INTEGER MATRIX EXACT COVERING SYSTEMS AND k-COLORED GENERALIZED FROBENIUS PARTITIONS

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Dedicated to Professors George Andrews and Bruce Berndt for their 85th birthdays

Abstract. In 1984, Andrews introduced the family of k-colored generalized Frobenius partition functions. For any $k \geq 1$, let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n, and let $\mathrm{C}\Phi_k(q)$ represent its generating function. Baruah and Sarmah derived integral expressions of $\mathrm{C}\Phi_k(q)$ with $k \in \{4, 5, 6\}$ by applying Cao's method of integer matrix exact covering systems. Subsequently, Chan, Wang and Yang systematically investigated the generating function of $c\phi_k(n)$ by utilizing the theory of modular forms. In this paper, we propose a unified framework to derive expressions for $\mathrm{C}\Phi_k(q)$ with integral coefficients, extending several known results about $\mathrm{C}\Phi_k(q)$.

1. Introduction

Throughout, we always assume that q is a complex number such that $|q| < 1$ and adopt the following standard notation:

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1;q)_\infty(a_2;q)_\infty \cdots (a_m;q)_\infty.$$

In his 1984 AMS Memoir, Andrews [1] introduced the notion of a generalized Frobenius partition of n, a two-rowed array of nonnegative integers of the form:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $n = r + \sum_{i=1}^{r}(a_i + b_i)$, and $a_i \geq a_{i+1}$, $b_i \geq b_{i+1}$. Andrews further introduced the notion of k-colored generalized Frobenius partitions. A k-colored generalized Frobenius partition is an array of the above form whose parts are taken from k copies of nonnegative integers. For any $k \geq 1$, let $c\phi_k(n)$ denote the number of k-colored

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generalized Frobenius partitions of n, and let $\mathrm{C}\Phi_k(q)$ represent its generating function, namely,

$$\mathrm{C}\Phi_k(q)=\sum_{n=0}^{\infty}c\phi_k(n)q^n.$$

Andrews [1, Theorem 5.2] proved that

$$\mathrm{C}\Phi_k(q)=\frac{1}{(q;q)_\infty^k}\sum_{m_1,m_2,\ldots,m_{k-1}=-\infty}^{\infty}q^{Q(m_1,m_2,\cdots,m_{k-1})},\tag{1.1}$$

where

$$Q(m_1,m_2,\cdots,m_{k-1})=\sum_{i=1}^{k-1}m_i^2+\sum_{1\le i<j\le k-1}m_im_j.\tag{1.2}$$

Building upon (1.1), Andrews [1, Corollary 5.2] derived alternative expressions for $\mathrm{C}\Phi_k(q)$ with $k\in\{2,3,5\}$, employing Jacobi's triple product identity (see [1, Eq. (3.1)] and the result of Kloosterman [14, pp. 358, 362] in the derivation. Specifically,

$$\mathrm{C}\Phi_2(q)=\frac{\Theta_3(q)}{(q;q)_\infty^2}=\frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4(q^4;q^4)_\infty},\tag{1.3}$$

$$\mathrm{C}\Phi_3(q)=\frac{1}{(q;q)_\infty^3}\big(\Theta_3(q)\Theta_3(q^3)+\Theta_2(q)\Theta_2(q^3)\big)\tag{1.4}$$

$$=\frac{1}{(q;q)_\infty^3}\left(1+6\sum_{j=0}^{\infty}\left(\frac{j}{3}\right)\frac{q^j}{1-q^j}\right),\tag{1.5}$$

$$\mathrm{C}\Phi_5(q)=\frac{1}{(q;q)_\infty^5}\left(1+25\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{q^j}{(1-q^j)^2}-5\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j}\right),\tag{1.6}$$

where

$$\Theta_2(q)=\sum_{j=-\infty}^{\infty}q^{(j+1/2)^2}=2q^{1/4}\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty},$$

$$\Theta_3(q)=\sum_{j=-\infty}^{\infty}q^{j^2}=\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2},$$

and $\left(\frac{\cdot}{p}\right)$ denotes the Kronecker symbol. Andrews [1, p. 26] mentioned that there exists a similar identity for the case $k=7$, but he did not give the concrete expression. The missing identity is

$$\mathrm{C}\Phi_7(q)=\frac{1}{(q;q)_\infty^7}\left(1+\frac{343}{8}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{q^j+q^{2j}}{(1-q^j)^3}-\frac{7}{8}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{j^2q^j}{1-q^j}\right), \tag{1.7}$$

later derived by Kolitsch [15, Lemma 2].

Recently, Baruah and Sarmah [2,3] established explicit expressions of $\mathrm{C}\Phi_k(q)$ with $k\in\{4,5,6\}$, applying the integer matrix exact cover systems developed by Cao [5]. They proved that

$$\mathrm{C}\Phi_4(q)=\frac{1}{(q;q)_\infty^4}\left(\Theta_3^3(q^2)+3\Theta_3(q^2)\Theta_2^2(q^2)\right), \tag{1.8}$$

$$\begin{aligned}\mathrm{C}\Phi_5(q)=\frac{1}{(q;q)_\infty^5}\Bigg(&\Theta_3(q^{10})\Theta_3^3(q^2)+3\Theta_3(q^{10})\Theta_3(q^2)\Theta_2^2(q^2)\\&+\frac{1}{2}\Theta_2(q^{5/2})\Theta_2^3(q^{1/2})+3\Theta_2(q^{10})\Theta_2(q^2)\Theta_3^2(q)+\Theta_2(q^{10})\Theta_2^3(q^2)\Bigg),\end{aligned} \tag{1.9}$$

$$\begin{aligned}\mathrm{C}\Phi_6(q)=\frac{1}{(q;q)_\infty^6}\Bigg(&\Theta_3^3(q)\Theta_3(q^2)\Theta_3(q^6)\\&+\frac{3}{4}\Theta_2^3(q^{1/2})\Theta_2(q)\Theta_2(q^{3/2})+\Theta_3^2(q)\Theta_2(q^2)\Theta_2(q^6)\Bigg).\end{aligned} \tag{1.10}$$

For $k>7$, it is not clear if new expressions associated with $\mathrm{C}\Phi_k(q)$ could be derived by utilizing the methods of Andrews and Baruah–Sarmah. Actually, Andrews [1, p. 15] commented that "as k increases the expressions quickly become long and messy".

Based on the theory of modular forms, Chan, Wang and Yang [7] conducted a systematic study of the expressions of $\mathrm{C}\Phi_k(q)$, uncovering remarkable modular and congruence properties for $\mathrm{C}\Phi_k(q)$ with $2\le k\le 17$. It is worth mentioning that some coefficients in the expressions of $\mathrm{C}\Phi_k(q)$ with $k\in\{10,12,14,15,17\}$ are not integers. Subsequent work by Jiang, Rolen and Woodbury [13] leveraged modular forms to explore alternative representations of the generating functions of $c\phi_k(n)$ as linear combinations of Dedekind's eta functions and Klein forms. However, the integrality of coefficients in these combinations remains an open question. Motivated by these discoveries, the authors [9] developed a general strategy to derive expressions of $\mathrm{C}\Phi_k(q)$ with guaranteed integral coefficients. As consequences, they proved some infinite families of congruences satisfied by $c\phi_k(n)$, valid for arbitrarily large k. While powerful, the method in [9] produces computationally cumbersome expressions for certain values of k. This paper presents a unified framework to derive integral-coefficient expressions for $\mathrm{C}\Phi_k(q)$, significantly addressing the limitations of prior methods.

The rest of this paper is structured as follows. In Section 2, we introduce some notation and compiles some essential lemmas that serve as the foundation for proving the main results. Section 3 is devoted to some novel expressions of $\mathrm{C}\Phi_k(q)$ with integral coefficients. Finally, we conclude with a discussion of congruence properties for $c\phi_k(n)$, including a conjectural congruence modulo 2187 for $c\phi_{18}(n)$.

2. Preliminary results

In this section, we collect some necessary notation and lemmas which will be used to prove the main results.

For notational convenience, we denote

$$J_{a,b} := (q^a, q^{b-a}, q^b; q^b)_\infty, \quad \overline{J}_{a,b} := (-q^a, -q^{b-a}, q^b; q^b)_\infty, \quad J_a := J_{a,3a} = (q^a; q^a)_\infty.$$

First, Ramanujan's theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a, -b, ab; ab)_\infty, \qquad |ab| < 1, \tag{2.1}$$

where the last identity is the celebrated Jacobi triple product identity [4, p. 35, Entry 19]. Two important special cases of $f(a,b)$ are, respectively, given by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{J_2^5}{J_1^2 J_4^2}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{J_2^2}{J_1}. \tag{2.3}$$

Next, we introduce the following auxiliary functions involving $\varphi(q)$ and $\psi(q)$, defined by

$$A(q) = \varphi^2(q)\varphi(q^2) + 8q\psi^2(q^2)\psi(q^4) \qquad \text{and} \qquad B(q) = \varphi^2(q) + 2\varphi^2(q^2). \tag{2.4}$$

For $0 < |z| < \infty$, we define

$$f_k(z) = f_k(z,q) := \left(\sum_{n=-\infty}^{\infty} z^n q^{n^2} \right)^k = \sum_{n=-\infty}^{\infty} c_{k,n}(q) z^n. \tag{2.5}$$

Based on (1.1), we [9, p. 6, Eq. (1.18)] found that

$$c_{k,0}(q) = \mathrm{CT}_z\big(f_k(z)\big) = (q^2;q^2)_\infty^k \mathrm{C\Phi}_k(q^2), \tag{2.6}$$

which means that once we find the constant term in the expression of $f_k(z)$, we can derive the function $\mathrm{C\Phi}_k(q)$. In our previous work [9], we deduced some expressions of $f_k(z)$ with some special values of k. Before stating them, we first recall the following notation [9]:

$$L_{a,b,c,d} = L_{a,b,c,d}(z,q) := \sum_{n=-\infty}^{\infty} z^{an+b} q^{cn^2+dn}.$$

The following lemmas play an important role in the proofs of the main results.

Lemma 2.1. [9, Lemma 2.2] *We have*

$$f_2(z) = \varphi(q^2)L_{2,0,2,0} + 2q\psi(q^4)L_{2,1,2,2}, \tag{2.7}$$

$$f_3(z) = a(q^2)L_{3,0,3,0} + 3q\frac{J_6^3}{J_2}\big(L_{3,1,3,2} + L_{3,-1,3,-2}\big), \tag{2.8}$$

where $a(q)$ is one of the Borwein cubic functions, given by (see [12])

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}. \tag{2.9}$$

Lemma 2.2. [9, Eq. (2.22)] *Let $A(q)$ and $B(q)$ be defined as in* (2.4)*. Then*

$$f_4(z) = A(q^2)L_{4,0,4,0} + 4q\psi^3(q^2)\big(L_{4,1,4,2} + L_{4,-1,4,-2}\big) + 2q^2B(q^2)\psi(q^8)L_{4,2,4,4}. \tag{2.10}$$

Lemma 2.3. [9, Eq. (2.38)]

$$\begin{aligned} f_6(z) = {} & \left(a^2(q^2)\varphi(q^6) + 18q^2\frac{J_4^2J_6^7J_{24}}{J_2^3J_8J_{12}}\right)L_{6,0,6,0} \\ & + \left(18q^3\psi(q^{12})\frac{J_6^6}{J_2^2} + 6qa(q^2)\frac{J_6^3J_8J_{12}^2}{J_2J_4J_{24}}\right)\big(L_{6,1,6,2} + L_{6,-1,6,-2}\big) \\ & + \left(9q^2\varphi(q^6)\frac{J_6^6}{J_2^2} + 6q^2a(q^2)\frac{J_4^2J_6^4J_{24}}{J_2^2J_8J_{12}}\right)\big(L_{6,2,6,4} + L_{6,-2,6,-4}\big) \\ & + \left(2q^3a^2(q^2)\psi(q^{12}) + 18q^3\frac{J_6^6J_8J_{12}^2}{J_2^2J_4J_{24}}\right)L_{6,3,6,6}. \end{aligned} \tag{2.11}$$

We also require the following identities involving $\varphi(q)$ and $\psi(q)$.

Lemma 2.4. [4, p. 40, Entry 25 (v) and (vi)]

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{2.12}$$

$$\varphi(-q) = \varphi(q^4) - 2q\psi(q^8), \tag{2.13}$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \tag{2.14}$$

$$\varphi^2(-q) = \varphi^2(q^2) - 4q\psi^2(q^4). \tag{2.15}$$

Finally, we need the following key identities involving Ramanujan's theta function $f(a,b)$.

Lemma 2.5. *We have*

$$f(a,b) = f\big(a^3b, ab^3\big) + af(b/a, a^5b^3), \tag{2.16}$$

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\big(b/c, ac^2d\big)f\big(b/d, acd^2\big), \tag{2.17}$$

where $ab = cd$.

Proof. The identity (2.16) follows from two identities in Berndt's book [4, Entry 30 (ii) and (iii)]. The identity (2.17) comes from Berndt's book [4, p. 45, Entry 29]. □

We conclude this section with a brief discussion on integer matrix exact covering systems developed by Cao [5], which serve as the theoretical backbone for our subsequent derivations. Recent applications in integer matrix exact covering systems are systematically reviewed in [6], where readers can find detailed technical discussions and comparative analyses.

An exact covering system is a partition of the integers into a finite set of arithmetic sequences. An integer matrix exact covering system is a partition of $\mathbb{Z}^n$ into a lattice and a finite number of its translates without overlap. For a given multi-variable sum

$$\sum_{x_1,x_2,\ldots,x_n=-\infty}^{\infty} f(x_1,x_2,\ldots,x_n),$$

change the variables from x_i to y_i $(i=1,2,\ldots,n)$ by the transformation $y=Ax$, where A is an $n\times n$ integer matrix, $\det A\neq 0$, $x=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$ and $y=\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}$. Then we have

$$x=A^{-1}y=\frac{1}{\det A}A^*y,$$

where A^* is the adjoint matrix of A. Set $B'=\operatorname{sgn}(\det A)A^*$ and $d'=|\det A|>0$ where $\operatorname{sgn}(\det A)$ denotes the sign of $\det A$. So,

$$x=\frac{1}{d'}B'y. \tag{2.18}$$

Since $\det A^*=(\det A)^{n-1}$, we have $B'=\pm(d')^{n-1}$. In addition, from $A\cdot A^*=\det A\cdot I$, it follows that $\det A\cdot(A^*)^{-1}=A$, which means that $d'(B')^{-1}$ is an integeral matrix.

Next, the identity (2.18) implies that

$$B'y\equiv 0 \pmod{d'}.$$

Then based on the definitions of B' and d', one can find that the greatest common divisor of all the elements of B' and d' is $d_{n-1}(A)$, which denotes the greatest common divisor of all the $(n-1)\times(n-1)$ determinantal minors of A.

Furthermore, let

$$B:=\frac{B'}{d_{n-1}(A)}=\frac{\operatorname{sgn}(\det A)A^*}{d_{n-1}(A)} \qquad\text{and}\qquad d:=\frac{d'}{d_{n-1}(A)}=\frac{|\det A|}{d_{n-1}(A)}.$$

Then B is also an integer matrix and d is a positive integer. Solve

$$By\equiv 0 \pmod{d}$$

to obtain the solutions $\{c_0, c_1, \ldots, c_{k-1}\}$ with $k = |\det B|$. Then as stated in [5], one can transform x to y by $x = By + \frac{1}{d}Bc_r$ $(r = 0, 1, 2, \ldots, |\det B| - 1)$ and $\{By + \frac{1}{d}Bc_r\}_{r=0}^{|\det B|-1}$ is an integer matrix exact covering system which transforms the variables from x_i to y_i $(i = 1, 2, \ldots, n)$.

In conclusion, to establish an integer matrix exact covering system, we first need to find an integer matrix B' and a positive integer d' such that $B' = \pm(d')^{n-1}$ and $d'(B')^{-1}$ is an integer matrix. Then to simplify calculations, we obtain B and d by dividing B' and d' by the greatest common divisor of all the elements of B' and d', respectively. Finally, solve $By \equiv 0 \pmod{d}$ to construct an integer matrix exact covering system $\{By + \frac{1}{d}Bc_r\}_{r=0}^{|\det B|-1}$.

In this paper, we always consider the following form

$$\begin{aligned} S :=& \sum_{m_1, m_2, \ldots, m_k=-\infty}^{\infty} q^{\sum_{i=1}^{k} m_i^2 + \sum_{1\le i<j\le k}^{k} m_i m_j} \\ &= \sum_{m_1, m_2, \ldots, m_k=-\infty}^{\infty} q^{(m_1, m_2, \ldots, m_k) D (m_1, m_2, \ldots, m_k)^T}, \end{aligned} \tag{2.19}$$

where

$$D = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{pmatrix}_{k\times k}.$$

By finding a proper invertible integer matrix B and a positive integer d, we may construct the corresponding integer matrix exact covering system $\{B\mathbf{n}+\frac{1}{d}Bc_r\}_{r=0}^{|\det B|-1}$ where $\mathbf{n} = (n_1, n_2, \ldots, n_k)^T$ and c_r $(r = 0, 1, \ldots, |\mathrm{det}B| - 1)$ are the solutions of $B\mathbf{n} \equiv 0 \pmod{d}$. Thus,

$$\begin{aligned} &\sum_{m_1, m_2, \ldots, m_k=-\infty}^{\infty} q^{(m_1, m_2, \ldots, m_k) D (m_1, m_2, \ldots, m_k)^T} \\ &= \sum_{r=0}^{|\det B|-1} \sum_{n_1, n_2, \ldots, n_k=-\infty}^{\infty} q^{(B\mathbf{n}+\frac{1}{d}Bc_r)^T D (B\mathbf{n}+\frac{1}{d}Bc_r)}. \end{aligned} \tag{2.20}$$

In [5], Cao particularly focused on a special kind of B satisfying a "generalized orthogonal" relation to express a product of n theta functions as a linear combination of products of theta functions. It should be pointed out that in this paper if the invertible matrix B satisfies that $B^T DB$ is a diagonal matrix, then by integer matrix exact covering systems, one can directly express the sum S defined in (2.19) as a linear combination of products of theta functions.

3. The integral expressions of $\mathrm{C}\Phi_k(q)$

In this section, we shall derive some integral expressions of $\mathrm{C}\Phi_k(q)$. According to (1.1), (1.2), (2.5), (2.6) and (2.20), one readily sees that the crucial step is to find an appropriate invertible integral matrix B such that B^TDB becomes a diagonal matrix.

Theorem 3.1. *We have*

$$\mathrm{C}\Phi_3(q) = \frac{1}{J_1^3}\left(\frac{J_2^5J_6^5}{J_1^2J_3^2J_4^2J_{12}^2} + 4q\frac{J_4^2J_{12}^2}{J_2J_6}\right). \tag{3.1}$$

It is evident that (3.1) is equivalent to (1.4), a result originally derived by Andrews [1] through a different way. The proof technique central to Theorem 3.1 will serve as a foundational tool in subsequent analyses. For completeness and to establish methodological consistency, we provide a detailed proof of (3.1) here.

Proof. It follows immediately from (1.1) that

$$\mathrm{C}\Phi_3(q) = \frac{1}{J_1^3}\sum_{m_1,m_2=-\infty}^{\infty} q^{m_1^2+m_2^2+m_1m_2}. \tag{3.2}$$

Next, we choose the following matrix:

$$B = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}.$$

According to (3.2), we further deduce that

$$\begin{aligned}\mathrm{C}\Phi_3(q) &= \frac{1}{J_1^3}\left(\sum_{n_1,n_2=-\infty}^{\infty} q^{3n_1^2+n_2^2} + \sum_{n_1,n_2=-\infty}^{\infty} q^{3n_1^2+3n_1+n_2^2+n_2+1}\right)\\ &= \frac{1}{J_1^3}\big(\varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6)\big) = \frac{1}{J_1^3}\left(\frac{J_2^5J_6^5}{J_1^2J_3^2J_4^2J_{12}^2} + 4q\frac{J_4^2J_{12}^2}{J_2J_6}\right),\end{aligned}$$

which is nothing but (3.1).

We therefore complete the proof of Theorem 3.1. □

Theorem 3.2. *We have*

$$\mathrm{C}\Phi_4(q) = \frac{1}{J_1^4}\left(\frac{J_2^8J_4}{J_1^4J_8^2} + 8q\frac{J_4^3J_8^2}{J_2^2}\right). \tag{3.3}$$

Baruah and Sarmah [2, Theorem 2.1] derived an equivalent formulation of (3.3) (see (1.4) above), while the same identity was established by the authors [9] through an alternative approach. For the completeness of the discussion, we present another proof of (3.3). Zhang and Wang [16, p. 127] later obtained another integral expression of (3.3) by employing some identities related to Ramanujan's classical theta functions. In the subsequent sections, we extensively utilize integer matrix exact covering systems to simplify many q-series multiple sums. Given the similarity of these simplifications to the proof of (3.1), we omit repetitive calculations and focus on presenting the final results.

Proof. From (1.1) we have

$$\mathrm{C}\Phi_4(q) = \frac{1}{J_1^4} \sum_{m_1,m_2,m_3=-\infty}^{\infty} q^{\sum_{i=1}^3 m_i^2 + \sum_{1\le i<j\le 3} m_i m_j}.$$

In this case, we adopt the following matrix:

$$B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

After simplification, we obtain that

$$\mathrm{C}\Phi_4(q) = \frac{1}{J_1^4}\left(\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_1(n_1,n_2,n_3)} + \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_2(n_1,n_2,n_3)} \right),$$

where

$$\begin{aligned} Q_1(n_1,n_2,n_3) &= 2n_1^2 + n_2^2 + n_3^2, \\ Q_2(n_1,n_2,n_3) &= 2n_1^2 + 2n_1 + n_2^2 + n_2 + n_3^2 + n_3 + 1. \end{aligned}$$

According to (2.2) and (2.3), we further get that

$$\mathrm{C}\Phi_4(q) = \frac{1}{J_1^4}\big(\varphi^2(q)\varphi(q^2) + 8q\psi^2(q^2)\psi(q^4)\big) = \frac{1}{J_1^4}\left(\frac{J_2^8 J_4}{J_1^4 J_8^2} + 8q\frac{J_4^3 J_8^2}{J_2^2} \right).$$

This competes the proof of Theorem 3.2. □

Theorem 3.3. *We have*

$$\begin{aligned} \mathrm{C}\Phi_5(q) = \frac{1}{J_1^5}\Bigg(&\frac{J_2^8 J_4 J_{20}^5}{J_1^4 J_8^2 J_{10}^2 J_{40}^2} + 8q\frac{J_2^6 J_{10}^2}{J_1^3 J_5} + 8q\frac{J_4^3 J_8^2 J_{20}^5}{J_2^2 J_{10}^2 J_{40}^2} \\ &+ 8q^3\frac{J_4^9 J_{40}^2}{J_2^4 J_8^2 J_{20}} + 4q^3\frac{J_2^{10} J_8^2 J_{40}^2}{J_1^4 J_4^5 J_{20}} \Bigg). \end{aligned} \tag{3.4}$$

Kolitsch [15] deduced the following integral expression of $\mathrm{C}\Phi_5(q)$:

$$\mathrm{C}\Phi_5(q) = \frac{1}{J_5} + 25q\frac{J_5^5}{J_1^6}. \tag{3.5}$$

Chan et al. [7] also established (3.5) by using the theory of modular forms.

Proof. According to (1.1), we have

$$\mathrm{C}\Phi_5(q) = \frac{1}{J_1^5} \sum_{m_1,m_2,m_3,m_4=-\infty}^{\infty} q^{\sum_{1\le i\le 4} m_i^2 + \sum_{1\le i<j\le 4} m_i m_j}.$$

This time we choose the following matrix:

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}.$$

Following a similar strategy as above, one can obtain (3.4) immediately. □

Theorem 3.4. *We have*

$$\mathrm{C}\Phi_6(q) = \frac{1}{J_1^6}\left(a(q^2)\frac{J_2^{15}}{J_1^6 J_4^6} + 24q\frac{J_2^5 J_4^2 J_6^2}{J_1^3 J_3} \right), \tag{3.6}$$

where $a(q)$ is defined as in (2.9).

The identity (3.6) was established by the authors [9, Eq. (3.8)] in a different way. Hirschhorn [11] derived another integral expression of $\mathrm{C}\Phi_6(q)$.

Proof. By (1.1), we have

$$\mathrm{C}\Phi_6(q) = \frac{1}{J_1^6} \sum_{m_1,m_2,\ldots,m_5=-\infty}^{\infty} q^{\sum_{1\le i\le 5} m_i^2 + \sum_{1\le i<j\le 5} m_i m_j}.$$

Next, we choose the following matrix:

$$B = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Upon simplification, we further deduce that

$$\begin{aligned} \mathrm{C}\Phi_6(q) = \frac{1}{J_1^6}\big\{ & a(q^2)\varphi^3(q) + 16q\varphi(q)\psi(q)\psi^2(q^2)\psi(q^3) \\ & + 8q\varphi(q)\psi^2(q^2)\big(\varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12})\big)\big\}. \end{aligned} \tag{3.7}$$

Thanks to [4, p. 69, Eq. (36.8)],

$$\varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}) = \psi(q)\psi(q^3). \tag{3.8}$$

Plugging (3.8) into (3.7), we conclude that

$$\begin{aligned}\mathrm{C}\Phi_6(q) &= \frac{1}{J_1^6}\big(a(q^2)\varphi^3(q) + 16q\varphi(q)\psi(q)\psi^2(q^2)\psi(q^3) \\ &\quad + 8q\varphi(q)\psi(q)\psi^2(q^2)\psi(q^3)\big) \\ &= \frac{1}{J_1^6}\left(a(q^2)\frac{J_2^{15}}{J_1^6J_4^6} + 24q\frac{J_2^5J_4^2J_6^2}{J_1^3J_3}\right). \end{aligned} \tag{3.9}$$

Thus, we complete the proof of Theorem 3.4. □

Theorem 3.5. *We have*

$$\begin{aligned}\mathrm{C}\Phi_7(q) &= \frac{1}{J_1^7}\left(a(q^2)\frac{J_2^{15}J_{42}^5}{J_1^6J_4^6J_{21}^2J_{84}^2} + 24q\frac{J_2^5J_4^2J_6^2J_{42}^5}{J_1^3J_3J_{21}^2J_{84}^2} + 12q\frac{J_2^{12}J_3^2J_{28}J_{42}^2}{J_1^6J_4^2J_6J_{14}J_{84}}\right. \\ &\quad + 48q^2\frac{J_4^6J_6^3J_{28}J_{42}^2}{J_2^4J_{14}J_{84}} + 24q^3\frac{J_2^6J_3^2J_4^2J_{14}^2J_{21}J_{84}}{J_1^4J_6J_7J_{28}J_{42}} + 6q^3\frac{J_2^{14}J_6^3J_{14}^2J_{21}J_{84}}{J_1^6J_4^6J_7J_{28}J_{42}} \\ &\quad \left. + 24q^6\frac{J_2^{11}J_6^2J_{84}^2}{J_1^5J_3J_4^2J_{42}} + 16q^6a(q^2)\frac{J_4^6J_{84}^2}{J_2^3J_{42}}\right), \end{aligned} \tag{3.10}$$

where $a(q)$ is defined as in (2.9).

Kolitsch [15] established the following integral expression of $\mathrm{C}\Phi_7(q)$:

$$\mathrm{C}\Phi_7(q) = \frac{1}{J_7} + 49q\frac{J_7^3}{J_1^4} + 343q^2\frac{J_7^7}{J_1^8}. \tag{3.11}$$

Chan et al. [7, Eq. (3.10)] also obtained (3.11) by employing the theory of modular forms.

Proof. In this case, we adopt the following matrix:

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

According to integer matrix exact covering systems, we deduce that

$$\begin{aligned}\mathrm{C}\Phi_7(q) &= \frac{1}{J_1^7}\sum_{m_1,m_2,\ldots,m_6=-\infty}^{\infty} q^{\sum_{1\le i\le 6} m_i^2 + \sum_{1\le i<j\le 6} m_im_j} \\ &= \frac{1}{J_1^7}\left\{\varphi^3(q)\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_3(n_1,n_2,n_3)}\right.\end{aligned}$$

$$
\begin{aligned}
&+ 6q\varphi^2(q)\psi(q^2) \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_4(n_1,n_2,n_3)} \\
&+ 12q^3\varphi(q)\psi^2(q^2) \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_5(n_1,n_2,n_3)} \\
&+ 8q^6\psi^3(q^2) \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_6(n_1,n_2,n_3)} \Bigg\},
\end{aligned} \tag{3.12}
$$

where

$$
\begin{aligned}
Q_3(n_1,n_2,n_3) &= 3\sum_{i=1}^{3} n_i^2 + 4\sum_{1\le i<j\le 3} n_in_j, \\
Q_4(n_1,n_2,n_3) &= 3\sum_{i=1}^{3} n_i^2 + 4\sum_{1\le i<j\le 3} n_in_j + 2n_1 + 2n_2 + 3n_3, \\
Q_5(n_1.n_2,n_3) &= 3\sum_{i=1}^{3} n_i^2 + 4\sum_{1\le i<j\le 3} n_in_j + 4n_1 + 5n_2 + 5n_3, \\
Q_6(n_1,n_2,n_3) &= 3\sum_{i=1}^{3} n_i^2 + 4\sum_{1\le i<j\le 3} n_in_j + 7n_1 + 7n_2 + 7n_3.
\end{aligned}
$$

In order to simplify the above q-series triple sums in (3.12), we choose the following matrix:

$$
B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}.
$$

Based on integer matrix exact covering systems, we deduce that

$$
\begin{aligned}
\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_3(n_1,n_2,n_3)} &= a(q^2)\varphi(q^{21}) + 2q^3\big(\varphi(q^2)\overline{J}_{2,12} + 2\psi(q^4)\overline{J}_{4,12}\big)\overline{J}_{7,42} \\
&= a(q^2)\varphi(q^{21}) + 6q^3\frac{J_6^3}{J_2}\overline{J}_{7,42} \\
&= a(q^2)\frac{J_{42}^5}{J_{21}^2J_{84}^2} + 6q^3\frac{J_6^3J_{14}^2J_{21}J_{84}}{J_2J_7J_{28}J_{42}},
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_4(n_1,n_2,n_3)} &= 2\psi(q)\big(\overline{J}_{5,12} + q\overline{J}_{1,12}\big)\overline{J}_{14,42} + 4q^5\psi(q)\psi(q^3)\psi(q^{42}) \\
&= 2\psi(q)\overline{J}_{1,3}\overline{J}_{14,42} + 4q^5\psi(q)\psi(q^3)\psi(q^{42})
\end{aligned}
$$

$$= 2\frac{J_2^3J_3^2J_{28}J_{42}^2}{J_1^2J_6J_{14}J_{84}} + 4q^5\frac{J_2^2J_6^2J_{84}^2}{J_1J_3J_{42}}, \tag{3.14}$$

$$\begin{aligned}
\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_5(n_1,n_2,n_3)} &= 2q^{-2}\varphi(q^{21})\big(\varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12})\big)\\
&\quad + 2\big(\varphi(q^2)\overline{J}_{4,12} + 2q\psi(q^4)\overline{J}_{2,12}\big)\overline{J}_{7,42}\\
&= 2q^{-2}\varphi(q^{21})\psi(q)\psi(q^3) + 2\psi(q)\overline{J}_{1,3}\overline{J}_{7,42}\\
&= 2q^{-2}\frac{J_2^2J_6^2J_{42}^5}{J_1J_3J_{21}^2J_{84}^2} + 2\frac{J_2^3J_3^2J_{14}^2J_{21}J_{84}}{J_1^2J_6J_7J_{28}J_{42}},
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{Q_6(n_1,n_2,n_3)} &= 2q^{-4}\big(\varphi(q^2)\overline{J}_{2,12} + 2\psi(q^4)\overline{J}_{4,12}\big)\overline{J}_{14,42} + 2a(q^2)\psi(q^{42})\\
&= 6q^{-4}\frac{J_6^3}{J_2}\overline{J}_{14,42} + 2a(q^2)\psi(q^{42})\\
&= 6q^{-4}\frac{J_6^3J_{28}J_{42}^2}{J_2J_{14}J_{84}} + 2a(q^2)\frac{J_{84}^2}{J_{42}}.
\end{aligned} \tag{3.16}$$

The derivations of the penultimate steps in (3.13)–(3.16) are established as follows.

i. For (3.13), the result follows directly from the identity [9, p. 17]:

$$\varphi(q)\overline{J}_{1,6} + 2\psi(q^2)\overline{J}_{2,6} = 3\frac{J_3^3}{J_1}. \tag{3.17}$$

ii. The penultimate step in (3.14) is an immediate consequence of (2.16).

iii. In (3.15), the transition relies on both (3.8) and the auxiliary identity

$$\varphi(q^2)\overline{J}_{4,12} + 2q\psi(q^4)\overline{J}_{2,12} = \psi(q)\overline{J}_{1,3}.$$

iv. The penultimate step in (3.16) follows from (3.17).

Now, (3.10) follows by substituting (3.13)–(3.16) into (3.12).

The proof of Theorem 3.5 is complete. □

Remark 3.6. While one might anticipate deriving a single matrix B to directly reduce $\mathrm{C}\Phi_7(q)$ to linear combinations of the product of theta functions $\varphi(q)$ and $\psi(q)$, the associated computational complexity proves prohibitively extensive. To address this challenge, we adopt a staged computational strategy, employing two judiciously selected matrices to partition the problem into tractable components. This approach not only mitigates computational burdens but also establishes a replicable framework that will be systematically applied throughout this study.

Theorem 3.7. *We have*

$$\mathrm{C}\Phi_8(q)=\frac{1}{J_1^8}\left(\frac{J_2^{16}J_8}{J_1^8J_{16}^2}+48q\frac{J_2^{11}J_4^2}{J_1^6}+8q^2\frac{J_2^{20}J_8^3J_{16}^2}{J_1^8J_4^{10}}\right.$$
$$\left.+64q^2\frac{J_4^4J_8^9}{J_2^4J_{16}^2}+32q^2\frac{J_4^{18}J_{16}^2}{J_2^8J_8^5}\right). \tag{3.18}$$

Proof. According to (2.5) and (2.7), we find that

$$\mathrm{CT}_z\big(f_8(z)\big)=\mathrm{CT}_z\big(f_2(z)^4\big)$$
$$=\frac{J_4^{20}}{J_2^8J_8^8}\sum_{n_1,n_2,n_3=-\infty}^{\infty}q^{4\sum_{i=1}^3 n_i^2+4\sum_{1\le i<j\le 3}n_in_j}$$
$$+24q^2\frac{J_4^8}{J_2^4}\sum_{n_1,n_2,n_3=-\infty}^{\infty}q^{Q_7(n_1,n_2,n_3)}+16q^4\frac{J_8^8}{J_4^4}\sum_{n_1,n_2,n_3=-\infty}^{\infty}q^{Q_8(n_1,n_2,n_3)}, \tag{3.19}$$

where

$$Q_7(n_1,n_2,n_3)=4\sum_{i=1}^3 n_i^2+4\sum_{1\le i<j\le 3}n_in_j+2n_1+2n_2+4n_3,$$
$$Q_8(n_1,n_2,n_3)=4\sum_{i=1}^3 n_i^2+4\sum_{1\le i<j\le 3}n_in_j+8n_1+8n_2+8n_3+4.$$

Based on (1.1) and (3.3), one readily sees that

$$\sum_{n_1,n_2,n_3=-\infty}^{\infty}q^{2\sum_{i=1}^3 n_i^2+2\sum_{1\le i<j\le 3}n_in_j}=\frac{J_4^8J_8}{J_2^4J_{16}^2}+8q^2\frac{J_8^3J_{16}^2}{J_4^2}. \tag{3.20}$$

Next, we adopt the following matrix to simplify the rest two q-series triple sums:

$$B=\begin{pmatrix}1&1&0\\-1&1&0\\0&-1&1\end{pmatrix}.$$

After simplification, we further obtain that

$$\sum_{n_1,n_2,n_3=-\infty}^{\infty}q^{Q_7(n_1,n_2,n_3)/2}=2\frac{J_4^4}{J_2^2}\left(\frac{J_8^5}{J_4^2J_{16}^2}+2q\frac{J_{16}^2}{J_8}\right)=2\frac{J_2^3J_4^2}{J_1^2}, \tag{3.21}$$
$$\sum_{n_1,n_2,n_3=-\infty}^{\infty}q^{Q_8(n_1,n_2,n_3)/2}=2\frac{J_4^{10}J_{16}^2}{J_2^4J_8^5}+4\frac{J_8^9}{J_4^4J_{16}^2}, \tag{3.22}$$

where the last step in (3.21) follows from (2.12).

Substituting (3.20)–(3.22) into (3.19) and applying (2.6), we obtain (3.18).

This finishes the proof of Theorem 3.7. □

For $C\Phi_9(q)$, an effective method is to utilize (2.6) and (2.8) for calculations, but this strategy has already been given in [9]. If we immediately apply the method of integer matrix exact covering systems to (1.1), the process will become exceptionally complicated and lengthy. Thus, we omit this case here.

Theorem 3.8. *We have*

$$C\Phi_{10}(q) = \frac{1}{J_1^{10}}\Bigg(\frac{J_2^{30}}{J_1^{10}J_4^{10}J_{10}} + 120q\frac{J_2^{16}J_{10}^5}{J_1^8 J_5^2 J_{20}^2} - 40q\frac{J_1^4 J_4^6 J_5^2}{J_2^2 J_{10}} + 80q^2\frac{J_2^{22}J_{20}^2}{J_1^{10}J_4^4 J_{10}} + 25q^2\frac{J_2^{24}J_{10}^5}{J_1^{10}J_4^{10}} + 640q^3\frac{J_4^{12}J_{20}^2}{J_1^2 J_2^2 J_{10}}\Bigg). \tag{3.23}$$

Proof. In view of (2.7), we obtain that

$$\begin{aligned}\mathrm{CT}_z\big(f_{10}(z)\big) &= \mathrm{CT}_z\big(f_2(z)^5\big)\\ &= \frac{J_4^{25}}{J_2^{10}J_8^{10}}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{4\sum_{i=1}^4 n_i^2 + 4\sum_{1\le i<j\le 4} n_i n_j}\\ &\quad + 40q^2\frac{J_4^{13}}{J_2^6 J_8^2}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_1(n_1,n_2,n_3,n_4)}\\ &\quad + 80q^4\frac{J_4 J_8^6}{J_2^2}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_2(n_1,n_2,n_3,n_4)},\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}Q_1(n_1,n_2,n_3,n_4) &= 4\sum_{i=1}^4 n_i^2 + 4\sum_{1\le i<j\le 4} n_i n_j + 4n_1 + 4n_2 + 6n_3 + 6n_4 + 2,\\ Q_2(n_1,n_2,n_3,n_4) &= 4\sum_{i=1}^4 n_i^2 + 4\sum_{1\le i<j\le 4} n_i n_j\\ &\quad + 10n_1 + 10n_2 + 10n_3 + 10n_4 + 8.\end{aligned}$$

According to (1.1) and (3.5),

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{2\sum_{i=1}^4 n_i^2 + 2\sum_{1\le i<j\le 4} n_i n_j} = \frac{J_2^5}{J_{10}} + 25q^2\frac{J_{10}^5}{J_2}. \tag{3.25}$$

We next turn to simplify the rest two q-series quadruple sums in (3.24). Here we adopt the following matrix:

$$B=\begin{pmatrix}1 & 1 & 1 & 0\\ 1 & -1 & 1 & 0\\ 1 & 0 & -1 & -1\\ 1 & 0 & -1 & 1\end{pmatrix}.$$

Upon simplification, we obtain that

$$\begin{aligned}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} & q^{Q_1(n_1,n_2,n_3,n_4)/2}\\ &= 2\psi^2(q^2)\big(\varphi(q^{20})+2q^5\psi(q^{40})\big)\big(\varphi(q^4)+2q\psi(q^8)\big)\\ &\quad + 2q\psi(q^2)\psi(q^{10})\big(\varphi^2(q^2)+4q\psi^2(q^4)\big)\\ &= 2\varphi(q)\varphi(q^5)\psi^2(q^2)+2q\varphi^2(q)\psi(q^2)\psi(q^{10})\\ &= 2\frac{J_2^3J_4^2J_{10}^5}{J_1^2J_5^2J_{20}^2}+2q\frac{J_2^9J_{20}^2}{J_1^4J_4^2J_{10}},\end{aligned}\tag{3.26}$$

$$\begin{aligned}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} & q^{Q_2(n_1,n_2,n_3,n_4)/2}\\ &= 2\varphi^2(q^2)\varphi(q^{20})\psi(q^8)+4\varphi(q^4)\varphi(q^{20})\psi^2(q^4)+8q\psi^3(q^2)\psi(q^{10})\\ &\quad + 2q^4\varphi^2(q^2)\varphi(q^4)\psi(q^{40})+16q^6\psi^2(q^4)\psi(q^8)\psi(q^{40}),\end{aligned}\tag{3.27}$$

where the penultimate step in (3.26) follows by using (2.12) and (2.14). We further simplify the right-hand side of (3.27). Based on (2.12)–(2.15), we find that

$$\begin{aligned}&2\varphi^2(q^2)\varphi(q^{20})\psi(q^8)+4\varphi(q^4)\varphi(q^{20})\psi^2(q^4)+8q\psi^3(q^2)\psi(q^{10})\\ &\quad + 2q^4\varphi^2(q^2)\varphi(q^4)\psi(q^{40})+16q^6\psi^2(q^4)\psi(q^8)\psi(q^{40})\\ &\quad= \frac{1}{8}q^{-1}\big(\varphi(q)+\varphi(-q)\big)\big(\varphi(q^5)+\varphi(-q^5)\big)\big(\varphi^2(q)-\varphi^2(-q)\big)\\ &\qquad + \frac{1}{8}q^{-1}\big(\varphi^2(q)+\varphi^2(-q)\big)\big(\varphi(q)+\varphi(-q)\big)\big(\varphi(q^5)-\varphi(-q^5)\big)\\ &\qquad + \frac{1}{8}q^{-1}\big(\varphi^2(q)+\varphi^2(-q)\big)\big(\varphi(q)-\varphi(-q)\big)\big(\varphi(q^5)+\varphi(-q^5)\big)\\ &\qquad + \frac{1}{8}q^{-1}\big(\varphi^2(q)-\varphi^2(-q)\big)\big(\varphi(q)-\varphi(-q)\big)\big(\varphi(q^5)-\varphi(-q^5)\big)\\ &\qquad + 8q\psi^3(q^2)\psi(q^{10})\\ &\quad= \frac{1}{2}q^{-1}\big(\varphi^3(q)\varphi(q^5)-\varphi^3(-q)\varphi(-q^5)\big)+8q\psi^3(q^2)\psi(q^{10})\end{aligned}$$

$$= \frac{1}{2}q^{-1}\frac{J_2^{15}J_{10}^5}{J_1^6J_4^6J_5^2J_{20}^2} - \frac{1}{2}q^{-1}\frac{J_1^6J_5^2}{J_2^3J_{10}} + 8q\frac{J_4^6J_{20}^2}{J_2^3J_{10}}.$$

That is,

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_2(n_1,n_2,n_3,n_4)/2} = \frac{1}{2}q^{-1}\frac{J_2^{15}J_{10}^5}{J_1^6J_4^6J_5^2J_{20}^2} - \frac{1}{2}q^{-1}\frac{J_1^6J_5^2}{J_2^3J_{10}} + 8q\frac{J_4^6J_{20}^2}{J_2^3J_{10}}. \tag{3.28}$$

Substituting (3.25) and (3.28) into (3.24) and applying (2.6), we obtain (3.23).

We therefore complete the proof of Theorem 3.8. □

While the application of integer matrix exact covering systems could theoretically yield an integral expression of $\mathrm{C}\Phi_{11}(q)$, such an approach would entail prohibitively complex computations due to the exponential growth in algorithmic intricacy and the emergence of dozens of non-trivial multisums in the resultant expression. In light of these computational challenges, we strategically omit this case from our current analysis. This methodological decision extends systematically to the generating functions $\mathrm{C}\Phi_{13}(q)$, $\mathrm{C}\Phi_{17}(q)$ and $\mathrm{C}\Phi_{19}(q)$, where analogous computational barriers would be encountered.

Theorem 3.9. *We have*

$$\begin{aligned}\mathrm{C}\Phi_{12}(q) = \frac{1}{J_1^{12}}\Bigg(& a(q^4)\frac{J_2^{24}J_4^3}{J_1^{12}J_8^6} + 120q\frac{J_2^{19}J_4^3J_6^5}{J_1^{10}J_3^2J_8^2J_{12}^2} + 480q^2\frac{J_4^{21}J_{12}^2}{J_1^4J_2J_6J_8^6}\\ &+ 264q^2\frac{J_2^{27}J_8^2J_{12}^2}{J_1^{12}J_4^7J_6} + 960q^2a(q^4)\frac{J_2^4J_4^7J_8^2}{J_1^4} + 512q^3a(q^4)\frac{J_4^9J_8^6}{J_2^6}\\ &+ 768q^3\frac{J_4^{23}J_{12}^2}{J_2^{11}J_6J_8^2} + 3840q^3\frac{J_2^3J_4^9J_8^2J_{12}^2}{J_1^4J_6}\Bigg),\end{aligned} \tag{3.29}$$

where $a(q)$ is defined as in (2.9).

Employing the method of integer matrix exact covering systems, the authors [8, Theorem 1.1] established another integral expression of $\mathrm{C}\Phi_{12}(q)$.

Proof. Applying (2.5) and (2.7) yields

$$\begin{aligned}\mathrm{CT}_z\big(f_{12}(z)\big) &= \mathrm{CT}_z\big(f_2(z)^6\big)\\ &= \frac{J_4^{30}}{J_2^{12}J_8^{12}}\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{4\sum_{i=1}^5 n_i^2+4\sum_{1\le i<j\le 5} n_in_j}\\ &\quad + 60q^2\frac{J_4^{18}}{J_2^8J_8^4}\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_1(n_1,n_2,\ldots,n_5)}\end{aligned}$$

$$+ 240q^4 \frac{J_4^6 J_8^4}{J_2^4} \sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_2(n_1,n_2,\ldots,n_5)}$$
$$+ 64q^6 \frac{J_8^{12}}{J_4^6} \sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_3(n_1,n_2,\ldots,n_5)}, \tag{3.30}$$

where

$$Q_1(n_1,n_2,\ldots,n_5) = 4\sum_{i=1}^{5} n_i^2 + 4\sum_{1\le i<j\le 5} n_in_j + 2n_1 + 2n_2 + 2n_3 + 2n_4 + 4n_5,$$
$$Q_2(n_1,n_2,\ldots,n_5) = 4\sum_{i=1}^{5} n_i^2 + 4\sum_{1\le i<j\le 5} n_in_j + 6n_1 + 6n_2 + 8n_3 + 8n_4 + 8n_5 + 4,$$
$$Q_3(n_1,n_2,\ldots,n_5) = 4\sum_{i=1}^{5} n_i^2 + 4\sum_{1\le i<j\le 5} n_in_j + 12n_1 + 12n_2 + 12n_3 + 12n_4 + 12n_5 + 12.$$

By (1.1) and (3.9), we deduce that

$$\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{2\sum_{1\le i\le 5} n_i^2 + 2\sum_{1\le i<j\le 5} n_in_j} = a(q^4)\frac{J_4^{15}}{J_2^6 J_8^6} + 24q^2 \frac{J_4^5 J_8^2 J_{12}^2}{J_2^3 J_6}. \tag{3.31}$$

Next, we turn to simplify another q-series quintuple sums in (3.30). For this purpose, we choose the following matrix:

$$B = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

After simplification, we obtain that

$$\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_1(n_1,n_2,\ldots,n_5)/2}$$
$$= 2\varphi(q^2)\psi^2(q)\psi(q^2)\big(\varphi(q^{12}) + 2q^3\psi(q^{24})\big)$$
$$+ 4q\psi^2(q)\psi(q^4)\psi(q^6)\big(\varphi(q^4) + 2q\psi(q^8)\big)$$
$$= 2\varphi(q^2)\varphi(q^3)\psi^2(q)\psi(q^2) + 4q\varphi(q)\psi^2(q)\psi(q^4)\psi(q^6)$$

$$= 2\frac{J_2 J_4^7 J_6^5}{J_1^2 J_3^2 J_8^2 J_{12}^2} + 4q\frac{J_2^9 J_8^2 J_{12}^2}{J_1^4 J_4^3 J_6}, \tag{3.32}$$

$$\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_2(n_1,n_2,\ldots,n_5)/2}$$
$$= 4a(q^4)\varphi(q^2)\psi^2(q^4) + 16q\varphi(q^2)\psi(q^2)\psi^2(q^4)\psi(q^6)$$
$$+ 2\varphi^3(q^2)\big(\varphi(q^{12})\psi(q^8) + q^2\varphi(q^4)\psi(q^{24})\big)$$
$$= 4a(q^4)\varphi(q^2)\psi^2(q^4) + 16q\varphi(q^2)\psi(q^2)\psi^2(q^4)\psi(q^6)$$
$$+ 2\varphi^3(q^2)\psi(q^2)\psi(q^6)$$
$$= 2\frac{J_4^{17} J_{12}^2}{J_2^7 J_6 J_8^6} + 4a(q^4)\frac{J_4^3 J_8^2}{J_2^2} + 16q\frac{J_4^5 J_8^2 J_{12}^2}{J_2^3 J_6}, \tag{3.33}$$

where the penultimate steps in (3.32) and (3.33) follow from (2.12) and (3.8), respectively. Similar to the proof of (3.33), one can obtain that

$$\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_3(n_1,n_2,\ldots,n_5)/2} = 12\frac{J_4^{11} J_{12}^2}{J_2^5 J_6 J_8^2} + 8a(q^4)\frac{J_8^6}{J_4^3}. \tag{3.34}$$

Now, (3.29) follows by substituting (3.31)–(3.34) into (3.30) and applying (2.6).

Therefore, we complete the proof of Theorem 3.9. □

Theorem 3.10. *We have*

$$\begin{aligned}
\mathrm{C}\Phi_{14}(q) = \frac{1}{J_1^{14}}\bigg(&\frac{J_2^{42}}{J_1^{14} J_4^{14} J_{14}} + 84q\frac{J_2^{23} J_7}{J_1^5 J_4^6} + 84q\frac{J_2^{39} J_6^2 J_{14} J_{21}^2}{J_1^{17} J_4^{11} J_7 J_{12} J_{42}} \\
&- 448q^2\frac{J_1^3 J_4^{10} J_7}{J_2} + 448q^2\frac{J_2^{15} J_4^5 J_6^2 J_{14} J_{21}^2}{J_1^9 J_7 J_{12} J_{42}} + 3360q^2\frac{J_2^{17} J_3^3 J_4^2 J_{14} J_{21}^2}{J_1^{10} J_7 J_{42}} \\
&+ 672q^2\frac{J_2^{24} J_3 J_{12} J_{14} J_{21}^2}{J_1^{12} J_4 J_6 J_7 J_{42}} + 49q^2\frac{J_2^{38} J_{14}^3}{J_1^{14} J_4^{14}} + 3584q^3\frac{J_3 J_4^{15} J_{12} J_{14} J_{21}^2}{J_1^4 J_6 J_7 J_{42}} \\
&+ 1792q^4\frac{J_2^{22} J_6^5 J_{42}^2}{J_1^{11} J_3^2 J_{12}^2 J_{21}} + 343q^4\frac{J_2^{34} J_{14}^7}{J_1^{14} J_4^{14}} + 168q^4\frac{J_2^{40} J_{12}^2 J_{42}^2}{J_1^{17} J_4^{12} J_6 J_{21}} \\
&+ 3584q^5\frac{J_4^{16} J_6^5 J_{42}^2}{J_1^3 J_2^2 J_3^2 J_{12}^2 J_{21}} + 5376q^5\frac{J_2^{16} J_4^4 J_{12}^2 J_{42}^2}{J_1^9 J_6 J_{21}}\bigg).
\end{aligned} \tag{3.35}$$

Proof. Combining (2.5) and (2.7) yields

$$\mathrm{CT}_z\big(f_{14}(z)\big) = \mathrm{CT}_z\big(f_2(z)^7\big)$$
$$= \frac{J_4^{35}}{J_2^{14} J_8^{14}} \sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{4\sum_{i=1}^{6} n_i^2 + 4\sum_{1\le i<j\le 6} n_i n_j}$$

$$+ 84q^2\frac{J_4^{23}}{J_2^{10}J_8^6}\sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{Q_1(n_1,n_2,\ldots,n_6)}$$

$$+ 560q^4\frac{J_4^{11}J_8^2}{J_2^6}\sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{Q_2(n_1,n_2,\ldots,n_6)}$$

$$+ 448q^6\frac{J_8^{10}}{J_2^2J_4}\sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{Q_3(n_1,n_2,\ldots,n_6)}, \tag{3.36}$$

where

$$\begin{aligned} Q_1(n_1,n_2,\ldots,n_6) &= 4\sum_{i=1}^{6} n_i^2 + 4\sum_{1\le i<j\le 6} n_in_j \\ &\quad + 4n_1+4n_2+4n_3+4n_4+6n_5+6n_6+2, \\ Q_2(n_1,n_2,\ldots,n_6) &= 4\sum_{i=1}^{6} n_i^2 + 4\sum_{1\le i<j\le 6} n_in_j \\ &\quad + 8n_1+8n_2+10n_3+10n_4+10n_5+10n_6+8, \\ Q_3(n_1,n_2,\ldots,n_6) &= 4\sum_{i=1}^{6} n_i^2 + 4\sum_{1\le i<j\le 6} n_in_j \\ &\quad + 14n_1+14n_2+14n_3+14n_4+14n_5+14n_6+18. \end{aligned}$$

According to (1.1) and (3.11), we find that

$$\sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{2\sum_{i=1}^{6} n_i^2+2\sum_{1\le i<j\le 6} n_in_j} = \frac{J_2^7}{J_{14}} + 49q^2J_2^3J_{14}^3 + 343q^4\frac{J_{14}^7}{J_2}. \tag{3.37}$$

To simplify the remaining three q-series sextuple sums in (3.36), we adopt the following matrix:

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -4 \\ -1 & 1 & 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & -1 & 1 & 0 & 3 \\ 0 & 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & -1 & -1 & 3 \end{pmatrix}.$$

In view of integer matrix exact covering systems, we obtain that

$$\sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{Q_1(n_1,n_2,\ldots,n_6)/2}$$

$$
\begin{aligned}
&= \varphi^2(q^2)\psi(q)\big\{2\varphi(q^4)\overline{J}_{10,24}\overline{J}_{35,84} + 2q^9\varphi(q^4)\overline{J}_{2,24}\overline{J}_{7,84}\\
&\qquad + 4q^3\psi(q^8)\overline{J}_{2,24}\overline{J}_{35,84} + 4q^8\psi(q^8)\overline{J}_{10,24}\overline{J}_{7,84}\\
&\qquad + 2q^3\psi(q^6)\psi(q^{21})\big(\varphi(q^4) + 2q\psi(q^8)\big)\big\}\\
&\quad + 4q^3\psi(q)\psi^2(q^2)\big\{\psi(q^{21})\big(\overline{J}_{3,8}\overline{J}_{9,24} + q^2\overline{J}_{1,8}\overline{J}_{3,24}\big)\\
&\qquad + q^{-2}\big(\overline{J}_{3,8}\overline{J}_{7,24} + q\overline{J}_{1,8}\overline{J}_{5,24}\big)\overline{J}_{35,84}\\
&\qquad + q^5\big(q^2\overline{J}_{3,8}\overline{J}_{1,24} + \overline{J}_{1,8}\overline{J}_{11,24}\big)\overline{J}_{7,84}\\
&\qquad + \psi(q^2)\psi(q^{21})\big(\varphi(q^{12}) + 2q^3\psi(q^{24})\big)\\
&\qquad + q^{-2}\psi(q^2)\overline{J}_{8,24}\big(\overline{J}_{8,24} + q\overline{J}_{4,24}\big)\big(\overline{J}_{35,84} + q^7\overline{J}_{7,84}\big)\big\}\\
&\quad + 4q^8\psi(q)\psi^2(q^4)\big\{\varphi(q^4)\overline{J}_{10,24}\overline{J}_{7,84} + q^{-5}\varphi(q^4)\overline{J}_{2,24}\overline{J}_{35,84}\\
&\qquad + 2q^{-6}\psi(q^8)\overline{J}_{10,24}\overline{J}_{35,84} + 2q^3\psi(q^8)\overline{J}_{2,24}\overline{J}_{7,84}\\
&\qquad + q^{-4}\psi(q^6)\psi(q^{21})\big(\varphi(q^4) + 2q\psi(q^8)\big)\\
&\qquad + q^{-4}\psi(q^{21})\big(q\overline{J}_{3,8}\overline{J}_{3,24} + \overline{J}_{1,8}\overline{J}_{9,24}\big)\\
&\qquad + q^{-6}\big(\overline{J}_{3,8}\overline{J}_{5,24} + \overline{J}_{1,8}\overline{J}_{7,24}\big)\overline{J}_{35,84}\\
&\qquad + \big(\overline{J}_{3,8}\overline{J}_{11,24} + q^3\overline{J}_{1,8}\overline{J}_{1,24}\big)\overline{J}_{7,84}\big\}\\
&= \varphi^2(q^2)\psi(q)\big\{2\varphi(q^4)\overline{J}_{10,24}\overline{J}_{35,84} + 2q^9\varphi(q^4)\overline{J}_{2,24}\overline{J}_{7,84}\\
&\qquad + 4q^3\psi(q^8)\overline{J}_{2,24}\overline{J}_{35,84} + 4q^8\psi(q^8)\overline{J}_{10,24}\overline{J}_{7,84}\\
&\qquad + 2q^3\varphi(q)\psi(q^6)\psi(q^{21})\big\}\\
&\qquad + 8q^3\psi(q)\psi^2(q^2)\big\{q^{-2}\psi(q^2)\overline{J}_{1,6}\overline{J}_{7,21} + \psi(q^2)\psi(q^{21})\varphi(q^3)\big\}\\
&\qquad + 8q^8\psi(q)\psi^2(q^4)\big\{\varphi(q^4)\overline{J}_{10,24}\overline{J}_{7,84} + q^{-5}\varphi(q^4)\overline{J}_{2,24}\overline{J}_{35,84}\\
&\qquad + 2q^{-6}\psi(q^8)\overline{J}_{10,24}\overline{J}_{35,84} + 2q^3\psi(q^8)\overline{J}_{2,24}\overline{J}_{7,84}\\
&\qquad + q^{-4}\varphi(q)\psi(q^6)\psi(q^{21})\big\},
\end{aligned}
\tag{3.38}
$$

where the penultimate step in (3.38) follows by utlizing (2.12) and a MAPLE package **thetaids** due to Frye and Garvan [10]. Next, by (2.14), we further deduce that

$$
\begin{aligned}
&2\varphi(q^4)\overline{J}_{10,24}\overline{J}_{35,84} + 2q^9\varphi(q^4)\overline{J}_{2,24}\overline{J}_{7,84} + 4q^3\psi(q^8)\overline{J}_{2,24}\overline{J}_{35,84}\\
&\quad + 4q^8\psi(q^8)\overline{J}_{10,24}\overline{J}_{7,84} + 2q^3\varphi(q)\psi(q^6)\psi(q^{21})\\
&\qquad = \frac{1}{4}\big(\varphi(q) + \varphi(-q)\big)\big(\overline{J}_{2,6} + J_{2,6}\big)\big(\overline{J}_{7,21} + J_{7,21}\big)\\
&\qquad\quad + \frac{1}{4}\big(\varphi(q) + \varphi(-q)\big)\big(\overline{J}_{2,6} - J_{2,6}\big)\big(\overline{J}_{7,21} - J_{7,21}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\big(\varphi(q)-\varphi(-q)\big)\big(\overline{J}_{2,6}-J_{2,6}\big)\big(\overline{J}_{7,21}+J_{7,21}\big)\\
&+\frac{1}{4}\big(\varphi(q)-\varphi(-q)\big)\big(\overline{J}_{2,6}+J_{2,6}\big)\big(\overline{J}_{7,21}-J_{7,21}\big)+2q^3\varphi(q)\psi(q^6)\psi(q^{21})\\
&=\varphi(-q)J_2J_7+\varphi(q)\overline{J}_{2,6}\overline{J}_{7,21}+2q^3\varphi(q)\psi(q^6)\psi(q^{21}). \qquad (3.39)
\end{aligned}
$$

Similar to (3.39), we obtain that

$$
\begin{aligned}
&\varphi(q^4)\overline{J}_{10,24}\overline{J}_{7,84}+q^{-5}\varphi(q^4)\overline{J}_{2,24}\overline{J}_{35,84}+2q^{-6}\psi(q^8)\overline{J}_{10,24}\overline{J}_{35,84}\\
&\quad+2q^3\psi(q^8)\overline{J}_{2,24}\overline{J}_{7,84}+q^{-4}\varphi(q)\psi(q^6)\psi(q^{21})\\
&\quad=-\frac{1}{2}q^{-7}\varphi(-q)J_2J_7+\frac{1}{2}q^{-7}\varphi(q)\overline{J}_{2,6}\overline{J}_{7,21}+q^{-4}\varphi(q)\psi(q^6)\psi(q^{21}). \qquad (3.40)
\end{aligned}
$$

Substituting (3.39) and (3.40) into (3.38), we find that

$$
\begin{aligned}
&\sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{Q_1(n_1,n_2,\ldots,n_6)/2}\\
&\quad=\varphi(-q)\psi(q)\big(\varphi^2(q^2)-4q\psi^2(q^4)\big)J_2J_7\\
&\qquad+\varphi(q)\psi(q)\big(\varphi(q^2)^2+4q\psi(q^4)^2\big)\overline{J}_{2,6}\overline{J}_{7,21}\\
&\qquad+2q^3\varphi(q)\psi(q)\psi(q^6)\psi(q^{21})\big(\varphi^2(q^2)+4q\psi^2(q^4)\big)\\
&\qquad+8q^3\psi(q)\psi^2(q^2)\big(q^{-2}\psi(q^2)\overline{J}_{1,6}\overline{J}_{7,21}+\varphi(q^3)\psi(q^2)\psi(q^{21})\big)\\
&\quad=\varphi^3(-q)\psi(q)J_2J_7+\varphi^3(q)\psi(q)\overline{J}_{2,6}\overline{J}_{7,21}\\
&\qquad+8q^3\psi(q)\psi^2(q^2)\big(q^{-2}\psi(q^2)\overline{J}_{1,6}\overline{J}_{7,21}+\varphi(q^3)\psi(q^2)\psi(q^{21})\big)\\
&\qquad+2q^3\varphi^3(q)\psi(q)\psi(q^6)\psi(q^{21})\\
&\quad=J_1^5J_7+\frac{J_2^{16}J_6^2J_{14}J_{21}^2}{J_1^7J_4^5J_7J_{12}J_{42}}+8q\frac{J_2J_3J_4^5J_{12}J_{14}J_{21}^2}{J_1^2J_6J_7J_{42}}\\
&\qquad+8q^3\frac{J_4^6J_6^5J_{42}^2}{J_1J_2J_3^2J_{12}^2J_{21}}+2q^3\frac{J_2^{17}J_{12}^2J_{42}^2}{J_1^7J_4^6J_6J_{21}}, \qquad (3.41)
\end{aligned}
$$

where the penultimate step can be obtained by using (2.14) and (2.15). Similarly, we can derive that

$$
\begin{aligned}
&\sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{Q_2(n_1,n_2,\ldots,n_6)/2}\\
&\quad=6\frac{J_2^6J_3^3J_{14}J_{21}^2}{J_1^4J_7J_{42}}+2q^2\frac{J_2^{11}J_6^5J_{42}^2}{J_1^5J_3^2J_4^2J_{12}^2J_{21}}+8q^3\frac{J_2^5J_4^2J_{12}^2J_{42}^2}{J_1^3J_6J_{21}}, \qquad (3.42)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n_1,n_2,\ldots,n_6=-\infty}^{\infty} q^{Q_3(n_1,n_2,\ldots,n_6)/2}\\
&\qquad = -q^{-1}J_1^5J_7+q^{-1}\frac{J_2^{16}J_6^2J_{14}J_{21}^2}{J_1^7J_4^5J_7J_{12}J_{42}}+8\frac{J_2J_3J_4^5J_{12}J_{14}J_{21}^2}{J_1^2J_6J_7J_{42}}\\
&\qquad\quad +8q^2\frac{J_4^6J_6^5J_{42}^2}{J_1J_2J_3^2J_{12}^2J_{21}}+2q^2\frac{J_2^{17}J_{12}^2J_{42}^2}{J_1^7J_4^6J_6J_{21}}.
\end{aligned}\tag{3.43}
$$

Now, (3.35) follows by plugging (3.37), (3.41)–(3.43) into (3.36) and applying (2.6).
This completes the proof of Theorem 3.10. □

Theorem 3.11. *We have*

$$
\begin{aligned}
&\mathrm{C}\Phi_{15}(q)\\
&=\frac{1}{J_1^{15}}\Big\{a^5(q)\frac{J_3^5}{J_{15}}+180qa^3(q)\frac{J_3^9J_4^5J_{60}^5}{J_1^3J_2^2J_8^2J_{30}^2J_{120}^2}+1080q^2a^2(q)\frac{J_3^9J_6J_8J_{12}^3J_{60}^5}{J_1^3J_2J_{24}J_{30}^2J_{120}^2}\\
&\quad+540q^2a^2(q)\frac{J_2J_3^8J_4J_6^5J_{24}J_{60}^5}{J_1^4J_8J_{12}^2J_{30}^2J_{120}^2}+2430q^2a(q)\frac{J_2^2J_3^{15}J_{10}J_{15}^2}{J_1^6J_5J_{30}}\\
&\quad+9720q^3\frac{J_3^{12}J_6^6J_{10}J_{15}^2}{J_1^5J_5J_{30}}+4860q^3a(q)\frac{J_3^{15}J_8^2J_{40}J_{60}^2}{J_1^5J_4J_{20}J_{120}}\\
&\quad+540q^3a^2(q)\frac{J_2^3J_3^{15}J_{30}^2}{J_1^6J_6^3J_{15}}+360q^3a^3(q)\frac{J_2^2J_3^9J_{30}^2}{J_1^4J_{15}}+25q^3a^5(q)\frac{J_{15}^5}{J_3}\\
&\quad+9720q^4\frac{J_3^{15}J_{12}^9J_{40}J_{60}^2}{J_1^5J_6^4J_{20}J_{24}^2J_{120}}+4860q^4\frac{J_3^{11}J_6^{10}J_{24}^2J_{40}J_{60}^2}{J_1^5J_{12}^5J_{20}J_{120}}\\
&\quad+2430q^5\frac{J_3^{11}J_6^8J_{12}J_{20}^2J_{30}J_{120}}{J_1^5J_{10}J_{24}^2J_{40}J_{60}}+2430q^5a(q)\frac{J_3^{15}J_4^5J_{20}^2J_{30}J_{120}}{J_1^5J_2^2J_8^2J_{10}J_{40}J_{60}}\\
&\quad+19440q^8\frac{J_3^{15}J_{12}^3J_{20}^2J_{24}^2J_{30}J_{120}}{J_1^5J_6^2J_{10}J_{40}J_{60}}+1080q^9a^2(q)\frac{J_2^2J_3^8J_6^4J_8J_{12}J_{120}^2}{J_1^4J_4^2J_{24}J_{60}}\\
&\quad+720q^9a^3(q)\frac{J_3^9J_8^2J_{120}^2}{J_1^3J_4J_{60}}+2160q^{10}a^2(q)\frac{J_3^9J_4^3J_6^2J_{24}J_{120}^2}{J_1^3J_2^2J_8J_{60}}\Big\},
\end{aligned}\tag{3.44}
$$

where $a(q)$ is defined as in (2.9).

Proof. Combining (2.5) and (2.8), we obtain that

$$
\begin{aligned}
\mathrm{CT}_z\big(f_{15}(z)\big)&=\mathrm{CT}_z\big(f_3(z)^5\big)\\
&=a^5(q^2)\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{6\sum_{i=1}^4 n_i^2+6\sum_{1\le i<j\le 4}n_in_j}
\end{aligned}
$$

$$+ 180q^2a^3(q^2)\frac{J_6^6}{J_2^2}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_3(n_1,n_2,n_3,n_4)}$$
$$+ 2430q^4a(q^2)\frac{J_6^{12}}{J_2^4}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_4(n_1,n_2,n_3,n_4)}$$
$$+ 540q^6a^2(q^2)\frac{J_6^9}{J_2^3}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_5(n_1,n_2,n_3,n_4)}$$
$$+ 2430q^{10}\frac{J_6^{15}}{J_2^5}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_6(n_1,n_2,n_3,n_4)}, \tag{3.45}$$

where

$$Q_3(n_1,n_2,n_3,n_4) = 6\sum_{i=1}^{4} n_i^2 + 6\sum_{1\le i<j\le 4} n_in_j - 2n_1 + 2n_4,$$
$$Q_4(n_1,n_2,n_3,n_4) = 6\sum_{i=1}^{4} n_i^2 + 6\sum_{1\le i<j\le 4} n_in_j + 2n_1 + 4n_2 + 4n_3,$$
$$Q_5(n_1,n_2,n_3,n_4) = 6\sum_{i=1}^{4} n_i^2 + 6\sum_{1\le i<j\le 4} n_in_j + 6n_1 + 8n_2 + 8n_3 + 8n_4,$$
$$Q_6(n_1,n_2,n_3,n_4) = 6\sum_{i=1}^{4} n_i^2 + 6\sum_{1\le i<j\le 4} n_in_j + 10n_1 + 10n_2 + 10n_3 + 10n_4.$$

First, from (1.1) and (3.5) we have

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{3\sum_{i=1}^{4} n_i^2+3\sum_{1\le i<j\le 4} n_in_j} = \frac{J_3^5}{J_{15}} + 25q^3\frac{J_{15}^5}{J_3}. \tag{3.46}$$

We next simplify the four q-series quadruple sums in (3.45). For this purpose, we choose the following matrix:

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}.$$

Based on (2.12) and (2.13), we obtain that

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_3(n_1,n_2,n_3,n_4)/2}$$

$$
\begin{aligned}
&= \varphi(q^3)\varphi(q^6)\varphi(q^{30})\overline{J}_{1,6} + 4q^2\varphi(q^{30})\psi(q^6)\psi(q^{12})\overline{J}_{2,6} \\
&\quad + 2q^2\psi(q^3)\psi(q^{15})\big(\varphi(q^3)\overline{J}_{2,6} + 2q\psi(q^6)\overline{J}_{1,6}\big) \\
&\quad + 4q^8\varphi(q^6)\psi(q^6)\psi(q^{60})\overline{J}_{2,6} + 4q^9\varphi(q^3)\psi(q^{12})\psi(q^{60})\overline{J}_{1,6} \\
&= \frac{J_2^2J_6^2J_{12}^4J_{60}^5}{J_1J_3J_4J_{24}^2J_{30}^2J_{120}^2} + 4q^2\frac{J_4J_6J_{24}^2J_{60}^5}{J_2J_{30}^2J_{120}^2} + 2q^2\frac{J_2^2J_3^3J_{30}^2}{J_1^2J_{15}} \\
&\quad + 4q^8\frac{J_4J_{12}^6J_{120}^2}{J_2J_6J_{24}^2J_{60}} + 4q^9\frac{J_2^2J_6^4J_{24}^2J_{120}^2}{J_1J_3J_4J_{12}^2J_{60}} \\
&= \frac{J_3^3J_4^5J_{60}^5}{J_1J_2^2J_8^2J_{30}^2J_{120}^2} + 2q^2\frac{J_2^2J_3^3J_{30}^2}{J_1^2J_{15}} + 4q^8\frac{J_3^3J_8^2J_{120}^2}{J_1J_4J_{60}},
\end{aligned} \tag{3.47}
$$

where the penultimate step in (3.47) follows by observing $2\psi(q) = f(1,q)$ and utilizing (2.17) and the last step in (3.47) follows by the package `thetaids`. Similarly, we deduce that

$$
\begin{aligned}
&\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_4(n_1,n_2,n_3,n_4)/2} \\
&= \overline{J}_{1,6}\overline{J}_{2,6}\big(\overline{J}_{5,12} + q\overline{J}_{1,12}\big)\big(\overline{J}_{25,60} + q^5\overline{J}_{5,60}\big) \\
&\quad + q\overline{J}_{2,6}^2\overline{J}_{2,12}\overline{J}_{20,60} + q\overline{J}_{1,6}^2\overline{J}_{4,12}\overline{J}_{20,60} \\
&\quad + q^3\overline{J}_{2,6}^2\overline{J}_{4,12}\overline{J}_{10,60} + q^4\overline{J}_{1,6}^2\overline{J}_{2,12}\overline{J}_{10,60} \\
&= \frac{J_2^2J_3^3J_{10}J_{15}^2}{J_1^2J_5J_{30}} + q\frac{J_4^4J_6^5J_{24}J_{40}J_{60}^2}{J_2^3J_8J_{12}^3J_{20}J_{120}} + q\frac{J_2^4J_3^2J_8J_{12}^4J_{40}J_{60}^2}{J_1^2J_4^3J_6^2J_{20}J_{24}J_{120}} \\
&\quad + q^3\frac{J_4J_6^4J_8J_{20}^2J_{30}J_{120}}{J_2^2J_{10}J_{24}J_{40}J_{60}} + q^4\frac{J_2^3J_3^2J_{12}J_{20}^2J_{24}J_{30}J_{120}}{J_1^2J_6J_8J_{10}J_{40}J_{60}} \\
&= \frac{J_2^2J_3^3J_{10}J_{15}^2}{J_1^2J_5J_{30}} + 2q\frac{J_3^3J_8^2J_{40}J_{60}^2}{J_1J_4J_{20}J_{120}} + q^3\frac{J_3^3J_4^5J_{20}^2J_{30}J_{120}}{J_1J_2^2J_8^2J_{10}J_{40}J_{60}},
\end{aligned} \tag{3.48}
$$

where the penultimate step in (3.48) follows by using (2.16) and the last step in (3.48) follows by the package `thetaids`. Similarly, we deduce that

$$
\begin{aligned}
&\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_5(n_1,n_2,n_3,n_4)/2} \\
&= 2q^{-1}\varphi(q^{30})\psi(q^6)\overline{J}_{2,6}\overline{J}_{4,12} + q^{-1}\varphi(q^3)\varphi(q^{30})\overline{J}_{1,6}\overline{J}_{2,12} \\
&\quad + \psi(q^{15})\big(\overline{J}_{5,12} + q\overline{J}_{1,12}\big)\big(\varphi(q^3)\overline{J}_{2,6} + 2q\psi(q^6)\overline{J}_{1,6}\big) \\
&\quad + 4q^7\psi(q^6)\psi(q^{60})\overline{J}_{2,6}\overline{J}_{2,12} + 2q^6\varphi(q^3)\psi(q^{60})\overline{J}_{1,6}\overline{J}_{4,12} \\
&= q^{-1}\varphi(q^3)\varphi(q^{30})\overline{J}_{1,6}\overline{J}_{2,12} + 2q^{-1}\varphi(q^{30})\psi(q^6)\overline{J}_{2,6}\overline{J}_{4,12}
\end{aligned}
$$

$$
\begin{aligned}
&+\psi(q^{15})\overline{J}_{1,3}^3+2q^6\varphi(q^3)\psi(q^{60})\overline{J}_{1,6}\overline{J}_{4,12}\\
&+4q^7\psi(q^6)\psi(q^{60})\overline{J}_{2,6}\overline{J}_{2,12}\\
&=q^{-1}\frac{J_2J_4J_6^5J_{24}J_{60}^5}{J_1J_3J_8J_{12}^2J_{30}^2J_{120}^2}+2q^{-1}\frac{J_6J_8J_{12}^3J_{60}^5}{J_2J_{24}J_{30}^2J_{120}^2}+\frac{J_2^3J_3^6J_{30}^2}{J_1^3J_6^3J_{15}}\\
&+2q^6\frac{J_2^2J_6^4J_8J_{12}J_{120}^2}{J_1J_3J_4^2J_{24}J_{60}}+4q^7\frac{J_4^3J_6^2J_{24}J_{120}^2}{J_2^2J_8J_{60}} \qquad (3.49)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_6(n_1,n_2,n_3,n_4)/2}\\
&=4q^{-2}\psi^3(q^3)\big(\overline{J}_{25,60}+q^5\overline{J}_{5,60}\big)+4q^{-1}\varphi(q^6)\psi^2(q^6)\overline{J}_{20,60}\\
&+2q^{-1}\varphi^2(q^3)\psi(q^{12})\overline{J}_{20,60}+\varphi^2(q^3)\varphi(q^6)\overline{J}_{10,60}\\
&+8q^3\psi^2(q^6)\psi(q^{12})\overline{J}_{10,60}\\
&=4q^{-2}\psi^3(q^3)\overline{J}_{5,15}+4q^{-1}\varphi(q^6)\psi^2(q^6)\overline{J}_{20,60}\\
&+2q^{-1}\varphi^2(q^3)\psi(q^{12})\overline{J}_{20,60}+\varphi^2(q^3)\varphi(q^6)\overline{J}_{10,60}\\
&+8q^3\psi^2(q^6)\psi(q^{12})\overline{J}_{10,60}\\
&=4q^{-2}\frac{J_6^6J_{10}J_{15}^2}{J_3^3J_5J_{30}}+4q^{-1}\frac{J_{12}^9J_{40}J_{60}^2}{J_6^4J_{20}J_{24}^2J_{120}}+2q^{-1}\frac{J_6^{10}J_{24}^2J_{40}J_{60}^2}{J_3^4J_{12}^5J_{20}J_{120}}\\
&+\frac{J_6^8J_{12}J_{20}^2J_{30}J_{120}}{J_3^4J_{10}J_{24}^2J_{40}J_{60}}+8q^3\frac{J_{12}^3J_{20}^2J_{24}^2J_{30}J_{120}}{J_6^2J_{10}J_{40}J_{60}}, \qquad (3.50)
\end{aligned}
$$

where the penultimate step in (3.49) follows by applying (2.16) and (2.17), and the penultimate step in (3.50) follows by employing (2.16).

Substituting (3.46)–(3.50) into (3.45) and applying (2.6), we obtain (3.44).

The proof of Theorem 3.11 is complete. □

Theorem 3.12. *We have*

$$
\begin{aligned}
\mathrm{C}\Phi_{16}(q)=\frac{1}{J_1^{16}}\bigg(&\frac{J_2^{32}J_{16}}{J_1^{16}J_{32}^2}+112q\frac{J_2^{30}}{J_1^8J_4^7}+112q\frac{J_2^{30}J_4^7}{J_1^{16}J_8^6}+1792q^2\frac{J_2^6J_4^{23}}{J_1^8J_8^6}\\
&+4480q^2\frac{J_2^{20}J_4^9}{J_1^{12}J_8^2}+3632q^2\frac{J_2^{34}J_8^2}{J_1^{16}J_4^5}-1792q^2J_2^6J_4^9\\
&+21504q^3\frac{J_2^{10}J_4^{11}J_8^2}{J_1^8}+8q^4\frac{J_2^{32}J_4^4J_{16}^3J_{32}^2}{J_1^{16}J_8^{10}}+64q^4\frac{J_2^{40}J_8^4J_{16}^9}{J_1^{16}J_4^{20}J_{32}^2}\\
&+32q^4\frac{J_2^{40}J_8^{18}J_{32}^2}{J_1^{16}J_4^{24}J_{16}^5}+4096q^4\frac{J_4^8J_8^{16}J_{16}}{J_2^8J_{32}^2}+12288q^4\frac{J_4^{27}J_8^2}{J_2^{14}}
\end{aligned}
$$

$$+ 512q^4\frac{J_4^{32}J_8^2J_{32}^2}{J_2^{16}J_{16}^5} + 1024q^4\frac{J_4^{36}J_{16}^9}{J_2^{16}J_8^{12}J_{32}^2} + 32768q^8\frac{J_4^{12}J_8^6J_{16}^3J_{32}^2}{J_2^8}\Bigg). \tag{3.51}$$

Proof. In view of (2.5) and (2.7), we deduce that

$$\begin{aligned}
\mathrm{CT}_z\big(f_{16}(z)\big) &= \mathrm{CT}_z\big(f_2(z)^8\big) \\
&= \frac{J_4^{40}}{J_2^{16}J_8^{16}} \sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{4\sum_{i=1}^7 n_i^2 + 4\sum_{1\le i<j\le 7} n_in_j} \\
&\quad + 112q^2\frac{J_4^{28}}{J_2^{12}J_8^8} \sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{Q_1(n_1,n_2,\ldots,n_7)} \\
&\quad + 1120q^4\frac{J_4^{16}}{J_2^8} \sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{Q_2(n_1,n_2,\ldots,n_7)} \\
&\quad + 1792q^6\frac{J_4^4J_8^8}{J_2^4} \sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{Q_3(n_1,n_2,\ldots,n_7)} \\
&\quad + 256q^8\frac{J_8^{16}}{J_4^8} \sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{Q_4(n_1,n_2,\ldots,n_7)},
\end{aligned} \tag{3.52}$$

where

$$\begin{aligned}
Q_1(n_1,n_2,\ldots,n_7) &= 4\sum_{i=1}^7 n_i^2 + 4\sum_{1\le i<j\le 7} n_in_j \\
&\quad + 2n_1 + 2n_2 + 2n_3 + 2n_4 + 2n_5 + 2n_6 + 4n_7, \\
Q_2(n_1,n_2,\ldots,n_7) &= 4\sum_{i=1}^7 n_i^2 + 4\sum_{1\le i<j\le 7} n_in_j \\
&\quad + 6n_1 + 6n_2 + 6n_3 + 6n_4 + 8n_5 + 8n_6 + 8n_7 + 4, \\
Q_3(n_1,n_2,\ldots,n_7) &= 4\sum_{i=1}^7 n_i^2 + 4\sum_{1\le i<j\le 7} n_in_j + 10n_1 + 10n_2 \\
&\quad + 12n_3 + 12n_4 + 12n_5 + 12n_6 + 12n_7 + 12, \\
Q_4(n_1,n_2,\ldots,n_7) &= 4\sum_{i=1}^7 n_i^2 + 4\sum_{1\le i<j\le 7} n_in_j + 16n_1 + 16n_2 \\
&\quad + 16n_3 + 16n_4 + 16n_5 + 16n_6 + 16n_7 + 24.
\end{aligned}$$

According to (1.1) and (3.18),

$$\sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{2\sum_{i=1}^{7} n_i^2+2\sum_{1\leq i<j\leq 7} n_in_j} = \frac{J_4^{16}J_{16}}{J_2^8J_{32}^2}+48q^2\frac{J_4^{11}J_8^2}{J_2^6}+64q^4\frac{J_8^4J_{16}^9}{J_4^4J_{32}^2} + 8q^4\frac{J_4^{20}J_{16}^3J_{32}^2}{J_2^8J_8^{10}}+32q^4\frac{J_8^{18}J_{32}^2}{J_4^8J_{16}^5}. \tag{3.53}$$

To simplify the remaining four q-series septuple sums, we adopt the following matrix:

$$B=\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & -1 & 1 & 0 & 0\\ -1 & 0 & 0 & 1 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 0 & -1 & 0\\ 1 & 0 & -1 & 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

We deduce that

$$\sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{Q_1(n_1,n_2,\ldots,n_7)/2} = J_1^4J_2^2J_4+\frac{J_2^2J_4^{15}}{J_1^4J_8^6}+12q\frac{J_2^6J_4^3J_8^2}{J_1^4}, \tag{3.54}$$

$$\sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{Q_2(n_1,n_2,\ldots,n_7)/2} = 4\frac{J_2^4J_4^9}{J_1^4J_8^2}+2\frac{J_2^{18}J_8^2}{J_1^8J_4^5}, \tag{3.55}$$

$$\sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{Q_3(n_1,n_2,\ldots,n_7)/2} = -q^{-1}J_1^4J_2^2J_4+q^{-1}\frac{J_2^2J_4^{15}}{J_1^4J_8^6}+12\frac{J_2^6J_4^3J_8^2}{J_1^4}, \tag{3.56}$$

$$\sum_{n_1,n_2,\ldots,n_7=-\infty}^{\infty} q^{Q_4(n_1,n_2,\ldots,n_7)/2} = 48\frac{J_4^{11}J_8^2}{J_2^6}+4\frac{J_4^{20}J_{16}^9}{J_2^8J_8^{12}J_{32}^2}+2\frac{J_4^{16}J_8^2J_{32}^2}{J_2^8J_{16}^5} + 16\frac{J_8^{16}J_{16}}{J_4^8J_{32}^2}+128q^4\frac{J_8^6J_{16}^3J_{32}^2}{J_4^4}. \tag{3.57}$$

It is worth mentioning that one needs to utilize (2.12)–(2.15) to derive (3.54)–(3.57). Finally, (3.51) follows by substituting (3.53)–(3.57) into (3.52) and applying (2.6).

We therefore complete the proof of Theorem 3.12. □

Theorem 3.13. *We have*

$$\mathrm{C}\Phi_{18}(q) = \frac{1}{J_1^{18}}\Bigg\{a^6(q)a(q^6)\frac{J_6^{15}}{J_3^6J_{12}^6}+270qa^4(q)a(q^6)\frac{J_2^2J_3^3J_6^9}{J_1^3J_4J_{12}^3}+7290q^2a^2(q)\frac{J_3^6J_6^{18}}{J_1^4J_2J_{12}^6}$$

$$+ 1080q^2a^3(q)\frac{J_3^8J_4J_6^6}{J_1^2J_2J_{12}} + 1080q^2a^3(q)\frac{J_3^8J_4J_6^6J_9^2}{J_1^4J_{12}J_{18}} + 1080q^2a^3(q)\frac{J_2J_3^6J_6^{12}}{J_1^4J_4J_{12}^3}$$
$$+ 14580q^3\frac{J_2J_3^{15}J_6^{12}}{J_1^7J_4J_{12}^3} + 58320q^3a(q)\frac{J_3^{14}J_4^3J_6^2J_9^2J_{12}}{J_1^6J_{18}}$$
$$+ 29160q^3a^2(q)\frac{J_2^3J_3^{12}J_6^2J_{12}^2}{J_1^6} + 2160q^3a^3(q)\frac{J_2^3J_3^{15}J_{12}^2J_{18}^2}{J_1^6J_6^4J_9}$$
$$+ 2160q^3a^4(q)\frac{J_3^6J_4^3J_6J_{12}J_{18}^2}{J_1^3J_9} + 24q^3a^6(q)\frac{J_6^5J_{12}^2J_{18}^2}{J_3^3J_9} + 29160q^4\frac{J_2^2J_3^{20}J_{12}^5}{J_1^6J_4J_6^3}$$
$$+ 13122q^4\frac{J_3^{16}J_9^4J_{18}^5}{J_1^6J_{36}^2} + 58320q^4\frac{J_3^{18}J_4J_6^5J_{18}^2}{J_1^7J_9J_{12}} + 29160q^4\frac{J_2^3J_3^{20}J_9^2J_{12}^5}{J_1^8J_4J_6^3J_{18}}$$
$$+ 43740q^4a(q)\frac{J_2^2J_3^{12}J_6^8J_{18}^3}{J_1^6J_4J_{12}^3} + 8748q^4a(q^3)\frac{J_3^{16}J_6^2J_9^4J_{36}}{J_1^6J_{12}J_{18}}\Bigg\}, \tag{3.58}$$

where $a(q)$ is defined as in (2.9).

Proof. In view of (2.8), we arrive at

$$\begin{aligned}
\mathrm{CT}_z\big(f_{18}(z)\big) &= \mathrm{CT}_z\big(f_3(z)^6\big) \\
&= a^6(q^2)\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{6\sum_{i=1}^5 n_i^2+6\sum_{1\le i<j\le 5} n_in_j} \\
&\quad + 270q^2a^4(q^2)\frac{J_6^6}{J_2^2}\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_4(n_1,n_2,\ldots,n_5)} \\
&\quad + 1080q^4a^3(q^2)\frac{J_6^9}{J_2^3}\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_5(n_1,n_2,\ldots,n_5)} \\
&\quad + 7290q^4a^2(q^2)\frac{J_6^{12}}{J_2^4}\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_6(n_1,n_2,\ldots,n_5)} \\
&\quad + 14580q^6a(q^2)\frac{J_6^{15}}{J_2^5}\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_7(n_1,n_2,\ldots,n_5)} \\
&\quad + 14580q^6\frac{J_6^{18}}{J_2^6}\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_8(n_1,n_2,\ldots,n_5)} \\
&\quad + 1458q^{14}\frac{J_6^{18}}{J_2^6}\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_9(n_1,n_2,\ldots,n_5)},
\end{aligned} \tag{3.59}$$

where

$$\begin{aligned}
Q_4(n_1,n_2,\ldots,n_5) &= 6\sum_{i=1}^{5} n_i^2 + 6\sum_{1\le i<j\le 4} n_in_j + 2n_1 + 2n_2 + 2n_3 + 2n_4 + 4n_5,\\
Q_5(n_1,n_2,\ldots,n_5) &= 6\sum_{i=1}^{5} n_i^2 + 6\sum_{1\le i<j\le 5} n_in_j - 4n_1 - 4n_2 - 4n_3 - 6n_4 - 6n_5,\\
Q_6(n_1,n_2,\ldots,n_5) &= 6\sum_{i=1}^{5} n_i^2 + 6\sum_{1\le i<j\le 4} n_in_j + 2n_1 + 2n_2 + 4n_3 + 4n_4,\\
Q_7(n_1,n_2,\ldots,n_5) &= 6\sum_{i=1}^{5} n_i^2 + 6\sum_{1\le i<j\le 4} n_in_j - 2n_1 - 4n_2 - 6n_3 - 6n_4 - 6n_5,\\
Q_8(n_1,n_2,\ldots,n_5) &= 6\sum_{i=1}^{5} n_i^2 + 6\sum_{1\le i<j\le 5} n_in_j + 4n_1 + 4n_2 + 4n_3,\\
Q_9(n_1,n_2,\ldots,n_5) &= 6\sum_{i=1}^{5} n_i^2 + 6\sum_{1\le i<j\le 5} n_in_j + 12n_1 + 12n_2\\
&\quad + 12n_3 + 12n_4 + 12n_5.
\end{aligned}$$

We next simplify the seven q-series quintuple sums on the right-hand side of (3.59). First, from (1.1) and (3.6) we note that

$$\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{3\sum_{i=1}^{5} n_i^2 + 3\sum_{1\le i<j\le 5} n_in_j} = a(q^6)\frac{J_6^{15}}{J_3^6J_{12}^6} + 24q^3\frac{J_6^5J_{12}^2J_{18}^2}{J_3^3J_9}. \tag{3.60}$$

By (2.5), we find that the coefficient of z^{-2} in the expression of $f_6(z)$ is given by

$$\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_9(n_1,n_2,\ldots,n_5)/3+4}. \tag{3.61}$$

According to (2.11) and (3.61), one sees that

$$\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_9(n_1,n_2,\ldots,n_5)/2} = 9q^{-3}\frac{J_9^4J_{18}^5}{J_3^2J_{36}^2} + 6q^{-3}a(q^3)\frac{J_6^2J_9^4J_{36}}{J_3^2J_{12}J_{18}}. \tag{3.62}$$

Next, we choose the following matrix:

$$B_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{pmatrix}.$$

After simplification, we obtain that

$$\begin{aligned} \sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} & q^{Q_6(n_1,n_2,\ldots,n_5)/2} \\ &= \varphi^3(q^3)\big(\overline{J}_{4,12}\overline{J}_{12,36} + q^2\overline{J}_{2,12}\overline{J}_{6,36}\big) \\ &\quad + 4q\varphi(q^3)\psi^2(q^6)\big(\overline{J}_{15,36} + q^3\overline{J}_{3,36}\big)\big(\overline{J}_{5,12} + q\overline{J}_{1,12}\big) \\ &\quad + 4q^2\varphi(q^3)\psi^2(q^6)\big(\overline{J}_{2,12}\overline{J}_{12,36} + q\overline{J}_{4,12}\overline{J}_{6,36}\big) \\ &= \frac{J_6^{18}}{J_2J_3^6J_{12}^6} + 4q\frac{J_2J_6^3J_9^2J_{12}^2}{J_1J_3J_{18}} + 4q^2\frac{J_2J_6^2J_{12}^2J_{18}^2}{J_1J_9} \\ &= \frac{J_6^{18}}{J_2J_3^6J_{12}^6} + 4q\frac{J_2^3J_6^2J_{12}^2}{J_1^2}, \end{aligned} \tag{3.63}$$

where the penultimate step follows by using (2.16) and the package `thetaids`, and the last step in (3.63) follows by utilizing the package `thetaids` again. Moreover, we derive that

$$\begin{aligned} \sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} & q^{Q_4(n_1,n_2,\ldots,n_5)/2} \\ &= a(q^6)\varphi^2(q^3)\overline{J}_{1,6} + 8q^2\psi^3(q^3)\psi(q^9)\overline{J}_{2,6} \\ &\quad + 8q^3\psi^2(q^6)\overline{J}_{1,6}\big(\varphi(q^{18})\psi(q^{12}) + q^3\varphi(q^6)\psi(q^{36})\big) \\ &= a(q^6)\frac{J_2^2J_6^9}{J_1J_3^3J_4J_{12}^3} + 8q^2\frac{J_4J_6^8J_{18}^2}{J_2J_3^3J_9J_{12}} + 8q^3\frac{J_2^2J_{12}^5J_{18}^2}{J_1J_4J_6J_9} \\ &= a(q^6)\frac{J_2^2J_6^9}{J_1J_3^3J_4J_{12}^3} + 8q^2\frac{J_4^3J_6J_{12}J_{18}^2}{J_1J_9}, \end{aligned} \tag{3.64}$$

where the penultimate step follows from (3.8) and the last step in (3.64) follows from the package `thetaids`. Following a similar strategy,

$$\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_7(n_1,n_2,\ldots,n_5)/2} = 4\frac{J_4^3J_6^2J_9^2J_{12}}{J_1J_3J_{18}} + 3q\frac{J_2^2J_6^8J_{18}^3}{J_1J_3^3J_4J_{12}^3}. \tag{3.65}$$

Similarly, we arrive at

$$\begin{aligned}
&\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_8(n_1,n_2,\ldots,n_5)/2}\\
&\quad= \varphi^2(q^3)\overline{J}_{1,6}\frac{J_6^3}{J_2} + 4q\psi^2(q^3)\psi(q^9)\overline{J}_{2,6}\overline{J}_{1,3}\\
&\qquad + 4q\psi^2(q^6)\overline{J}_{1,6}\big(\overline{J}_{5,12}\overline{J}_{15,36} + q^4\overline{J}_{1,12}\overline{J}_{3,36}\big)\\
&\quad= \varphi^2(q^3)\overline{J}_{1,6}\frac{J_6^3}{J_2} + 4q\psi^2(q^3)\psi(q^9)\overline{J}_{1,3}\overline{J}_{2,6}\\
&\qquad + q\psi^2(q^6)\overline{J}_{1,6}\big\{\big(\overline{J}_{1,3}+J_1\big)\big(\overline{J}_{3,9}+J_3\big) + \big(\overline{J}_{1,3}-J_1\big)\big(\overline{J}_{3,9}-J_3\big)\big\}\\
&\quad= \varphi^2(q^3)\overline{J}_{1,6}\frac{J_6^3}{J_2} + 4q\psi^2(q^3)\psi(q^9)\overline{J}_{1,3}\overline{J}_{2,6}\\
&\qquad + 2q\psi^2(q^6)\overline{J}_{1,6}\big(\overline{J}_{1,3}\overline{J}_{3,9} + J_1J_3\big)\\
&\quad= \frac{J_2J_6^{12}}{J_1J_3^3J_4J_{12}^3} + 4q\frac{J_4J_6^5J_{18}^2}{J_1J_9J_{12}} + 2q\frac{J_2^3J_3^2J_9^2J_{12}^5}{J_1^2J_4J_6^3J_{18}} + 2q\frac{J_2^2J_3^2J_{12}^5}{J_4J_6^3}, \qquad (3.66)
\end{aligned}$$

where the second step follows from the following two identities:

$$\begin{aligned}
J_{1,3} &= \overline{J}_{5,12} - q\overline{J}_{1,12},\\
\overline{J}_{1,3} &= \overline{J}_{5,12} + q\overline{J}_{1,12},
\end{aligned}$$

which are derived by (2.16). Finally, we obtain that

$$\begin{aligned}
\sum_{n_1,n_2,\ldots,n_5=-\infty}^{\infty} q^{Q_5(n_1,n_2,\ldots,n_5)/2} &= \frac{J_4J_6^6J_9^2}{J_1J_3J_{12}J_{18}} + \frac{J_2J_6^{12}}{J_1J_3^3J_4J_{12}^3}\\
&\quad + \frac{J_1J_4J_6^6}{J_2J_3J_{12}} + 2q\frac{J_2^3J_3^6J_{12}^2J_{18}^2}{J_1^3J_6^4J_9}. \qquad (3.67)
\end{aligned}$$

Now, (3.58) follows by combining (3.59), (3.60) and (3.62)–(3.67), and applying (2.6).
We therefore complete the proof of Theorem 3.13. □

Theorem 3.14. *We have*

$$\begin{aligned}
\mathrm{C}\Phi_{20}(q) = \frac{1}{J_1^{20}}\Big\{&A^5(q)\frac{J_4^5}{J_{20}} + 160qA^3(q)\frac{J_2^{12}J_4J_{10}^2}{J_1^4J_{20}} + 160qA^3(q)\frac{J_2^{14}J_4^5J_{20}^5}{J_1^8J_8^2J_{10}^2J_{40}^2}\\
&+ 40q^2A^3(q)B^2(q)\frac{J_2^6J_8^4J_{10}^2}{J_1^3J_4^2J_5} + 1920q^2A^2(q)B(q)\frac{J_2^{16}J_8^4J_{20}^5}{J_1^8J_4^2J_{10}^2J_{40}^2}\\
&+ 7680q^2A(q)\frac{J_2^{20}J_4^9J_{20}^5}{J_1^{12}J_8^2J_{10}^2J_{40}^2} + 40q^2A^3(q)B^2(q)\frac{J_1^3J_4J_5J_8^4J_{20}}{J_2^3J_{10}}
\end{aligned}$$

$$+ 7680q^3A(q)B^2(q)\frac{J_2^{12}J_4^2J_8^4J_{10}^2}{J_1^7J_5} + 10240q^3A(q)\frac{J_2^{24}J_8^6J_{20}^5}{J_1^{12}J_4^3J_{10}^2J_{40}^2}$$
$$+ 20480q^3B(q)\frac{J_2^{30}J_8^2J_{10}^2}{J_1^{15}J_4J_5} + 80q^3A(q)B^4(q)\frac{J_2^6J_8^8J_{10}^2}{J_1^3J_4^4J_5}$$
$$- 1280q^3B^3(q)\frac{J_2^{15}J_5J_8^6J_{20}}{J_1^5J_4^4J_{10}} + 1280q^3B^3(q)\frac{J_2^{24}J_8^6J_{10}^2}{J_1^{11}J_4^7J_5}$$
$$- 80q^3A(q)B^4(q)\frac{J_1^3J_5J_8^8J_{20}}{J_2^3J_4J_{10}} + 1920q^4A^2(q)B(q)\frac{J_2^{14}J_4^4J_{40}^2}{J_1^8J_{20}}$$
$$+ 640q^4A^3(q)\frac{J_2^{16}J_8^2J_{40}^2}{J_1^8J_4J_{20}} + 2560q^4A(q)\frac{J_2^{18}J_4^{15}J_{40}^2}{J_1^{12}J_8^6J_{20}}$$
$$+ 25q^4A^5(q)\frac{J_{20}^5}{J_4} + 30720q^5A(q)\frac{J_2^{22}J_4^3J_8^2J_{40}^2}{J_1^{12}J_{20}}\Bigg\}, \tag{3.68}$$

where $A(q)$ and $B(q)$ are defined as in (2.4).

Proof. By means of (2.10), we find that

$$\begin{aligned}
\mathrm{CT}_z\big(f_{20}(z)\big) &= \mathrm{CT}_z\big(f_4(z)^5\big)\\
&= A^5(q^2)\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4} n_in_j}\\
&\quad + 320q^2A^3(q^2)\frac{J_4^{12}}{J_2^6}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_7(n_1,n_2,n_3,n_4)}\\
&\quad + 7680q^4A(q^2)\frac{J_4^{24}}{J_2^{12}}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_8(n_1,n_2,n_3,n_4)}\\
&\quad + 2560q^4A(q^2)\frac{J_4^{24}}{J_2^{12}}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_9(n_1,n_2,n_3,n_4)}\\
&\quad + 1920q^4A^2(q^2)B(q^2)\frac{J_4^{12}J_{16}^2}{J_2^6J_8}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{10}(n_1,n_2,\ldots,n_4)}\\
&\quad + 40q^4A^3(q^2)B^2(q^2)\frac{J_{16}^4}{J_8^2}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{11}(n_1,n_2,n_3,n_4)}\\
&\quad + 20480q^6B(q^2)\frac{J_4^{24}J_{16}^2}{J_2^{12}J_8}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{12}(n_1,n_2,n_3,n_4)}
\end{aligned}$$

$$
\begin{aligned}
&+3840q^6A(q^2)B^2(q^2)\frac{J_4^{12}J_{16}^4}{J_2^6J_8^2}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty}q^{Q_{13}(n_1,n_2,n_3,n_4)}\\
&+80q^8A(q^2)B^4(q^2)\frac{J_{16}^8}{J_8^4}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty}q^{Q_{14}(n_1,n_2,n_3,n_4)}\\
&+2560q^8B^3(q^2)\frac{J_4^{12}J_{16}^6}{J_2^6J_8^3}\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty}q^{Q_{15}(n_1,n_2,n_3,n_4)}, \qquad (3.69)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_7(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j+2n_1-2n_4,\\
Q_8(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j-2n_1-2n_2+2n_3+2n_4,\\
Q_9(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j\\
&\quad+10n_1+10n_2+10n_3+10n_4+4,\\
Q_{10}(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j+4n_1+4n_2+6n_3+6n_4,\\
Q_{11}(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j+4n_1+4n_2+4n_3+8n_4,\\
Q_{12}(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j+2n_1+6n_2+6n_3+6n_4,\\
Q_{13}(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j+2n_1+4n_2+6n_3+8n_4,\\
Q_{14}(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j\\
&\quad+20n_1+20n_2+20n_3+20n_4+16,\\
Q_{15}(n_1,n_2,n_3,n_4)&=8\sum_{i=1}^4 n_i^2+8\sum_{1\le i<j\le 4}n_in_j
\end{aligned}
$$

$$+ 8n_1 + 14n_2 + 14n_3 + 14n_4 + 6.$$

Next, we simplify the ten q-series quadruple sums in (3.69). To this end, from (3.25) we have

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{4\sum_{i=1}^4 n_i^2+4\sum_{1\le i<j\le 4} n_in_j} = \frac{J_4^5}{J_{20}} + 25q^4\frac{J_{20}^5}{J_4}. \tag{3.70}$$

To simplify the remaining nine q-series quadruple sums, we adopt the following matrix:

$$B_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}.$$

Upon simplification, we obtain that

$$\begin{aligned}
&\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_7(n_1,n_2,n_3,n_4)/2} \\
&= 2q^3\psi(q^{20})\overline{J}_{1,8}\overline{J}_{3,8}\big(\overline{J}_{6,16} + q\overline{J}_{2,16}\big) \\
&\quad + \varphi(q^{40})\overline{J}_{3,8}^2\overline{J}_{6,16} + q^2\varphi(q^{40})\overline{J}_{1,8}^2\overline{J}_{2,16} \\
&\quad + 2q^{11}\psi(q^{80})\overline{J}_{3,8}^2\overline{J}_{2,16} + 2q^{11}\psi(q^{80})\overline{J}_{1,8}^2\overline{J}_{6,16} \\
&= 2q^3\psi^2(q)\psi(q^4)\psi(q^{20}) \\
&\quad + \frac{1}{4}\varphi(q^{40})\big(J_{1,4}J_{2,4} + \overline{J}_{1,4}\overline{J}_{2,4}\big)\big(J_{1,4} + \overline{J}_{1,4}\big) \\
&\quad + \frac{1}{4}\varphi(q^{40})\big(-J_{1,4}J_{2,4} + \overline{J}_{1,4}\overline{J}_{2,4}\big)\big(\overline{J}_{1,4} - J_{1,4}\big) \\
&\quad + \frac{1}{2}q^{10}\psi(q^{80})\big(J_{1,4}J_{2,4} + \overline{J}_{1,4}\overline{J}_{2,4}\big)\big(\overline{J}_{1,4} - J_{1,4}\big) \\
&\quad + \frac{1}{2}q^{10}\psi(q^{80})\big(-J_{1,4}J_{2,4} + \overline{J}_{1,4}\overline{J}_{2,4}\big)\big(J_{1,4} + \overline{J}_{1,4}\big) \\
&= 2q^3\psi^2(q)\psi(q^4)\psi(q^{20}) + \frac{1}{2}J_{1,4}^2J_{2,4}\big(\varphi(q^{40}) - 2q^{10}\psi(q^{80})\big) \\
&\quad + \frac{1}{2}\overline{J}_{1,4}^2\overline{J}_{2,4}\big(\varphi(q^{40}) + 2q^{10}\psi(q^{80})\big) \\
&= 2q^3\psi^2(q)\psi(q^4)\psi(q^{20}) + \frac{1}{2}\varphi(-q^{10})J_{1,4}^2J_{2,4} + \frac{1}{2}\varphi(q^{10})\overline{J}_{1,4}^2\overline{J}_{2,4} \\
&= \frac{J_2^2J_4^5J_{20}^5}{2J_1^2J_8^2J_{10}^2J_{40}^2} + \frac{J_1^2J_4J_{10}^2}{2J_{20}} + 2q^3\frac{J_2^4J_8^2J_{40}^2}{J_1^2J_4J_{20}},
\end{aligned} \tag{3.71}$$

where the second step follows by using (2.16), the third step follows from (2.16) and (2.17), and the penultimate step in (3.71) follows from (2.12) and (2.13). Similarly, one can simplify the remaining q-series quadruple sums. For conciseness, we present only the finalized expressions herein, with the detailed derivations omitted for brevity.

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_8(n_1,n_2,n_3,n_4)/2} = \frac{J_4^9 J_{20}^5}{J_2^4 J_8^2 J_{10}^2 J_{40}^2} + 4q^3 \frac{J_4^3 J_8^2 J_{40}^2}{J_2^2 J_{20}}, \tag{3.72}$$

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_9(n_1,n_2,n_3,n_4)/2} = 4q \frac{J_8^6 J_{20}^5}{J_4^3 J_{10}^2 J_{40}^2} + q^2 \frac{J_4^{15} J_{40}^2}{J_2^6 J_8^6 J_{20}}, \tag{3.73}$$

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{10}(n_1,n_2,n_3,n_4)/2} = \frac{J_2^4 J_8^2 J_{20}^5}{J_1^2 J_4 J_{10}^2 J_{40}^2} + q^2 \frac{J_2^2 J_4^5 J_{40}^2}{J_1^2 J_8^2 J_{20}}, \tag{3.74}$$

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{11}(n_1,n_2,n_3,n_4)/2} = \frac{J_2^6 J_{10}^2}{J_1^3 J_5} + \frac{J_1^3 J_4^3 J_5 J_{20}}{J_2^3 J_{10}}, \tag{3.75}$$

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{12}(n_1,n_2,n_3,n_4)/2} = \frac{J_2^6 J_{10}^2}{J_1^3 J_5}, \tag{3.76}$$

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{13}(n_1,n_2,n_3,n_4)/2} = 2\frac{J_4^4 J_{10}^2}{J_1 J_5}, \tag{3.77}$$

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{14}(n_1,n_2,n_3,n_4)/2} = q^{-1} \frac{J_2^6 J_{10}^2}{J_1^3 J_5} - q^{-1} \frac{J_1^3 J_4^3 J_5 J_{20}}{J_2^3 J_{10}}, \tag{3.78}$$

$$\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{Q_{15}(n_1,n_2,n_3,n_4)/2} = \frac{1}{2} q^{-1} \frac{J_2^{12} J_{10}^2}{J_1^5 J_4^4 J_5} - \frac{1}{2} q^{-1} \frac{J_1 J_2^3 J_5 J_{20}}{J_4 J_{10}}. \tag{3.79}$$

Now, (3.68) follows by substituting (3.70)–(3.79) into (3.69) and utilizing (2.6).

This finishes the proof of Theorem 3.14. □

4. Concluding remarks

In this work, we present a novel unified methodology for deriving the integral expression of $C\Phi_k(q)$, the generating function of k-colored generalized Frobenius partitions of n. The present results, when synthesized with prior work [2,3,8], demonstrate the emerging significance of integer matrix exact covering systems—a computational framework pioneered in [5] that is now proving its full potential in partition theory. While congruence properties of $c\phi_k(n)$ have been extensively documented for small values of k, we point out that this remarkable arithmetic behavior persists for substantially larger values of k. We only present the following example:

$$c\phi_{18}(3n+2) \equiv 0 \pmod{2187}. \tag{4.1}$$

It is noteworthy that the modulus 2187 in (4.1) is best possible, as evidenced by the explicit calculation $c\phi_{18}(2) = 24057$. We defer the proof of (4.1) to the interested reader.

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