. Counting the dots in successive columns, we obtain the conjugate partition of λ denoted by λ'. The Durfee square of λ is the largest square that could fit in its Ferrers graph. Then each partition λ can be written as a triple (d, π, σ), where d is the side length of the Durfee square, $\pi = (\lambda_{d+1}, \lambda_{d+2}, \cdots)$ is the subpartition below the Durfee square of λ, and $\sigma = (\lambda_1 - d, \lambda_2 - d, \cdots, \lambda_d - d)$. In 1998, Nolan et al. [22] gave the generating function and an alternative description of basis partitions from the viewpoint of Ferrers graphs.

Definition 1.4. [22, Theorem 3] A partition $\lambda = (d, \pi, \sigma)$ is said to be a basis partition if π and σ' do not have parts in common.

In 2023, Fu [17] derived the refinements of the above three theorems due to Andrews by exhibiting bijections.

Definition 1.5. [17] Let $L_d(n, k, \ell)$ denote the number of two-color partitions of n in $\mathcal{L}_d$ with k red parts and ℓ green parts.

Theorem 1.6. *[17, Theorem 1.7] For non-negative integers n, k, l, let $A(n, k, \ell)$ be the number of two-color partitions of n, such that for a certain non-negative integer j, each partition is consisted of $k + j$ distinct red parts and ℓ distinct even green parts, wherein exactly k red parts are larger than ℓ. Then $L_1(n, k, \ell) = A(n, k, \ell)$.*

Theorem 1.7. *[17, Theorem 1.9] Let $B(n, k, \ell)$ be the number of basis partitions $(k + \ell, \pi, \sigma)$ of n, such that π has exactly ℓ distinct parts. Then $L_2(n, k, \ell) = B(n, k, \ell)$.*

Theorem 1.8. *[17, Theorem 1.11] Let $D(n, k, \ell)$ be the number of partitions of n into $k+2\ell$ distinct parts such that the Durfee square is of side $k + \ell$. Then $L_3(n, k, \ell) = D(n, k, \ell)$.*

In this paper, with the aid of linked partition ideals, we provide a uniform method to give the analytic proofs of the following theorems.

Theorem 1.9. *Let $L_1(n, m)$ denote the number of two-color partitions of n in $\mathcal{L}_1$ with m green parts. Let $A(n, m)$ be the number of two-color partitions of n with distinct red parts and m distinct even green parts. Then $L_1(n, m) = A(n, m)$.*

Theorem 1.10. *Let $\widetilde{B}(n, k, \ell)$ be the number of partitions $(k + \ell, \pi, \sigma)$ of n, such that π is a distinct partition with ℓ parts. Then $L_2(n, k, \ell) = \widetilde{B}(n, k, \ell)$.*

Theorem 1.11. *Let $L_3(n, m)$ denote the number of partitions of n in $\mathcal{L}_3$, such that the sum of the number of red parts and twice the number of green parts is m. Let $D(n, m)$ be the number of distinct partitions of n with m parts. Then $L_3(n, m) = D(n, m)$.*

Theorem 1.12. *For $d \geqslant 3$, let $\overline{L}_d^e(n)$ (resp. $\overline{L}_d^o(n)$) denote the number of partitions of n in $\mathcal{L}_d$ with parts $\geqslant d$ except for 1_r and $(d-1)_r$, such that the sum of the number of parts and the number of green parts is even (resp. odd). Then for any non-negative integer j,*

$$\overline{L}_d^e(n) - \overline{L}_d^o(n) = \begin{cases} (-1)^j, & \text{if } n = j(dj \pm (d-2))/2, \\ 0, & \text{otherwise.} \end{cases}$$

The paper is organized as follows. In Section 2, some preliminaries are provided. In Section 3, in view of linked partition ideals, we give the analytic proofs of Theorems 1.9-1.12. Section 4 is devoted to the combinatorial proofs of the main theorems. Finally, we conclude the paper in Section 5.

2. Preliminaries

In this section, we present some preliminaries.

Let a and q be complex numbers with $|q| < 1$. Then for any positive integer n, the q-shifted factorials are defined as [18]

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

Lemma 2.1. *[18, Equation (1.3.2)] For $|z| < 1$,*

$$\sum_{n\geqslant 0} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.1}$$

Setting $a = 0$ in (2.1) yields that

$$\sum_{n\geqslant 0} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}. \tag{2.2}$$

Moreover, replacing z by $-z/a$ in (2.1) and then letting $a \to \infty$, we derive that

$$\sum_{n\geqslant 0} \frac{z^n q^{\binom{n}{2}}}{(q;q)_n} = (-z;q)_\infty. \tag{2.3}$$

The Gaussian polynomial [6] is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, & \text{if } \ 0 \leqslant m \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.2. *[6, Equation (3.3.6)] We have*

$$(z;q)_n = \sum_{j=0}^{n} (-1)^j z^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q. \tag{2.4}$$

Lemma 2.3. *[6, Theorem 9.2] We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n q^{n(3n+1)/2}(xq;q)_n(1-xq^{2n+1})}{(q;q)_n} = (xq;q)_\infty. \tag{2.5}$$

In [3], Andrews derived the following q-series from a well-poised basic hypergeometric series. Then in [6], these q-series were used to study Rogers–Ramanujan identities. For further applications, see [2, 4, 10, 21]. Here we use the definitions which are stated in [6].

Definition 2.4. [6, Section 7.2] For $|x| < |q|^{-1}$,

$$H_{k,i}(a;x;q) := \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n-in} a^n (1-x^i q^{2ni})(axq^{n+1};q)_\infty (a^{-1};q)_n}{(q;q)_n (xq^n;q)_\infty}, \tag{2.6}$$

$$J_{k,i}(a;x;q) := H_{k,i}(a;xq;q) - xqaH_{k,i-1}(a;xq;q). \tag{2.7}$$

From (2.6), it is obvious that

$$H_{k,0}(a;x;q) = 0. \tag{2.8}$$

Lemma 2.5. *[6, Section 7.2] We have*

$$H_{k,-i}(a;x;q) = -x^{-i} H_{k,i}(a;x;q), \tag{2.9}$$

$$H_{k,i}(a;x;q) - H_{k,i-1}(a;x;q) = x^{i-1} J_{k,k-i+1}(a;x;q). \tag{2.10}$$

Lemma 2.6. *For $|x| < |q|^{-1}$,*

$$H_{\frac{1}{2},\frac{1}{2}}(0;x;q) = \frac{1}{(-x^{1/2};q)_\infty}. \tag{2.11}$$

Proof. First, setting $k = i = 1/2$ and $a = 0$ in (2.10) yields that

$$x^{-\frac{1}{2}} J_{\frac{1}{2},1}(0;x;q) = H_{\frac{1}{2},\frac{1}{2}}(0;x;q) - H_{\frac{1}{2},-\frac{1}{2}}(0;x;q) = (1+x^{-\frac{1}{2}}) H_{\frac{1}{2},\frac{1}{2}}(0;x;q),$$

where the last step follows by (2.9). It is plain that

$$\begin{aligned}
(1+x^{\frac{1}{2}}) H_{\frac{1}{2},\frac{1}{2}}(0;x;q) &= J_{\frac{1}{2},1}(0;x;q) \\
&= H_{\frac{1}{2},1}(0;xq;q) \qquad \text{(by (2.7) with } (k,i,a) \to (1/2,1,0)) \\
&= J_{\frac{1}{2},\frac{1}{2}}(0;xq;q) \qquad \text{(by (2.8) and (2.10) with } (k,i,a) \to (1/2,1,0)) \\
&= H_{\frac{1}{2},\frac{1}{2}}(0;xq^2;q) \qquad \text{(by (2.7) with } (k,i,a) \to (1/2,1/2,0)).
\end{aligned}$$

Hence, iterating the above identity, we complete the proof. □

Lemma 2.7. *For $|x| < |q|^{-1}$,*

$$\sum_{n_1,n_2 \geqslant 0} \frac{x^{n_1+2n_2} q^{2\binom{n_1}{2}+n_1n_2+2n_1+2n_2} (1+xq^{n_1+1})}{(q;q)_{n_1}(q;q)_{n_2}} = \frac{1}{(xq;q)_\infty}. \tag{2.12}$$

Proof. By using (2.2) and setting $n_1 \to n$ on the left-hand side of (2.12), we obtain that

$$\sum_{n_1,n_2\geqslant 0} \frac{x^{n_1+2n_2} q^{2\binom{n_1}{2}+n_1n_2+2n_1+2n_2}(1+xq^{n_1+1})}{(q;q)_{n_1}(q;q)_{n_2}} = \sum_{n\geqslant 0} \frac{x^n q^{n^2+n}(1+xq^{n+1})}{(q;q)_n(x^2q^{n+2};q)_\infty}. \tag{2.13}$$

Then setting $a=0$, $i=k=1/2$ and $x \to x^2q^2$ in (2.6) yields that

$$\begin{aligned}\sum_{n\geqslant 0} \frac{(-1)^n x^n q^{n^2+n}(1-xq^{n+1})}{(q;q)_n(x^2q^{n+2};q)_\infty} &= H_{\frac{1}{2},\frac{1}{2}}(0;x^2q^2;q) \\ &= \frac{1}{(-xq;q)_\infty},\end{aligned} \tag{2.14}$$

where we derive the last step by setting $x \to x^2q^2$ in (2.11). Therefore, combining (2.13) and (2.14), we complete the proof. $\square$

The definition of linked partition ideals was first introduced by Andrews in [5]. It has been silent for many years, and only recently come into spotlight [8, 9, 12–16, 19]. Much lately, Andrews and Chern [8] generalized the definition for overpartitions. In this paper, we define the linked partition ideals of two-color partitions. Let μ be a two-color partition. Then $\phi^m(\mu)$ denotes the two-color partition given by adding m to each part of μ with color preserved. For two two-color partitions μ and ν, the operation $\mu \oplus \nu$ yields a two-color partition including all the parts in μ and ν.

Definition 2.8. Assume that

(1) $\Pi = \{\pi_1, \pi_2, \ldots, \pi_K\}$ is a finite set of two-color partitions, where $\pi_1 = \varnothing$;
(2) for each $\pi_a \in \Pi$, there exists a corresponding linking set $\mathcal{L}(\pi_a) \subseteq \Pi$, with especially, $\mathcal{L}(\pi_1) = \mathcal{L}(\varnothing) = \Pi$ and $\pi_1 = \varnothing \in \mathcal{L}(\pi_k)$ for any $1 \leqslant k \leqslant K$;
(3) and there is a positive integer T, referred to as the *modulus*, which is greater than or equal to the largest part among all two-color partitions in Π.

A *span one linked partition ideal* $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, T)$ is the collection of all two-color partitions of the form

$$\begin{aligned}\lambda &= \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \cdots \oplus \phi^{NT}(\lambda_N) \oplus \phi^{(N+1)T}(\pi_1) \oplus \phi^{(N+2)T}(\pi_1) \oplus \cdots \\ &= \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \cdots \oplus \phi^{NT}(\lambda_N),\end{aligned} \tag{2.15}$$

where $\lambda_i \in \mathcal{L}(\lambda_{i-1})$ for each i and λ_N is not the empty partition.

Notice that $\mathscr{I}$ includes the empty partition which corresponds to $\phi^0(\pi_1) \oplus \phi^T(\pi_1) \oplus \cdots$. It is obvious that each summand $\phi^{iT}(\lambda_i)$ in (2.15) consists of parts ranging in size from $iT+1$ to $iT+T$, indicating that no part appears in two different summands simultaneously.

For any given positive integer d, we define the span one linked partition ideal

$$\mathscr{I}^d := \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, d),$$

where $\Pi = \{\pi_1 = \varnothing, \pi_2 = 1_r, \pi_3 = 1_g, \pi_4 = 2_r, \pi_5 = 2_g, \dots, \pi_{2d} = d_r, \pi_{2d+1} = d_g\}$ and

$$\begin{cases} \mathcal{L}(\pi_1) = \{\pi_1, \pi_2, \pi_3, \pi_4, \dots, \pi_{2d}, \pi_{2d+1}\}, \\ \mathcal{L}(\pi_2) = \mathcal{L}(\pi_3) = \{\pi_1, \pi_2, \pi_4, \pi_5, \dots, \pi_{2d}, \pi_{2d+1}\}, \\ \mathcal{L}(\pi_4) = \mathcal{L}(\pi_5) = \{\pi_1, \pi_4, \pi_6, \pi_7, \dots, \pi_{2d}, \pi_{2d+1}\}, \\ \cdots \\ \mathcal{L}(\pi_{2d-2}) = \mathcal{L}(\pi_{2d-1}) = \{\pi_1, \pi_{2d-2}, \pi_{2d}, \pi_{2d+1}\}, \\ \mathcal{L}(\pi_{2d}) = \mathcal{L}(\pi_{2d+1}) = \{\pi_1, \pi_{2d}\}. \end{cases}$$

Let $\mathscr{L}_d$ to be the set of two-color partitions into numerically distinct parts with the added condition that red parts are at least d larger than the next largest part and green parts are at least $d+1$ larger than the next largest part. Now, we derive the following lemma related to $\mathscr{L}_d$.

Lemma 2.9. *For any positive integer d, $\mathscr{L}_d$ is equinumerous with the span one linked partition ideal $\mathscr{I}^d$.*

Proof. It can be easily verified that all two-color partitions in $\mathscr{I}^d$ satisfy the conditions for $\mathscr{L}_d$. On the other hand, for a given positive integer d, decompose each two-color partition in $\mathscr{L}_d$ into blocks $\mathbf{B}_0, \mathbf{B}_1, \dots$, such that all parts between $di+1$ and $di+d$ belong to the block $\mathbf{B}_i$. It is evident that $\phi^{-di}(\mathbf{B}_i)$ exclusively belongs to Π. Furthermore, if $\phi^{-di}(\mathbf{B}_i)$ is π_1 (i.e., $\mathbf{B}_i$ is empty), then $\phi^{-d(i+1)}(\mathbf{B}_{i+1})$ can be any element from Π. If $\phi^{-di}(\mathbf{B}_i)$ is π_2 or π_3 (i.e., $\mathbf{B}_i$ is either $(di+1)_r$ or $(di+1)_g$), then $\mathbf{B}_{i+1}$ cannot be $(di+d+1)_g$ due to the second condition of $\mathscr{L}_d$. Consequently, $\phi^{-d(i+1)}(\mathbf{B}_{i+1})$ cannot be π_3. Since similar arguments can be applied to other possibilities of $\phi^{-di}(\mathbf{B}_i)$, the details are omitted. Therefore, we complete the proof. □

Next, we define the generating functions for partitions in $\mathscr{I}^d$ according to the first decomposed block:

$$G_i(x) = G_i(x, y, z, q) := \sum_{\substack{\lambda \in \mathscr{I}^d \\ \lambda_0 = \pi_i}} x^{\sharp(\lambda)} y^{\sharp_r(\lambda)} z^{\sharp_g(\lambda)} q^{|\lambda|},$$

where $\sharp_g(\lambda)$ (resp. $\sharp_r(\lambda)$) denotes the number of green (resp. red) parts in λ. It is plain that

$$G_i(x) = x^{\sharp(\pi_i)} y^{\sharp_r(\pi_i)} z^{\sharp_g(\pi_i)} q^{|\pi_i|} \sum_{j : \pi_j \in \mathcal{L}(\pi_i)} G_j(xq^d).$$

Hence,

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ \vdots \\ G_{2d}(x) \\ G_{2d+1}(x) \end{pmatrix} = W \cdot A \cdot \begin{pmatrix} G_1(xq^d) \\ G_2(xq^d) \\ G_3(xq^d) \\ \vdots \\ G_{2d}(xq^d) \\ G_{2d+1}(xq^d) \end{pmatrix}, \tag{2.16}$$

where

$$W = \mathrm{diag}(1, xyq, xzq, xyq^2, xzq^2, \dots, xyq^d, xzq^d) \tag{2.17}$$

and

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{2.18}$$

Chern [12] introduced a crucial recurrence relation for a family of q-multi-summations, and later he gave a refinement of the relation in [14]. Let R and J be positive integers. Then fix a symmetric matrix $\underline{\boldsymbol{\alpha}} = (\alpha_{i,j}) \in \mathbf{Mat}_{R\times R}(\mathbb{N})$, a vector $\underline{\boldsymbol{A}} = (A_r) \in \mathbb{N}^R_{>0}$ and J vectors $\underline{\boldsymbol{\gamma}_j} = (\gamma_{j,r}) \in \mathbb{N}^R_{\geqslant 0}$ for $j = 1, 2, \dots, J$. Let $x_1, x_2, \dots, x_J$ and q be indeterminates such that the following q-multi-summation $H(\underline{\boldsymbol{\beta}}) = H(\beta_1, \dots, \beta_R)$ for $\underline{\boldsymbol{\beta}} \in \mathbb{Z}^R$ converges:

$$H(\underline{\boldsymbol{\beta}}) := \sum_{n_1,\dots,n_R \geqslant 0} \frac{x_1^{\sum_{r=1}^R \gamma_{1,r} n_r} \cdots x_J^{\sum_{r=1}^R \gamma_{J,r} n_r} q^{\sum_{r=1}^R \alpha_{r,r}\binom{n_r}{2} + \sum_{1\leqslant i<j\leqslant R} \alpha_{i,j} n_i n_j + \sum_{r=1}^R \beta_r n_r}}{(q^{A_1}; q^{A_1})_{n_1} \cdots (q^{A_R}; q^{A_R})_{n_R}}. \tag{2.19}$$

The recurrence relation for $H(\underline{\boldsymbol{\beta}})$ is as follows.

Lemma 2.10. *[14, Lemma 2.1] For $1 \leqslant r \leqslant R$,*

$$\begin{aligned} H(\beta_1, \dots, \beta_r, \dots, \beta_R) =& H(\beta_1, \dots, \beta_r + A_r, \dots, \beta_R) \\ &+ x_1^{\gamma_{1,r}} \cdots x_J^{\gamma_{J,r}} q^{\beta_r} H(\beta_1 + \alpha_{r,1}, \dots, \beta_r + \alpha_{r,r}, \dots, \beta_R + \alpha_{r,R}). \end{aligned}$$

In [14], Chern illustrated the relation with a binary tree, in which the coordinate β_r is displayed in boldface. See Figure 1.

$$
\begin{tikzcd}[column sep=small]
& H(\beta_1,\dots,\boldsymbol{\beta_r},\dots,\beta_R) \arrow[dl, dash, "1"'] \arrow[dr, dash, "x_1^{\gamma_{1,r}}\cdots x_J^{\gamma_{J,r}}q^{\beta_r}"] & \\
H(\beta_1,\dots,\beta_r+A_r,\dots,\beta_R) & & H(\beta_1+\alpha_{r,1},\dots,\beta_r+\alpha_{r,r},\dots,\beta_R+\alpha_{r,R})
\end{tikzcd}
$$

Figure 1. Node $H(\beta_1,\dots,\beta_r,\dots,\beta_R)$ and its children

3. Proofs of the main results

Based on the definition of $H(\underline{\boldsymbol{\beta}})$ in (2.19), we set $R=2$ and $J=3$. Then choose

$$
\underline{\alpha}=\begin{pmatrix} d & d \\ d & d+1\end{pmatrix}, \qquad \underline{A}=(1,1), \qquad \begin{array}{ll} x_1=x, & \gamma_1=(1,1), \\ x_2=y, & \gamma_2=(1,0), \\ x_3=z, & \gamma_3=(0,1). \end{array}
$$

Thus, we have

$$
H(\beta_1,\beta_2)=\sum_{n_1,n_2\geqslant 0}\frac{x^{n_1+n_2}y^{n_1}z^{n_2}q^{d\binom{n_1}{2}+(d+1)\binom{n_2}{2}+dn_1n_2+\beta_1n_1+\beta_2n_2}}{(q;q)_{n_1}(q;q)_{n_2}}. \tag{3.1}
$$

Let $\mathscr{I}_S$ denote the subset of partitions in $\mathscr{I}$ such that parts from S are forbidden. For example, if $S=\{1,2_g\}$, then 1_r, 1_g and 2_g are not the parts of the partitions in $\mathscr{I}_{\{1,2_g\}}$. Define

$$
F_1(x):=\sum_{\lambda\in\mathscr{I}^d}x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,2,3,4,\dots,2d,2d+1\}}G_i(x),
$$

$$
F_2(x)=F_3(x):=\sum_{\lambda\in\mathscr{I}^d_{\{1_g\}}}x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,2,4,5,\dots,2d,2d+1\}}G_i(x), \tag{3.2}
$$

$$
F_4(x)=F_5(x):=\sum_{\lambda\in\mathscr{I}^d_{\{1,2_g\}}}x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,4,6,7,\dots,2d,2d+1\}}G_i(x), \tag{3.3}
$$

$$
F_6(x)=F_7(x):=\sum_{\lambda\in\mathscr{I}^d_{\{1,2,3_g\}}}x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,6,8,9,\dots,2d,2d+1\}}G_i(x),
$$

$$
\cdots
$$

$$
F_{2d-2}(x)=F_{2d-1}(x):=\sum_{\lambda\in\mathscr{I}^d_{\{1,2,\dots,(d-2),(d-1)_g\}}}x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,2d-2,2d,2d+1\}}G_i(x), \tag{3.4}
$$

$$
F_{2d}(x)=F_{2d+1}(x):=\sum_{\lambda\in\mathscr{I}^d_{\{1,2,\dots,(d-1),d_g\}}}x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,2d\}}G_i(x).
$$

It is obvious that

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_{2d+1}(x) \end{pmatrix} = A \cdot \begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_{2d+1}(x) \end{pmatrix} = A \cdot W \cdot A \cdot \begin{pmatrix} G_1(xq^d) \\ G_2(xq^d) \\ \vdots \\ G_{2d+1}(xq^d) \end{pmatrix},$$

where the last equality follows from (2.16). Thus, we have

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_{2d+1}(x) \end{pmatrix} = A \cdot W \cdot \begin{pmatrix} F_1(xq^d) \\ F_2(xq^d) \\ \vdots \\ F_{2d+1}(xq^d) \end{pmatrix}. \tag{3.5}$$

Meanwhile, note that $G_i(0) = 1$ if $i = 1$ and $G_i(0) = 0$ otherwise. So,

$$F_1(0) = F_2(0) = \cdots = F_{2d+1}(0) = 1.$$

Then starting from $H(1,1)$, we repeatedly apply Lemma 2.10. The process is illustrated by the binary tree shown in Figure 2, where the bolded number in each step acts as β_r in Lemma 2.10. As a result, we derive the following relation.

$$\begin{pmatrix} H(1,1) \\ H(1,2) \\ H(1,2) \\ H(2,3) \\ H(2,3) \\ \vdots \\ H(i,i+1) \\ H(i,i+1) \\ \vdots \\ H(d,d+1) \\ H(d,d+1) \end{pmatrix} = A \cdot W \cdot \begin{pmatrix} H(d+1,d+1) \\ H(d+1,d+2) \\ H(d+1,d+2) \\ H(d+2,d+3) \\ H(d+2,d+3) \\ \vdots \\ H(d+i,d+i+1) \\ H(d+i,d+i+1) \\ \vdots \\ H(2d,2d+1) \\ H(2d,2d+1) \end{pmatrix}, \tag{3.6}$$

in which W and A are the same as (2.17) and (2.18), respectively. Therefore, the vector on the left-hand side of (3.5) and that on the left-hand side of (3.6) satisfy the same recurrence relation. Furthermore, taking $x = 0$ in the H-vector on the left-hand side of (3.6) gives $(1, 1, 1, \ldots, 1, 1)^T$. So, these two vectors also have the same initial condition. Thus, we

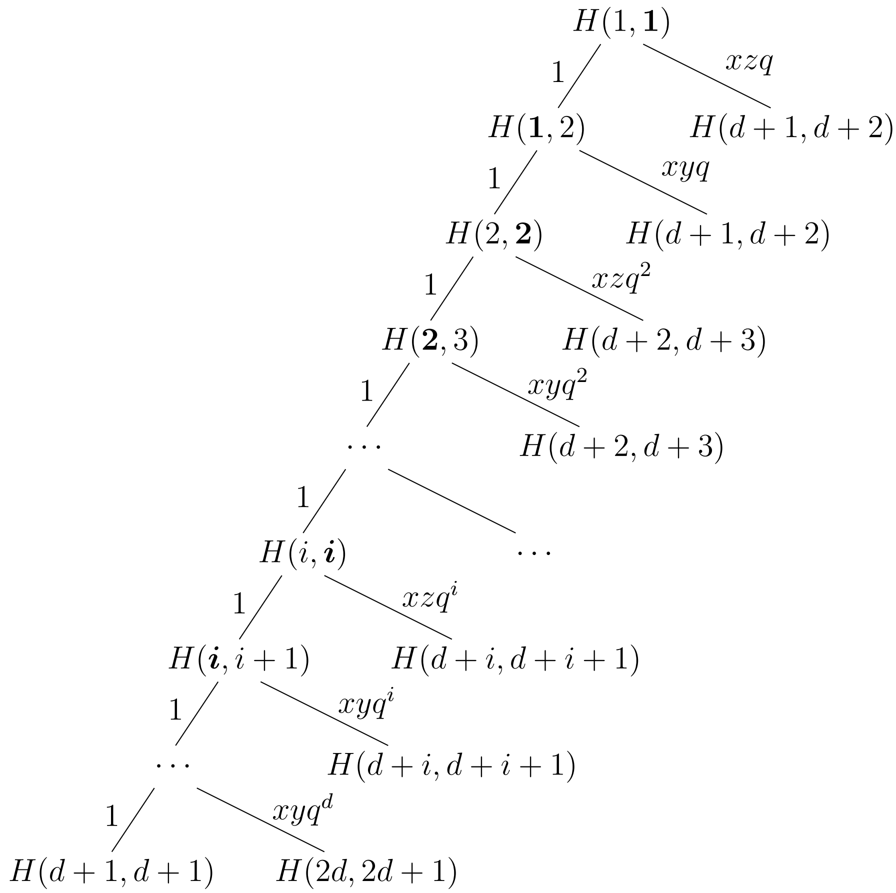

FIGURE 2. The binary tree for arbitrary d

derive that

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ F_5(x) \\ \vdots \\ F_{2d-2}(x) \\ F_{2d-1}(x) \\ F_{2d}(x) \\ F_{2d+1}(x) \end{pmatrix} = \begin{pmatrix} H(1,1) \\ H(1,2) \\ H(1,2) \\ H(2,3) \\ H(2,3) \\ \vdots \\ H(d-1,d) \\ H(d-1,d) \\ H(d,d+1) \\ H(d,d+1) \end{pmatrix}. \tag{3.7}$$

Next, in view of (3.7), we give the proofs of Theorem 1.9-1.12.

Proof of Theorem 1.9. Based on the definition of $L_1(n,m)$, we have

$$\sum_{m,n\geqslant 0} L_1(n,m)z^m q^n = \sum_{\lambda\in\mathscr{I}^1_{\{1_g\}}} z^{\sharp_g(\lambda)}q^{|\lambda|}. \tag{3.8}$$

Next, combining (3.1), (3.2) and (3.7) yields that

$$\sum_{\lambda\in\mathscr{I}^1_{\{1_g\}}} x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|} = F_2(x) = H(1,2) = \sum_{n_1,n_2\geqslant 0} \frac{x^{n_1+n_2}y^{n_1}z^{n_2}q^{\binom{n_1}{2}+2\binom{n_2}{2}+n_1n_2+n_1+2n_2}}{(q;q)_{n_1}(q;q)_{n_2}}.$$

Letting $x=y=1$ in the above identity, we have

$$\begin{aligned}
\sum_{\lambda\in\mathscr{I}^1_{\{1_g\}}} z^{\sharp_g(\lambda)}q^{|\lambda|} &= \sum_{n_1,n_2\geqslant 0} \frac{z^{n_2}q^{\binom{n_1}{2}+2\binom{n_2}{2}+n_1n_2+n_1+2n_2}}{(q;q)_{n_1}(q;q)_{n_2}}\\
&= \sum_{n_2\geqslant 0} \frac{z^{n_2}q^{2\binom{n_2}{2}+2n_2}}{(q;q)_{n_2}} \times (-q^{n_2+1};q)_\infty\\
&= (-q;q)_\infty \sum_{n_2\geqslant 0} \frac{z^{n_2}q^{2\binom{n_2}{2}+2n_2}}{(q^2;q^2)_{n_2}}\\
&= (-q;q)_\infty(-zq^2;q^2)_\infty\\
&= \sum_{m,n\geqslant 0} A(n,m)z^m q^n,
\end{aligned}$$

where the second and penultimate steps follow by (2.3). Finally, combining (3.8) and the above identity, we complete the proof. □

Proof of Theorem 1.10. According to Definition 1.5, we have

$$\sum_{n,k,\ell\geqslant 0} L_2(n,k,\ell)x^{k+\ell}z^\ell q^n = \sum_{\lambda\in\mathscr{I}^2_{\{1_g\}}} x^{\sharp(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|}. \tag{3.9}$$

Next, in view of (3.1), (3.2) and (3.7), we obtain that

$$\sum_{\lambda\in\mathscr{I}^2_{\{1_g\}}} x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|} = F_2(x) = H(1,2) = \sum_{n_1,n_2\geqslant 0} \frac{x^{n_1+n_2}y^{n_1}z^{n_2}q^{2\binom{n_1}{2}+3\binom{n_2}{2}+2n_1n_2+n_1+2n_2}}{(q;q)_{n_1}(q;q)_{n_2}}.$$

Letting $y=1$, $N_1=n_1+n_2$ and $N_2=n_2$, we get

$$\begin{aligned}
\sum_{\lambda\in\mathscr{I}^2_{\{1_g\}}} x^{\sharp(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|} &= \sum_{N_1,N_2\geqslant 0} \frac{x^{N_1}z^{N_2}q^{N_1^2+\binom{N_2}{2}+N_2}}{(q;q)_{N_1-N_2}(q;q)_{N_2}}\\
&= \sum_{N_1\geqslant 0} \frac{x^{N_1}q^{N_1^2}}{(q;q)_{N_1}} \sum_{N_2\geqslant 0} z^{N_2}q^{\binom{N_2}{2}+N_2} {N_1 \brack N_2}_q
\end{aligned}$$

$$= \sum_{N_1 \geqslant 0} \frac{x^{N_1} q^{N_1^2} (-zq;q)_{N_1}}{(q;q)_{N_1}}$$
$$= \sum_{n,k,\ell \geqslant 0} \widetilde{B}(n,k,\ell) x^{k+\ell} z^{\ell} q^n,$$

where the penultimate step follows by (2.4). Therefore, combining (3.9) and the above identity, we complete the proof. $\square$

Recall the following functional operator $\mathcal{B}$ defined on $\mathbb{C}(q)[[x]]$:

$$\mathcal{B}\left(\sum_{n \geqslant 0} c_n x^n\right) := \sum_{n \geqslant 0} c_n q^{\binom{n}{2}} x^n,$$

where the coefficients c_n are in $\mathbb{C}(q)$. This operator can be considered as a specialization of the q-Borel operators, and for more applications, one can see [8, 13].

Next, we provide two proofs of Theorem 1.11.

First proof of Theorem 1.11. From the definition of $L_3(n,m)$, it follows that

$$\sum_{n,m \geqslant 0} L_3(n,m) x^m q^n = \sum_{\lambda \in \mathscr{I}^3_{\{1_g, 2_g\}}} x^{\sharp(\lambda) + \sharp_g(\lambda)} q^{|\lambda|}.$$

Next, based on whether 1_r appears or not, we divide the set $\mathscr{I}^3_{\{1_g,2_g\}}$ into two disjoint subsets, A^3 and B^3, where A^3 is $\mathscr{I}^3_{\{1,2_g\}}$, and B^3 denotes the set of the partitions in $\mathscr{I}^3$ with 1_r as the smallest part. Notice that by the definition of $\mathscr{I}^3$, for any partition in B^3, removing the part 1_r and subtracting 3 from each remaining part results in a partition that belongs precisely to $\mathscr{I}^3_{\{1_g\}}$. Then we have

$$\begin{aligned}
&\sum_{\lambda \in \mathscr{I}^3_{\{1_g,2_g\}}} x^{\sharp(\lambda)} y^{\sharp_r(\lambda)} z^{\sharp_g(\lambda)} q^{|\lambda|} \\
&= \sum_{\lambda \in \mathscr{I}^3_{\{1,2_g\}}} x^{\sharp(\lambda)} y^{\sharp_r(\lambda)} z^{\sharp_g(\lambda)} q^{|\lambda|} + xyq \sum_{\lambda \in \mathscr{I}^3_{\{1_g\}}} x^{\sharp(\lambda)} y^{\sharp_r(\lambda)} z^{\sharp_g(\lambda)} q^{|\lambda| + 3\sharp(\lambda)} \\
&= F_4(x) + xyqF_2(xq^3) \qquad \text{(by (3.2) and (3.3))} \\
&= H(2,3) + xyqH(4,5) \qquad \text{(by (3.5)-(3.7))} \\
&= \sum_{n_1,n_2 \geqslant 0} \frac{x^{n_1+n_2} y^{n_1} z^{n_2} q^{3\binom{n_1}{2} + 4\binom{n_2}{2} + 3n_1n_2 + 2n_1 + 3n_2}}{(q;q)_{n_1}(q;q)_{n_2}} \times (1 + xyq^{2n_1+2n_2+1}), \qquad (3.10)
\end{aligned}$$

where the final step follows from (3.1). Setting $y = 1$ and $z = x$ in (3.10), we have

$$\begin{aligned}
\sum_{\lambda \in \mathscr{I}^3_{\{1_g,2_g\}}} x^{\sharp(\lambda)+\sharp_g(\lambda)} q^{|\lambda|} &= \sum_{n_1,n_2 \geqslant 0} \frac{x^{n_1+2n_2} q^{3\binom{n_1}{2} + 4\binom{n_2}{2} + 3n_1n_2 + 2n_1 + 3n_2}}{(q;q)_{n_1}(q;q)_{n_2}} \times \left(1 + xq^{2n_1+2n_2+1}\right) \\
&= \mathcal{B}\left(L.H.S\ (2.12)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{B}\left(R.H.S\ (2.12)\right)\\
&= \mathcal{B}\left(\sum_{m\geqslant 0} \frac{(xq)^m}{(q;q)_m}\right)\\
&= \sum_{m\geqslant 0} \frac{(xq)^m q^{\binom{m}{2}}}{(q;q)_m}\\
&= (-xq;q)_\infty\\
&= \sum_{n,m\geqslant 0} D(n,m)x^m q^n,
\end{aligned}
$$

where the fourth step follows by (2.2), and we obtain the penultimate step by using (2.3).

Finally, by assembling the pieces, we complete the proof. $\square$

Second proof of Theorem 1.11. Letting $y=1$, $z=x$, $N_1=n_1+n_2$ and $N_2=n_2$ in (3.10), we have

$$
\begin{aligned}
\sum_{\lambda\in\mathscr{I}^3_{\{1_g,2_g\}}} x^{\sharp(\lambda)+\sharp_g(\lambda)}q^{|\lambda|} &= \sum_{N_1,N_2\geqslant 0} \frac{x^{N_1+N_2}q^{3\binom{N_1}{2}+\binom{N_2}{2}+2N_1+N_2}(1+xq^{2N_1+1})}{(q;q)_{N_1-N_2}(q;q)_{N_2}}\\
&= \sum_{N_1\geqslant 0} \frac{x^{N_1}q^{3\binom{N_1}{2}+2N_1}(1+xq^{2N_1+1})}{(q;q)_{N_1}} \sum_{N_2\geqslant 0} x^{N_2}q^{\binom{N_2}{2}+N_2}\begin{bmatrix}N_1\\N_2\end{bmatrix}_q\\
&= \sum_{N_1\geqslant 0} \frac{x^{N_1}q^{3\binom{N_1}{2}+2N_1}(1+xq^{2N_1+1})(-xq;q)_{N_1}}{(q;q)_{N_1}}\\
&= (-xq;q)_\infty,
\end{aligned}
$$

where the penultimate step follows by (2.4) with $z=-xq$, and we derive the last step by utilizing (2.5) with $x\to -x$. Thus, we prove the theorem. $\square$

Proof of Theorem 1.12. From the definitions of $\overline{L}^{\mathrm{e}}_d(n)$ and $\overline{L}^{\mathrm{o}}_d(n)$, it follows that

$$
\sum_{n\geqslant 0}\left(\overline{L}^{\mathrm{e}}_d(n)-\overline{L}^{\mathrm{o}}_d(n)\right)q^n = \sum_{\lambda\in\mathscr{I}^d_{\{1_g,2,3,\ldots,(d-2),(d-1)_g\}}} (-1)^{\sharp(\lambda)+\sharp_g(\lambda)}q^{|\lambda|}. \tag{3.11}
$$

Depending on whether 1_r appears or not, the set $\mathscr{I}^d_{\{1_g,2,3,\ldots,(d-2),(d-1)_g\}}$ can be divided into two disjoint subsets, A^d and B^d, where A^d is $\mathscr{I}^d_{\{1,2,3,\ldots,(d-2),(d-1)_g\}}$, and B^d is the set of the partitions in $\mathscr{I}^d$ with 1_r as the smallest part. Then according to the definition of $\mathscr{I}^d$, for any partition in B^d, removing the part 1_r and subtracting d from each remaining part results in a partition in $\mathscr{I}^d_{\{1_g\}}$. So, we obtain that

$$
\sum_{\lambda\in\mathscr{I}^d_{\{1_g,2,3,\ldots,(d-2),(d-1)_g\}}} x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|}
$$

$$
\begin{aligned}
&= \sum_{\lambda\in\mathscr{I}^d_{\{1,2,3,\ldots,(d-2),(d-1)_g\}}} x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|} + xyq\sum_{\lambda\in\mathscr{I}^d_{\{1_g\}}} x^{\sharp(\lambda)}y^{\sharp_r(\lambda)}z^{\sharp_g(\lambda)}q^{|\lambda|+d\sharp(\lambda)}\\
&= F_{2d-2}(x) + xyqF_2(xq^d) \qquad \text{(by (3.2) and (3.4))}\\
&= H(d-1,d) + xyqH(d+1,d+2) \qquad \text{(by (3.5)-(3.7))}\\
&= \sum_{n_1,n_2\geqslant 0} \frac{x^{n_1+n_2}y^{n_1}z^{n_2}q^{d\binom{n_1}{2}+(d+1)\binom{n_2}{2}+dn_1n_2+(d-1)n_1+dn_2}}{(q;q)_{n_1}(q;q)_{n_2}} \times (1+xyq^{2n_1+2n_2+1}),
\end{aligned}
$$

where we derive the last step by (3.1). Then setting $x=z=-1$, $y=1$, $N_1=n_1+n_2$ and $N_2=n_2$ in the above identity, we have

$$
\begin{aligned}
&\sum_{\lambda\in\mathscr{I}^d_{\{1_g,2,3,\ldots,(d-2),(d-1)_g\}}} (-1)^{\sharp(\lambda)+\sharp_g(\lambda)}q^{|\lambda|}\\
&= \sum_{N_1,N_2\geqslant 0} \frac{(-1)^{N_1+N_2}q^{d\binom{N_1}{2}+\binom{N_2}{2}+(d-1)N_1+N_2}(1-q^{2N_1+1})}{(q;q)_{N_1-N_2}(q;q)_{N_2}}\\
&= \sum_{N_1\geqslant 0} \frac{(-1)^{N_1}q^{d\binom{N_1}{2}+(d-1)N_1}(1-q^{2N_1+1})}{(q;q)_{N_1}} \sum_{N_2\geqslant 0}(-1)^{N_2}q^{\binom{N_2}{2}+N_2}\begin{bmatrix}N_1\\N_2\end{bmatrix}_q\\
&= \sum_{N_1\geqslant 0}(-1)^{N_1}q^{d\binom{N_1}{2}+(d-1)N_1}(1-q^{2N_1+1})\\
&= \sum_{j=-\infty}^{\infty}(-1)^j q^{j(dj-d+2)/2}. \qquad (3.12)
\end{aligned}
$$

where the third step follows by (2.4) with $z=q$. Thus, combining (3.11) and (3.12), we have

$$
\sum_{n\geqslant 0}\left(\overline{L}^{\mathrm{e}}_d(n)-\overline{L}^{\mathrm{o}}_d(n)\right)q^n = 1+\sum_{j=1}^{\infty}(-1)^jq^{j(dj-d+2)/2}+\sum_{j=1}^{\infty}(-1)^jq^{j(dj+d-2)/2},
$$

which implies the theorem. □

4. Bijective proofs of the main results

In this section, we provide the combinatorial proofs of Theorems 1.9-1.12.

Proof of Theorem 1.9. In the survey [23, Section 4.4.1], a nice bijection between $\mathcal{G}_{n,m}$ and $\mathcal{E}_{n,m}$ given by Alladi and Gordon [1] was presented, where $\mathcal{G}_{n,m}$ is the set of all MacMahon diagrams with n squares, and m marked squares, such that all rows have distinct length, and every row with a marked square has a gap (say that the ith row in a diagram $[\lambda]$ has a gap if $\lambda_i-\lambda_{i+1}\geqslant 2$), and $\mathcal{E}_{n,m}$ is the set of pairs of distinct partitions (σ,τ), such that $|\sigma|+|\tau|=n$, $\sharp(\sigma)=m$, and the parts in σ are even. It is obvious that $L_1(n,m)=|\mathcal{G}_{n,m}|$ and $A(n,m)=|\mathcal{E}_{n,m}|$. So, this bijection implies the combinatorial proof of Theorem 1.9. □

Proof of Theorem 1.10. Let $\widetilde{\mathcal{B}}(n,k,\ell)$ denote the set of partitions of n such that the side length of Durfee square is $k+\ell$, and the subpartition below Durfee square is a distinct partition with ℓ parts. Let $\mathcal{L}_2(n,k,\ell)$ denote the set of partitions of n in $\mathcal{L}_2$ with k red parts and ℓ green parts. Now, we construct a bijection between $\widetilde{\mathcal{B}}(n,k,\ell)$ and $\mathcal{L}_2(n,k,\ell)$.

First, we define a map θ from $\widetilde{\mathcal{B}}(n,k,\ell)$ to $\mathcal{L}_2(n,k,\ell)$. For a partition $\lambda \in \widetilde{\mathcal{B}}(n,k,\ell)$, let $k+\ell = d$. Rewrite λ as (d,π,σ), in which d denotes the side length of the Durfee square D, π is the subpartition below D, and σ is the subpartition on the right of D. Note that if $\pi = (\pi_1,\pi_2,\cdots,\pi_\ell)$, then $d \geqslant \pi_1 > \pi_2 > \cdots > \pi_\ell$. In the following, we insert π into σ. First, insert π_1 to the $(d+1-\pi_1)$th row in σ. If $\sigma_{d+1-\pi_1}+\pi_1 \leqslant \sigma_{d-\pi_1}$, we finish the insertion of π_1. Otherwise, we exchange the positions of $\sigma_{d+1-\pi_1}+\pi_1$ and $\sigma_{d-\pi_1}$. If $\sigma_{d+1-\pi_1}+\pi_1 \leqslant \sigma_{d-\pi_1-1}$, we finish the insertion. Otherwise, we exchange the two parts. Repeat the operation until we find a right place for $\sigma_{d+1-\pi_1}+\pi_1$. Then we change the last dot in $\sigma_{d+1-\pi_1}+\pi_1$ to a hollow dot. The same action is performed to $\pi_2,\pi_3,\ldots,\pi_\ell$ in turn. After finish the insertion of π, we get a partition $\widetilde{\mu}$. For any part which has a hollow dot in the Ferrers graph, we color it green, and the rest parts are colored red. Notice that each green part in $\widetilde{\mu}$ is larger than the part below.

Since $d^2 = (2d-1)+(2d-3)+\cdots+1$, put the partition $(2d-1,2d-3,\ldots,3,1)$ to the left of $\widetilde{\mu}$ such that the first row of the two partitions should be aligned. Keeping the colors in $\widetilde{\mu}$, we obtain a new partition μ. It is obvious that the difference between any two parts in μ is at least two, and if μ_i is green, $\mu_i-\mu_{i+1} > 2$. So, $\mu \in \mathcal{L}_2(n,k,\ell)$.

The inverse of θ is defined as follows. For a two-color partition μ in $\mathcal{L}_2(n,k,\ell)$, we represent μ as a 2-indented Ferrers graph and change the last dot to a hollow dot for any green part. The 2-indented Ferrers graphs discussed here are Ferrers graphs where each row is indented by 2 units relative to the row above, rather than being left-aligned. Note that $\sharp(\mu) = k+\ell = d$. Then let $\widetilde{\mu}$ denote the partition obtained from μ by dissecting the staircase $(2d-1,2d-3,\cdots,3,1)$, which can be rewritten as a Durfee square of side d.

Next, we construct a partition π from the parts with hollow dots in $\widetilde{\mu}$. Assume that the lowest hollow dot of $\widetilde{\mu}$ is in the ith row. If $\widetilde{\mu}_i-(d+1-i) \geqslant \widetilde{\mu}_{i+1}$, then we find π_ℓ, i.e. $\pi_\ell = d+1-i$. If $\widetilde{\mu}_i-(d+1-i) < \widetilde{\mu}_{i+1}$, then put $\widetilde{\mu}_i$ in the $(i+1)$th row, $\widetilde{\mu}_{i+1}$ in the ith row and compare $\widetilde{\mu}_i-(d+1-(i+1))$ with $\widetilde{\mu}_{i+2}$. Next, if $\widetilde{\mu}_i-(d-i) \geqslant \widetilde{\mu}_{i+2}$, then we get $\pi_\ell = d-i$. Otherwise put $\widetilde{\mu}_i$ in the $(i+2)$th row, $\widetilde{\mu}_{i+2}$ in the $(i+1)$th row and compare $\widetilde{\mu}_i-(d+1-(i+2))$ with $\widetilde{\mu}_{i+3}$. Repeat the procedure above until we get π_ℓ. By applying the same operation to other parts with hollow dot from bottom to top, we get $\pi_{\ell-1},\ldots,\pi_2,\pi_1$ in turn. It is clear that every part in π is distinct and no more than d. Together with the Durfee square, we finally get a partition in $\widetilde{\mathcal{B}}(n,k,\ell)$.

□

Example 4.1. For $\mu = (12_g,8_g,5_r,2_g)$ in $\mathcal{L}_2(27,1,3)$, we give the graphical representation of the bijection step by step. See Figure 3. The corresponding partition is $\lambda = (4,(3,2,1),(2,2,1)) \in \widetilde{\mathcal{B}}(27,1,3)$. We can get μ from λ by the map θ.

In the following, we provide two bijective proofs of Theorem 1.11. The first one is based on the above proof of Theorem 1.10.

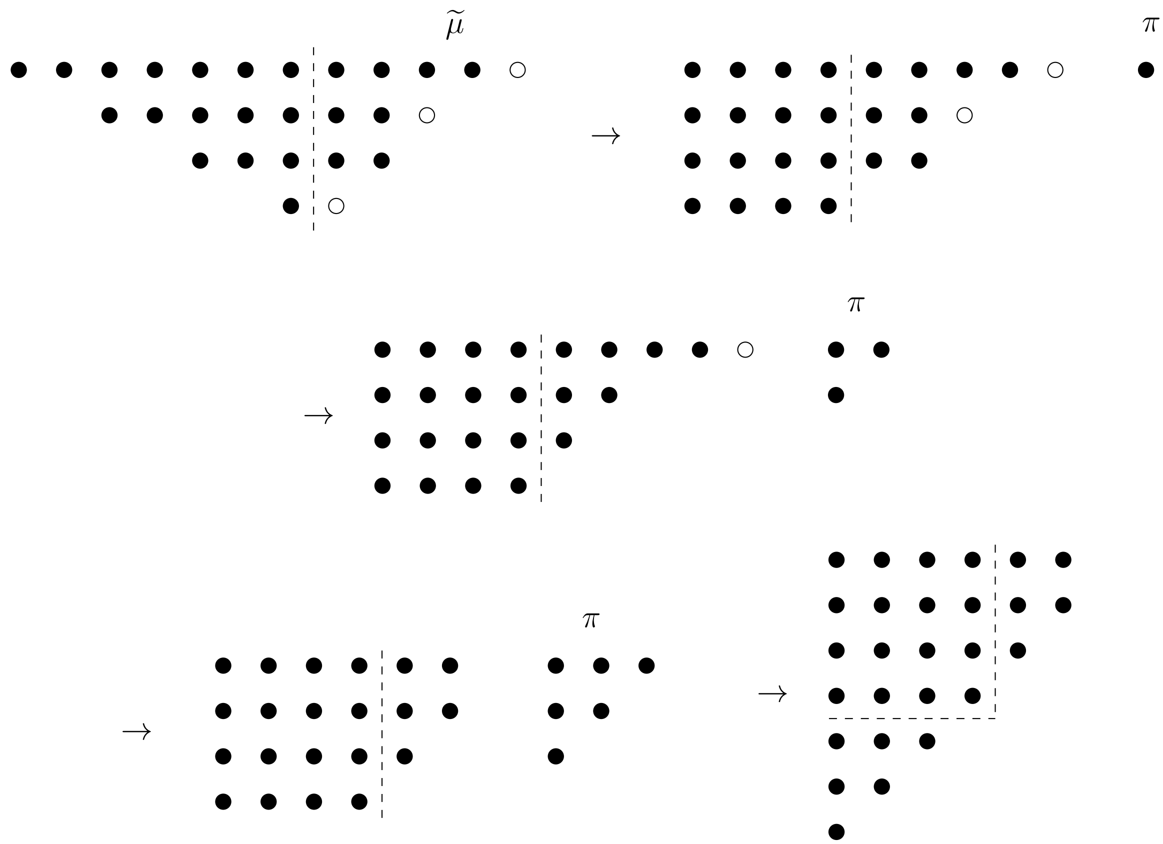

FIGURE 3. The graphical representation of the map from $\mathcal{L}_2(27,1,3)$ to $\widetilde{\mathcal{B}}(27,1,3)$

First proof of Theorem 1.11. Let $\mathcal{L}_3(n,k,\ell)$ denote the set of partitions of n in $\mathcal{L}_3$ with k red parts and ℓ green parts. Notice that $\mathcal{L}_3(n,k,\ell) \subset \mathcal{L}_2(n,k,\ell)$. Let $\mathcal{D}(n,k,\ell)$ denote the set of partitions of n into $k+2\ell$ distinct parts and the side length of Durfee square is $k+\ell$. Thus, we have $\mathcal{D}(n,k,\ell) \subset \widetilde{\mathcal{B}}(n,k,\ell)$. In the proof of Theorem 1.10, when map $\mathcal{L}_2(n,k,\ell)$ to $\widetilde{\mathcal{B}}(n,k,\ell)$, we take out a staircase $(2d-1, 2d-3, \ldots, 3, 1)$ and change it to a Durfee square of side d. For a partition ν in $\mathcal{L}_3(n,k,\ell)$, we find a staircase $(3d-2, 3d-5, \ldots, 4, 1)$, and change it to $(2d-1, 2d-2, \cdots, d+1, d)$. Then by taking out the staircase from ν, we obtain a partition $\widetilde{\nu}$. Consider the parts with hollow dot in $\widetilde{\nu}$ from bottom to top. If the part under consideration is in the ith row, to derive a distinct partition, we need to consider the following two cases.

(1) If $\nu_d \neq 1$, compare $\widetilde{\nu}_i - (d+1-i)$ with $\widetilde{\nu}_{i+1}$, $\widetilde{\nu}_i - (d-i)$ with $\widetilde{\nu}_{i+2}$, and so on. The operation is the same as that described in the above proof.
(2) If $\nu_d = 1$, compare $\widetilde{\nu}_i - (d-i)$ with $\widetilde{\nu}_{i+1}$, $\widetilde{\nu}_i - (d-1-i)$ with $\widetilde{\nu}_{i+2}$, and so on.

Then based on the bijection in the proof of Theorem 1.10, we construct a new bijection between $\mathcal{L}_3(n,k,\ell)$ and $\mathcal{D}(n,k,\ell)$. $\square$

Example 4.2. For $\nu = (13_g, 9_r, 6_g, 1_r)$ in $\mathcal{L}_3(29,2,2)$, we show the map in Figure 4. The corresponding partition $(9,7,6,4,2,1)$ is just in $\mathcal{D}(29,2,2)$.

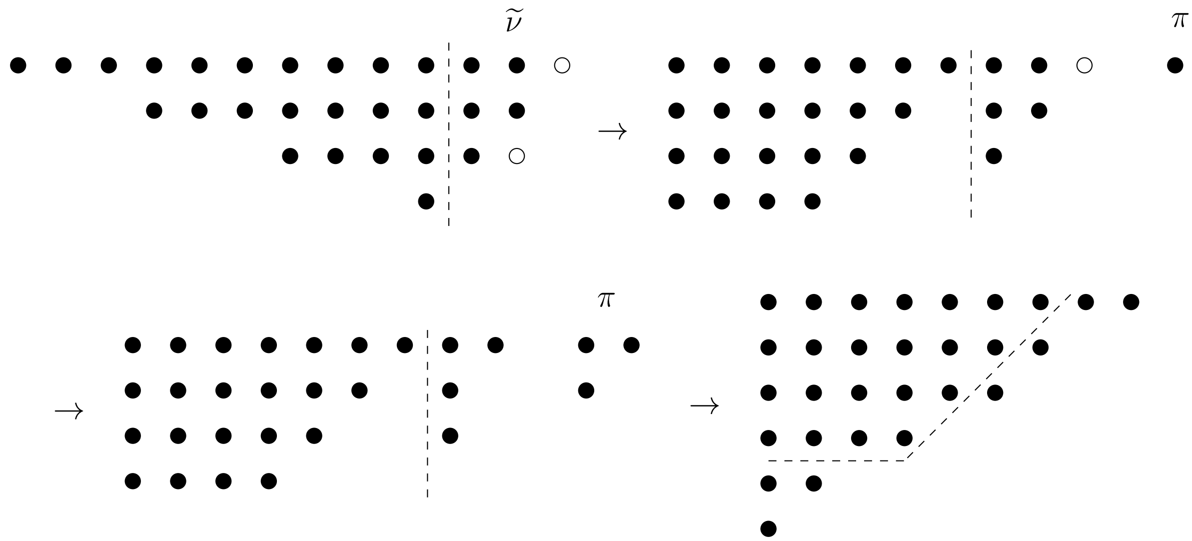

FIGURE 4. The graphical representation of the map from $\mathcal{L}_3(29,2,2)$ to $\mathcal{D}_3(29,2,2)$

Note that the above bijection is different from the one given by Fu [17]. For example, In [17], the partitions $(12_r, 5_g, 1_r)$ and $(9_r, 6_r, 3_g)$ are mapped to $(9,4,3,2)$ and $(6,5,4,3)$, respectively. Whereas, according to our bijection, these two partitions are mapped to $(10,4,3,1)$ and $(7,6,4,1)$, respectively.

Next, inspired by the work of Bressoud [11], we provide the second bijective proof of Theorem 1.11.

Second proof of Theorem 1.11. Let $\mathcal{L}_3(n,m)$ denote the set of partitions of n in $\mathcal{L}_3$, such that the sum of the number of red parts and twice the number of green parts is m. Let $\mathcal{D}(n,m)$ be the set of distinct partitions of n with m parts.

We first give a map ξ from $\mathcal{L}_3(n,m)$ to $\mathcal{D}(n,m)$. For a partition $\tau \in \mathcal{L}_3(n,m)$, list all parts in a column in decreasing order for convenience. Next, we find the smallest green part. If the part is $2k+1$ (resp. $2k+2$), rewrite it to $(k+1,k)$ (resp. $(k+2,k)$). Then if $k \leqslant$ the part below, we add three to the part below, change the pair to $(k, k-2)$ (resp. $(k,k-1)$), and then switch the positions of these two rows. We show the operation as follows.

$$\begin{array}{cc} k+1 & k \\ b & \end{array} \quad \xrightarrow{k \leqslant b} \quad \begin{array}{cc} b+3 & \\ k & k-2 \end{array}$$

or

$$\begin{array}{cc} k+2 & k \\ b & \end{array} \quad \xrightarrow{k \leqslant b} \quad \begin{array}{cc} b+3 & \\ k & k-1. \end{array}$$

Notice that in the first case $(k+1)+k-b>3$, and in the second case $(k+2)+k-b>3$. Continue to compare the smaller part of this new pair with the part below until the smaller one is larger than the part below. This is for the case that the part below is a single number.

When we meet two pairs which are in adjacent rows. For example,

$$\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2. \end{array}$$

If $a_2 \leqslant b_1$, we do the following adjustments.

$$\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array} \quad \xrightarrow[a_2 \leqslant b_1]{a_1-a_2=1} \quad \begin{array}{cc} b_1+3 & a_2 \\ a_2-2 & b_2 \end{array} \quad = \quad \begin{array}{cc} b_1+3 & a_2 \\ a_1-3 & b_2 \end{array}$$

or

$$\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array} \quad \xrightarrow[a_2 \leqslant b_1]{a_1-a_2=2} \quad \begin{array}{cc} b_1+3 & a_2 \\ a_2-1 & b_2 \end{array} \quad = \quad \begin{array}{cc} b_1+3 & a_2 \\ a_1-3 & b_2. \end{array}$$

Otherwise, we stop moving the pair (a_1, a_2).

Repeat above operation for all green parts in increasing order. Once we complete the process, collecting all the numbers, we get a distinct partition ω with $(\sharp(\tau)+\sharp_g(\tau))$ parts. So, $\omega \in \mathcal{D}(n,m)$.

Conversely, for a partition ω in $\mathcal{D}(n,m)$, put all the parts of ω in a column. Then starting from the largest part, let the first two adjacent parts with the difference less than three be a pair and put them in a row.

Let $(c+1,c)$ (resp. $(c+2,c)$) be a pair under consideration, and let d be the part above it. If the sum of the pair is larger than $d-3$, we subtract three from the part above, add three to the smaller part of the pair, and switch their positions. The operation is stated as follows.

$$\begin{array}{cc} d & \\ c+1 & c \end{array} \quad \xrightarrow{2c+1>d-3} \quad \begin{array}{cc} c+3 & c+1 \\ d-3 & \end{array}$$

or

$$\begin{array}{cc} d & \\ c+2 & c \end{array} \quad \xrightarrow{2c+2>d-3} \quad \begin{array}{cc} c+3 & c+2 \\ d-3. & \end{array}$$

Notice that in the first case $d-(c+1) \geqslant 3$, and in the second case $d-(c+2) \geqslant 3$. Repeat the operation until the sum of the pair is less than or equal to the part above reduced three, or there is a green part above it, or there is nothing above it. Merge the pair together and color it green. Then we move to find next two adjacent parts with the difference less than three under this green part and repeat the same procedure. Once we complete the process for all pairs, color the remaining parts red. Thus, we obtain a two-color partition in $\mathcal{L}_3(n,m)$. $\square$

Example 4.3. For $\tau = (16_g, 12_r, 8_g, 4_g) \in \mathcal{L}_3(40,7)$, we have $\xi(\tau) = \omega = (15,9,6,4,3,2,1) \in \mathcal{D}_3(40,7)$.

$$\tau = \begin{array}{cc} 16_g & \\ 12_r & \\ 8_g & \\ 4_g & \end{array} \mapsto \begin{array}{cc} 16_g & \\ 12_r & \\ 8_g & \\ 3 & 1 \end{array} \mapsto \begin{array}{cc} 16_g & \\ 12_r & \\ 5 & 3 \\ 3 & 1 \end{array} \mapsto \begin{array}{cc} 16_g & \\ 12_r & \\ 6 & 3 \\ 2 & 1 \end{array} \mapsto \begin{array}{cc} 9 & 7 \\ 12_r & \\ 6 & 3 \\ 2 & 1 \end{array} \mapsto \begin{array}{cc} 15_r & \\ 7 & 6 \\ 6 & 3 \\ 2 & 1 \end{array} \mapsto \begin{array}{cc} 15 & \\ 9 & 6 \\ 4 & 3 \\ 2 & 1 \end{array}.$$

The inverse of ξ on ω is shown as follows.

$$\omega = \begin{array}{cc} 15 & \\ 9 & \\ 6 & 4 \\ 3 & \\ 2 & \\ 1 & \end{array} \mapsto \begin{array}{cc} 15 & \\ 7 & 6 \\ 6 & \\ 3 & \\ 2 & \\ 1 & \end{array} \mapsto \begin{array}{cc} 9 & 7 \\ 12 & \\ 6 & \\ 3 & \\ 2 & \\ 1 & \end{array} \mapsto \begin{array}{cc} 16_g & \\ 12 & \\ 6 & \\ 3 & 2 \\ 1 & \end{array} \mapsto \begin{array}{cc} 16_g & \\ 12 & \\ 5 & 3 \\ 3 & \\ 1 & \end{array} \mapsto \begin{array}{cc} 16_g & \\ 12 & \\ 8_g & \\ 3 & 1 \end{array} \mapsto \begin{array}{cc} 16_g & \\ 12_r & \\ 8_g & \\ 4_g & \end{array}.$$

Proof of Theorem 1.12. Let $\overline{\mathcal{L}}_d(n)$ denote the set of partitions of n in $\mathcal{L}_d$ with parts $\geqslant d$ except for 1_r and $(d-1)_r$. If $n=0$, there is only empty partition. Thus, Theorem 1.12 is true. For $n>0$, we provide an involution ψ on $\overline{\mathcal{L}}_d(n)$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j) \in \overline{\mathcal{L}}_d(n)$, let λ_i^r (resp. λ_i^g) denote the ith part of λ with red (resp. green) color. Assume that λ_s^g is the smallest green part. Then we find the smallest red part denoted by λ_t^r which satisfies $\lambda_t^r \geqslant d$ for $t=j$ or $\lambda_t^r - \lambda_{t+1} > d$ for $t<j$. If λ_s^g or λ_t^r does not exist, then treat it as $+\infty$. Next, we make the following changes.

(1) If $\lambda_s^g < \lambda_t^r$, then change λ_s^g to a red part.
(2) If $\lambda_s^g > \lambda_t^r$, then change λ_t^r to a green part.

It is clear that $\psi^2(\lambda) = \lambda$ and $\sharp(\lambda) + \sharp_g(\lambda) = \sharp(\psi(\lambda)) + \sharp_g(\psi(\lambda)) \pm 1$.

When neither λ_s^g nor λ_t^r exists, it is easy to see that all the parts in λ are red, $\lambda_j < d$, and the difference between any two parts is d. So, when $n = j(dj-d+2)/2$, there is only one partition that cannot be mapped by ψ, which is $((dj-d+1)_r, (dj-2d+1)_r, \ldots, (d+1)_r, 1_r)$; and if $n = j(dj+d-2)/2$, the partition is $((dj-1)_r, (dj-d-1)_r, \ldots, (2d-1)_r, (d-1)_r)$.

Consequently, if $n \neq j(dj \pm (d-2))/2$, we have $\overline{L}_d^{\mathrm{e}}(n) = \overline{L}_d^{\mathrm{o}}(n)$; and if $n = j(dj \pm (d-2))/2$, then $\overline{L}_d^{\mathrm{e}}(n) = \overline{L}_d^{\mathrm{o}}(n) + (-1)^j$. Therefore, we complete the proof. $\square$

5. Concluding remarks

Notice that summing over k and change ℓ to m, we obtain Theorem 1.9 from Theorem 1.6. For a partition $(k+\ell, \pi, \sigma)$ mentioned in Theorem 1.10, moving the parts in σ', which also appear in π, to π, we derive a basis partition described in Theorem 1.7. So, $\widetilde{B}(n,k,\ell) = B(n,k,\ell)$. Finally, setting $m = k+2\ell$ in Theorem 1.8 yields Theorem 1.11. Although Theorems 1.9-1.11 can be obtained from Theorems 1.6-1.8, respectively, we want to show that linked partition ideals can be used to construct a uniform method to derive some partition identities. Theorem 1.12 is an example. So, the readers who are interested in this tool may find more partition identities by following this line.

It should be claimed that the bijective proofs of Theorems 1.10 and 1.11 presented in this paper are different from those given by Fu [17].

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