GENERALIZATIONS OF ROGERS–RAMANUJAN TYPE IDENTITIES

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Abstract. The Rogers–Ramanujan identities were first proved by Rogers in 1894 and later rediscovered by Ramanujan around 1913. During the study of these two identities, many Rogers–Ramanujan type identities have been found, and these identities have always been of great interest to mathematicians. In this paper, by means of properties of Appell–Lerch sums in combination with Bailey pairs and Bailey's lemma, we derive some generalizations of Rogers–Ramanujan type identities, which include some known identities and also imply some new ones.

1. Introduction

Here and throughout the paper, we assume that $|q| < 1$. The q-shifted factorials are defined by [15]

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \qquad \text{and} \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

where n is a non-negative integer. For convenience, we use the compact notation:

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

where n is a non-negative integer or infinity.

Jacobi's triple product identity [15, Eq. (1.6.1)] is given by

$$j(x;q) := (x;q)_\infty(q/x;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}} x^n. \tag{1.1}$$

Let a be a nonnegative integer and m be a positive integer. Then

$$J_{a,m} := j(q^a;q^m), \qquad \overline{J}_{a,m} := j(-q^a;q^m), \qquad J_m := J_{m,3m} = (q^m;q^m)_\infty.$$

The Rogers–Ramanujan identities are stated as

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{1.2}$$

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$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty},$$

which were first discovered and proved by Rogers [30] as early as 1894. Unfortunately, few people paid attention to his work and this work has been neglected for almost two decades. Around 1913, Ramanujan [29] independently rediscovered these two identities, and the rediscovery brought belated attention to Rogers' work. Many mathematicians proved the above two identities by using various approaches. In particular, Bailey [4, 5] used a technique that was later named "Bailey's lemma" by Andrews [2] to prove them and discovered many identities similar to them, which are called Rogers–Ramanujan type identities. Around 1950, Slater [36] obtained a list of 130 Rogers–Ramanujan type identities by using Bailey's lemma and various Bailey pairs. In 2008, McLaughlin, Sills and Zimmer [26] published an article containing a more complete list of Rogers–Ramanujan type identities. Moreover, Chu and Zhang [9] provided a list of 200 Rogers–Ramanujan type identities by applying a bilateral Bailey lemma in 2009. For more information on Rogers–Ramanujan type identities, one can refer to Sills' excellent book [34].

It is worth mentioning that generalizations of Rogers–Ramanujan type identities have received a great deal of attention in the literature. For instance, the following bivariate identity was recorded by Ramanujan in his lost notebook [3, p. 99, Entry 5.3.1]:

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{2n^2}}{(q^2;q^2)_{2n}}=\frac{j(xq^3;q^6)}{J_2}. \tag{1.3}$$

Setting special values of x yields some Rogers–Ramanujan type identities. For example, $x=1$ gives

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2q^{2n^2}}{(q^2;q^2)_{2n}}=\frac{J_{3,6}}{J_2},$$

which appears in [3, p. 102, Entry 5.3.3]. Inspired by the above work, McLaughlin and Sills [25] found several bivariate Rogers–Ramanujan identities similar to (1.3).

Furthermore, Andrews [2] considered a two-variable generalization $f(q,t)$ of Rogers–Ramanujan series $\sum_{n=0}^{\infty}f(q)$, where $f(q,t)$ is a generating function in t of a sequence of polynomials $D_n(q)$ such that $lim_{n\to\infty}D_n(q)=\sum_{n=0}^{\infty}f(q)$ and $f(q,t)$ satisfies a first-order nonhomogeneous q-difference equation. For example, the two-variable generalization $f(q,t)$ of the left-hand side of (1.2) is stated as

$$\sum_{n=0}^{\infty}\frac{t^{2n}q^{n^2}}{(t;q)_{n+1}}.$$

Andrews noticed that $f(q,-1)$ is one of Ramanujan's fifth order mock theta functions. He further examined the two-variable generalizations of several Rogers–Ramanujan type series and found that $f(q,-1)$ is a false theta function. Later, Bowman et al. [6] presented another possibility. They studied the two-variable generalizations of four Rogers–Ramanujan type series due to Dyson [4, p. 433, Eqs. (B1)-(B4)] and discovered that $f(q,-1)$ can be represented as a sum of modular forms rather than as a mock or a false theta function. They proved their results by utilizing Bailey's lemma and Bailey pairs. One of the results is given below.

$$\sum_{n=0}^{\infty}\frac{(-q^3;q^3)_n q^{n^2+n}}{(-q;q)_n(q;q)_{2n+1}}=\frac{J_9-2q^3\overline{J}_{27,108}+2q^9\overline{J}_{9,108}}{J_1}.$$

For more research on generalizations of Rogers–Ramanujan type identities, one can see [7,14,16,18,31–33].

In 2014, Hickerson and Mortenson [19] gave the following definition of Appell–Lerch sums to further study mock theta functions:

Definition 1.1. Let $x,\ z\in\mathbb{C}^*:=\mathbb{C}\backslash\{0\}$ with neither z nor xz an integral power of q. Then

$$m(x,q,z):=\frac{1}{j(z;q)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}z^n}{1-q^{n-1}xz}. \tag{1.4}$$

Changing n to $n+1$ in (1.4) provides another useful form of $m(x,q,z)$ [19, Eq. (3.1)]:

$$m(x,q,z)=\frac{-z}{j(z;q)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}z^n}{1-q^n xz}. \tag{1.5}$$

For more applications of Appell–Lerch sums, one can see [8,11–13,17,20,22,27,28,37].

The object of this paper is to use Appell–Lerch sums in combination with Bailey pairs and Bailey's lemma to derive more generalizations of Rogers–Ramanujan type identities. In addition to some known Rogers–Ramanujan type identities as special cases, we also find some new ones. The main results are stated as follows:

Theorem 1.2. *We have*

$$\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_n q^{n^2+n}}{(q;q)_{2n}}=\frac{j(xq;q^3)+xj(xq^2;q^3)}{(1+x)J_1}. \tag{1.6}$$

Remark 1.3. From the above theorem, we obtain the following Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2+n}}{(q;q)_{2n}}=\frac{J_4J_6^5}{J_2^2J_3^2J_{12}^2}, \tag{1.7}$$

$$\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n q^{n^2+3n}}{(q;q)_n(q;q)_{2n+2}}=\frac{J_{3,27}}{J_1}, \tag{1.8}$$

$$\sum_{n=0}^{\infty}\frac{(-1;q^3)_n q^{n^2+n}}{(-1;q)_n(q;q)_{2n}}=\frac{J_{1,9}J_{7,18}}{J_1J_{18}}. \tag{1.9}$$

Notice that the identity (1.7) is due to Ramanujan, see [3, p. 86, Entry 4.2.10]. The identity (1.8) is due to Dyson and appears in [4, p. 433, Eq. (B1)]. The identity (1.9) appears in [23, p. 158, Eq. (P12)] and was rediscovered by McLaughlin and Sills [24, p. 766, Eq. (1.3)].

Corollary 1.4. *For any integer $m \geq 0$,*

$$\sum_{n=0}^{\infty} \frac{(-q^{3m+1}, -q^{-3m-1}; q)_n q^{n^2+n}}{(q;q)_{2n}} = \frac{q^{-m(3m+1)/2}(\overline{J}_{1,3} - q^{2m+1}\overline{J}_{0,3})}{(1-q^{3m+1})J_1}, \tag{1.10}$$

$$\sum_{n=0}^{\infty} \frac{(-q^{3m+2}, -q^{-3m-2}; q)_n q^{n^2+n}}{(q;q)_{2n}} = \frac{q^{-3m(m+1)/2}(\overline{J}_{0,3} - q^{2m+1}\overline{J}_{1,3})}{(1-q^{3m+2})J_1}, \tag{1.11}$$

$$\sum_{n=0}^{\infty} \frac{(-q^{3m+3}, -q^{-3m-3}; q)_n q^{n^2+n}}{(q;q)_{2n}} = \frac{q^{-m(3m+5)/2-1}(1 - q^{2m+2})\overline{J}_{1,3}}{(1-q^{3m+3})J_1}. \tag{1.12}$$

Theorem 1.5. *We have*

$$\sum_{n=0}^{\infty} \frac{(x, x^{-1}; q)_n q^{n^2}}{(q;q)_{2n}} = \frac{j(xq^2;q^3) + xj(xq;q^3)}{(1+x)J_1}. \tag{1.13}$$

Remark 1.6. Theorem 1.5 implies the following Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(q;q)_{2n}} = \frac{J_2^3}{J_1^2 J_4}, \tag{1.14}$$

$$\sum_{n=0}^{\infty} \frac{(q^3;q^3)_n q^{n^2+2n}}{(q;q)_n (q;q)_{2n+2}} = \frac{J_{6,27}}{J_1}, \tag{1.15}$$

$$\sum_{n=0}^{\infty} \frac{(-1;q^3)_n q^{n^2}}{(-1;q)_n (q;q)_{2n}} = \frac{J_{2,9} J_{5,18}}{J_1 J_{18}}. \tag{1.16}$$

The identity (1.14) is a special case of the q-Gauss sum [15, Appendix (II.8)] and first appeared in [36, p. 156, Eq. (47)]. The identity (1.15) is due to Dyson and appears in [4, p. 433, Eq. (B2)]. The identity (1.16) was presented in [23, p. 159, Eq. (P12 bis)] and rediscovered by McLaughlin and Sills [24, p. 766, Eq. (1.4)].

Corollary 1.7. *For any integer $m \geq 0$,*

$$\sum_{n=0}^{\infty} \frac{(-q^{3m+1}, -q^{-3m-1}; q)_n q^{n^2}}{(q;q)_{2n}} = \frac{q^{-3m(m+1)/2}(\overline{J}_{0,3} - q^{4m+1}\overline{J}_{1,3})}{(1-q^{3m+1})J_1},$$

$$\sum_{n=0}^{\infty} \frac{(-q^{3m+2}, -q^{-3m-2}; q)_n q^{n^2}}{(q;q)_{2n}} = \frac{q^{-m(3m+5)/2-1}(\overline{J}_{1,3} - q^{4m+3}\overline{J}_{0,3})}{(1-q^{3m+2})J_1},$$

$$\sum_{n=0}^{\infty} \frac{(-q^{3m+3}, -q^{-3m-3}; q)_n q^{n^2}}{(q;q)_{2n}} = \frac{q^{-m(3m+7)/2-2}(1 - q^{4m+4})\overline{J}_{1,3}}{(1-q^{3m+3})J_1}.$$

Theorem 1.8. *We have*

$$\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+2n}}{(q;q)_{2n+1}}$$
$$=\frac{x^2q(1-x^{-4})j(x;q^3)-x^{-1}q(1-x^2q^{-1})j(xq;q^3)+(1-x^2q)j(xq^2;q^3)}{(1+x)(1-x^2q)(1-x^{-2}q)J_1}. \tag{1.17}$$

Corollary 1.9. *We have*

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n^2+2n}}{(q;q)_{2n+1}}=\frac{J_2^3}{(1+q)J_1^2J_4}, \tag{1.18}$$

$$3\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}q^{n^2+2n}}{(q;q)_{n-1}(q;q)_{2n+1}}=\frac{-3qJ_9J_{18}+(2+q)\overline{J}_{1,9}J_{7,18}-(1-q)\overline{J}_{2,9}J_{5,18}}{(1+q+q^2)J_1J_{18}}-\frac{1}{1-q}, \tag{1.19}$$

$$\sum_{n=0}^{\infty}\frac{(-1;q^3)_nq^{n^2+2n}}{(-1;q)_n(q;q)_{2n+1}}=\frac{qJ_3J_{3,18}+J_{2,9}J_{5,18}+q^2J_{4,9}J_{1,18}}{(1+q+q^2)J_1J_{18}}. \tag{1.20}$$

Theorem 1.10. *We have*

$$\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+3n}}{(q;q)_{2n+1}}$$
$$=\frac{(1+q)(x-x^{-1})j(x;q^3)-q(1-x^2q^{-1})j(xq;q^3)-xq(1-x^{-2}q^{-1})j(xq^2;q^3)}{(1+x)(1-x^2q)(1-x^{-2}q)J_1}. \tag{1.21}$$

Corollary 1.11. *We have*

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n^2+3n}}{(q;q)_{2n+1}}=\frac{2J_{3,12}-J_{5,12}+qJ_{1,12}}{(1+q)J_1}, \tag{1.22}$$

$$3\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}q^{n^2+3n}}{(q;q)_{n-1}(q;q)_{2n+1}}=\frac{3(1+q)J_9J_{18}-(1-q)\overline{J}_{1,9}J_{7,18}-(1+2q)\overline{J}_{2,9}J_{5,18}}{(1+q+q^2)J_1J_{18}}-\frac{1}{1-q}, \tag{1.23}$$

$$\sum_{n=0}^{\infty}\frac{(-1;q^3)_nq^{n^2+3n}}{(-1;q)_n(q;q)_{2n+1}}=\frac{(1+q)J_3J_{3,18}-qJ_{1,9}J_{7,18}+qJ_{4,9}J_{1,18}}{(1+q+q^2)J_1J_{18}}. \tag{1.24}$$

Remark 1.12. Combining (1.19) and (1.23), we obtain the following identity:

$$\sum_{n=0}^{\infty}\frac{(q^3;q^3)_nq^{n^2+4n+3}}{(q;q)_n^2(q^{n+2};q)_{n+2}}=\frac{\overline{J}_{1,9}J_{7,18}+q\overline{J}_{2,9}J_{5,18}-(1+2q)J_9J_{18}}{(1+q+q^2)J_1J_{18}}.$$

Theorem 1.13. *We have*

$$\sum_{n=0}^{\infty}\frac{(x,x^{-1};q^2)_nq^{n^2}}{(q;q^2)_n(q^4;q^4)_n}=\frac{j(xq^3;q^4)+xj(xq;q^4)}{(1+x)J_{1,4}}. \tag{1.25}$$

Remark 1.14. The above theorem implies the following Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty} \frac{(-1;q^4)_n q^{n^2}}{(q;q^2)_n (q^4;q^4)_n} = \frac{\overline{J}_{1,8} J_{3,8}}{J_{1,4} J_{4,16}}, \tag{1.26}$$

$$\sum_{n=0}^{\infty} \frac{(q^6;q^6)_n q^{n^2+2n}}{(q;q)_{2n+1} (q^4;q^4)_{n+1}} = \frac{J_{9,36}}{J_{1,4}}, \tag{1.27}$$

$$\sum_{n=0}^{\infty} \frac{(-1;q^6)_n q^{n^2}}{(-1;q^2)_n (q;q^2)_n (q^4;q^4)_n} = \frac{J_{3,12} J_{6,24}}{J_{24} J_{1,4}}. \tag{1.28}$$

The identities (1.26)-(1.28) appear in [36, p. 158, Eq. (66)], [4, p. 434, Eq. (C2)] and [24, p.767, Eq. (1.13)], respectively.

Theorem 1.15. *We have*

$$\sum_{n=0}^{\infty} \frac{(x, x^{-1};q^2)_n q^{n^2+2n}}{(q;q^2)_n (q^4;q^4)_n} = \frac{j(xq;q^4) + xj(xq^3;q^4)}{(1+x) J_{1,4}}. \tag{1.29}$$

Remark 1.16. The above theorem implies the following Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty} \frac{(-1;q^4)_n q^{n^2+2n}}{(q;q^2)_n (q^4;q^4)_n} = \frac{J_{1,8} \overline{J}_{3,8}}{J_{1,4} J_{4,16}}, \tag{1.30}$$

$$\sum_{n=0}^{\infty} \frac{(q^6;q^6)_n q^{n^2+4n}}{(q;q)_{2n+1} (q^4;q^4)_{n+1}} = \frac{J_{3,36}}{J_{1,4}}, \tag{1.31}$$

$$\sum_{n=0}^{\infty} \frac{(-1;q^6)_n q^{n^2+2n}}{(-1;q^2)_n (q;q^2)_n (q^4;q^4)_n} = \frac{J_{1,12} J_{10,24}}{J_{24} J_{1,4}}. \tag{1.32}$$

The identities (1.30)-(1.32) appear in [36, p. 158, Eq. (67)], [4, p. 434, Eq. (C1)] and [24, p.767, Eq. (1.12)], respectively.

Theorem 1.17. *We have*

$$\sum_{n=0}^{\infty} \frac{(xq, x^{-1}q;q^2)_n q^{2n^2+2n}}{(q^2;q^2)_{2n+1}} = \frac{j(xq;q^6) - x^2 j(xq^5;q^6)}{(1-x^2) J_2}. \tag{1.33}$$

Remark 1.18. The above theorem implies the following Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+n}}{(q;q)_{2n+1}} = \frac{J_4}{J_1}, \tag{1.34}$$

$$\sum_{n=0}^{\infty} \frac{(q^3;q^6)_n q^{2n^2+2n}}{(q;q^2)_n (q^2;q^2)_{2n+1}} = \frac{\overline{J}_{5,18} J_{8,36}}{J_2 J_{36}}. \tag{1.35}$$

The identities (1.34) and (1.35) appear in [36, p. 157, Eq. (51)] and [36, p. 166, Eq. (124)], respectively.

Theorem 1.19. *We have*

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{2n^2+4n}}{(q^2;q^2)_{2n+1}}=\frac{xq^{-1}(j(xq;q^6)-j(xq^5;q^6))}{(1-x^2)J_2}. \tag{1.36}$$

Remark 1.20. The above theorem implies the following Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2+2n}}{(q;q)_{2n+1}}=\frac{J_{2,12}}{J_1}, \tag{1.37}$$

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_nq^{2n^2+4n}}{(q;q^2)_n(q^2;q^2)_{2n+1}}=\frac{\overline{J}_{7,18}J_{4,36}}{J_2J_{36}}. \tag{1.38}$$

The identities (1.37) and (1.38) appear in [3, p. 65, Entry 3.4.4] and [36, p. 166, Eq. (125)], respectively.

Theorem 1.21. *We have*

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{n^2+n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}=\frac{j(xq;q^4)-x^2j(xq^3;q^4)}{(1-x^2)J_{2,4}}. \tag{1.39}$$

Remark 1.22. The above theorem implies the following Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^4)_nq^{n^2+n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}=\frac{J_{6,16}}{J_{2,4}}, \tag{1.40}$$

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_nq^{n^2+n}}{(q;q)_{2n}(q^2;q^4)_{n+1}}=\frac{\overline{J}_{3,12}J_{6,24}}{J_{2,4}J_{24}}. \tag{1.41}$$

The identities (1.40) and (1.41) appear in [3, p. 34, Entry 1.7.8] and [36, p. 163, Eq. (107)], respectively.

Theorem 1.23. *We have*

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{n^2+3n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}=\frac{xq^{-1}(j(xq;q^4)-j(xq^3;q^4))}{(1-x^2)J_{2,4}}. \tag{1.42}$$

Remark 1.24. The above theorem implies the following Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^4)_nq^{n^2+3n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}=\frac{J_{2,16}}{J_{2,4}}, \tag{1.43}$$

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_nq^{n^2+3n}}{(q;q)_{2n}(q^2;q^4)_{n+1}}=\frac{\overline{J}_{5,12}J_{2,24}}{J_{2,4}J_{24}}. \tag{1.44}$$

The identities (1.43) and (1.44) appear in [3, p. 32, Entry 1.7.6] and [24, p. 768, Eq. (1.25)], respectively.

Remark 1.25. It is worth noting that by means of our method, one can also derive (1.3) and the following two-variable Rogers–Ramanujan type identities:

$$\sum_{n=0}^{\infty}\frac{(-xq,-x^{-1}q;q^2)_nq^{n^2}}{(q;q^2)_n(q^4;q^4)_n}=\frac{j(-xq^2;q^4)}{J_{1,4}}, \tag{1.45}$$

$$\sum_{n=0}^{\infty}\frac{(x;q)_{n+1}(x^{-1}q;q)_nq^{n^2+n}}{(q;q)_{2n+1}}=\frac{j(x;q^3)}{J_1}, \tag{1.46}$$

$$\sum_{n=0}^{\infty}\frac{(-xq,-x^{-1}q;q^2)_nq^{n^2+2n}}{(q;q^2)_{n+1}(q^4;q^4)_n}=\frac{j(-x;q^4)}{(1+x)J_{1,4}}, \tag{1.47}$$

$$\sum_{n=0}^{\infty}\frac{(x;q)_{n+1}(-q,x^{-1}q;q)_nq^{(n^2+n)/2}}{(q;q)_{2n+1}}=\frac{j(x;q^2)}{J_{1,2}}. \tag{1.48}$$

The above identities appear in [3, p. 103, Entry 5.3.5] and [25, p. 593, Eqs. (1.4)-(1.6)].

In addition, we claim that some new identities similar to those in Corollaries 1.4 and 1.7 can also be found from Theorems 1.8, 1.10, 1.13, 1.15, 1.17, 1.19, 1.21 and 1.23.

An outline of this paper is as follows. In Section 2, we provide some preliminaries. In Section 3, utilizing Bailey pairs, Bailey's lemma and properties of Appell–Lerch sums, we prove the main results.

2. Preliminaries

In this section, we present some preliminary results.

Throughout the paper, we always utilize the following identities [19] without mention:

$$J_{1,2}=\frac{J_1^2}{J_2},\quad \overline{J}_{1,3}=\frac{J_2J_3^2}{J_1J_6},\quad J_{1,4}=\frac{J_1J_4}{J_2},\quad \overline{J}_{1,4}=\frac{J_2^2}{J_1},\quad \overline{J}_{1,6}=\frac{J_2^2J_3J_{12}}{J_1J_4J_6}.$$

Lemma 2.1. *[19, Eqs. (2.2a), (2.2b), (2.2d), (2.2f) and (2.4b)] For generic $x,y\in\mathbb{C}^*$,*

$$j(q^nx;q)=(-1)^nq^{-n(n-1)/2}x^{-n}j(x;q),\quad n\in\mathbb{Z}, \tag{2.1}$$

$$j(x;q)=j(q/x;q)=-xj(x^{-1};q), \tag{2.2}$$

$$j(x,q)=J_1j(x;q^n)j(xq;q^n)\cdots j(xq^{n-1};q^n)/J_n^n,\quad n\geq 1, \tag{2.3}$$

$$j(x;q)=\sum_{k=0}^{m-1}(-1)^kq^{k(k-1)/2}x^kj((-1)^{m+1}q^{m(m-1)/2+mk}x^m;q^{m^2}),\quad m\geq 1, \tag{2.4}$$

$$j(x;q)j(y;q)=j(-xy;q^2)j(-qx^{-1}y;q^2)-xj(-qxy;q^2)j(-x^{-1}y;q^2). \tag{2.5}$$

According to [19], the term "generic" indicates that the parameters do not cause poles in Appell–Lerch sums or in quotients of theta functions.

Lemma 2.2. *[19, Eq. (3.7)] For generic $x, z_0, z_1 \in \mathbb{C}^*$,*

$$m(x,q,z_1) - m(x,q,z_0) = \frac{z_0 J_1^3 j(z_1/z_0;q) j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}. \tag{2.6}$$

Lemma 2.3. *[10, Eq. (1.7)] (The quintuple product identity)*

$$j(z^3q;q^3) + zj(z^{-3}q;q^3) = \frac{j(-z;q)j(z^2q;q^2)}{J_2}. \tag{2.7}$$

Lemma 2.4. *For any integers k, s and t with $k>0$, there holds*

$$\sum_{n=-\infty}^{\infty} \frac{q^{kn^2+sn}}{1-xq^{kn+t}}$$
$$= j(-q^{k-s};q^{2k})m(-x^2q^{2t-s+k}, q^{2k}, -q^{s-k}) + xq^{t-s}j(-q^s;q^{2k})m(-x^2q^{2t-s}, q^{2k}, -q^s). \tag{2.8}$$

Proof. Observe that

$$\sum_{n=-\infty}^{\infty} \frac{q^{kn^2+sn}}{1-xq^{kn+t}} = \sum_{n=-\infty}^{\infty} \frac{(1+xq^{kn+t})q^{kn^2+sn}}{1-x^2q^{2kn+2t}} = \sum_{n=-\infty}^{\infty} \frac{q^{kn^2+sn}}{1-x^2q^{2kn+2t}} + \sum_{n=-\infty}^{\infty} \frac{xq^{kn^2+(k+s)n+t}}{1-x^2q^{2kn+2t}}$$
$$= q^{k-s}j(-q^{s-k};q^{2k})m(-x^2q^{2t+k-s}, q^{2k}, -q^{s-k}) + xq^{t-s}j(-q^s;q^{2k})m(-x^2q^{2t-s}, q^{2k}, -q^s),$$

where the last equality follows from (1.5). Therefore, combining (2.2) and the above identity, we complete the proof. □

Lemma 2.5. *We have*

$$\frac{\overline{J}_{1,6}j(-x^2q^2;q^6) + x\overline{J}_{2,6}j(-x^2q^5;q^6)}{j(x^2;q^6)j(x^2q^4;q^6)} = \frac{J_3^4}{J_6^2 j(x;q^3)j(xq^2;q^3)}, \tag{2.9}$$
$$\frac{\overline{J}_{2,6}j(-x^2q;q^6) + x\overline{J}_{1,6}j(-x^2q^4;q^6)}{j(x^2;q^6)j(x^2q^2;q^6)} = \frac{J_3^4}{J_6^2 j(x;q^3)j(xq;q^3)}, \tag{2.10}$$
$$\frac{\overline{J}_{2,6}j(-x^2q^3;q^6) + x^{-1}q\overline{J}_{1,6}j(-x^2;q^6)}{j(x^2q^2;q^6)j(x^2q^4;q^6)} = \frac{J_3^4}{J_6^2 j(xq;q^3)j(xq^2;q^3)}, \tag{2.11}$$
$$\frac{\overline{J}_{4,12}j(-x^2q^8;q^{12}) + x^{-1}q\overline{J}_{2,12}j(-x^2q^2;q^{12})}{j(x^2q^6;q^{12})j(x^2q^{10};q^{12})} = \frac{J_6^4}{J_{12}^2 j(xq^3;q^6)j(xq^5;q^6)}, \tag{2.12}$$
$$\frac{\overline{J}_{4,12}j(-x^2q^4;q^{12}) + xq\overline{J}_{2,12}j(-x^2q^{10};q^{12})}{j(x^2q^2;q^{12})j(x^2q^6;q^{12})} = \frac{J_6^4}{J_{12}^2 j(xq;q^6)j(xq^3;q^6)}, \tag{2.13}$$
$$\frac{\overline{J}_{2,12}j(-x^2q^6;q^{12}) + x^{-1}q\overline{J}_{4,12}j(-x^2;q^{12})}{j(x^2q^2;q^{12})j(x^2q^{10};q^{12})} = \frac{J_6^4}{J_{12}^2 j(xq;q^6)j(xq^5;q^6)}. \tag{2.14}$$

Proof. Setting $(q,x,y) \to (q^3, -x, -xq^2)$, $(q^3,-x,-xq)$ and $(q^3, -x^{-1}q, -xq)$ in (2.5), respectively, we obtain that

$$j(-x;q^3)j(-xq^2;q^3) = \overline{J}_{1,6}j(-x^2q^2;q^6) + x\overline{J}_{2,6}j(-x^2q^5;q^6),$$
$$j(-x;q^3)j(-xq;q^3) = \overline{J}_{2,6}j(-x^2q;q^6) + x\overline{J}_{1,6}j(-x^2q^4;q^6),$$

$$j(-x^{-1}q;q^3)j(-xq;q^3)=\overline{J}_{2,6}j(-x^2q^3;q^6)+x^{-1}q\overline{J}_{1,6}j(-x^2;q^6).$$

Combining the above identities and (2.2), we have

$$\frac{\overline{J}_{1,6}j(-x^2q^2;q^6)+x\overline{J}_{2,6}j(-x^2q^5;q^6)}{j(x^2;q^6)j(x^2q^4;q^6)}=\frac{j(-x;q^3)j(-xq^2;q^3)}{j(x^2;q^6)j(x^2q^4;q^6)},$$
$$\frac{\overline{J}_{2,6}j(-x^2q;q^6)+x\overline{J}_{1,6}j(-x^2q^4;q^6)}{j(x^2;q^6)j(x^2q^2;q^6)}=\frac{j(-x;q^3)j(-xq;q^3)}{j(x^2;q^6)j(x^2q^2;q^6)},$$
$$\frac{\overline{J}_{2,6}j(-x^2q^3;q^6)+x^{-1}q\overline{J}_{1,6}j(-x^2;q^6)}{j(x^2q^2;q^6)j(x^2q^4;q^6)}=\frac{j(-xq;q^3)j(-xq^2;q^3)}{j(x^2q^2;q^6)j(x^2q^4;q^6)}.$$

So, the above three identities imply (2.9)-(2.11), respectively. Next, setting $(q,x)\to(q^2,xq)$ in (2.11) yields (2.12). Then setting $(q,x)\to(q^2,x/q)$ in (2.11) and $(q,x)\to(q^2,xq)$ in (2.9), respectively, and then using (2.2), we derive (2.13) and (2.14). □

Next, we recall some facts related to Bailey pairs.

Definition 2.6. [2] A pair of sequences $(\alpha_n,\beta_n)_{n\geq 0}$ is called a Bailey pair relative to (a,q) if α_n and β_n satisfy the following identity. For $n\geq 0$,

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}.$$

Lemma 2.7. *[5, Eq. (3.1)] For a Bailey pair (α_n,β_n) relative to (a,q), there holds*

$$\sum_{n=0}^{\infty}(\rho_1)_n(\rho_2)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n=\frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n}{(aq/\rho_1)_n(aq/\rho_2)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n\alpha_n. \tag{2.15}$$

The cases $(a,\rho_1,\rho_2)=(1,x,x^{-1})$, (q,x,x^{-1}), $(q,xq^{1/2},x^{-1}q^{1/2})$ and $(q^2,xq,x^{-1}q)$ of (2.15) are recorded as the following four lemmas, which play a central role in this paper.

Lemma 2.8. *For a Bailey pair (α_n,β_n) relative to $(1,q)$,*

$$\sum_{n=0}^{\infty}(x,x^{-1};q)_nq^n\beta_n=\frac{(x,x^{-1};q)_\infty}{J_1^2}\sum_{n=0}^{\infty}\frac{q^n\alpha_n}{(1-xq^n)(1-x^{-1}q^n)}. \tag{2.16}$$

Lemma 2.9. *For a Bailey pair (α_n,β_n) relative to (q,q),*

$$\sum_{n=0}^{\infty}(x,x^{-1};q)_nq^{2n}\beta_n=\frac{(x,x^{-1};q)_\infty}{(q^2;q)_\infty^2}\sum_{n=0}^{\infty}\frac{q^{2n}\alpha_n}{(1-xq^n)(1-x^{-1}q^n)(1-xq^{n+1})(1-x^{-1}q^{n+1})}. \tag{2.17}$$

Lemma 2.10. *For a Bailey pair (α_n,β_n) relative to (q,q),*

$$\sum_{n=0}^{\infty}(xq^{1/2},x^{-1}q^{1/2};q)_nq^n\beta_n=\frac{(xq^{1/2},x^{-1}q^{1/2};q)_\infty}{(q,q^2;q)_\infty}\sum_{n=0}^{\infty}\frac{q^n\alpha_n}{(1-xq^{n+1/2})(1-x^{-1}q^{n+1/2})}. \tag{2.18}$$

Lemma 2.11. *For a Bailey pair (α_n, β_n) relative to (q^2, q),*

$$\sum_{n=0}^{\infty}(xq, x^{-1}q; q)_n q^n \beta_n = \frac{(xq, x^{-1}q; q)_\infty}{(q, q^3; q)_\infty} \sum_{n=0}^{\infty} \frac{q^n \alpha_n}{(1 - xq^{n+1})(1 - x^{-1}q^{n+1})}. \tag{2.19}$$

The following Bailey pairs given by Slater [35] are used in the proofs of the main results:

Lemma 2.12. *[35, A(5)] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to $(1, q)$, where*

$$\alpha_n = \begin{cases} 1, & n = 0, \\ -q^{3m^2-m}, & n = 3m-1, \\ q^{3m^2-m} + q^{3m^2+m}, & n = 3m, m \geq 1, \\ -q^{3m^2+m}, & n = 3m+1, \end{cases} \qquad \beta_n = \frac{q^{n^2}}{(q;q)_{2n}}.$$

Lemma 2.13. *[35, A(7)] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to $(1, q)$, where*

$$\alpha_n = \begin{cases} 1, & n = 0, \\ -q^{3m^2-4m+1}, & n = 3m-1, \\ q^{3m^2-2m} + q^{3m^2+2m}, & n = 3m, m \geq 1, \\ -q^{3m^2+4m+1}, & n = 3m+1, \end{cases} \qquad \beta_n = \frac{q^{n^2-n}}{(q;q)_{2n}}.$$

Lemma 2.14. *[35, A(6)] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q, q), where*

$$\alpha_n = \begin{cases} q^{3m^2+m}, & n = 3m-1, \\ q^{3m^2-m}, & n = 3m, \\ -q^{3m^2+m} - q^{3m^2+5m+2}, & n = 3m+1, \end{cases} \qquad \beta_n = \frac{q^{n^2}}{(q^2;q)_{2n}}.$$

Lemma 2.15. *[35, A(8)] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q, q), where*

$$\alpha_n = \begin{cases} q^{3m^2-2m}, & n = 3m-1, \\ q^{3m^2+2m}, & n = 3m, \\ -q^{3m^2+4m+1} - q^{3m^2+2m}, & n = 3m+1, \end{cases} \qquad \beta_n = \frac{q^{n^2+n}}{(q^2;q)_{2n}}.$$

Lemma 2.16. *[35, p. 470] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to $(1, q)$, where*

$$\alpha_n = \begin{cases} 1, & n = 0, \\ (-1)^m q^{m^2}(q^{3m/2} + q^{-3m/2}), & n = 2m, m \geq 1, \\ (-1)^m q^{m^2}(q^{-m/2-1/2} - q^{5m/2+1}), & n = 2m+1, \end{cases} \qquad \beta_n = \frac{q^{n^2/2-n}}{(q^{1/2};q)_n (q^2;q^2)_n}.$$

Lemma 2.17. *[35, p. 470] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to $(1, q)$, where*

$$\alpha_n = \begin{cases} 1, & n = 0, \\ (-1)^m q^{m^2}(q^{m/2} + q^{-m/2}), & n = 2m, m \geq 1, \\ (-1)^{m+1} q^{m^2}(q^{m/2} - q^{3m/2+1/2}), & n = 2m+1, \end{cases} \qquad \beta_n = \frac{q^{n^2/2}}{(q^{1/2};q)_n (q^2;q^2)_n}.$$

There is a printing error in the above Bailey pair mentioned in Slater's paper [35].

Lemma 2.18. *[35, C(6)] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q, q), where*

$$\alpha_n = \begin{cases} (-1)^m q^{m^2-m}, & n = 2m, \\ (-1)^{m+1} q^{m^2+3m+2}, & n = 2m+1, \end{cases} \qquad \beta_n = \frac{q^{(n^2-n)/2}}{(q;q)_n (q^3;q^2)_n}.$$

Lemma 2.19. *[35, C(7)] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q, q), where*

$$\alpha_n = \begin{cases} (-1)^m q^{m^2+m}, & n = 2m, \\ (-1)^{m+1} q^{m^2+m}, & n = 2m+1, \end{cases} \qquad \beta_n = \frac{q^{(n^2+n)/2}}{(q;q)_n (q^3;q^2)_n}.$$

Lemma 2.20. *[35, G(5)] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q, q), where*

$$\alpha_n = \begin{cases} q^{m^2-m/2}(1-q^{2m+1/2})/(1-q^{1/2}), & n = 2m, \\ q^{m^2+m/2}(1-q^{-2m+1/2})/(1-q^{1/2}), & n = 2m-1, \end{cases} \qquad \beta_n = \frac{(-1)^n q^{n^2/2}}{(-q^{3/2};q)_n (q^2;q^2)_n}.$$

There is a printing error in the above Bailey pair mentioned in Slater's paper [35].

The following Bailey pairs are due to Ji and Zhao [21]:

Lemma 2.21. *[21, Lemma 2.3] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q, q), where*

$$\alpha_n = \begin{cases} q^{3m^2-2m}(1-q^{6m+1})/(1-q), & n = 3m, \\ 0, & n = 3m+1, \\ -q^{3m^2+2m}(1-q^{6m+5})/(1-q), & n = 3m+2, \end{cases} \qquad \beta_n = \frac{q^{n^2-n}}{(q;q)_{2n}}.$$

Lemma 2.22. *[21, Lemma 2.5] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q, q), where*

$$\alpha_n = \frac{(-1)^n q^{(n^2-3n)/4}(1-q^{2n+1})}{1-q}, \quad \beta_n = \frac{(-1)^n q^{n^2/2-n}(q^{1/2};q)_n}{(q;q)_{2n}}.$$

Lemma 2.23. *[21, Lemma 2.4] The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q^2, q), where*

$$\alpha_n = \begin{cases} (-1)^m q^{m^2-m}(1-q^{4m+2})/(1-q^2), & n = 2m, \\ 0, & n = 2m+1, \end{cases} \qquad \beta_n = \frac{(-q;q)_n q^{(n^2-n)/2}}{(q^2;q)_{2n}}.$$

We also need the following Bailey pair, which is a variant of Slater's Bailey pair A(6).

Lemma 2.24. *The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q^2, q), where*

$$\alpha_n = \begin{cases} q^{3m^2-m}(1-q^{6m+2})/(1-q^2), & n = 3m, \\ 0, & n = 3m-1, \\ q^{3m^2+m}(1-q^{-6m+2})/(1-q^2), & n = 3m-2, \end{cases} \qquad \beta_n = \frac{q^{n^2}}{(q^2;q)_{2n}}.$$

Proof. According to [35, Eq. (3.7)], we have

$$\begin{aligned}\frac{q^{n^2}}{(q^2;q)_{2n}} &= \sum_{m=-[n/3]}^{[n/3]} \frac{q^{3m^2-m}(1-q^{6m+2})}{(q;q)_{n-3m}(q^2;q)_{n+3m+1}} \\ &= \sum_{m=0}^{[n/3]} \frac{q^{3m^2-m}(1-q^{6m+2})}{(q;q)_{n-3m}(q^2;q)_{n+3m+1}} + \sum_{m=1}^{[n/3]} \frac{q^{3m^2+m}(1-q^{-6m+2})}{(q;q)_{n+3m}(q^2;q)_{n-3m+1}} \\ &= \sum_{m=0}^{[n/3]} \frac{q^{3m^2-m}(1-q^{6m+2})}{(1-q^2)(q;q)_{n-3m}(q^3;q)_{n+3m}} + \sum_{m=1}^{[n/3]} \frac{q^{3m^2+m}(1-q^{-6m+2})}{(1-q^2)(q^3;q)_{n+3m-2}(q;q)_{n-3m+2}}.\end{aligned}$$

Thus, we complete the proof. ∎

3. Proofs of the main results

In this section, we prove the main results.

Proof of Theorem 1.2. Substituting the Bailey pair in Lemma 2.12 into (2.16), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(x,x^{-1};q)_n q^{n^2+n}}{(q;q)_{2n}} &= \frac{(x,x^{-1};q)_\infty}{J_1^2} \Bigg(\frac{1}{(1-x)(1-x^{-1})} - \sum_{n=1}^{\infty} \frac{q^{3n^2+2n-1}}{(1-xq^{3n-1})(1-x^{-1}q^{3n-1})} \\ &\quad + \sum_{n=1}^{\infty} \frac{q^{3n^2+2n}+q^{3n^2+4n}}{(1-xq^{3n})(1-x^{-1}q^{3n})} - \sum_{n=0}^{\infty} \frac{q^{3n^2+4n+1}}{(1-xq^{3n+1})(1-x^{-1}q^{3n+1})} \Bigg) \\ &= \frac{(x,x^{-1};q)_\infty}{J_1^2} \Bigg(\sum_{n=0}^{\infty} \frac{q^{3n^2+2n}}{(1-xq^{3n})(1-x^{-1}q^{3n})} + \sum_{n=-\infty}^{-1} \frac{q^{3n^2-4n}}{(1-xq^{-3n})(1-x^{-1}q^{-3n})} \\ &\quad - \sum_{n=-\infty}^{-1} \frac{q^{3n^2-2n-1}}{(1-xq^{-3n-1})(1-x^{-1}q^{-3n-1})} - \sum_{n=0}^{\infty} \frac{q^{3n^2+4n+1}}{(1-xq^{3n+1})(1-x^{-1}q^{3n+1})} \Bigg) \\ &= \frac{(x,x^{-1};q)_\infty}{J_1^2} \Bigg(\sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{(1-xq^{3n})(1-x^{-1}q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+4n+1}}{(1-xq^{3n+1})(1-x^{-1}q^{3n+1})} \Bigg).\end{aligned} \tag{3.1}$$

Notice that

$$\frac{1}{(1-xz)(1-x^{-1}z)} = \frac{1}{1-x^2}\left(\frac{1}{1-x^{-1}z} - \frac{x^2}{1-xz}\right). \tag{3.2}$$

So, combining (3.1) and (3.2) yields that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(x,x^{-1};q)_n q^{n^2+n}}{(q;q)_{2n}} &= \frac{(x,x^{-1};q)_\infty}{(1-x^2)J_1^2} \Bigg(\sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{1-x^{-1}q^{3n}} \\ &\quad - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+4n+1}}{1-x^{-1}q^{3n+1}} - \sum_{n=-\infty}^{\infty} \frac{x^2q^{3n^2+2n}}{1-xq^{3n}} + \sum_{n=-\infty}^{\infty} \frac{x^2q^{3n^2+4n+1}}{1-xq^{3n+1}} \Bigg)\end{aligned}$$

$$
\begin{aligned}
&= \frac{(x,x^{-1};q)_\infty}{(1-x^2)J_1^2}\left(\overline{J}_{1,6}\left(m(-x^{-2}q,q^6,-q^{-1})-m(-x^{-2}q,q^6,-q)\right)\right.\\
&\quad + x^{-1}q^{-2}\overline{J}_{2,6}\left(m(-x^{-2}q^{-2},q^6,-q^2)-m(-x^{-2}q^{-2},q^6,-q^4)\right)\\
&\quad - x^2\overline{J}_{1,6}\left(m(-x^2q,q^6,-q^{-1})-m(-x^2q,q^6,-q)\right)\\
&\quad \left.-x^3q^{-2}\overline{J}_{2,6}\left(m(-x^2q^{-2},q^6,-q^2)-m(-x^2q^{-2},q^6,-q^4)\right)\right),
\end{aligned}
$$

where the last equality follows from (2.2) and (2.8). Then in view of (2.2) and (2.6), we derive that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+n}}{(q;q)_{2n}} &= \frac{-x(x,x^{-1};q)_\infty J_2J_6^3}{(1-x^2)J_1^2}\left(\frac{xj(-x^2q^5;q^6)}{\overline{J}_{1,6}j(x^2;q^6)j(x^2q^4;q^6)}\right.\\
&\quad + \frac{j(-x^2q^2;q^6)}{\overline{J}_{2,6}j(x^2;q^6)j(x^2q^4;q^6)}+\frac{xj(-x^2q;q^6)}{\overline{J}_{1,6}j(x^2;q^6)j(x^2q^2;q^6)}+\left.\frac{x^2j(-x^2q^4;q^6)}{\overline{J}_{2,6}j(x^2;q^6)j(x^2q^2;q^6)}\right)\\
&= \frac{-x(x,x^{-1};q)_\infty J_2J_6^3}{(1-x^2)J_1^2\overline{J}_{1,6}\overline{J}_{2,6}}\left(\frac{\overline{J}_{1,6}j(-x^2q^2;q^6)+x\overline{J}_{2,6}j(-x^2q^5;q^6)}{j(x^2;q^6)j(x^2q^4;q^6)}\right.\\
&\quad +\left.\frac{x\left(\overline{J}_{2,6}j(-x^2q;q^6)+x\overline{J}_{1,6}j(-x^2q^4;q^6)\right)}{j(x^2;q^6)j(x^2q^2;q^6)}\right).
\end{aligned}
$$

Substituting (2.9) and (2.10) into the above identity yields that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+n}}{(q;q)_{2n}} &= \frac{-x(x,x^{-1};q)_\infty J_2J_3^4J_6(j(xq;q^3)+xj(xq^2;q^3))}{(1-x^2)J_1^2\overline{J}_{1,6}\overline{J}_{2,6}j(x;q^3)j(xq;q^3)j(xq^2;q^3)}\\
&= \frac{J_2J_3J_6(j(xq;q^3)+xj(xq^2;q^3))}{(1+x)J_1^2\overline{J}_{1,6}\overline{J}_{2,6}},
\end{aligned}
$$

where the last equality follows from (2.3). Hence, the above identity implies (1.6). □

Next, we illustrate how to deduce (1.7)-(1.9) from (1.6).

Proof of (1.7). Let i denote the imaginary unit. Then $i^2=-1$. Setting $x=i$ in (1.6) yields that

$$
\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n^2+n}}{(q;q)_{2n}} = \frac{j(iq;q^3)+ij(iq^2;q^3)}{(1+i)J_1} = \frac{j(iq;q^3)+ij(-iq;q^3)}{(1+i)J_1}, \tag{3.3}
$$

where the last step follows from (2.2). According to (1.1), we have

$$
\begin{aligned}
j(iq;q^3) &= \sum_{n=-\infty}^{\infty}(-1)^ni^nq^{\frac{3n^2-n}{2}} = \sum_{n=-\infty}^{\infty}(-1)^nq^{6n^2-n} - i\sum_{n=-\infty}^{\infty}(-1)^nq^{6n^2+5n+1}\\
&= J_{5,12}-iqJ_{1,12},\\
j(-iq;q^3) &= \sum_{n=-\infty}^{\infty}i^nq^{\frac{3n^2-n}{2}} = \sum_{n=-\infty}^{\infty}(-1)^nq^{6n^2-n} + i\sum_{n=-\infty}^{\infty}(-1)^nq^{6n^2+5n+1}\\
&= J_{5,12}+iqJ_{1,12}.
\end{aligned}
$$

Then substituting the above two identities into (3.3), we obtain that

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2+n}}{(q;q)_{2n}}=\frac{J_{5,12}-qJ_{1,12}}{J_1}.$$

Setting $m=2$ and replacing (x,q) by $(q,-q^3)$ in (2.4) yields that

$$j(q;-q^3)=J_{5,12}-qJ_{1,12}.$$

Combining the above two identities, we deduce (1.7). □

Proof of (1.8). Let ω be a cube root of unity. So, $\omega^3=1$ and $1+\omega+\omega^2=0$. Observe that

$$(1-\omega)(1-\omega^{-1})=3$$

and for $n\geq 1$,

$$(\omega q,\omega^{-1}q,q;q)_{n-1}=\prod_{k=1}^{n-1}(1-\omega q^k)(1-\omega^{-1}q^k)(1-q^k)=\prod_{k=1}^{n-1}(1-q^{3k})=(q^3;q^3)_{n-1}.$$

So, for $n\geq 1$,

$$(\omega,\omega^{-1};q)_n=3\frac{(q^3;q^3)_{n-1}}{(q;q)_{n-1}}.$$

Setting $x=\omega$ in (1.6), we deduce that

$$1+3\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}q^{n^2+n}}{(q;q)_{n-1}(q;q)_{2n}}=\frac{j(\omega q;q^3)+\omega j(\omega q^2;q^3)}{(1+\omega)J_1}=\frac{j(\omega q;q^3)+\omega j(\omega^2 q;q^3)}{(1+\omega)J_1},\tag{3.4}$$

where the last equality follows from (2.2). Based on (1.1), we have

$$\begin{aligned}
j(\omega q;q^3)&=\sum_{n=-\infty}^{\infty}(-1)^n\omega^n q^{\frac{3n^2-n}{2}}\\
&=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{27n^2-3n}{2}}-\omega\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{27n^2+15n+2}{2}}-\omega^2\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{27n^2-21n+4}{2}}\\
&=J_{12,27}-\omega qJ_{6,27}-\omega^2q^2J_{3,27},\\
j(\omega^2 q;q^3)&=\sum_{n=-\infty}^{\infty}(-1)^n\omega^{2n} q^{\frac{3n^2-n}{2}}\\
&=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{27n^2-3n}{2}}-\omega^2\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{27n^2+15n+2}{2}}-\omega\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{27n^2-21n+4}{2}}\\
&=J_{12,27}-\omega^2qJ_{6,27}-\omega q^2J_{3,27}.
\end{aligned}$$

Then substituting the above two identities into (3.4) implies that

$$1+3\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}q^{n^2+n}}{(q;q)_{n-1}(q;q)_{2n}}=\frac{J_{12,27}-qJ_{6,27}+2q^2J_{3,27}}{J_1}. \tag{3.5}$$

Since

$$J_1=\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{3n^2-n}{2}}=J_{12,27}-qJ_{6,27}-q^2J_{3,27}, \tag{3.6}$$

combining (3.5) and (3.6), we prove (1.8). $\square$

Proof of (1.9). Setting $x=-\omega$ in (1.6) and observing that for $n\geq 0$,

$$(-\omega,-\omega^{-1},-1;q)_n=\prod_{k=0}^{n-1}(1+\omega q^k)(1+\omega^{-1}q^k)(1+q^k)=\prod_{k=0}^{n-1}(1+q^{3k})=(-1;q^3)_n,$$

we derive that

$$\sum_{n=0}^{\infty}\frac{(-1;q^3)_nq^{n^2+n}}{(-1;q)_n(q;q)_{2n}}=\frac{j(-\omega q;q^3)-\omega j(-\omega q^2;q^3)}{(1-\omega)J_1}=\frac{j(-\omega q;q^3)-\omega j(-\omega^2 q;q^3)}{(1-\omega)J_1}. \tag{3.7}$$

From (1.1), it follows that

$$j(-\omega q;q^3)=\overline{J}_{12,27}+\omega q\overline{J}_{6,27}+\omega^2q^2\overline{J}_{3,27},$$
$$j(-\omega^2 q;q^3)=\overline{J}_{12,27}+\omega^2 q\overline{J}_{6,27}+\omega q^2\overline{J}_{3,27}.$$

Then substituting the above two identities into (3.7), we obtain that

$$\sum_{n=0}^{\infty}\frac{(-1;q^3)_nq^{n^2+n}}{(-1;q)_n(q;q)_{2n}}=\frac{\overline{J}_{12,27}-q\overline{J}_{6,27}}{J_1}. \tag{3.8}$$

Notice that replacing (z,q) by $(-q,q^9)$ in (2.7) yields that

$$\overline{J}_{12,27}-q\overline{J}_{6,27}=\frac{J_{1,9}J_{7,18}}{J_{18}}. \tag{3.9}$$

Hence, combining (3.8) and (3.9), we complete the proof of (1.9). $\square$

Proof of Corollary 1.4. For $m\geq 0$, setting $x=-q^{3m+1}$, $-q^{3m+2}$ and $-q^{3m+3}$ in (1.6), respectively, we arrive at

$$\sum_{n=0}^{\infty}\frac{(-q^{3m+1},-q^{-3m-1};q)_nq^{n^2+n}}{(q;q)_{2n}}=\frac{j(-q^{3m+2};q^3)-q^{3m+1}j(-q^{3m+3};q^3)}{(1-q^{3m+1})J_1}, \tag{3.10}$$

$$\sum_{n=0}^{\infty}\frac{(-q^{3m+2},-q^{-3m-2};q)_nq^{n^2+n}}{(q;q)_{2n}}=\frac{j(-q^{3m+3};q^3)-q^{3m+2}j(-q^{3m+4};q^3)}{(1-q^{3m+2})J_1}, \tag{3.11}$$

$$\sum_{n=0}^{\infty}\frac{(-q^{3m+3},-q^{-3m-3};q)_nq^{n^2+n}}{(q;q)_{2n}}=\frac{j(-q^{3m+4};q^3)-q^{3m+3}j(-q^{3m+5};q^3)}{(1-q^{3m+3})J_1}. \tag{3.12}$$

Combining (2.1) and (2.2) implies that

$$\begin{aligned}
j(-q^{3m+2};q^3)&=j(q^{3m}(-q^2);q^3)=q^{-m(3m+1)/2}\overline{J}_{1,3},\\
j(-q^{3m+3};q^3)&=j(q^{3m}(-q^3);q^3)=q^{-3m(m+1)/2}\overline{J}_{0,3},\\
j(-q^{3m+4};q^3)&=j(q^{3(m+1)}(-q);q^3)=q^{-m(3m+5)/2-1}\overline{J}_{1,3},\\
j(-q^{3m+5};q^3)&=j(q^{3(m+1)}(-q^2);q^3)=q^{-m(3m+7)/2-2}\overline{J}_{1,3}.
\end{aligned}$$

Then substituting the above identities into (3.10)-(3.12), we derive (1.10)-(1.12). Therefore, we complete the proof. $\square$

Proof of Theorem 1.5. Substituting the Bailey pair in Lemma 2.13 into (2.16), we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2}}{(q;q)_{2n}}&=\frac{(x,x^{-1};q)_\infty}{J_1^2}\Bigg(\frac{1}{(1-x)(1-x^{-1})}-\sum_{n=1}^{\infty}\frac{q^{3n^2-n}}{(1-xq^{3n-1})(1-x^{-1}q^{3n-1})}\\
&\quad+\sum_{n=1}^{\infty}\frac{q^{3n^2+n}+q^{3n^2+5n}}{(1-xq^{3n})(1-x^{-1}q^{3n})}-\sum_{n=0}^{\infty}\frac{q^{3n^2+7n+2}}{(1-xq^{3n+1})(1-x^{-1}q^{3n+1})}\Bigg)\\
&=\frac{(x,x^{-1};q)_\infty}{J_1^2}\Bigg(\sum_{n=0}^{\infty}\frac{q^{3n^2+n}}{(1-xq^{3n})(1-x^{-1}q^{3n})}+\sum_{n=-\infty}^{-1}\frac{q^{3n^2-5n}}{(1-xq^{-3n})(1-x^{-1}q^{-3n})}\\
&\quad-\sum_{n=1}^{\infty}\frac{q^{3n^2-n}}{(1-xq^{3n-1})(1-x^{-1}q^{3n-1})}-\sum_{n=-\infty}^{0}\frac{q^{3n^2-7n+2}}{(1-xq^{-3n+1})(1-x^{-1}q^{-3n+1})}\Bigg)\\
&=\frac{(x,x^{-1};q)_\infty}{J_1^2}\Bigg(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+n}}{(1-xq^{3n})(1-x^{-1}q^{3n})}-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2-n}}{(1-xq^{3n-1})(1-x^{-1}q^{3n-1})}\Bigg)\\
&=\frac{(x,x^{-1};q)_\infty}{(1-x^2)J_1^2}\Bigg(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+n}}{1-x^{-1}q^{3n}}-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2-n}}{1-x^{-1}q^{3n-1}}\\
&\quad-\sum_{n=-\infty}^{\infty}\frac{x^2q^{3n^2+n}}{1-xq^{3n}}+\sum_{n=-\infty}^{\infty}\frac{x^2q^{3n^2-n}}{1-xq^{3n-1}}\Bigg),
\end{aligned}$$

where the last step follows from (3.2). Then utilizing (2.2) and (2.8), we derive that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2}}{(q;q)_{2n}}&=\frac{(x,x^{-1};q)_\infty}{(1-x^2)J_1^2}\left(\overline{J}_{2,6}\left(m(-x^{-2}q^2,q^6,-q^{-2})-m(-x^{-2}q^2,q^6,-q^{-4})\right)\right.\\
&\quad+x^{-1}q^{-1}\overline{J}_{1,6}\left(m(-x^{-2}q^{-1},q^6,-q)-m(-x^{-2}q^{-1},q^6,-q^{-1})\right)\\
&\quad-x^2\overline{J}_{2,6}\left(m(-x^2q^2,q^6,-q^{-2})-m(-x^2q^2,q^6,-q^{-4})\right)
\end{aligned}$$

$$
\begin{aligned}
&-x^3q^{-1}\overline{J}_{1,6}\left(m(-x^2q^{-1},q^6,-q)-m(-x^2q^{-1},q^6,-q^{-1})\right)\Big)\\
=&\frac{-x(x,x^{-1};q)_\infty J_2J_6^3}{(1-x^2)J_1^2}\left(\frac{xj(-x^2q^4;q^6)}{\overline{J}_{2,6}j(x^2;q^6)j(x^2q^2;q^6)}+\frac{j(-x^2q;q^6)}{\overline{J}_{1,6}j(x^2;q^6)j(x^2q^2;q^6)}\right.\\
&\left.+\frac{xj(-x^2q^2;q^6)}{\overline{J}_{2,6}j(x^2;q^6)j(x^2q^4;q^6)}+\frac{x^2j(-x^2q^5;q^6)}{\overline{J}_{1,6}j(x^2;q^6)j(x^2q^4;q^6)}\right)\\
=&\frac{-x(x,x^{-1};q)_\infty J_2J_6^3}{(1-x^2)J_1^2\overline{J}_{1,6}\overline{J}_{2,6}}\left(\frac{\overline{J}_{2,6}j(-x^2q;q^6)+x\overline{J}_{1,6}j(-x^2q^4;q^6)}{j(x^2;q^6)j(x^2q^2;q^6)}\right.\\
&\left.+\frac{x\left(\overline{J}_{1,6}j(-x^2q^2;q^6)+x\overline{J}_{2,6}j(-x^2q^5;q^6)\right)}{j(x^2;q^6)j(x^2q^4;q^6)}\right),
\end{aligned}\tag{3.13}
$$

where the second equality follows from (2.2) and (2.6). Next, applying (2.3), (2.9) and (2.10) to (3.13), we derive that

$$
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2}}{(q;q)_{2n}}=\frac{J_2J_3J_6(j(xq^2;q^3)+xj(xq;q^3))}{(1+x)J_1^2\overline{J}_{1,6}\overline{J}_{2,6}},
$$

which implies (1.13). Therefore, we complete the proof. □

Proofs of (1.14)-(1.16). The proofs of (1.14)-(1.16) are similar to those of (1.7)-(1.9). Notice that the following two identities are used in the proofs of (1.14) and (1.16):

$$
J_{5,12}+qJ_{1,12}=\frac{\overline{J}_{1,4}J_{2,8}}{J_8},\tag{3.14}
$$

$$
\overline{J}_{12,27}-q^2\overline{J}_{3,27}=\frac{J_{2,9}J_{5,18}}{J_{18}}.\tag{3.15}
$$

Here (3.14) and (3.15) are obtained by setting $(z,q)\to(q,q^4)$ and $(-q^2,q^9)$ in (2.7), respectively. □

Proof of Corollary 1.7. Similar to the proof of Corollary 1.4, Corollary 1.7 can be deduced from (1.13) immediately. □

Proof of Theorem 1.8. Substituting the Bailey pair in Lemma 2.14 into (2.17) and then multiplying by $1/(1-q)$ on both sides, we derive that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+2n}}{(q;q)_{2n+1}}\\
&=\frac{(x,x^{-1};q)_\infty}{(q^2;q)_\infty J_1}\left(\sum_{n=1}^{\infty}\frac{q^{3n^2+7n-2}}{(1-xq^{3n-1})(1-x^{-1}q^{3n-1})(1-xq^{3n})(1-x^{-1}q^{3n})}\right.\\
&\quad+\sum_{n=0}^{\infty}\frac{q^{3n^2+5n}}{(1-xq^{3n})(1-x^{-1}q^{3n})(1-xq^{3n+1})(1-x^{-1}q^{3n+1})}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{n=0}^{\infty}\frac{q^{3n^2+7n+2}+q^{3n^2+11n+4}}{(1-xq^{3n+1})(1-x^{-1}q^{3n+1})(1-xq^{3n+2})(1-x^{-1}q^{3n+2})}\Bigg)\\
&=\frac{(x,x^{-1};q)_\infty}{(q^2;q)_\infty J_1}\Bigg(\sum_{n=-\infty}^{-1}\frac{q^{3n^2-7n-2}}{(1-xq^{-3n-1})(1-x^{-1}q^{-3n-1})(1-xq^{-3n})(1-x^{-1}q^{-3n})}\\
&+\sum_{n=0}^{\infty}\frac{q^{3n^2+5n}}{(1-xq^{3n})(1-x^{-1}q^{3n})(1-xq^{3n+1})(1-x^{-1}q^{3n+1})}\\
&-\sum_{n=1}^{\infty}\frac{q^{3n^2+n-2}}{(1-xq^{3n-2})(1-x^{-1}q^{3n-2})(1-xq^{3n-1})(1-x^{-1}q^{3n-1})}\\
&-\sum_{n=-\infty}^{0}\frac{q^{3n^2-11n+4}}{(1-xq^{-3n+1})(1-x^{-1}q^{-3n+1})(1-xq^{-3n+2})(1-x^{-1}q^{-3n+2})}\Bigg)\\
&=\frac{(x,x^{-1};q)_\infty}{(q^2;q)_\infty J_1}\Bigg(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+5n}}{(1-xq^{3n})(1-x^{-1}q^{3n})(1-xq^{3n+1})(1-x^{-1}q^{3n+1})}\\
&-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+n-2}}{(1-xq^{3n-2})(1-x^{-1}q^{3n-2})(1-xq^{3n-1})(1-x^{-1}q^{3n-1})}\Bigg).
\end{aligned}
\tag{3.16}
$$

Notice that

$$
\begin{aligned}
&\frac{1}{(1-xz)(1-x^{-1}z)(1-xzq)(1-x^{-1}zq)}\\
&=\frac{z^{-1}}{(1-q)(x-x^{-1})}\left(\frac{1}{(1-xz)(1-x^{-1}zq)}-\frac{1}{(1-x^{-1}z)(1-xzq)}\right)\\
&=\frac{z^{-1}}{(1-q)(x-x^{-1})}\\
&\times\left(\frac{1}{1-x^2q^{-1}}\left(\frac{1}{1-x^{-1}zq}-\frac{x^2q^{-1}}{1-xz}\right)-\frac{1}{1-x^{-2}q^{-1}}\left(\frac{1}{1-xzq}-\frac{x^{-2}q^{-1}}{1-x^{-1}z}\right)\right).
\end{aligned}
\tag{3.17}
$$

Then by (3.16) and (3.17), we derive that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+2n}}{(q;q)_{2n+1}}&=\frac{(x,x^{-1};q)_\infty}{(x-x^{-1})J_1^2}\Bigg(\frac{1}{1-x^2q^{-1}}\Bigg(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+2n}}{1-x^{-1}q^{3n+1}}\\
&-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2-2n}}{1-x^{-1}q^{3n-1}}-x^2q^{-1}\left(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+2n}}{1-xq^{3n}}-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2-2n}}{1-xq^{3n-2}}\right)\Bigg)\\
&+\frac{1}{1-x^{-2}q^{-1}}\Bigg(x^{-2}q^{-1}\left(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+2n}}{1-x^{-1}q^{3n}}-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2-2n}}{1-x^{-1}q^{3n-2}}\right)
\end{aligned}
$$

$$-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+2n}}{1-xq^{3n+1}}+\sum_{n=-\infty}^{\infty}\frac{q^{3n^2-2n}}{1-xq^{3n-1}}\Bigg)\Bigg)$$

$$=\frac{(x,x^{-1};q)_\infty}{(x-x^{-1})J_1^2}\left(\frac{1}{1-x^2q^{-1}}\left(\overline{J}_{1,6}\left(m(-x^{-2}q^3,q^6,-q^{-1})-m(-x^{-2}q^3,q^6,-q^{-5})\right)\right.\right.$$
$$+x^{-1}q^{-1}\overline{J}_{2,6}\left(m(-x^{-2},q^6,-q^2)-m(-x^{-2},q^6,-q^{-2})\right)$$
$$-x^2q^{-1}\overline{J}_{1,6}\left(m(-x^2q,q^6,-q^{-1})-m(-x^2q,q^6,-q^{-5})\right)$$
$$\left.-x^3q^{-3}\overline{J}_{2,6}\left(m(-x^2q^{-2},q^6,-q^2)-m(-x^2q^{-2},q^6,-q^{-2})\right)\right)$$
$$+\frac{1}{1-x^{-2}q^{-1}}\left(x^{-2}q^{-1}\overline{J}_{1,6}\left(m(-x^{-2}q,q^6,-q^{-1})-m(-x^{-2}q,q^6,-q^{-5})\right)\right.$$
$$+x^{-3}q^{-3}\overline{J}_{2,6}\left(m(-x^{-2}q^{-2},q^6,-q^2)-m(-x^{-2}q^{-2},q^6,-q^{-2})\right)$$
$$-\overline{J}_{1,6}\left(m(-x^2q^3,q^6,-q^{-1})-m(-x^2q^3,q^6,-q^{-5})\right)$$
$$\left.\left.-xq^{-1}\overline{J}_{2,6}\left(m(-x^2,q^6,-q^2)-m(-x^2,q^6,-q^{-2})\right)\right)\right),$$

where the last equality follows from (2.2) and (2.8). According to (2.2) and (2.6), the above can be stated as

$$\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+2n}}{(q;q)_{2n+1}}=\frac{(x,x^{-1};q)_\infty J_2J_6^3}{(x-x^{-1})J_1^2}\left(\frac{1}{1-x^2q^{-1}}\left(\frac{j(-x^2q^3;q^6)}{\overline{J}_{1,6}j(x^2q^2;q^6)j(x^2q^4;q^6)}\right.\right.$$
$$\left.+\frac{x^{-1}qj(-x^2;q^6)}{\overline{J}_{2,6}j(x^2q^2;q^6)j(x^2q^4;q^6)}-\frac{x^2q^{-1}j(-x^2q;q^6)}{\overline{J}_{1,6}j(x^2;q^6)j(x^2q^2;q^6)}-\frac{x^3q^{-1}j(-x^2q^4;q^6)}{\overline{J}_{2,6}j(x^2;q^6)j(x^2q^2;q^6)}\right)$$
$$+\frac{1}{1-x^{-2}q^{-1}}\left(-\frac{q^{-1}j(-x^2q^5;q^6)}{\overline{J}_{1,6}j(x^2;q^6)j(x^2q^4;q^6)}-\frac{x^{-1}q^{-1}j(-x^2q^2;q^6)}{\overline{J}_{2,6}j(x^2;q^6)j(x^2q^4;q^6)}\right.$$
$$\left.\left.-\frac{j(-x^2q^3;q^6)}{\overline{J}_{1,6}j(x^2q^2;q^6)j(x^2q^4;q^6)}-\frac{x^{-1}qj(-x^2;q^6)}{\overline{J}_{2,6}j(x^2q^2;q^6)j(x^2q^4;q^6)}\right)\right)$$
$$=\frac{(x,x^{-1};q)_\infty J_2J_6^3}{(x-x^{-1})J_1^2\overline{J}_{1,6}\overline{J}_{2,6}}\left(\frac{1}{1-x^2q^{-1}}\left(\frac{\overline{J}_{2,6}j(-x^2q^3;q^6)+x^{-1}q\overline{J}_{1,6}j(-x^2;q^6)}{j(x^2q^2;q^6)j(x^2q^4;q^6)}\right.\right.$$
$$\left.-\frac{x^2q^{-1}(\overline{J}_{2,6}j(-x^2q;q^6)+x\overline{J}_{1,6}j(-x^2q^4;q^6))}{j(x^2;q^6)j(x^2q^2;q^6)}\right)$$
$$+\frac{1}{1-x^{-2}q^{-1}}\left(-\frac{x^{-1}q^{-1}(\overline{J}_{1,6}j(-x^2q^2;q^6)+x\overline{J}_{2,6}j(-x^2q^5;q^6))}{j(x^2;q^6)j(x^2q^4;q^6)}\right.$$
$$\left.\left.-\frac{\overline{J}_{2,6}j(-x^2q^3;q^6)+x^{-1}q\overline{J}_{1,6}j(-x^2;q^6)}{j(x^2q^2;q^6)j(x^2q^4;q^6)}\right)\right). \tag{3.18}$$

Then applying (2.9)-(2.11) to (3.18), we obtain that

$$\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+2n}}{(q;q)_{2n+1}}=\frac{(x,x^{-1};q)_\infty J_2J_3^4J_6}{(x-x^{-1})J_1^2\overline{J}_{1,6}\overline{J}_{2,6}}\left(\frac{1}{1-x^2q^{-1}}\left(\frac{1}{j(xq;q^3)j(xq^2;q^3)}\right.\right.$$

$$-\frac{x^2q^{-1}}{j(x;q^3)j(xq;q^3)}\Bigg)+\frac{1}{1-x^{-2}q^{-1}}\left(-\frac{x^{-1}q^{-1}}{j(x;q^3)j(xq^2;q^3)}-\frac{1}{j(xq;q^3)j(xq^2;q^3)}\right)\Bigg)$$
$$=\frac{(x,x^{-1};q)_\infty J_2J_3^4J_6}{(x-x^{-1})(1-x^2q)(1-x^{-2}q)J_1^2\overline{J}_{1,6}\overline{J}_{2,6}}$$
$$\times\frac{x^2q(1-x^{-4})j(x;q^3)-x^{-1}q(1-x^2q^{-1})j(xq;q^3)+(1-x^2q)j(xq^2;q^3)}{j(x;q^3)j(xq;q^3)j(xq^2;q^3)}.$$

Hence, combining (2.3) and the above identity, we complete the proof. □

Proof of Corollary 1.9. First, notice that from (1.1), it follows that

$$j(i;q^3)=(1-i)\sum_{n=-\infty}^{\infty}(-1)^nq^{6n^2-3n}=(1-i)J_{3,12},$$
$$j(\omega;q^3)=(1-\omega)\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{27n^2-9n}{2}}=(1-\omega)J_9,$$
$$j(-\omega;q^3)=(1+\omega)\sum_{n=-\infty}^{\infty}q^{\frac{27n^2-9n}{2}}+\omega^2\sum_{n=-\infty}^{\infty}q^{\frac{27n^2-27n+6}{2}}=(1+\omega)(\overline{J}_{9,27}-q^3\overline{J}_{0,27}).$$

The proof of (1.18) is similar to that of (1.7). Setting $x=i$ in (1.17), we obtain that

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n^2+2n}}{(q;q)_{2n+1}}=\frac{J_{5,12}+qJ_{1,12}}{(1+q)J_1}.$$

Then substituting (3.14) into the above identity implies (1.18).

To prove (1.19), observe that

$$(1-\omega q)(1-\omega^2q)=1+q+q^2.$$

Similar to the proof of (1.8), setting $x=\omega$ in (1.17) yields that

$$\frac{1}{1-q}+3\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}q^{n^2+2n}}{(q;q)_{n-1}(q;q)_{2n+1}}$$
$$=\frac{-3qJ_9+(2+q)(J_{12,27}+qJ_{6,27})-(1-q)(J_{12,27}+q^2J_{3,27})}{(1+q+q^2)J_1}. \tag{3.19}$$

Replacing (z,q) by (q,q^9) and (q^2,q^9) in (2.7), respectively, we have

$$J_{12,27}+qJ_{6,27}=\frac{\overline{J}_{1,9}J_{7,18}}{J_{18}}, \tag{3.20}$$
$$J_{12,27}+q^2J_{3,27}=\frac{\overline{J}_{2,9}J_{5,18}}{J_{18}}. \tag{3.21}$$

So, substituting (3.20) and (3.21) into (3.19), we obtain (1.19).

The proof of (1.20) is similar to that of (1.9). Setting $x=-\omega$ in (1.17) yields that

$$\sum_{n=0}^{\infty}\frac{(-1;q^3)_nq^{n^2+2n}}{(-1;q)_n(q;q)_{2n+1}}=\frac{q(\overline{J}_{9,27}-q^3\overline{J}_{0,27})+(\overline{J}_{12,27}-q^2\overline{J}_{3,27})+q^2(\overline{J}_{6,27}-q\overline{J}_{3,27})}{(1+q+q^2)J_1}. \quad (3.22)$$

Then replacing (z,q) by $(-q^3,q^9)$ and $(-q^4,q^9)$ in (2.7), respectively, we have

$$\overline{J}_{9,27}-q^3\overline{J}_{0,27}=\frac{J_3J_{3,18}}{J_{18}}, \quad (3.23)$$

$$\overline{J}_{6,27}-q\overline{J}_{3,27}=\frac{J_{4,9}J_{1,18}}{J_{18}}. \quad (3.24)$$

Hence, substituting (3.15), (3.23) and (3.24) into (3.22), we complete the proof. □

Proof of Theorem 1.10. Substituting the Bailey pair in Lemma 2.15 into (2.17) and then multiplying by $1/(1-q)$ on both sides, we derive that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+3n}}{(q;q)_{2n+1}}\\
&=\frac{(x,x^{-1};q)_\infty}{(q^2;q)_\infty J_1}\left(\sum_{n=1}^{\infty}\frac{q^{3n^2+4n-2}}{(1-xq^{3n-1})(1-x^{-1}q^{3n-1})(1-xq^{3n})(1-x^{-1}q^{3n})}\right.\\
&\quad+\sum_{n=0}^{\infty}\frac{q^{3n^2+8n}}{(1-xq^{3n})(1-x^{-1}q^{3n})(1-xq^{3n+1})(1-x^{-1}q^{3n+1})}\\
&\quad\left.-\sum_{n=0}^{\infty}\frac{q^{3n^2+10n+3}+q^{3n^2+8n+2}}{(1-xq^{3n+1})(1-x^{-1}q^{3n+1})(1-xq^{3n+2})(1-x^{-1}q^{3n+2})}\right)\\
&=\frac{(x,x^{-1};q)_\infty}{(q^2;q)_\infty J_1}\left(\sum_{n=-\infty}^{-1}\frac{q^{3n^2-4n-2}}{(1-xq^{-3n-1})(1-x^{-1}q^{-3n-1})(1-xq^{-3n})(1-x^{-1}q^{-3n})}\right.\\
&\quad+\sum_{n=0}^{\infty}\frac{q^{3n^2+8n}}{(1-xq^{3n})(1-x^{-1}q^{3n})(1-xq^{3n+1})(1-x^{-1}q^{3n+1})}\\
&\quad-\sum_{n=1}^{\infty}\frac{q^{3n^2+4n-4}}{(1-xq^{3n-2})(1-x^{-1}q^{3n-2})(1-xq^{3n-1})(1-x^{-1}q^{3n-1})}\\
&\quad\left.-\sum_{n=-\infty}^{0}\frac{q^{3n^2-8n+2}}{(1-xq^{-3n+1})(1-x^{-1}q^{-3n+1})(1-xq^{-3n+2})(1-x^{-1}q^{-3n+2})}\right)\\
&=\frac{(x,x^{-1};q)_\infty}{(q^2;q)_\infty J_1}\left(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+8n}}{(1-xq^{3n})(1-x^{-1}q^{3n})(1-xq^{3n+1})(1-x^{-1}q^{3n+1})}\right.
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+4n-4}}{(1-xq^{3n-2})(1-x^{-1}q^{3n-2})(1-xq^{3n-1})(1-x^{-1}q^{3n-1})}\Bigg)\\
&=\frac{(x,x^{-1};q)_\infty}{(x-x^{-1})J_1^2}\left(\frac{1}{1-x^2q^{-1}}\left(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+5n}}{1-x^{-1}q^{3n+1}}-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+n-2}}{1-x^{-1}q^{3n-1}}\right.\right.\\
&\left.\left.-x^2q^{-1}\left(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+5n}}{1-xq^{3n}}-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+n-2}}{1-xq^{3n-2}}\right)\right)\right.\\
&+\frac{1}{1-x^{-2}q^{-1}}\left(x^{-2}q^{-1}\left(\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+5n}}{1-x^{-1}q^{3n}}-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+n-2}}{1-x^{-1}q^{3n-2}}\right)\right.\\
&\left.\left.-\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+5n}}{1-xq^{3n+1}}+\sum_{n=-\infty}^{\infty}\frac{q^{3n^2+n-2}}{1-xq^{3n-1}}\right)\right),
\end{aligned}
$$

where the last step follows from (3.17). In view of (2.2) and (2.8), we rewrite the above as

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+3n}}{(q;q)_{2n+1}}\\
&=\frac{(x,x^{-1};q)_\infty}{(x-x^{-1})J_1^2}\left(\frac{1}{1-x^2q^{-1}}\left(q^{-2}\overline{J}_{2,6}\left(m(-x^{-2},q^6,-q^2)-m(-x^{-2},q^6,-q^{-2})\right)\right.\right.\\
&+x^{-1}q^{-4}\overline{J}_{1,6}\left(m(-x^{-2}q^{-3},q^6,-q^5)-m(-x^{-2}q^{-3},q^6,-q)\right)\\
&-x^2q^{-3}\overline{J}_{2,6}\left(m(-x^2q^{-2},q^6,-q^2)-m(-x^2q^{-2},q^6,-q^{-2})\right)\\
&\left.-x^3q^{-6}\overline{J}_{1,6}\left(m(-x^2q^{-5},q^6,-q^5)-m(-x^2q^{-5},q^6,-q)\right)\right)\\
&+\frac{1}{1-x^{-2}q^{-1}}\left(x^{-2}q^{-3}\overline{J}_{2,6}\left(m(-x^{-2}q^{-2},q^6,-q^2)-m(-x^{-2}q^{-2},q^6,-q^{-2})\right)\right.\\
&+x^{-3}q^{-6}\overline{J}_{1,6}\left(m(-x^{-2}q^{-5},q^6,-q^5)-m(-x^{-2}q^{-5},q^6,-q)\right)\\
&-q^{-2}\overline{J}_{2,6}\left(m(-x^2,q^6,-q^2)-m(-x^2,q^6,-q^{-2})\right)\\
&\left.\left.-xq^{-4}\overline{J}_{1,6}\left(m(-x^2q^{-3},q^6,-q^5)-m(-x^2q^{-3},q^6,-q)\right)\right)\right)\\
&=\frac{(x,x^{-1};q)_\infty J_2J_6^3}{(x-x^{-1})J_1^2}\left(\frac{1}{1-x^2q^{-1}}\left(\frac{j(-x^2;q^6)}{\overline{J}_{2,6}j(x^2q^2;q^6)j(x^2q^4;q^6)}+\frac{xq^{-1}j(-x^2q^3;q^6)}{\overline{J}_{1,6}j(x^2q^2;q^6)j(x^2q^4;q^6)}\right.\right.\\
&\left.-\frac{x^2q^{-1}j(-x^2q^4;q^6)}{\overline{J}_{2,6}j(x^2;q^6)j(x^2q^2;q^6)}-\frac{xq^{-1}j(-x^2q;q^6)}{\overline{J}_{1,6}j(x^2;q^6)j(x^2q^2;q^6)}\right)\\
&+\frac{1}{1-x^{-2}q^{-1}}\left(-\frac{q^{-1}j(-x^2q^2;q^6)}{\overline{J}_{2,6}j(x^2;q^6)j(x^2q^4;q^6)}-\frac{xq^{-1}j(-x^2q^5;q^6)}{\overline{J}_{1,6}j(x^2;q^6)j(x^2q^4;q^6)}\right.\\
&\left.\left.-\frac{x^{-2}j(-x^2;q^6)}{\overline{J}_{2,6}j(x^2q^2;q^6)j(x^2q^4;q^6)}-\frac{x^{-1}q^{-1}j(-x^2q^3;q^6)}{\overline{J}_{1,6}j(x^2q^2;q^6)j(x^2q^4;q^6)}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(x,x^{-1};q)_\infty J_2J_6^3}{(x-x^{-1})J_1^2\overline{J}_{1,6}\overline{J}_{2,6}}\left(\frac{1}{1-x^2q^{-1}}\left(\frac{xq^{-1}\left(\overline{J}_{2,6}j(-x^2q^3;q^6)+x^{-1}q\overline{J}_{1,6}j(-x^2;q^6)\right)}{j(x^2q^2;q^6)j(x^2q^4;q^6)}\right.\right.\\
&\quad\left.-\frac{xq^{-1}(\overline{J}_{2,6}j(-x^2q;q^6)+x\overline{J}_{1,6}j(-x^2q^4;q^6))}{j(x^2;q^6)j(x^2q^2;q^6)}\right)\\
&\quad+\frac{1}{1-x^{-2}q^{-1}}\left(-\frac{q^{-1}\left(\overline{J}_{1,6}j(-x^2q^2;q^6)+x\overline{J}_{2,6}j(-x^2q^5;q^6)\right)}{j(x^2;q^6)j(x^2q^4;q^6)}\right.\\
&\quad\left.\left.-\frac{x^{-1}q^{-1}\left(\overline{J}_{2,6}j(-x^2q^3;q^6)+x^{-1}q\overline{J}_{1,6}j(-x^2;q^6)\right)}{j(x^2q^2;q^6)j(x^2q^4;q^6)}\right)\right), \qquad (3.25)
\end{aligned}
$$

where the second equality follows from (2.2) and (2.6). Then substituting (2.9)-(2.11) into (3.25), we derive that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q)_nq^{n^2+3n}}{(q;q)_{2n+1}} &= \frac{(x,x^{-1};q)_\infty J_2J_3^4J_6}{(x-x^{-1})J_1^2\overline{J}_{1,6}\overline{J}_{2,6}}\left(\frac{1}{1-x^2q^{-1}}\left(\frac{xq^{-1}}{j(xq;q^3)j(xq^2;q^3)}\right.\right.\\
&\quad\left.-\frac{xq^{-1}}{j(x;q^3)j(xq;q^3)}\right)+\frac{1}{1-x^{-2}q^{-1}}\left(-\frac{q^{-1}}{j(x;q^3)j(xq^2;q^3)}-\frac{x^{-1}q^{-1}}{j(xq;q^3)j(xq^2;q^3)}\right)\Bigg)\\
&= \frac{(x,x^{-1};q)_\infty J_2J_3^4J_6}{(x-x^{-1})(1-x^2q)(1-x^{-2}q)J_1^2\overline{J}_{1,6}\overline{J}_{2,6}}\\
&\quad\times\frac{(1+q)(x-x^{-1})j(x;q^3)-q(1-x^2q^{-1})j(xq;q^3)-xq(1-x^{-2}q^{-1})j(xq^2;q^3)}{j(x;q^3)j(xq;q^3)j(xq^2;q^3)}.
\end{aligned}
$$

Therefore, combining (2.3) and the above identity, we complete the proof. □

Proof of Corollary 1.11. The proofs of (1.22)-(1.24) are similar to those of (1.7)-(1.9). We deduce (1.22) by setting $x=i$ in (1.21). Next, setting $x=\omega$ in (1.21) yields that

$$
\begin{aligned}
&\frac{1}{1-q}+3\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}q^{n^2+3n}}{(q;q)_{n-1}(q;q)_{2n+1}}\\
&\quad=\frac{3(1+q)J_9-(1-q)(J_{12,27}+qJ_{6,27})-(1+2q)(J_{12,27}+q^2J_{3,27})}{(1+q+q^2)J_1}.
\end{aligned}
$$

Then substituting (3.20) and (3.21) into the above identity, we obtain (1.23).

To prove (1.24), setting $x=-\omega$ in (1.21), we obtain that

$$
\sum_{n=0}^{\infty}\frac{(-1;q^3)_nq^{n^2+3n}}{(-1;q)_n(q;q)_{2n+1}}=\frac{(1+q)(\overline{J}_{9,27}-q^3\overline{J}_{0,27})-q(\overline{J}_{12,27}-q\overline{J}_{6,27})+q(\overline{J}_{6,27}-q\overline{J}_{3,27})}{(1+q+q^2)J_1}.
$$

Therefore, substituting (3.9), (3.23) and (3.24) into the above identity, we complete the proof. □

Proof of Theorem 1.13. Substituting the Bailey pair in Lemma 2.16 into (2.16) and replacing q by q^2, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q^2)_nq^{n^2}}{(q;q^2)_n(q^4;q^4)_n}&=\frac{(x,x^{-1};q^2)_\infty}{J_2^2}\Bigg(\frac{1}{(1-x)(1-x^{-1})}\\
&\quad+\sum_{n=1}^{\infty}\frac{(-1)^n(q^{2n^2+7n}+q^{2n^2+n})}{(1-xq^{4n})(1-x^{-1}q^{4n})}+\sum_{n=0}^{\infty}\frac{(-1)^n(q^{2n^2+3n+1}-q^{2n^2+9n+4})}{(1-xq^{4n+2})(1-x^{-1}q^{4n+2})}\Bigg)\\
&=\frac{(x,x^{-1};q^2)_\infty}{J_2^2}\Bigg(\sum_{n=0}^{\infty}\frac{(-1)^nq^{2n^2+n}}{(1-xq^{4n})(1-x^{-1}q^{4n})}+\sum_{n=-\infty}^{-1}\frac{(-1)^nq^{2n^2-7n}}{(1-xq^{-4n})(1-x^{-1}q^{-4n})}\\
&\quad-\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n^2-n}}{(1-xq^{4n-2})(1-x^{-1}q^{4n-2})}-\sum_{n=-\infty}^{0}\frac{(-1)^nq^{2n^2-9n+4}}{(1-xq^{-4n+2})(1-x^{-1}q^{-4n+2})}\Bigg)\\
&=\frac{(x,x^{-1};q^2)_\infty}{J_2^2}\Bigg(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n}}{(1-xq^{4n})(1-x^{-1}q^{4n})}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2-n}}{(1-xq^{4n-2})(1-x^{-1}q^{4n-2})}\Bigg)\\
&=\frac{(x,x^{-1};q^2)_\infty}{(1-x^2)J_2^2}\Bigg(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n}}{1-x^{-1}q^{4n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2-n}}{1-x^{-1}q^{4n-2}}\\
&\quad-\sum_{n=-\infty}^{\infty}\frac{(-1)^nx^2q^{2n^2+n}}{1-xq^{4n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nx^2q^{2n^2-n}}{1-xq^{4n-2}}\Bigg),
\end{aligned}$$

where the last step follows from (3.2). Then utilizing (1.5) and (2.2), we derive that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(x,x^{-1};q^2)_nq^{n^2}}{(q;q^2)_n(q^4;q^4)_n}&=\frac{(x,x^{-1};q^2)_\infty J_{1,4}}{(1-x^2)J_2^2}\Big(m(x^{-1}q,q^4,q^{-1})-m(x^{-1}q,q^4,q^{-3})\\
&\quad-x^2\big(m(xq,q^4,q^{-1})-m(xq,q^4,q^{-3})\big)\Big)\\
&=\frac{-x(x,x^{-1};q^2)_\infty J_4^3J_{2,4}(j(xq^3;q^4)+xj(xq;q^4))}{(1-x^2)J_2^2J_{1,4}j(x;q^4)j(xq^2;q^4)},
\end{aligned}$$

where the last equality follows from (2.2) and (2.6). Hence, the above identity implies (1.25). $\square$

Next, we illustrate how to deduce (1.26)-(1.28) from (1.25).

Proof of (1.26). Setting $x=i$ in (1.25) yields that

$$\sum_{n=0}^{\infty}\frac{(-1;q^4)_nq^{n^2}}{(q;q^2)_n(q^4;q^4)_n}=\frac{j(iq^3;q^4)+ij(iq;q^4)}{(1+i)J_{1,4}}=\frac{j(-iq;q^4)+ij(iq;q^4)}{(1+i)J_{1,4}}, \tag{3.26}$$

where the last equality follows from (2.2). According to (1.1), we have

$$j(iq;q^4)=J_{6,16}-iqJ_{2,16},$$
$$j(-iq;q^4)=J_{6,16}+iqJ_{2,16}.$$

Then substituting the above two identities into (3.26), we obtain that

$$\sum_{n=0}^{\infty}\frac{(-1;q^4)_n q^{n^2}}{(q;q^2)_n(q^4;q^4)_n}=\frac{J_{6,16}+qJ_{2,16}}{J_{1,4}}.$$

Setting $m=2$ and replacing (x,q) by $(-q,-q^4)$ in (2.4) yields that

$$j(-q;-q^4)=J_{6,16}+qJ_{2,16}.$$

Combining the above two identities, we deduce (1.26). $\square$

Proof of (1.27)**.** Setting $x=\omega$ in (1.25), we deduce that

$$1+3\sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}q^{n^2}}{(q;q)_{2n-1}(q^4;q^4)_n}=\frac{j(\omega q^3;q^4)+\omega j(\omega q;q^4)}{(1+\omega)J_{1,4}}=\frac{j(\omega^2 q;q^4)+\omega j(\omega q;q^4)}{(1+\omega)J_{1,4}}, \tag{3.27}$$

where the last equality follows from (2.2). Based on (1.1), we have

$$j(\omega q;q^4)=J_{15,36}-\omega qJ_{9,36}-\omega^2q^3J_{3,36},$$
$$j(\omega^2 q;q^4)=J_{15,36}-\omega^2 qJ_{9,36}-\omega q^3J_{3,36}.$$

Then substituting the above two identities into (3.27) implies that

$$1+3\sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}q^{n^2}}{(q;q)_{2n-1}(q^4;q^4)_n}=\frac{J_{15,36}-q^3J_{3,36}+2qJ_{9,36}}{J_{1,4}}. \tag{3.28}$$

Since

$$J_{1,4}=\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2-n}=J_{15,36}-qJ_{9,36}-q^3J_{3,36}, \tag{3.29}$$

combining (3.28) and (3.29), we prove (1.27). $\square$

Proof of (1.28)**.** Setting $x=-\omega$ in (1.25), we derive that

$$\sum_{n=0}^{\infty}\frac{(-1;q^6)_n q^{n^2}}{(-1;q^2)_n(q;q^2)_n(q^4;q^4)_n}=\frac{j(-\omega q^3;q^4)-\omega j(-\omega q;q^4)}{(1-\omega)J_{1,4}}=\frac{j(-\omega^2 q;q^3)-\omega j(-\omega q;q^4)}{(1-\omega)J_{1,4}}. \tag{3.30}$$

From (1.1), it follows that

$$j(-\omega q;q)=\overline{J}_{15,36}+\omega q\overline{J}_{9,36}+\omega^2q^3\overline{J}_{3,36},$$
$$j(-\omega^2 q;q)=\overline{J}_{15,36}+\omega^2 q\overline{J}_{9,36}+\omega q^3\overline{J}_{3,36}.$$

Then substituting the above two identities into (3.30), we obtain that

$$\sum_{n=0}^{\infty}\frac{(-1;q^6)_n q^{n^2}}{(-1;q^2)_n(q;q^2)_n(q^4;q^4)_n}=\frac{\overline{J}_{15,36}-q^3\overline{J}_{3,36}}{J_{1,4}}. \tag{3.31}$$

Notice that replacing (z,q) by $(-q^3,q^{12})$ in (2.7) yields that

$$\overline{J}_{15,36}-q^3\overline{J}_{3,36}=\frac{J_{3,12}J_{6,24}}{J_{24}}. \tag{3.32}$$

Hence, combining (3.31) and (3.32), we complete the proof of (1.28). □

Proof of Theorem 1.15. Substituting the Bailey pair in Lemma 2.17 into (2.16) and replacing q by q^2, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(x,x^{-1};q^2)_nq^{n^2+2n}}{(q;q^2)_n(q^4;q^4)_n}&=\frac{(x,x^{-1};q^2)_\infty}{J_2^2}\Bigg(\frac{1}{(1-x)(1-x^{-1})}\\&\quad+\sum_{n=1}^{\infty}\frac{(-1)^n(q^{2n^2+5n}+q^{2n^2+3n})}{(1-xq^{4n})(1-x^{-1}q^{4n})}-\sum_{n=0}^{\infty}\frac{(-1)^n(q^{2n^2+5n+2}-q^{2n^2+7n+3})}{(1-xq^{4n+2})(1-x^{-1}q^{4n+2})}\Bigg)\\&=\frac{(x,x^{-1};q^2)_\infty}{J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+3n}}{(1-xq^{4n})(1-x^{-1}q^{4n})}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n-1}}{(1-xq^{4n-2})(1-x^{-1}q^{4n-2})}\right)\\&=\frac{(x,x^{-1};q^2)_\infty}{(1-x^2)J_2^2}\Bigg(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+3n}}{1-x^{-1}q^{4n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n-1}}{1-x^{-1}q^{4n-2}}\\&\quad-\sum_{n=-\infty}^{\infty}\frac{(-1)^nx^2q^{2n^2+3n}}{1-xq^{4n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nx^2q^{2n^2+n-1}}{1-xq^{4n-2}}\Bigg),\end{aligned}$$

where the last step follows from (3.2). Then utilizing (1.5) and (2.2), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(x,x^{-1};q^2)_nq^{n^2+2n}}{(q;q^2)_n(q^4;q^4)_n}&=\frac{-q^{-1}(x,x^{-1};q^2)_\infty J_{1,4}}{(1-x^2)J_2^2}\big(m(x^{-1}q^{-1},q^4,q)-m(x^{-1}q^{-1},q^4,q^{-1})\\&\qquad-x^2\left(m(xq^{-1},q^4,q)-m(xq^{-1},q^4,q^{-1})\right)\big)\\&=\frac{-x(x,x^{-1};q^2)_\infty J_4^3J_{2,4}(j(xq;q^4)+xj(xq^3;q^4))}{(1-x^2)J_2^2J_{1,4}j(x;q^4)j(xq^2;q^4)},\end{aligned}$$

where the last equality follows from (2.2) and (2.6). Hence, the above identity implies (1.29). □

Proofs of (1.30)-(1.32). The proofs of (1.30)-(1.32) are similar to those of (1.26)-(1.28). Notice that the following identities are used in the proofs of (1.30) and (1.32), respectively:

$$j(q;-q^4)=J_{6,16}-qJ_{2,16},$$

$$\overline{J}_{15,36}-q\overline{J}_{9,36}=\frac{J_{1,12}J_{10,24}}{J_{24}}.$$

The first identity is obtained by setting $m=2$ and replacing (x,q) by $(q,-q^4)$ in (2.4) and the second identity is obtained by setting $(z,q)=(-q,q^{12})$ in (2.7). □

Proof of Theorem 1.17. Substituting the Bailey pair in Lemma 2.14 into (2.18), multiplying by $1/(1-q)$ on both sides and then replacing q by q^2, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{2n^2+2n}}{(q^2;q^2)_{2n+1}}=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=1}^{\infty}\frac{q^{6n^2+8n-2}}{(1-xq^{6n-1})(1-x^{-1}q^{6n-1})}\right.\\
&\quad\left.+\sum_{n=0}^{\infty}\frac{q^{6n^2+4n}}{(1-xq^{6n+1})(1-x^{-1}q^{6n+1})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+8n+2}+q^{6n^2+16n+6}}{(1-xq^{6n+3})(1-x^{-1}q^{6n+3})}\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{6n^2+4n}}{(1-xq^{6n+1})(1-x^{-1}q^{6n+1})}-\sum_{n=-\infty}^{\infty}\frac{q^{6n^2-4n}}{(1-xq^{6n-3})(1-x^{-1}q^{6n-3})}\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{6n^2+4n}}{1-x^{-1}q^{6n+1}}-\sum_{n=-\infty}^{\infty}\frac{q^{6n^2-4n}}{1-x^{-1}q^{6n-3}}\right.\\
&\quad\left.-\sum_{n=-\infty}^{\infty}\frac{x^2q^{6n^2+4n}}{1-xq^{6n+1}}+\sum_{n=-\infty}^{\infty}\frac{x^2q^{6n^2-4n}}{1-xq^{6n-3}}\right),
\end{aligned}$$

where the last step follows from (3.2). Then utilizing (2.2) and (2.8), we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{2n^2+2n}}{(q^2;q^2)_{2n+1}}\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\Big(\overline{J}_{2,12}\left(m(-x^{-2}q^4,q^{12},-q^{-2})-m(-x^{-2}q^4,q^{12},-q^{-10})\right)\\
&\quad+x^{-1}q^{-3}\overline{J}_{4,12}\left(m(-x^{-2}q^{-2},q^{12},-q^4)-m(-x^{-2}q^{-2},q^{12},-q^{-4})\right)\\
&\quad-x^2\overline{J}_{2,12}\left(m(-x^2q^4,q^{12},-q^{-2})-m(-x^2q^4,q^{12},-q^{-10})\right)\\
&\quad-x^3q^{-3}\overline{J}_{4,12}\left(m(-x^2q^{-2},q^{12},-q^4)-m(-x^2q^{-2},q^{12},-q^{-4})\right)\Big)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty J_4J_{12}^3}{(1-x^2)J_2^2\overline{J}_{2,12}\overline{J}_{4,12}}\left(\frac{\overline{J}_{4,12}j(-x^2q^8;q^{12})+x^{-1}q\overline{J}_{2,12}j(-x^2q^2;q^{12})}{j(x^2q^6;q^{12})j(x^2q^{10};q^{12})}\right.\\
&\quad\left.-\frac{x^2\left(\overline{J}_{4,12}j(-x^2q^4;q^{12})+xq\overline{J}_{2,12}j(-x^2q^{10};q^{12})\right)}{j(x^2q^2;q^{12})j(x^2q^6;q^{12})}\right),
\end{aligned}$$

where the last equality follows from (2.2) and (2.6). Next, applying (2.12) and (2.13) to the above identity, we have

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{2n^2+2n}}{(q^2;q^2)_{2n+1}}=\frac{J_4J_6J_{12}(j(xq;q^6)-x^2j(xq^5;q^6))}{(1-x^2)J_2^2\overline{J}_{2,12}\overline{J}_{4,12}},$$

which implies (1.33). Therefore, we complete the proof. $\square$

Proofs of (1.34) and (1.35). The following identities are used in the proofs of (1.34) and (1.35):

$$\begin{aligned} j(iq;q^6) &= J_{8,24} - iqJ_{4,24}, \\ j(-iq;q^6) &= J_{8,24} + iqJ_{4,24}, \\ j(wq;q^6) &= J_{21,54} - \omega qJ_{15,54} - \omega^2 q^5 J_{3,54}, \\ j(w^2q;q^6) &= J_{21,54} - \omega^2 qJ_{15,54} - \omega q^5 J_{3,54}, \\ J_{21,54} + q^5 J_{3,54} &= \frac{\overline{J}_{5,18}J_{8,36}}{J_{36}}, \end{aligned}$$

where the first four identities are due to (1.1) and the last identity is obtained by setting $(z,q)=(q^5,q^{18})$ in (2.7). We omit the details. □

Proof of Theorem 1.19. Substituting the Bailey pair in Lemma 2.15 into (2.18), multiplying by $1/(1-q)$ on both sides and then replacing q by q^2, we have

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_n q^{2n^2+4n}}{(q^2;q^2)_{2n+1}} &= \frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=1}^{\infty}\frac{q^{6n^2+2n-2}}{(1-xq^{6n-1})(1-x^{-1}q^{6n-1})}\right. \\ &\quad \left.+\sum_{n=0}^{\infty}\frac{q^{6n^2+10n}}{(1-xq^{6n+1})(1-x^{-1}q^{6n+1})} - \sum_{n=0}^{\infty}\frac{q^{6n^2+14n+4}+q^{6n^2+10n+2}}{(1-xq^{6n+3})(1-x^{-1}q^{6n+3})}\right) \\ &= \frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{6n^2+10n}}{(1-xq^{6n+1})(1-x^{-1}q^{6n+1})} - \sum_{n=-\infty}^{\infty}\frac{q^{6n^2+2n-4}}{(1-xq^{6n-3})(1-x^{-1}q^{6n-3})}\right) \\ &= \frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{6n^2+10n}}{1-x^{-1}q^{6n+1}} - \sum_{n=-\infty}^{\infty}\frac{q^{6n^2+2n-4}}{1-x^{-1}q^{6n-3}}\right. \\ &\quad \left.-\sum_{n=-\infty}^{\infty}\frac{x^2q^{6n^2+10n}}{1-xq^{6n+1}} + \sum_{n=-\infty}^{\infty}\frac{x^2q^{6n^2+2n-4}}{1-xq^{6n-3}}\right), \end{aligned}$$

where the last step follows from (3.2). Then utilizing (2.2) and (2.8), we deduce that

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_n q^{2n^2+4n}}{(q^2;q^2)_{2n+1}} \\ &= \frac{q^{-4}(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\Big(\overline{J}_{4,12}\left(m(-x^{-2}q^{-2},q^{12},-q^4) - m(-x^{-2}q^{-2},q^{12},-q^{-4})\right) \\ &\quad + x^{-1}q^{-5}\overline{J}_{2,12}\left(m(-x^{-2}q^{-8},q^{12},-q^{10}) - m(-x^{-2}q^{-8},q^{12},-q^2)\right) \\ &\quad - x^2\overline{J}_{4,12}\left(m(-x^2q^{-2},q^{12},-q^4) - m(-x^2q^{-2},q^{12},-q^{-4})\right) \\ &\quad - x^3q^{-5}\overline{J}_{2,12}\left(m(-x^2q^{-8},q^{12},-q^{10}) - m(-x^2q^{-8},q^{12},-q^2)\right)\Big) \end{aligned}$$

$$= \frac{xq^{-1}(xq,x^{-1}q;q^2)_\infty J_4J_{12}^3}{(1-x^2)J_2^2\overline{J}_{2,12}\overline{J}_{4,12}}\left(\frac{\overline{J}_{4,12}j(-x^2q^8;q^{12})+x^{-1}q\overline{J}_{2,12}j(-x^2q^2;q^{12})}{j(x^2q^6;q^{12})j(x^2q^{10};q^{12})}\right.$$
$$\left.-\frac{\overline{J}_{4,12}j(-x^2q^4;q^{12})+xq\overline{J}_{2,12}j(-x^2q^{10};q^{12})}{j(x^2q^2;q^{12})j(x^2q^6;q^{12})}\right),$$

where the last equality follows from (2.2) and (2.6). Next, applying (2.12) and (2.13) to the above identity, we have

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{2n^2+4n}}{(q^2;q^2)_{2n+1}} = \frac{xq^{-1}J_4J_6J_{12}(j(xq;q^6)-j(xq^5;q^6))}{(1-x^2)J_2^2\overline{J}_{2,12}\overline{J}_{4,12}},$$

which implies (1.36). Therefore, we complete the proof. □

Proofs of (1.37) and (1.38). The proofs of (1.37) and (1.38) are similar to those of (1.34) and (1.35). The following identity is used in the proof of (1.38):

$$J_{15,54}-q^4J_{3,54} = \frac{\overline{J}_{7,18}J_{4,36}}{J_{36}},$$

which is obtained by setting $(z,q)=(q^7,q^{18})$ in (2.7) and then applying (2.2). We omit the details. □

Proof of Theorem 1.21. We rewrite the Bailey pair in Lemma 2.18 as

$$\alpha_n = \begin{cases} q^{4m^2-2m}, & n=4m,\\ -q^{4m^2+6m+2}, & n=4m+1,\\ -q^{4m^2+2m}, & n=4m+2,\\ q^{4m^2+10m+6}, & n=4m+3,\end{cases}\qquad \beta_n = \frac{q^{(n^2-n)/2}}{(q;q)_n(q^3;q^2)_n}.$$

Substituting the above Bailey pair into (2.18), multiplying by $1/(1-q)$ on both sides and then replacing q by q^2, we have

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{n^2+n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}$$
$$= \frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=0}^{\infty}\frac{q^{8n^2+4n}}{(1-xq^{8n+1})(1-x^{-1}q^{8n+1})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+20n+6}}{(1-xq^{8n+3})(1-x^{-1}q^{8n+3})}\right.$$
$$\left.-\sum_{n=0}^{\infty}\frac{q^{8n^2+12n+4}}{(1-xq^{8n+5})(1-x^{-1}q^{8n+5})}+\sum_{n=0}^{\infty}\frac{q^{8n^2+28n+18}}{(1-xq^{8n+7})(1-x^{-1}q^{8n+7})}\right)$$
$$= \frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{8n^2+4n}}{(1-xq^{8n+1})(1-x^{-1}q^{8n+1})}-\sum_{n=-\infty}^{\infty}\frac{q^{8n^2-4n}}{(1-xq^{8n-3})(1-x^{-1}q^{8n-3})}\right)$$
$$= \frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{8n^2+4n}}{1-x^{-1}q^{8n+1}}-\sum_{n=-\infty}^{\infty}\frac{q^{8n^2-4n}}{1-x^{-1}q^{8n-3}}\right.$$

$$- \sum_{n=-\infty}^{\infty} \frac{x^2 q^{8n^2+4n}}{1-xq^{8n+1}} + \sum_{n=-\infty}^{\infty} \frac{x^2 q^{8n^2-4n}}{1-xq^{8n-3}} \Bigg),$$

where the last step follows from (3.2). Then utilizing (2.2) and (2.8), we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(xq, x^{-1}q; q^2)_n q^{n^2+n}}{(q^2;q^2)_n (q^2;q^4)_{n+1}} \\
&= \frac{(xq, x^{-1}q; q^2)_\infty \overline{J}_{4,16}}{(1-x^2)J_2^2} \big(m(-x^{-2}q^6, q^{16}, -q^{-4}) - m(-x^{-2}q^6, q^{16}, -q^{-12}) \\
&\quad + x^{-1}q^{-3} \big(m(-x^{-2}q^{-2}, q^{16}, -q^4) - m(-x^{-2}q^{-2}, q^{16}, -q^{-4}) \big) \\
&\quad - x^2 \big(m(-x^2q^6, q^{16}, -q^{-4}) - m(-x^2q^6, q^{16}, -q^{-12}) \big) \\
&\quad - x^3q^{-3} \big(m(-x^2q^{-2}, q^{16}, -q^4) - m(-x^2q^{-2}, q^{16}, -q^{-4}) \big) \big) \\
&= \frac{(xq, x^{-1}q; q^2)_\infty J_{16}^3 J_{8,16}}{(1-x^2)J_2^2 \overline{J}_{4,16}} \left(\frac{j(-x^2q^{10}; q^{16}) + x^{-1}qj(-x^2q^2; q^{16})}{j(x^2q^6; q^{16}) j(x^2q^{14}; q^{16})} \right. \\
&\quad \left. - \frac{x^2 (j(-x^2q^6; q^{16}) + xqj(-x^2q^{14}; q^{16}))}{j(x^2q^2; q^{16}) j(x^2q^{10}; q^{16})} \right), \qquad (3.33)
\end{aligned}$$

where the last equality follows from (2.2) and (2.6). Setting $m = 2$ and $(q, x) \to (q^4, -x^{-1}q)$, $(q^4, -xq)$ in (2.4), respectively, we have

$$j(-x^{-1}q; q^4) = j(-x^{-2}q^6; q^{16}) + x^{-1}qj(-x^{-2}q^{14}; q^{16}), \qquad (3.34)$$

$$j(-xq; q^4) = j(-x^2q^6; q^{16}) + xqj(-x^2q^{14}; q^{16}). \qquad (3.35)$$

So, combining (2.2) and (3.33)-(3.35), we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(xq, x^{-1}q; q^2)_n q^{n^2+n}}{(q^2;q^2)_n (q^2;q^4)_{n+1}} &= \frac{(xq, x^{-1}q; q^2)_\infty J_8 J_{16} J_{8,16}}{(1-x^2) J_2^2 \overline{J}_{4,16}} \left(\frac{j(-xq^3; q^4)}{j(x^2q^6; q^8)} - \frac{x^2 j(-xq; q^4)}{j(x^2q^2; q^8)} \right) \\
&= \frac{J_{16} J_{8,16} (j(xq; q^4) - x^2 j(xq^3; q^4))}{(1-x^2) J_2^2 \overline{J}_{4,16}},
\end{aligned}$$

which implies (1.39). Therefore, we complete the proof. □

Proofs of (1.40) and (1.41). The proofs of (1.40) and (1.41) are similar to those of (1.26) and (1.27). The following identity is used in the proof of (1.41):

$$J_{15,36} + q^3 J_{3,36} = \frac{\overline{J}_{3,12} J_{6,24}}{J_{24}},$$

where the above identity is obtained by setting $(z, q) = (q^3, q^{12})$ in (2.7). We omit the details. □

Proof of Theorem 1.23. We rewrite the Bailey pair in Lemma 2.19 as

$$\alpha_n=\begin{cases} q^{4m^2+2m}, & n=4m,\\ -q^{4m^2+2m}, & n=4m+1,\\ -q^{4m^2+6m+2}, & n=4m+2,\\ q^{4m^2+6m+2}, & n=4m+3,\end{cases}\qquad \beta_n=\frac{q^{(n^2+n)/2}}{(q;q)_n(q^3;q^2)_n}.$$

Substituting the above Bailey pair into (2.18), multiplying by $1/(1-q)$ on both sides and then replacing q by q^2, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{n^2+3n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=0}^{\infty}\frac{q^{8n^2+12n}}{(1-xq^{8n+1})(1-x^{-1}q^{8n+1})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+12n+2}}{(1-xq^{8n+3})(1-x^{-1}q^{8n+3})}\right.\\
&\quad\left.-\sum_{n=0}^{\infty}\frac{q^{8n^2+20n+8}}{(1-xq^{8n+5})(1-x^{-1}q^{8n+5})}+\sum_{n=0}^{\infty}\frac{q^{8n^2+20n+10}}{(1-xq^{8n+7})(1-x^{-1}q^{8n+7})}\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{8n^2+12n}}{(1-xq^{8n+1})(1-x^{-1}q^{8n+1})}-\sum_{n=-\infty}^{\infty}\frac{q^{8n^2+20n+8}}{(1-xq^{8n+5})(1-x^{-1}q^{8n+5})}\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{8n^2+12n}}{1-x^{-1}q^{8n+1}}-\sum_{n=-\infty}^{\infty}\frac{q^{8n^2+20n+8}}{1-x^{-1}q^{8n+5}}\right.\\
&\quad\left.-\sum_{n=-\infty}^{\infty}\frac{x^2q^{8n^2+12n}}{1-xq^{8n+1}}+\sum_{n=-\infty}^{\infty}\frac{x^2q^{8n^2+20n+8}}{1-xq^{8n+5}}\right),
\end{aligned}$$

where the last step follows from (3.2). Then utilizing (2.1), (2.2) and (2.8), we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_nq^{n^2+3n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}\\
&=\frac{q^{-4}(xq,x^{-1}q;q^2)_\infty\overline{J}_{4,16}}{(1-x^2)J_2^2}\Big(m(-x^{-2}q^{-2},q^{16},-q^4)-m(-x^{-2}q^{-2},q^{16},-q^{12})\\
&\quad+x^{-1}q^{-7}\left(m(-x^{-2}q^{-10},q^{16},-q^{12})-m(-x^{-2}q^{-10},q^{16},-q^{20})\right)\\
&\quad-x^2\left(m(-x^2q^{-2},q^{16},-q^4)-m(-x^2q^{-2},q^{16},-q^{12})\right)\\
&\quad-x^3q^{-7}\left(m(-x^2q^{-10},q^{16},-q^{12})-m(-x^2q^{-10},q^{16},-q^{20})\right)\Big)\\
&=\frac{xq^{-1}(xq,x^{-1}q;q^2)_\infty J_{16}^3J_{8,16}}{(1-x^2)J_2^2\overline{J}_{4,16}}\left(\frac{j(-x^2q^{10};q^{16})+x^{-1}qj(-x^2q^2;q^{16})}{j(x^2q^6;q^{16})j(x^2q^{14};q^{16})}\right.\\
&\quad\left.-\frac{j(-x^2q^6;q^{16})+xqj(-x^2q^{14};q^{16})}{j(x^2q^2;q^{16})j(x^2q^{10};q^{16})}\right),
\end{aligned}\tag{3.36}$$

where the last equality follows from (2.2) and (2.6). Then combining (2.2) and (3.34)-(3.36), we obtain that

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}=\frac{xq^{-1}(xq,x^{-1}q;q^2)_\infty J_8J_{16}J_{8,16}}{(1-x^2)J_2^2\overline{J}_{4,16}}\left(\frac{j(-xq^3;q^4)}{j(x^2q^6;q^8)}-\frac{j(-xq;q^4)}{j(x^2q^2;q^8)}\right)$$
$$=\frac{xq^{-1}J_{16}J_{8,16}(j(xq;q^4)-j(xq^3;q^4))}{(1-x^2)J_2^2\overline{J}_{4,16}},$$

which implies (1.42). Therefore, we complete the proof. □

Proofs of (1.43) and (1.44). The proofs of (1.43) and (1.44) are similar to those of (1.26) and (1.27). The following identity is used in the proof of (1.44):

$$J_{9,36}-q^2J_{3,36}=\frac{\overline{J}_{5,12}J_{2,24}}{J_{24}},$$

where the above identity is obtained by setting $(z,q)=(q^5,q^{12})$ in (2.7) and then applying (2.2). We omit the details. □

Proofs of (1.3) and (1.45)-(1.48). Substituting the Bailey pair in Lemma 2.21 into (2.18) and then replacing q by q^2, we have

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_n q^{2n^2}}{(q^2;q^2)_{2n}}$$
$$=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=0}^{\infty}\frac{q^{6n^2+2n}(1-q^{12n+2})}{(1-xq^{6n+1})(1-x^{-1}q^{6n+1})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+10n+4}(1-q^{12n+10})}{(1-xq^{6n+5})(1-x^{-1}q^{6n+5})}\right)$$
$$=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{6n^2+2n}}{(1-xq^{6n+1})(1-x^{-1}q^{6n+1})}-\sum_{n=-\infty}^{\infty}\frac{q^{6n^2-2n}}{(1-xq^{6n-1})(1-x^{-1}q^{6n-1})}\right)$$
$$=\frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{6n^2+2n}}{1-x^{-1}q^{6n+1}}-\sum_{n=-\infty}^{\infty}\frac{q^{6n^2-2n}}{1-x^{-1}q^{6n-1}}\right.$$
$$\left.-\sum_{n=-\infty}^{\infty}\frac{x^2q^{6n^2+2n}}{1-xq^{6n+1}}+\sum_{n=-\infty}^{\infty}\frac{x^2q^{6n^2-2n}}{1-xq^{6n-1}}\right),$$

where the last step follows from (3.2). Then utilizing (2.2) and (2.8), we deduce that

$$\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_n q^{2n^2}}{(q^2;q^2)_{2n}}$$
$$=\frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\big(\overline{J}_{4,12}\left(m(-x^{-2}q^6,q^{12},-q^{-4})-m(-x^{-2}q^6,q^{12},-q^{-8})\right)$$
$$+x^{-1}q^{-1}\overline{J}_{2,12}\left(m(-x^{-2},q^{12},-q^2)-m(-x^{-2},q^{12},-q^{-2})\right)\big)$$

$$
\begin{aligned}
&- x^2\overline{J}_{4,12}\left(m(-x^2q^6,q^{12},-q^{-4})-m(-x^2q^6,q^{12},-q^{-8})\right)\\
&-x^3q^{-1}\overline{J}_{2,12}\left(m(-x^2,q^{12},-q^2)-m(-x^2,q^{12},-q^{-2})\right)\Big)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty J_4J_{12}^3\left(\overline{J}_{2,12}j(-x^2q^6;q^{12})+x^{-1}q\overline{J}_{4,12}j(-x^2;q^{12})\right)}{J_2^2\overline{J}_{2,12}\overline{J}_{4,12}j(x^2q^2;q^{12})j(x^2q^{10};q^{12})},
\end{aligned}
$$

where the last equality follows from (2.2) and (2.6). Next, applying (2.14) to the above identity, we have

$$
\sum_{n=0}^\infty\frac{(xq,x^{-1}q;q^2)_nq^{2n^2}}{(q^2;q^2)_{2n}}=\frac{J_4J_6J_{12}j(xq^3;q^6)}{J_2^2\overline{J}_{2,12}\overline{J}_{4,12}},
$$

which implies (1.3).

Next, we rewrite the Bailey pair in Lemma 2.22 as

$$
\alpha_n=\begin{cases} q^{m^2-3m/2}(1-q^{4m+1})/(1-q), & n=2m,\\ -q^{m^2-m/2-1/2}(1-q^{4m+3})/(1-q), & n=2m+1,\end{cases}\qquad \beta_n=\frac{(-1)^nq^{n^2/2-n}(q^{1/2};q)_n}{(q;q)_{2n}}.
$$

Substituting the above Bailey pair into (2.18) and then replacing q by q^2, we have

$$
\begin{aligned}
&\sum_{n=0}^\infty\frac{(q,xq,x^{-1}q;q^2)_n(-1)^nq^{n^2}}{(q^2;q^2)_{2n}}\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=0}^\infty\frac{q^{2n^2+n}(1-q^{8n+2})}{(1-xq^{4n+1})(1-x^{-1}q^{4n+1})}-\sum_{n=0}^\infty\frac{q^{2n^2+3n+1}(1-q^{8n+6})}{(1-xq^{4n+3})(1-x^{-1}q^{4n+3})}\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=-\infty}^\infty\frac{q^{2n^2+n}}{(1-xq^{4n+1})(1-x^{-1}q^{4n+1})}-\sum_{n=-\infty}^\infty\frac{q^{2n^2-n}}{(1-xq^{4n-1})(1-x^{-1}q^{4n-1})}\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\left(\sum_{n=-\infty}^\infty\frac{q^{2n^2+n}}{1-x^{-1}q^{4n+1}}-\sum_{n=-\infty}^\infty\frac{q^{2n^2-n}}{1-x^{-1}q^{4n-1}}\right.\\
&\quad\left.-\sum_{n=-\infty}^\infty\frac{x^2q^{2n^2+n}}{1-xq^{4n+1}}+\sum_{n=-\infty}^\infty\frac{x^2q^{2n^2-n}}{1-xq^{4n-1}}\right),
\end{aligned}
$$

where the last step follows from (3.2). Then utilizing (1.5) and (2.2), we deduce that

$$
\begin{aligned}
&\sum_{n=0}^\infty\frac{(q,xq,x^{-1}q;q^2)_n(-1)^nq^{n^2}}{(q^2;q^2)_{2n}}\\
&=\frac{(xq,x^{-1}q;q^2)_\infty\overline{J}_{1,4}}{(1-x^2)J_2^2}\left(m(-x^{-1}q^2,q^4,-q^{-1})-m(-x^{-1}q^2,q^4,-q^{-3})\right.\\
&\quad\left.-x^2\left(m(-xq^2,q^4,-q^{-1})-m(-xq^2,q^4,-q^{-3})\right)\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty J_4^3J_{2,4}j(-xq^2;q^4)}{J_2^2\overline{J}_{1,4}j(xq;q^4)j(xq^3;q^4)}
\end{aligned}
$$

$$= \frac{J_1 j(-xq^2;q^4)}{J_2^2},$$

where the second equality follows from (2.2) and (2.6). Setting $q \to -q$, we have

$$\sum_{n=0}^{\infty} \frac{(-q,-xq,-x^{-1}q;q^2)_n q^{n^2}}{(q^2;q^2)_{2n}} = \frac{j(-xq^2;q^4)}{J_{1,4}},$$

which implies (1.45).

Substituting the Bailey pair in Lemma 2.24 into (2.19) and then multiplying by $1/(1-q)$ on both sides, we obtain that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(xq,x^{-1}q;q)_n q^{n^2+n}}{(q;q)_{2n+1}} \\
&= \frac{(xq,x^{-1}q;q)_\infty}{J_1^2} \left(\sum_{n=0}^{\infty} \frac{q^{3n^2+2n}(1-q^{6n+2})}{(1-xq^{3n+1})(1-x^{-1}q^{3n+1})} + \sum_{n=1}^{\infty} \frac{q^{3n^2+4n-2}(1-q^{-6n+2})}{(1-xq^{3n-1})(1-x^{-1}q^{3n-1})} \right) \\
&= \frac{(xq,x^{-1}q;q)_\infty}{J_1^2} \left(\sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{(1-xq^{3n+1})(1-x^{-1}q^{3n+1})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2-2n}}{(1-xq^{3n-1})(1-x^{-1}q^{3n-1})} \right) \\
&= \frac{(xq,x^{-1}q;q)_\infty}{(1-x^2)J_1^2} \left(\sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{1-x^{-1}q^{3n+1}} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2-2n}}{1-x^{-1}q^{3n-1}} \right. \\
&\quad \left. - \sum_{n=-\infty}^{\infty} \frac{x^2 q^{3n^2+2n}}{1-xq^{3n+1}} + \sum_{n=-\infty}^{\infty} \frac{x^2 q^{3n^2-2n}}{1-xq^{3n-1}} \right),
\end{aligned}$$

where the last step follows from (3.2). Then utilizing (2.2) and (2.8), we derive that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(xq,x^{-1}q;q)_n q^{n^2+n}}{(q;q)_{2n+1}} \\
&= \frac{(xq,x^{-1}q;q)_\infty}{(1-x^2)J_1^2} \Big(\overline{J}_{1,6} \left(m(-x^{-2}q^3,q^6,-q^{-1}) - m(-x^{-2}q^3,q^6,-q^{-5}) \right) \\
&\quad + x^{-1}q^{-1}\overline{J}_{2,6} \left(m(-x^{-2},q^6,-q^2) - m(-x^{-2},q^6,-q^{-2}) \right) \\
&\quad - x^2\overline{J}_{1,6} \left(m(-x^2q^3,q^6,-q^{-1}) - m(-x^2q^3,q^6,-q^{-5}) \right) \\
&\quad - x^3q^{-1}\overline{J}_{2,6} \left(m(-x^2,q^6,-q^2) - m(-x^2,q^6,-q^{-2}) \right) \Big) \\
&= \frac{(xq,x^{-1}q;q)_\infty J_2 J_6^3 (\overline{J}_{2,6} j(-x^2q^3;q^6) + x^{-1}q\overline{J}_{1,6} j(-x^2;q^6))}{J_1^2 \overline{J}_{1,6} \overline{J}_{2,6} j(x^2q^2;q^6) j(x^2q^4;q^6)},
\end{aligned}$$

where the second equality follows from (2.2) and (2.6). Then combining (2.11) and the above identity, we derive that

$$\sum_{n=0}^{\infty} \frac{(xq,x^{-1}q;q)_n q^{n^2+n}}{(q;q)_{2n+1}} = \frac{(xq^3,x^{-1}q^3;q^3)_\infty J_2 J_3^2 J_6}{J_1^2 \overline{J}_{1,6} \overline{J}_{2,6}}.$$

Multiplying by $(1-x)$ on both sides, we complete the proof of (1.46).

Substituting the Bailey pair in Lemma 2.20 into (2.18), replacing q by q^2 and then multiplying by $1/(1+q)$ on both sides, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_n(-1)^nq^{n^2+2n}}{(-q;q^2)_{n+1}(q^4;q^4)_n}\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=0}^{\infty}\frac{q^{2n^2+3n}(1-q^{4n+1})}{(1-xq^{4n+1})(1-x^{-1}q^{4n+1})}+\sum_{n=1}^{\infty}\frac{q^{2n^2+5n-2}(1-q^{-4n+1})}{(1-xq^{4n-1})(1-x^{-1}q^{4n-1})}\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+3n}}{(1-xq^{4n+1})(1-x^{-1}q^{4n+1})}-\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n-1}}{(1-xq^{4n-1})(1-x^{-1}q^{4n-1})}\right)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty}{(1-x^2)J_2^2}\left(\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+3n}}{1-x^{-1}q^{4n+1}}-\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n-1}}{1-x^{-1}q^{4n-1}}\right.\\
&\qquad\left.-\sum_{n=-\infty}^{\infty}\frac{x^2q^{2n^2+3n}}{1-xq^{4n+1}}+\sum_{n=-\infty}^{\infty}\frac{x^2q^{2n^2+n-1}}{1-xq^{4n-1}}\right),
\end{aligned}$$

where the last step follows from (3.2). Then utilizing (1.5) and (2.2), we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(xq,x^{-1}q;q^2)_n(-1)^nq^{n^2+2n}}{(-q;q^2)_{n+1}(q^4;q^4)_n}\\
&=\frac{q^{-1}(xq,x^{-1}q;q^2)_\infty\overline{J}_{1,4}}{(1-x^2)J_2^2}\big(m(-x^{-1},q^4,-q)-m(-x^{-1},q^4,-q^{-1})\\
&\quad-x^2\big(m(-x,q^4,-q)-m(-x,q^4,-q^{-1})\big)\\
&=\frac{(xq,x^{-1}q;q^2)_\infty J_4^3J_{2,4}j(-x;q^4)}{(1+x)J_2^2\overline{J}_{1,4}j(xq;q^4)j(xq^3;q^4)}\\
&=\frac{j(-x;q^4)}{(1+x)\overline{J}_{1,4}},
\end{aligned}$$

where the second equality follows from (2.2) and (2.6). Setting $q\to -q$ yields (1.47).

Finally, we rewrite the Bailey pair in Lemma 2.23 as

$$\alpha_n=\begin{cases}q^{4m^2-2m}(1-q^{8m+2})/(1-q^2), & n=4m,\\ 0, & n=4m+1,\\ -q^{4m^2+2m}(1-q^{8m+6})/(1-q^2), & n=4m+2,\\ 0, & n=4m+3,\end{cases}\qquad \beta_n=\frac{(-q;q)_nq^{(n^2-n)/2}}{(q^2;q)_{2n}}.$$

Substituting the above Bailey pair into (2.19) and then multiplying by $1/(1-q)$ on both sides, we have

$$\sum_{n=0}^{\infty}\frac{(-q,xq,x^{-1}q;q)_nq^{(n^2+n)/2}}{(q;q)_{2n+1}}$$

$$= \frac{(xq, x^{-1}q; q)_\infty}{J_1^2} \left(\sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1-q^{8n+2})}{(1-xq^{4n+1})(1-x^{-1}q^{4n+1})} - \sum_{n=0}^{\infty} \frac{q^{4n^2+6n+2}(1-q^{8n+6})}{(1-xq^{4n+3})(1-x^{-1}q^{4n+3})} \right)$$
$$= \frac{(xq, x^{-1}q; q)_\infty}{J_1^2} \left(\sum_{n=-\infty}^{\infty} \frac{q^{4n^2+2n}}{(1-xq^{4n+1})(1-x^{-1}q^{4n+1})} - \sum_{n=-\infty}^{\infty} \frac{q^{4n^2-2n}}{(1-xq^{4n-1})(1-x^{-1}q^{4n-1})} \right)$$
$$= \frac{(xq, x^{-1}q; q)_\infty}{(1-x^2)J_1^2} \left(\sum_{n=-\infty}^{\infty} \frac{q^{4n^2+2n}}{1-x^{-1}q^{4n+1}} - \sum_{n=-\infty}^{\infty} \frac{q^{4n^2-2n}}{1-x^{-1}q^{4n-1}} \right.$$
$$\left. - \sum_{n=-\infty}^{\infty} \frac{x^2q^{4n^2+2n}}{1-xq^{4n+1}} + \sum_{n=-\infty}^{\infty} \frac{x^2q^{4n^2-2n}}{1-xq^{4n-1}} \right),$$

where the last step follows from (3.2). Then utilizing (2.2) and (2.8), we derive that

$$\sum_{n=0}^{\infty} \frac{(-q, xq, x^{-1}q; q)_n q^{(n^2+n)/2}}{(q;q)_{2n+1}}$$
$$= \frac{(xq, x^{-1}q; q)_\infty \overline{J}_{2,8}}{(1-x^2)J_1^2} \left(m(-x^{-2}q^4, q^8, -q^{-2}) - m(-x^{-2}q^4, q^8, -q^{-6}) \right.$$
$$+ x^{-1}q^{-1}\left(m(-x^{-2}, q^8, -q^2) - m(-x^{-2}, q^8, -q^{-2}) \right)$$
$$- x^2\left(m(-x^2q^4, q^8, -q^{-2}) - m(-x^2q^4, q^8, -q^{-6}) \right)$$
$$\left. - x^3q^{-1}\left(m(-x^2, q^8, -q^2) - m(-x^2, q^8, -q^{-2}) \right) \right)$$
$$= \frac{(xq, x^{-1}q; q)_\infty J_8^3 J_{4,8}(j(-x^2q^4; q^8) + x^{-1}qj(-x^2; q^8))}{J_1^2 \overline{J}_{2,8} j(x^2q^2; q^8) j(x^2q^6; q^8)},$$

where the last equality follows from (2.2) and (2.6). Setting $m = 2$ and $(q, x) \to (q^2, -x^{-1}q)$ in (2.4) yields that

$$j(-x^{-1}q; q^2) = j(-x^{-2}q^4; q^8) + x^{-1}qj(-x^{-2}q^8; q^8).$$

So, combining (2.2) and the above two identities, we obtain that

$$\sum_{n=0}^{\infty} \frac{(-q, xq, x^{-1}q; q)_n q^{(n^2+n)/2}}{(q;q)_{2n+1}} = \frac{(xq^2, x^{-1}q^2; q^2)_\infty J_2^2}{J_1^2}.$$

Multiplying by $(1-x)$ on both sides, we complete the proof of (1.48). $\square$

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