THE BRIGGS INEQUALITY OF BOROS-MOLL SEQUENCES

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Abstract. Briggs conjectured that if a polynomial $a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients has only negative zeros, then

$$a_k^2(a_k^2 - a_{k-1}a_{k+1}) > a_{k-1}^2(a_{k+1}^2 - a_k a_{k+2})$$

for any $1 \le k \le n-1$. The Boros-Moll sequence $\{d_i(m)\}_{i=0}^m$ arises in the study of evaluation of certain quartic integral, and a lot of interesting inequalities for this sequence have been obtained. In this paper we show that the Boros-Moll sequence $\{d_i(m)\}_{i=0}^m$, its normalization $\{d_i(m)/i!\}_{i=0}^m$, and its transpose $\{d_i(m)\}_{m\ge i}$ satisfy the Briggs inequality. For the first two sequences, we prove the Briggs inequality by using a lower bound for $(d_{i-1}(m)d_{i+1}(m))/d_i^2(m)$ due to Chen and Gu and an upper bound due to Zhao. For the transposed sequence, we derive the Briggs inequality by establishing its strict ratio-log-convexity. As a consequence, we also obtain the strict log-convexity of the sequence $\{\sqrt[n]{d_i(i+n)}\}_{n\ge 1}$ for $i \ge 1$.

1. Introduction

In the study of positive irreducibility of binding polynomials, Briggs [12] proposed two inequality conjectures concerning elementary symmetric functions over a set of positive numbers. Suppose that $X = \{x_1, x_2, \ldots, x_n\}$ is a set of n variables. Recall that the k-th elementary symmetric function is defined by

$$e_k(X) = \sum_{1\le j_1<j_2<\cdots<j_k\le n} x_{j_1}x_{j_2}\cdots x_{j_k}, \qquad 1 \le k \le n.$$

By convention, we set $e_0 = 1$ and $e_k = 0$ if $k < 0$ or $k > n$. Briggs [12] conjectured that if X is a set of n positive numbers, then for $1 \le k \le n-1$ the following two inequalities hold:

$$e_{k-1}e_{k+2}^2 + e_k^2 e_{k+3} + e_{k+1}^3 > e_{k+1}\left(e_{k-1}e_{k+3} + 2e_k e_{k+2}\right), \tag{1.1}$$

$$e_k^2(e_k^2 - e_{k-1}e_{k+1}) > e_{k-1}^2(e_{k+1}^2 - e_k e_{k+2}). \tag{1.2}$$

It is worth noting that the expression

$$(x + x_1)(x + x_2)\cdots(x + x_n) = \sum_{k=0}^n e_{n-k}x^k$$

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allows us to formulate Briggs' conjecture in the following manner: If a polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ with positive coefficients has only real zeros, then for $1 \le k \le n-1$ we have

$$a_{k-1}a_{k+2}^2 + a_k^2 a_{k+3} + a_{k+1}^3 > a_{k+1}\left(a_{k-1}a_{k+3} + 2a_k a_{k+2}\right), \tag{1.3}$$

$$a_k^2(a_k^2 - a_{k-1}a_{k+1}) > a_{k-1}^2(a_{k+1}^2 - a_k a_{k+2}). \tag{1.4}$$

We would like to point out that the inequality (1.3) is closely related to the notion of 2-log-concavity. Recall that a sequence $\{a_k\}_{k\ge0}$ with real numbers is said to be *log-concave* if

$$a_k^2 - a_{k-1}a_{k+1} \ge 0 \tag{1.5}$$

for any $k \ge 1$. The sequence $\{a_k\}_{k\ge0}$ is called log-convex if $a_k^2 - a_{k-1}a_{k+1} \le 0$ for $k \ge 1$. A polynomial is said to be log-concave (resp. log-convex) if its coefficient sequence is log-concave (resp. log-convex). For more information on log-concave and log-convex sequences, see Brändén [10], Brenti [11] and Stanley [34]. For a sequence $\{a_n\}_{n\ge0}$, define an operator $\mathcal{L}$ by $\mathcal{L}(\{a_n\}_{n\ge0}) = \{b_n\}_{n\ge0}$, where $b_n = a_n^2 - a_{n-1}a_{n+1}$ for $n \ge 0$, with the convention that $a_{-1} = 0$. A sequence $\{a_n\}_{n\ge0}$ is said to be k-log-concave (resp. strictly k-log-concave) if the sequence $\mathcal{L}^j\left(\{a_n\}_{n\ge0}\right)$ is nonnegative (resp. positive) for each $1 \le j \le k$, and $\{a_n\}_{n\ge0}$ is said to be ∞-log-concave if $\mathcal{L}^k\left(\{a_n\}_{n\ge0}\right)$ is nonnegative for any $k \ge 1$. The (strict) k-log-convexity is defined in a similar manner. The notion of infinite log-concavity was introduced by Boros and Moll [8]. We see that (1.3) can be reformulated as

$$\begin{vmatrix} a_{k+1} & a_{k+2} & a_{k+3} \\ a_k & a_{k+1} & a_{k+2} \\ a_{k-1} & a_k & a_{k+1} \end{vmatrix} > 0. \tag{1.6}$$

By Newton's inequality, if $f(x) = \sum_{k=0}^{n} a_k x^k$ has only real zeros, then the sequence $\{a_k\}_{k\ge0}$ is log-concave. Due to the positivity and log-concavity of the coefficients a_k, the inequality (1.6) amounts to say that the sequence $\{a_k\}_{k\ge0}$ is 2-log-concave. It is also natural to study (1.3) and (1.4) for infinite sequences $\{a_k\}_{k\ge0}$. Notably, the 2-log-concavity property was verified for various sequences. For instance, Hou and Zhang [25] and Jia and Wang [26] independently proved it for the partition function $\{p(i)\}_{i\ge221}$. Mukherjee [33] explored the sequence of overpartition functions $\{\overline{p}(i)\}_{i\ge42}$. Yang [39] investigated the sequences of differences $\{p(i) - p(i-1)\}_{i\ge71}$ and $\{\overline{p}(i) - \overline{p}(i-1)\}_{i\ge8}$. Additionally, Yang [38] examined the sequence of broken k-diamond partition function $\{\Delta_k(i)\}_{i\ge12}$ for $k = 1$ or 2.

It is known that a real-rooted polynomial with nonnegative coefficients, more generally a multiplier sequence, is 2-log-concave due to Aissen, Edrei, Schoenberg and Whitney [1]. See also Craven and Csordas [20, Theorem 2.13] for the case of multiplier sequence. Hence, the conjectured inequality (1.3) for a real-rooted polynomial with positive coefficients is already established by allowing equality. In fact, the strict inequality can also be proved by using Newton's inequality and Brändén's theorem [9] (see also [13, Theorem 1.2]).

However, the conjectured inequality (1.4) for a real-rooted polynomial with positive coefficients is still open. From now on we will call (1.4) the *Briggs inequality*. Recently, Liu and Zhang [29] proved the Briggs inequality for both the partition function sequence $\{p(i)\}_{i\ge114}$ and the overpartition function sequence $\{\overline{p}(i)\}_{i\ge18}$. The objective of this paper is to prove the Briggs inequality for the Boros-Moll sequence, the normalized Boros-Moll sequence and the transposed Boros-Moll sequence, which we shall recall below.

The Boros-Moll sequences arise in the study of the following quartic integral

$$\int_0^{\infty} \frac{1}{(t^4+2xt^2+1)^{m+1}} dt$$

for $x > -1$ and $m \in \mathbb{N}$. Boros and Moll [4, 7] proved that this integral is equal to $\frac{\pi}{2^{m+3/2}(x+1)^{m+1/2}} P_m(x)$, where

$$P_m(x) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (x+1)^k. \tag{1.7}$$

The coefficients of x^i in $P_m(x)$, denoted by $d_i(m)$, are called the Boros-Moll numbers. The polynomials $P_m(x)$ are called the Boros-Moll polynomials, and the sequences $\{d_i(m)\}_{i=0}^m$ are called the Boros-Moll sequences. Clearly, one sees from (1.7) that

$$d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i} \tag{1.8}$$

for $0 \le i \le m$. See [3, 5, 6, 7, 8, 14, 17, 23, 24, 31] for more information on these sequences.

Boros and Moll [5] showed that the sequence $\{d_i(m)\}_{i=0}^m$ is unimodal with the maximum term located in the middle for $m \ge 2$, see also [2, 6]. Moll [31] conjectured a stronger property that the Boros-Moll sequences $\{d_i(m)\}_{i=0}^m$ are log-concave, which was proved by Kauers and Paule [28] with a computer algebra method. Chen, Pang and Qu [16] also gave a combinatorial proof for this conjecture by building a structure of partially 2-colored permutations. Boros and Moll [8] also conjectured that the sequence $\{d_i(m)\}_{i=0}^m$ is ∞-log-concave, and this conjecture is still open.

Motivated by Newton's inequality and the infinite log-concavity conjecture on binomial numbers proposed by Boros and Moll [8], Fisk [22], McNamara and Sagan [30] and Stanley (see [9]) independently gave a general conjecture, which states that if a polynomial $\sum_{k=0}^{n} a_k x^k$ has only real and negative zeros, then so does the polynomial $\sum_{k=0}^{n} (a_k^2 - a_{k-1}a_{k+1}) z^k$ where $a_{-1} = a_{n+1} = 0$. This conjecture was confirmed by Brändén [9]. Brändén's Theorem provides an approach to ∞-log-concavity of a sequence by relating real-rooted polynomials to higher-order log-concavity.

Although, as shown by Boros and Moll [5], the polynomials $P_m(x)$ are not real-rooted in general, Brändén introduced two polynomials

$$Q_m(x) = \sum_{i=0}^{m} \frac{d_i(m)}{i!} x^i, \quad R_m(x) = \sum_{i=0}^{m} \frac{d_i(m)}{(i+2)!} x^i, \quad m \ge 1,$$

derived from $P_m(x)$ and conjectured their real-rootedness [9, Conjectures 8.5 & 8.6]. Brändén's conjectures were proved by Chen, Dou and Yang [13]. As noted by Brändén [9], based on two theorems of Craven and Csordas [21] on iterated Turán inequalities, the real-rootedness of $Q_m(x)$ and $R_m(x)$ imply, respectively, the 2-log-concavity and the 3-log-concavity of $P_m(x)$. In another direction, Chen and Xia [19] showed an analytic proof of the

2-log-concavity of the Boros-Moll sequences by founding an intermediate function $f(m,i)$, so that the quartic inequalities for the 2-log-concavity is reduced to quadratic inequalities.

Since $d_i(m)$ has two parameters, it is natural to consider properties for the sequences $\{d_i(m)\}_{m\geq i}$, which was called *transposed Boros-Moll sequences* in [41] where the extended ultra log-concavity and the asymptotic ratio-log-convexity for $\{d_i(m)\}_{m\geq i}$ was proved, and the asymptotic log-convexity for $\{\sqrt[n]{d_i(i+n)}\}_{n\geq i^2}$ was also obtained. Note that recently, Jiang and Wang [27] proved the 2-log-concavity, the higher order Turán inequality and the Laguerre inequality of lower order for the sequences $\{d_i(m)\}_{m\geq i}$.

As mentioned above, both the Boros-Moll sequence $\{d_i(m)\}_{i=0}^m$ and its transpose $\{d_i(m)\}_{m\geq i}$ are 2-log-concave, i.e., both of them satisfy (1.3). This motivated us to study whether they also satisfy the Briggs inequality (1.4). On the other hand, since Briggs' conjecture on (1.4) is still open for real polynomials with only negative zeros, it is also desirable to know whether $Q_m(x)$ and $R_m(x)$ satisfy the Briggs inequality. For this purpose, we define the normalized Boros-Moll sequence $\{\tilde{d}_{i,k}(m)\}_{i=1}^m$ as $\{d_i(m)/(i+k)!\}_{i=1}^m$ for any $k\geq 0$. Our main results are stated as follows.

Theorem 1.1. *For each $m\geq 2$, the Boros-Moll sequence $\{d_i(m)\}_{i=1}^m$ satisfies the Briggs inequality. That is, for any $m\geq 2$ and $1\leq i\leq m-1$,*

$$d_i^2(m)(d_i^2(m)-d_{i-1}(m)d_{i+1}(m))>d_{i-1}^2(m)(d_{i+1}^2(m)-d_i(m)d_{i+2}(m)). \tag{1.9}$$

Theorem 1.2. *For any $k\geq 0$ and $m\geq 2$, the normalized Boros-Moll sequence $\{\tilde{d}_{i,k}(m)\}_{i=1}^m$ satisfies the Briggs inequality. That is, for any $m\geq 2$, $k\geq 0$ and $1\leq i\leq m-1$,*

$$\tilde{d}_{i,k}^2(m)\big(\tilde{d}_{i,k}^2(m)-\tilde{d}_{i-1,k}(m)\tilde{d}_{i+1,k}(m)\big)>\tilde{d}_{i-1,k}^2(m)\big(\tilde{d}_{i+1,k}^2(m)-\tilde{d}_{i,k}(m)\tilde{d}_{i+2,k}(m)\big). \tag{1.10}$$

Theorem 1.3. *For each $i\geq 1$, the transposed Boros-Moll sequence $\{d_i(m)\}_{m\geq i}$ satisfies the Briggs inequality. That is, for any $i\geq 1$ and $m\geq i+1$,*

$$d_i^2(m)(d_i^2(m)-d_i(m-1)d_i(m+1))>d_i^2(m-1)(d_i^2(m+1)-d_i(m)d_i(m+2)). \tag{1.11}$$

However, for $i=0$, the inverse relation of (1.11) holds.

In the remainder of this paper, we first complete the proofs of Theorems 1.1 and 1.2, in Section 2, by employing a lower bound for $(d_{i-1}(m)d_{i+1}(m))/d_i^2(m)$ given by Chen and Gu [14] and an upper bound given by Zhao [40]. In Section 3 we establish the strict ratio-log-convexity of the transposed Boros-Moll sequence $\{d_i(m)\}_{m\geq i}$, which is the key ingredient of the proof of Theorem 1.3. The complete proof of Theorem 1.3 will be presented in Section 4. At last, in Section 5, we derive the strict log-convexity of the sequence $\{\sqrt[n]{d_i(i+n)}\}_{n\geq 1}$ for each $i\geq 1$ from the strict ratio-log-convexity of $\{d_i(m)\}_{m\geq i}$.

2. Proofs of Theorems 1.1 and 1.2

This section is devoted to proving the Briggs inequality for the Boros-Moll sequence $\{d_i(m)\}_{i=1}^m$ and the normalized Boros-Moll sequence $\{\tilde{d}_{i,k}(m)\}_{i=1}^m$.

Let us first prove Theorem 1.1. In order to do so, we need proper bounds for $(d_{i-1}(m)d_{i+1}(m))/d_i^2(m)$. Chen and Gu [14] established the following inequality in studying the reverse ultra log-concavity of the Boros-Moll polynomials.

Theorem 2.1. ([14, Theorem 1.1]) *For $m \geq 2$ and $1 \leq i \leq m-1$, there holds*

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m-i+1)(i+1)}{(m-i)i}. \tag{2.1}$$

The following result was obtained by Zhao [40], which was used to give a new proof of the higher order Turán inequality for the Boros-Moll sequence.

Theorem 2.2. ([40, Theorem 3.1]) *For each $m \geq 2$ and $1 \leq i \leq m-1$, we have*

$$\frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} > \frac{(m-i+1)(i+1)(m+i^2)}{(m-i)i(m+i^2+1)}. \tag{2.2}$$

We are now in a stage to show a proof of Theorem 1.1.

Proof of Theorem 1.1. By (1.8), it is clear that $d_i(m) > 0$ for $m \geq 1$ and $1 \leq i \leq m$. So, for any $m \geq 2$ and $1 \leq i \leq m-1$, the inequality (1.9) can be rewritten as

$$\big(u_{i+1}(m) - 1\big)u_i(m) - 1 + \frac{1}{u_i(m)} > 0, \tag{2.3}$$

where

$$u_i(m) = \frac{d_{i-1}(m)d_{i+1}(m)}{d_i^2(m)}. \tag{2.4}$$

We aim to prove (2.3) for $m \geq 2$ and $1 \leq i \leq m-1$. As will be seen, the bounds given by Theorems 2.1 and 2.2 are crucial to the proof.

For any $1 \leq i \leq m-1$ and $m \geq 2$, it is routine to verify that

$$(m-i+1)(i+1)(m+i^2) > (m-i)i(m+i^2+1)$$

and hence by (2.2) we have

$$0 < u_i(m) < 1. \tag{2.5}$$

It should be noted that (2.5) is also implied by a result due to Chen and Xia [18, Theorem 1.1].

By Theorems 2.1 and 2.2, we have for $m \geq 2$ and $1 \leq i \leq m-1$,

$$f_i(m) < u_i(m) < g_i(m), \tag{2.6}$$

where

$$f_i(m) = \frac{(m-i)i}{(m-i+1)(i+1)}, \quad g_i(m) = \frac{(m-i)i(m+i^2+1)}{(m-i+1)(i+1)(m+i^2)}. \tag{2.7}$$

It follows from (2.5) and (2.6) that for $m \geq 2$ and $1 \leq i \leq m-2$,

$$(u_{i+1}(m) - 1)u_i(m) - 1 + \frac{1}{u_i(m)} > (f_{i+1}(m) - 1)g_i(m) - 1 + \frac{1}{g_i(m)}. \tag{2.8}$$

Denote by $E_i(m)$ the right-hand side of (2.8). Observe that

$$E_i(m) = \frac{A}{i(i+1)(i+2)(m-i)(m-i+1)(m+i^2)(m+i^2+1)}, \tag{2.9}$$

where

$$A = i^5(2m^2 - i^2 + 10m - 4i) + 5i^3m^2(m-i) + i^3m(13m-3i) + 3im^3(m-i) + 4i^5 + 4i^4 + 4i^3m + 4i^2m^2 + 2im^3 + 2m^4 + i^3 + 5i^2m + m^2(4m-i) + 2m^2.$$

For $m \geq 2$ and $1 \leq i \leq m-2$, it is clear that $A > 0$, and hence $E_i(m) > 0$. Then we obtain (2.3) by (2.8) for $m \geq 2$ and $1 \leq i \leq m-2$.

It remains to prove (2.3) for $m \geq 2$ and $i = m-1$. In this case, note that

$$u_{i+1}(m) = u_m(m) = \frac{d_{m-1}(m)d_{m+1}(m)}{d_m^2(m)} = f_{i+1}(m) = f_m(m) = 0.$$

Clearly, the inequality in (2.8) still holds for $m \geq 2$ and $i = m-1$. It is easily checked that $E_i(m) > 0$ for $m \geq 2$ and $i = m-1$. Again, we arrive at (2.3) by (2.8). This completes the proof. □

We proceed to prove Theorem 1.2.

Proof of Theorem 1.2. Fix $k \geq 0$ throughout this proof. Similar to the proof of Theorem 1.1, for any $m \geq 2$ and $1 \leq i \leq m-1$, we reformulate the inequality (1.10) as

$$\big(u_{i+1,k}(m) - 1\big)u_{i,k}(m) - 1 + \frac{1}{u_{i,k}(m)} > 0, \tag{2.10}$$

where

$$u_{i,k}(m) = \frac{i+k}{i+k+1}u_i(m). \tag{2.11}$$

We aim to prove (2.10) for $m \geq 2$ and $1 \leq i \leq m-1$. By (2.5) and (2.11), we see that

$$0 < u_{i,k}(m) < 1. \tag{2.12}$$

Moreover, from the relations (2.6) and (2.11) we have that for $m \geq 2$ and $1 \leq i \leq m-1$,

$$f_{i,k}(m) < u_{i,k}(m) < g_{i,k}(m), \tag{2.13}$$

where

$$f_{i,k}(m) = \frac{i+k}{i+k+1}f_i(m), \quad g_{i,k}(m) = \frac{i+k}{i+k+1}g_i(m). \tag{2.14}$$

Combining (2.12) and (2.13), we deduce that for $m \geq 2$ and $1 \leq i \leq m-2$,

$$\big(u_{i+1,k}(m) - 1\big)u_{i,k}(m) - 1 + \frac{1}{u_{i,k}(m)} > \big(f_{i+1,k}(m) - 1\big)g_{i,k}(m) - 1 + \frac{1}{g_{i,k}(m)}. \tag{2.15}$$

Let $E_{i,k}(m)$ denote the right-hand side of (2.15). Observe that

$$E_{i,k}(m) = \frac{B_0 + B_1 \cdot k + B_2 \cdot k^2 + A \cdot k^3}{i(i+1)(i+2)(m-i)(m-i+1)(m+i^2)(m+i^2+1)(i+k)(i+k+1)(i+k+2)},$$

where A remains the same as (2.9), and B_0, B_1, B_2 are defined as follows:

$$B_0 = 9i^8m^2 + 19i^6m^3 - 14i^6m^2 + 42i^5m^3 + 40i^5m^2 + 10i^4m^4 + 29i^4m^3 + 69i^4m^2 + 32i^3m^4 + 30i^3m^3 + 46i^3m^2 + 37i^2m^4 + 42i^2m^3 + 17i^2m^2 - 10i^9m - 4i^8m + 27i^7m + 15i^6m$$

$$
\begin{aligned}
&+ 7i^5m + 21i^4m + 28i^3m + 8i^2m + 2i^{10} - 5i^9 - 22i^8 - 12i^7 + 12i^6 + 17i^5 + 4i^4 \\
&+ 20im^4 + 32im^3 + 12im^2 + 4m^4 + 8m^3 + 4m^2 \\
B_1 &= i^7(17m^2 - 14im + 24m - 24i) + i^4(51m^3 - 38i^3) + i^5m^2(37m - 11i) + 59i^6m + 20i^3m^4 \\
&+ 3i^5m^2 + 8i^6 + 8i^5m + 110i^4m^2 + 16i^3m^3 + 34i^5 + 25i^4m + 87i^3m^2 + 42i^2m^3 + 37im^4 \\
&+ 16i^4 + 50i^3m + 28i^2m^2 + 54im^3 + 48i^2m^4 + 10m^4 + 24i^2m + 17im^2 + 20m^3 + 10m^2 \\
B_2 &= i^6(10m^2 - 4im - 3i^2 + 34m - 21i) + i^5(21m - 8i) + i^4m^2(23m - 14i) + 30i^4m^2 \\
&+ 10i^3m^3 + 13i^2m^4 + 20i^5 + i^4m + 62i^3m^2 + i^2m^3 + 20im^4 + 15i^4 + 27i^3m + 17i^2m^2 \\
&+ 24im^3 + 8m^4 + i^3 + 21i^2m + 4im^2 + 16m^3 + 8m^2.
\end{aligned}
$$

For any $m \geq 2$ and $1 \leq i \leq m-2$, it is clear that $A,\ B_1,\ B_2 > 0$. To show $B_0 > 0$, notice that

$$B_0 = (m-i)C + D,$$

where

$$
\begin{aligned}
C &= i^8(9m + 15 - i) + i^7(19m + 65) + i^6\left(19m^2 + 38m + 116\right) + i^5\left(52m^2 + 101m + 143\right) \\
&+ i^4\left(10m^3 + 61m^2 + 136m + 129\right) + i^3\left(32m^3 + 67m^2 + 108m + 81\right) \\
&+ i^2\left(37m^3 + 62m^2 + 53m + 28\right) + 4i\left(5m^3 + 9m^2 + 5m + 1\right) + 4m(m+1)^2, \\
D &= 4i^2 + 28i^3 + 85i^4 + 146i^5 + 155i^6 + 104i^7 + 43i^8 + 10i^9 + i^{10}.
\end{aligned}
$$

Hence, $E_{i,k}(m) > 0$. Then (2.10) follows from (2.15) for $k \geq 0$, $m \geq 2$ and $1 \leq i \leq m-2$.

It remains to verify (2.10) for $k \geq 0$, $m \geq 2$ and $i = m-1$. In this case, note that

$$u_{i+1,k}(m) = u_{m,k}(m) = \frac{m+k}{m+k+1}u_m(m) = f_{i+1,k}(m) = f_{m,k}(m) = \frac{m+k}{m+k+1}f_m(m) = 0.$$

Clearly, the inequality (2.15) still holds for $k \geq 0$, $m \geq 2$ and $i = m-1$. It is easy to verify that $E_{m-1,k}(m) > 0$ for $k \geq 0$ and $m \geq 2$. Again, we arrive at (2.10) by (2.15). This completes the proof. □

3. Strict ratio-log-convexity of $\{d_i(m)\}_{m\geq i}$

The aim of this section is to prove the strict ratio-log-convexity of transposed Boros-Moll sequences $\{d_i(m)\}_{m\geq i}$ for $i \geq 1$. As will be seen in Section 4, this property plays a key role in our proof of Theorem 1.3.

Chen, Guo and Wang [15] initiated the study of ratio log-behavior of combinatorial sequences. Recall that a real sequence $\{a_n\}_{n\geq 0}$ is called ratio log-concave (resp. ratio log-convex) if the sequence $\{a_n/a_{n-1}\}_{n\geq 1}$ is log-concave (resp. log-convex). They also showed that the ratio log-concavity (resp. ratio log-convexity) of a sequence $\{a_n\}_{n\geq N}$ implies the strict log-concavity (resp. strict log-convexity) of the sequence $\{\sqrt[n]{a_n}\}_{n\geq N}$ under certain initial condition [15]. By applying these criteria, they confirmed a conjecture of Sun [36] on the log-concavity of the sequence $\{\sqrt[n]{D_n}\}_{n\geq 1}$ for the Domb numbers D_n.

Partial progress has been made on the ratio log-behaviour of the sequence $\{d_i(m)\}_{m\geq i}$. Zhao [41, Theorem 6.1] proved that the sequence $\{d_i(m)\}_{m\geq i}$ is strictly ratio log-concave for $i = 0$, while it is strictly ratio log-convex for each $1 \le i \le 135$. For $i \ge 136$, Zhao [41, Theorem 6.2] proved an asymptotic result for this property for $m \ge \lfloor(\sqrt{2}/4)i^{3/2} - 15i/32\rfloor + 2$. With a sharper bound for $d_i(m)/d_i(m-1)$, we obtain the following exact result for this property.

Theorem 3.1. *The transposed Boros-Moll sequences $\{d_i(m)\}_{m\ge i}$ are strictly ratio-log-convex for any $i \ge 1$. That is, for each $i \ge 1$ and $m \ge i+2$,*

$$\left(\frac{d_i(m)}{d_i(m-1)}\right)^2 < \left(\frac{d_i(m-1)}{d_i(m-2)}\right)\left(\frac{d_i(m+1)}{d_i(m)}\right). \tag{3.1}$$

We shall prove Theorem 3.1 by applying the following criterion for the ratio log-convexity of a sequence obtained by Sun and Zhao [35], which was deduced along with the spirit of Chen, Guo, and Wang [15, Sectioin 4].

Theorem 3.2. ([35, Theorem 4.2]) *Let $\{S_n\}_{n\ge 0}$ be a positive sequence satisfying the following recurrence relation,*

$$S_n = \mathfrak{a}(n)S_{n-1} + \mathfrak{b}(n)S_{n-2}, \quad n \ge 2,$$

with real $\mathfrak{a}(n)$ and $\mathfrak{b}(n)$. Suppose $\mathfrak{a}(n) > 0$ and $\mathfrak{b}(n) < 0$ for $n \ge N$ where N is a nonnegative integer. If there exists a function $g(n)$ such that for all $n \ge N+2$,

(i) $\frac{\mathfrak{a}(n)}{2} \le g(n) \le \frac{S_n}{S_{n-1}}$;
(ii) $4g^3(n) - 3\mathfrak{a}(n)g^2(n) - \mathfrak{a}(n+1)\mathfrak{b}(n) \ge 0$;
(iii) $g^4(n) - \mathfrak{a}(n)g^3(n) - \mathfrak{a}(n+1)\mathfrak{b}(n)g(n) - \mathfrak{b}(n)\mathfrak{b}(n+1) \ge 0$,

then $\{S_n\}_{n\ge N}$ is ratio log-convex, that is, for $n \ge N+2$,

$$(S_n/S_{n-1})^2 \le (S_{n-1}/S_{n-2})(S_{n+1}/S_n). \tag{3.2}$$

Remark 3.3. *By the proof of Theorem 3.2, it is easy to see that the inequality in (3.2) holds strictly if the inequality in condition (iii) is strict.*

To apply Theorem 3.2, we shall employ the following recursion found by Kauers and Paule [28] with a computer algebraic system, and independently obtained by Moll [32] via the WZ-method [37].

Theorem 3.4. ([28, Eq. (8)]) *For $m \ge 2$ and $1 \le i \le m-1$, there holds*

$$d_i(m) = \frac{8m^2 - 8m - 4i^2 + 3}{2m(m-i)}d_i(m-1) - \frac{(4m-5)(4m-3)(m-1+i)}{4m(m-1)(m-i)}d_i(m-2). \tag{3.3}$$

The following lower bound for the ratio $d_i(m+1)/d_i(m)$ derived by Zhao [40] is crucial to our proof of Theorem 3.1.

Theorem 3.5. ([40, Theorem 2.1]) *For any $m \ge 2$ and $1 \le i \le m-1$,we have*

$$\frac{d_i(m+1)}{d_i(m)} > L(m,i), \tag{3.4}$$

where

$$L(m,i)=\frac{4m^2+7m-2i^2+3}{2(m+1)(m-i+1)}+\frac{i\sqrt{4i^4+8i^2m+5i^2+m}}{2(m+1)(m-i+1)\sqrt{m+i^2}}. \tag{3.5}$$

We are now ready to present a proof of Theorem 3.1.

Proof of Theorem 3.1. Fix $i\geq 1$ throughout this proof. To apply Theorem 3.2, let $S_n=d_i(n)$ and set $N=i$. By Theorem 3.4, we have

$$d_i(n)=\mathfrak{a}(n)d_i(n-1)+\mathfrak{b}(n)d_i(n-2),$$

for $n\geq i+2$, where

$$\mathfrak{a}(n)=\frac{8n^2-8n-4i^2+3}{2n(n-i)},\qquad \mathfrak{b}(n)=-\frac{(4n-5)(4n-3)(n-1+i)}{4n(n-1)(n-i)}. \tag{3.6}$$

Clearly, $\mathfrak{a}(n)>0$ and $\mathfrak{b}(n)<0$ for $n\geq i+2$.

For any $n\geq i+2$, we first prove the conditions (i) and (ii) in Theorem 3.2. For this purpose, let

$$g(n)=L(n-1,i)=\frac{4n^2-2i^2-n}{2n(n-i)}+\frac{i\sqrt{4i^4+8i^2n-3i^2+n-1}}{2n(n-i)\sqrt{i^2+n-1}}. \tag{3.7}$$

By Theorem 3.5, we have

$$g(n)<\frac{d_i(n)}{d_i(n-1)},$$

for $n\geq i+2$. On the other hand, we see that

$$g(n)-\frac{\mathfrak{a}(n)}{2}=\frac{(6n-3)\Delta_1+2i\Delta_2}{4n(n-i)\Delta_1}>0, \tag{3.8}$$

for $n\geq i+2$, where

$$\Delta_1=\sqrt{i^2+n-1},\quad \Delta_2=\sqrt{4i^4+8i^2n-3i^2+n-1}. \tag{3.9}$$

So the inequalities in condition (i) of Theorem 3.2 are confirmed.

We proceed to check the condition (ii). Direct computation gives that

$$\begin{aligned}&4g^3(n)-3\mathfrak{a}(n)g^2(n)-\mathfrak{a}(n+1)\mathfrak{b}(n)\\ &=\frac{\Delta_1(F_1(n-i-1)+\hat{F}_1)+\Delta_2(F_2(n-i-1)i+\hat{F}_2)}{8(i^2+n-1)^{\frac{3}{2}}n^3(n-i)^3(n+1-i)(n^2-1)},\end{aligned} \tag{3.10}$$

where Δ_1, Δ_2 are given by (3.9), and

$$\begin{aligned}F_1&=240n^7+(240i^2+256i-364)n^6+(160i^4+256i^3-164i^2-32i+32)n^5\\&+(280i^4+352i^3+676i^2-392i+166)n^4+32(2n^2-i^2)i^6n^2\\&+(64i^6+448i^5+1004i^4+104i^3-64i^2+256i-83)n^3\\&+(128i^7+456i^6+1168i^5+798i^4+984i^3-203i^2-18i+9)n^2\\&+(64i^8+576i^7+1568i^6+2440i^5+1596i^4+442i^3+103i^2)n\end{aligned}$$

$$
\begin{aligned}
&\quad + 64i^9 + 672i^8 + 2208i^7 + 3996i^6 + 3748i^5 + 2251i^4 + 560i^3 + 79i^2,\\
\hat{F}_1 &= 64i^{10} + 704i^9 + 2912i^8 + 6152i^7 + 7796i^6 + 6074i^5 + 2736i^4 + 648i^3 + 70i^2,\\
F_2 &= 144n^6 + 36n^5 + 18(6i^2 + 16i - 13)n^4 + (288i^3 + 202i^2 + 72i - 18)n^3\\
&\quad + (144n^2 - 40i^2)i^2n^3 + (16i^6 + 232i^4 + 504i^3 + 400i^2 - 180i + 90)n^2\\
&\quad + (32i^6 + 208i^5 + 800i^4 + 764i^3 + 230i^2 + 36i - 18)n\\
&\quad + 4i^2(8i^5 + 56i^4 + 242i^3 + 395i^2 + 281i + 44),\\
\hat{F}_2 &= 4i^3(8i^6 + 68i^5 + 294i^4 + 643i^3 + 670i^2 + 315i + 54).
\end{aligned}
$$

Clearly, Δ_1, Δ_2, $\hat{F}_1$, $\hat{F}_2$ and the denominator of the right-hand side of (3.10) are positive for $n \geq i+2$. It is easy to see that $F_1 > 0$, $F_2 > 0$ for $n \geq i+2$. It follows that (3.10) is positive for $n \geq i+2$, which leads to the condition (ii).

It remains to prove that the conditions (iii) in Theorem 3.2 holds for $n \geq i+2$. For convenience, define a function

$$h(x) := x^4 - \mathfrak{a}(n)x^3 - \mathfrak{a}(n+1)\mathfrak{b}(n)x - \mathfrak{b}(n)\mathfrak{b}(n+1), \tag{3.11}$$

for $x \in \mathbb{R}$. By computation we get

$$h(g(n)) = \frac{G_1 + \Delta_1\Delta_2 G_2}{16(n^2-1)(n+1-i)(i^2+n-1)^2 n^4 (n-i)^4}, \tag{3.12}$$

where Δ_1, Δ_2 are given by (3.9), and

$$
\begin{aligned}
G_1 &= (512i - 128)n^{10} + (1024i^3 - 768i^2 - 1664i + 400)n^9\\
&\quad + (512i^5 - 928i^4 - 2304i^3 + 2880i^2 + 1392i - 328)n^8\\
&\quad + (-352i^6 - 864i^5 + 4424i^4 - 192i^3 - 3584i^2 + 776i - 173)n^7\\
&\quad + (-64i^8 - 160i^7 + 2840i^6 - 1896i^5 - 6252i^4 + 4448i^3 + 922i^2 - 1765i + 410)n^6\\
&\quad + (64i^9 + 688i^8 - 1528i^7 - 4340i^6 + 6276i^5 + 2279i^4 - 4694i^3 + 1440i^2 + 885i - 224)n^5\\
&\quad + (64i^{10} - 496i^9 - 1336i^8 + 4396i^7 + 1592i^6 - 6645i^5 + 1812i^4 + 2068i^3\\
&\qquad - 1075i^2 - 139i + 46)n^4\\
&\quad + (-64i^{11} - 80i^{10} + 1320i^9 + 332i^8 - 4396i^7 + 1554i^6 + 3155i^5\\
&\qquad - 1916i^4 - 305i^3 + 149i^2 + 3i - 3)n^3\\
&\quad + (144i^{11} + 48i^{10} - 1412i^9 + 672i^8 + 2152i^7 - 1615i^6 - 579i^5 + 599i^4 - 54i^3 + 45i^2)n^2\\
&\quad + (-192i^{11} + 88i^{10} + 740i^9 - 551i^8 - 537i^7 + 471i^6 - 20i^5 + i^4 + 9i^3 - 9i^2)n\\
&\quad + 104i^{11} - 104i^{10} - 189i^9 + 189i^8 + 66i^7 - 66i^6 + 19i^5 - 19i^4\\
G_2 &= 240in^8 + (240i^3 + 16i^2 - 604i)n^7 + (32i^5 + 16i^4 - 676i^3 + 76i^2 + 396i)n^6\\
&\quad + (-32i^6 - 280i^5 + 276i^4 + 864i^3 - 392i^2 + 134i)n^5\\
&\quad + (-32i^7 + 184i^6 + 564i^5 - 932i^4 - 310i^3 + 482i^2 - 249i)n^4\\
&\quad + (32i^8 + 40i^7 - 492i^6 - 112i^5 + 914i^4 - 379i^3 - 191i^2 + 92i)n^3
\end{aligned}
$$

$$+ (-72i^8 - 24i^7 + 484i^6 - 276i^5 - 325i^4 + 296i^3 + 9i^2 - 9i)n^2$$
$$+ (96i^8 - 44i^7 - 208i^6 + 159i^5 + 29i^4 - 32i^3)n - 52i^8 + 52i^7 + 49i^6 - 49i^5 + 3i^4 - 3i^3.$$

Clearly, the denominator of (3.12) is positive for $n \geq i + 2$. It suffices to show that

$$G_1 + \Delta_1\Delta_2 G_2 > 0, \qquad \text{for } n \geq i + 2. \tag{3.13}$$

Observe that

$$\begin{aligned} \Delta_1\Delta_2 &= \sqrt{i^2 + n - 1}\sqrt{(4i^2 + 1)(i^2 + n - 1) + 4i^2 n} \\ &= 2i(i^2 + n - 1)\sqrt{1 + \frac{1}{4i^2} + \frac{n}{i^2 + n - 1}} \\ &> 2i(i^2 + n - 1), \end{aligned} \tag{3.14}$$

for $n \geq i + 2$. Also note that

$$G_2 = i(n - i - 1)H_1 + H_2,$$

where

$$\begin{aligned} H_1 = {} & 240n^7 + (240i^2 + 256i - 364)n^6 + (256i^3 - 180i^2 - 32i + 32)n^5 \\ & + (8i^4 + 352i^3 + 652i^2 - 392i + 166)n^4 + 32(n^2 - i^2)i^4 n^3 \\ & + (192i^5 + 924i^4 + 72i^3 - 50i^2 + 256i - 83)n^3 \\ & + (200i^6 + 624i^5 + 884i^4 + 936i^3 - 173i^2 - 18i + 9)n^2 \\ & + (128i^7 + 800i^6 + 1992i^5 + 1544i^4 + 438i^3 + 105i^2)n \\ & + 128i^8 + 1024i^7 + 2748i^6 + 3328i^5 + 2141i^4 + 572i^3 + 73i^2, \\ H_2 = {} & 2i^3(64i^7 + 576i^6 + 1860i^5 + 3064i^4 + 2759i^3 + 1332i^2 + 324i + 35). \end{aligned}$$

Clearly, $H_1 > 0$, $H_2 > 0$, and hence $G_2 > 0$ for $n \geq i + 2$. It follows from (3.14) that

$$G_1 + \Delta_1\Delta_2 G_2 > G_1 + 2i(i^2 + n - 1)G_2. \tag{3.15}$$

It remains to show that

$$G_1 + 2i(i^2 + n - 1)G_2 > 0, \quad n \geq i + 2. \tag{3.16}$$

To this end, we rewrite the left hand side as

$$G_1 + 2i(i^2 + n - 1)G_2 = (n - i - 2)K_1 + K_2,$$

where

$$\begin{aligned} K_1 = {} & (512i - 128)n^9 + (1024i^3 + 224i^2 - 768i + 144)n^8 + (512i^5 + 1056i^4 + 872i^2 - 40)n^7 \\ & + (704i^6 + 1280i^5 + 3496i^4 + 800i^3 + 160i^2 + 736i - 253)n^6 \\ & + (512i^7 + 3552i^6 + 4832i^5 + 5412i^4 + 5272i^3 + 1454i^2 - 546i - 96)n^5 \\ & + (576i^8 + 4032i^7 + 11012i^6 + 18152i^5 + 16295i^4 + 9052i^3 + 3036i^2 - 303i - 416)n^4 \\ & + (512i^9 + 5120i^8 + 20256i^7 + 39796i^6 + 50610i^5 + 42818i^4 + 21862i^3 \\ & \quad + 5376i^2 - 1161i - 786)n^3 \end{aligned}$$

$$
\begin{aligned}
&+(512i^{10}+6272i^9+30476i^8+79692i^7+130670i^6+144333i^5\\
&\qquad+107116i^4+49195i^3+9538i^2-3105i-1575)n^2\\
&+(512i^{11}+7296i^{10}+42912i^9+140724i^8+290172i^7+405520i^6+395929i^5\\
&\qquad+263352i^4+107856i^3+16034i^2-7785i-3150)n\\
&+512i^{12}+8320i^{11}+57504i^{10}+226576i^9+571579i^8+985899i^7\\
&\qquad+1206960i^6+1055138i^5+634619i^4+231755i^3+24274i^2-18720i-6300,\\
K_2=\,&512i^{13}+9344i^{12}+74144i^{11}+341584i^{10}+1024744i^9+2129044i^8+3178732i^7\\
&+3469084i^6+2744908i^5+1500980i^4+487784i^3+29828i^2-43740i-12600.
\end{aligned}
$$

It is straightforward to check that $K_1 > 0$, $K_2 > 0$ for $n \geq i+2$, and hence

$$G_1 + 2i(i^2+n-1)G_2 > 0, \quad n \geq i+2.$$

So (3.16) is proved. Combining (3.12), (3.15) and (3.16), we have $h(g(n)) > 0$ for $n \geq i+2$, that is, the condition (iii) in Theorem 3.2 holds for $n \geq i+2$. It follows from Theorem 3.2 and Remark 3.3 that (3.2) holds strictly for $S_n = d_i(n)$. Therefore, we obtain the strict ratio-log-convexity of the transposed Boros-Moll sequence $\{d_i(m)\}_{m\geq i}$ for each $i \geq 1$. □

4. Proof of Theorem 1.3

In this section we proceed to prove the Briggs inequality for the transposed Boros-Moll sequence $\{d_i(m)\}_{m\geq i}$ for $i \geq 1$. Comparing with the proof of Theorem 1.1, it is natural to consider the equivalent form of (1.11), that is, for $i \geq 1$ and $m \geq i+1$,

$$(v_i(m+1)-1)v_i(m) - 1 + \frac{1}{v_i(m)} > 0, \tag{4.1}$$

where $v_i(m) = (d_i(m-1)d_i(m+1))/d_i^2(m)$. However, numerical experiment shows that the left-hand side of (4.1) trends to zero very fast, and the known bounds for $v_i(m)$ are not sufficiently sharp for the proof of (4.1). In order to prove Theorem 1.3, we find the following sufficient conditions for the Briggs inequality.

Theorem 4.1. *Let $\{a_n\}_{n\geq 0}$ be a sequence with positive numbers. Let N_0 be a positive integer. If the following conditions are satisfied:*

(i) *$\{a_n\}_{n\geq N_0}$ is strictly log-concave,*
(ii) *$\{a_n\}_{n\geq N_0}$ is strictly ratio-log-convex,*

then the Briggs inequality holds for $\{a_n\}_{n\geq N_0}$. That is, for $n \geq N_0+1$,

$$a_n^2(a_n^2 - a_{n-1}a_{n+1}) > a_{n-1}^2(a_{n+1}^2 - a_n a_{n+2}). \tag{4.2}$$

Moreover, if $\{a_n\}_{n\geq N_0}$ is strictly 2-log-convex and condition (ii) holds, then the Briggs inequality (4.2) holds for $n \geq N_0+2$.

Proof. Let $\{a_n\}_{n\geq 0}$ be a sequence with $a_n > 0$ for all n. Let $N_0 > 0$ be an integer. We first prove the first result that conditions (i) and (ii) imply (4.2) for $n \geq N_0+1$. By condition

(i), the sequence $\{a_n\}_{n\geq N_0}$ is strictly log-concave, that is, for $n \geq N_0+1$,

$$a_n^2 - a_{n-1}a_{n+1} > 0. \tag{4.3}$$

Then we have for $n \geq N_0+1$,

$$\frac{a_n^2}{a_{n-1}^2} > \frac{a_{n+1}^2}{a_n^2}. \tag{4.4}$$

On the other hand, by condition (ii), the sequence $\{a_n\}_{n\geq N_0}$ is strictly ratio-log-convex, implying that

$$\frac{a_{n-2}a_n}{a_{n-1}^2} < \frac{a_{n-1}a_{n+1}}{a_n^2}, \quad n \geq N_0+2. \tag{4.5}$$

Hence for $n \geq N_0+1$,

$$\frac{a_{n-1}a_{n+1}}{a_n^2} < \frac{a_n a_{n+2}}{a_{n+1}^2}. \tag{4.6}$$

It follows from (4.6) and (4.3) that

$$1 - \frac{a_{n-1}a_{n+1}}{a_n^2} > 1 - \frac{a_n a_{n+2}}{a_{n+1}^2} > 0, \tag{4.7}$$

for $n \geq N_0+1$. Combining (4.4) and (4.7), we obtain

$$\frac{a_n^2}{a_{n-1}^2}\left(1 - \frac{a_{n-1}a_{n+1}}{a_n^2}\right) > \frac{a_{n+1}^2}{a_n^2}\left(1 - \frac{a_n a_{n+2}}{a_{n+1}^2}\right). \tag{4.8}$$

Multiply $a_n^2 a_{n-1}^2$ on both sides of (4.8). Then we have (4.2) for $n \geq N_0+1$, and hence the first result is proved.

We proceed to show the second statement that the strict 2-log-convexity of $\{a_n\}_{n\geq N_0}$ together with condition (ii) imply the Briggs inequality (4.2) for $n \geq N_0+2$. Assume that $\{a_n\}_{n\geq N_0}$ is strictly 2-log-convex, that is, for $n \geq N_0+2$,

$$\frac{a_n^2 - a_{n-1}a_{n+1}}{a_{n-1}^2 - a_{n-2}a_n} < \frac{a_{n+1}^2 - a_n a_{n+2}}{a_n^2 - a_{n-1}a_{n+1}}. \tag{4.9}$$

To make the proof more concise, let

$$c_n = \frac{a_{n-1}a_{n+1}}{a_n^2}, \quad n \geq 1. \tag{4.10}$$

Thus, (4.9) can be rewritten as

$$\frac{1-c_n}{1-c_{n-1}} < c_n^2 \frac{1-c_{n+1}}{1-c_n}, \tag{4.11}$$

for $n \geq N_0+2$. Clearly, $\{a_n\}_{n\geq N_0}$ is strictly log-convex. Then we have $c_n > 1$ for $n \geq N_0+1$. So, the inequality (4.11) is equivalent to

$$(c_{n+1}-1)c_n^2 > \frac{(c_n-1)^2}{c_{n-1}-1}, \quad n \geq N_0+2. \tag{4.12}$$

On the other hand, by condition (ii), we have (4.5), which can be restated as

$$c_{n-1} < c_n, \quad n \geq N_0 + 2. \tag{4.13}$$

Combining (4.12) and (4.13), one can easily obtain that for $n \geq N_0 + 2$,

$$(c_{n+1} - 1)c_n^2 > \frac{(c_n - 1)^2}{c_n - 1} = c_n - 1. \tag{4.14}$$

Substituting (4.10) into (4.14) leads to the desired inequality (4.2) for $n \geq N_0 + 2$. □

To prove Theorem 1.3, we also need the following result due to Jiang and Wang [27], where the log-concavity of the sequences $\{d_i(m)\}_{m\geq i}$ for $i \geq 1$ was established.

Theorem 4.2. ([27, Theorem 3.1]) *For any $i \geq 1$, the sequence $\{d_i(m)\}_{m\geq i}$ is log-concave.*

Notice that by the proof of Theorem 4.2, it is clear that the sequence $\{d_i(m)\}_{m\geq i}$ is strictly log-concave for any $i \geq 1$. We would like to point out that Zhao [41] obtained an inequality in studying the extended reverse ultra log-concavity of the transposed Boros-Moll sequences $\{d_i(m)\}_{m\geq i}$, which implies the strict log-concavity of $\{d_i(m)\}_{m\geq i}$ for $i \geq 1$.

Theorem 4.3. ([41, Theorem 3.2]) *For each $i \geq 0$ and $m \geq i + 1$, we have*

$$\frac{d_i^2(m)}{d_i(m-1)d_i(m+1)} > \frac{(m-i+1)m^3}{(m-i)(m+1)(m^2+1)}. \tag{4.15}$$

Denote by $R(i,m)$ the right-hand side of (4.15). Clearly, $R(1,m) = m^4/(m^4-1) > 1$ for $m \geq 2$, and it is easily checked that $R(i,m) > (m^2+1)/m^2 > 1$ for $i \geq 2$ and $m \geq i+1$.

We are now in a position to give a proof of Theorem 1.3.

Proof of Theorem 1.3. Fix $i \geq 1$. To apply Theorem 4.1, set $a_n = d_i(n)$ and let $N_0 = i$. By Theorem 4.2 or Theorem 4.3, the sequences $\{d_i(n)\}_{n\geq i}$ are strictly log-concave for $i \geq 1$. By Theorem 3.1, the transposed Boros-Moll sequences $\{d_i(n)\}_{n\geq i}$ are strictly ratio-log-convex for any $i \geq 1$. Thus, for any given $i \geq 1$, it follows from Theorem 4.1 that (4.2) holds for $n \geq i+1$, or equivalently, (1.11) holds for $m \geq i+1$.

It remains to prove the reverse Briggs inequality for $i = 0$. That is, for $m \geq 1$,

$$d_0^2(m)(d_0^2(m) - d_0(m-1)d_0(m+1)) < d_0^2(m-1)(d_0^2(m+1) - d_0(m)d_0(m+2)). \tag{4.16}$$

Observe that (4.16) holds if and only if

$$\delta_0(m) := r_0^2(m)\left(1 - \frac{r_0(m+1)}{r_0(m)}\right) - r_0^2(m+1)\left(1 - \frac{r_0(m+2)}{r_0(m+1)}\right) < 0, \tag{4.17}$$

for $m \geq 1$, where $r_0(m) = d_0(m)/d_0(m-1)$. Recall that for $i \geq 0$ and $m \geq i$, Kauers and Paule [28, Eq. (6)] showed a recurrence relation as follows.

$$d_i(m+1) = \frac{m+i}{m+1}d_{i-1}(m) + \frac{4m+2i+3}{2(m+1)}d_i(m). \tag{4.18}$$

By setting $i = 0$ in (4.18), we have

$$r_0(m) = \frac{4m-1}{2m}, \quad m \geq 1.$$

It is easily checked that

$$\delta_0(m) = -\frac{8m^2+5m-2}{4m^2(m+1)^2(m+2)} < 0,$$

for $m \geq 1$, and hence (4.17) is true. This completes the proof. □

5. Strict log-convexity of $\{\sqrt[n]{d_i(i+n)}\}_{n\geq 1}$

In this section, we show the strict log-convexity of the sequence $\{\sqrt[n]{d_i(i+n)}\}_{n\geq 1}$ for $i \geq 1$, by applying Theorem 3.1 and a sufficient condition given by Chen, Guo and Wang [15]. Note that Zhao [41] proved an asymptotic result of this property for $i \geq 136$.

Theorem 5.1. ([41, Theorems 7.1 & 7.2]) *The sequence $\{\sqrt[n]{d_i(i+n)}\}_{n\geq 1}$ is strictly log-concave for $i = 0$, and is strictly log-convex for each $1 \leq i \leq 135$. For $i \geq 136$, the sequences $\{\sqrt[n]{d_i(i+n)}\}_{n\geq i^2}$ are strictly log-convex.*

The main result of this section is as follows.

Theorem 5.2. *The sequence $\{\sqrt[n]{d_i(i+n)}\}_{n\geq 1}$ is strictly log-convex for each $i \geq 1$. That is, for any $i \geq 1$ and $n \geq 1$, we have*

$$\frac{\sqrt[n+1]{d_i(i+n+1)}}{\sqrt[n]{d_i(i+n)}} < \frac{\sqrt[n+2]{d_i(i+n+2)}}{\sqrt[n+1]{d_i(i+n+1)}}. \tag{5.1}$$

Chen, Guo and Wang [15] established the following criterion which indicates that ratio log-convexity of a sequence $\{S_n\}$ together with certain initial condition imply log-convexity of the sequence $\{\sqrt[n]{S_n}\}$.

Theorem 5.3. ([15, Theorem 3.6]) *Let $\{S_n\}_{n\geq 0}$ be a positive sequence. If the sequence $\{S_n\}_{n\geq N}$ is ratio log-convex and*

$$\frac{\sqrt[N+1]{S_{N+1}}}{\sqrt[N]{S_N}} < \frac{\sqrt[N+2]{S_{N+2}}}{\sqrt[N+1]{S_{N+1}}} \tag{5.2}$$

for some positive integer N, then the sequence $\{\sqrt[n]{S_n}\}_{n\geq N}$ is strictly log-convex.

We are now ready to present a proof of Theorem 5.2.

Proof of Theorem 5.2. For $i \geq 1$, setting $S_n = d_i(i+n)$ and $N = 1$ in Theorem 5.3. By Theorem 3.1, the sequence $\{d_i(i+n)\}_{n\geq 0}$ is strictly ratio log-convex for each $i \geq 1$. It suffices to verify the condition (5.2), or equivalently,

$$\frac{d_i(i+2)}{d_i(i+1)\cdot\sqrt[3]{d_i(i+3)}} < 1, \quad i \geq 1. \tag{5.3}$$

For convenience, denote by $\mathfrak{r}(i)$ the left-hand side of (5.3).

Notice that it was proved in [41, Lemma 7.5] that the sequence $\{\mathfrak{r}(i)\}_{i\geq 0}$ is strictly decreasing. That is, for $i \geq 0$,

$$\frac{d_i(i+2)}{d_i(i+1)\sqrt[3]{d_i(i+3)}} > \frac{d_{i+1}(i+3)}{d_{i+1}(i+2)\sqrt[3]{d_{i+1}(i+4)}}. \tag{5.4}$$

To be self-contained, we make a brief overview of the proof for (5.4). Since $d_i(m) > 0$ for $m \geq i \geq 0$, the inequality (5.4) holds if and only if

$$\frac{d_i(i+2)d_{i+1}(i+2)}{d_i(i+1)d_{i+1}(i+3)} > \frac{\sqrt[3]{d_i(i+3)}}{\sqrt[3]{d_{i+1}(i+4)}}. \tag{5.5}$$

Applying (1.8), it is easy to obtain the expressions of the terms in (5.5). For example,

$$d_i(i+1) = \frac{(2i+3)(2i+1)(2i)!}{2^{i+1}(i+1)(i!)^2}.$$

Denote by V_1 and V_2, respectively, the LHS and RHS of (5.5). We have checked that

$$V_1^3 - V_2^3 = \frac{(i+1)\sum_{k=0}^{18} s_k i^k}{(i+2)^6(2i+3)^3(4i^2+26i+43)^3(4i^2+30i+59)(2i+9)(2i+5)},$$

where $s_k > 0$ are integers for $k = 0, 1, \ldots, 18$. Clearly, $V_1^3 - V_2^3 > 0$ for $i \geq 0$, which leads to (5.5), as well as (5.4).

Using (1.8) again, it is easily verified that

$$\mathfrak{r}(1) = \frac{d_1(3)}{d_1(2) \cdot \sqrt[3]{d_1(4)}} = \frac{43 \cdot 885^{\frac{2}{3}} \cdot 32^{\frac{1}{3}}}{13275} = 0.9 + \epsilon_1, \quad 0 < \epsilon_1 < 10^{-1}.$$

Thus, $\mathfrak{r}(1) < 1$. By (5.4), we see that $\mathfrak{r}(i) < 1$ for all $i \geq 1$, and hence (5.3) is proved. That is, for any $i \geq 1$, the condition in (5.2) is satisfied for $N = 1$. It follows from Theorem 5.3 that the sequence $\{\sqrt[n]{d_i(i+n)}\}_{n\geq 1}$ is strictly log-convex for each $i \geq 1$. The proof is complete. □

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