r-dynamic colorings and the spectral radius in graphs

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Abstract

Let $\chi(G)$ and $\rho(G)$ be the chromatic number and spectral radius of G, respectively. In 1967, Wilf proved that for a graph G, we have $\chi(G) \leq 1 + \rho(G)$. An r-dynamic k-coloring of a graph G is a proper k-coloring of G such that every vertex v in $V(G)$ has neighbors in at least $\min\{d(v), r\}$ different color classes. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the least k such that G has such a k-coloring. Note that $\chi(G) = \chi_1(G)$ and $\chi_r(G) \leq 1 + r\Delta(G)$ (*) ([11, 16]). By the inequality (*), we observe that for a positive integer $r \geq 2$ and a connected graph G, we have $\chi_r(G) \leq 1 + r\rho^2(G)$.

In this paper, for a positive integer $k > r^2$, we provide graphs $H_{k,r}$ with $\chi_r(H_{k,r}) = \Theta(\rho^2(H_{k,r}))$ to show that the bound is almost sharp. When $r = 2$, we prove that $\chi_r(G) \leq 1 + \rho^2(G)$; equality holds only when $G = P_1, P_2, P_3$, or C_5. For $r = 3$ and $\Delta(G) \leq 4$, we prove that $\chi_r(G) \leq 10$; equality holds when G is the Petersen graph. When $r = 3$ and $\Delta(G) \geq 5$, we prove that $\chi_r(G) \leq 2\Delta(G) + 1$, which implies $\chi_r(G) \leq 1 + 2\rho^2(G)$. The graph $H_{k,3}$ guarantees that $\chi_3(G) \leq 2\Delta(G) + 1$ is sharp.

Keywords: r-dynamic coloring, Spectral radius, Steiner System

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1 Introduction

For undefined terms of graph theory, see West [19]. For basic properties of spectral graph theory, see Brouwer and Haemers [4] or Godsil and Royle [8].

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In this paper, we consider a finite, simple, and connected graph. A *k-coloring* of G is a function from $V(G)$ to S, where $|S| = k$; it is *proper* if adjacent vertices receive different colors. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum k such that G has a proper k-coloring. An *r-dynamic k-coloring* is a proper k-coloring of G such that for each vertex v in $V(G)$, at least $\min\{r, d(v)\}$ colors are used on the neighborhood $N(v)$. The *r-dynamic chromatic number* of a graph G, written $\chi_r(G)$, which was introduced by Montgomery [15], is the minimum k such that G admits such a proper k-coloring. In the thesis, he proved that $\chi_2(G) \leq 3 + \Delta(G)$. Later, Lai, Montgomery, and Poon [13] proved that if $\Delta(G) \geq 4$, $\chi_2(G) \leq 1 + \Delta(G)$. For a positive integer $r \geq 2$, Jahanbekam, Kim, O, and West [11] proved that

$$\chi_r(G) \leq 1 + r\Delta(G). \tag{1}$$

For $r = 3$ and a $K_{1,3}$-free graph G, Lai and Li [14] improved the bound; $\chi_3(G) \leq \max\{3 + \Delta(G), 7\}$. Asayama, Kawasaki, Kim, Nakamoto, and Ozeki [3] gave a sharp upper bound for $\chi_3(G)$ in a planar triangulation; $\chi_3(G) \leq 5$. In this paper, we show that for $\Delta(G) \leq 4$, we have $\chi_3(G) \leq 10$ and for $\Delta(G) \geq 5$, we have $\chi_3(G) \leq 1 + 2\Delta(G)$, which gives a relation between $\chi_3(G)$ and the spectral radius of G (see Section 3). For history and recent results on the theory of r-dynamic colorings, we refer the reader to an excellent survey [5] by Chen, Fan, Lai, and Xu.

For a graph G with $V(G) = \{v_1, \ldots, v_n\}$, the *adjacency matrix* of G, written $A(G)$, is defined to be an $n \times n$ matrix whose (i, j)-entry equals 1 if v_i and v_j are adjacent and 0 otherwise. The *eigenvalues* of G are the eigenvalues of its adjacency matrix. Since $A(G)$ is real and symmetric, its eigenvalues are real and can be arranged in a non-increasing order as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let $\rho(G)$ be the *spectral radius* of G, that is, $\rho(G) = \max\{|\lambda_i(G)| : 1 \leq i \leq n\}$. By the Perron-Frobenius Theorem (see [4, 8]), we have $\rho(G) = \lambda_1(G)$.

In 1967, Wilf [20] gave a relation between the chromatic number and spectral radius of G; $\chi(G) \leq 1 + \rho(G)$. Note that $\chi(G) = \chi_1(G)$. Thus, it is natural to ask whether for $r \geq 2$, there is such a relation between the r-dynamic chromatic number and the spectral radius of a graph. By the inequality (1) and Theorem 2.3, we observe that for a positive integer $r \geq 2$, we have

$$\chi_r(G) \leq 1 + r\rho^2(G). \tag{2}$$

Note that since we have $\chi_r(H) = \Theta(\rho^2(H))$ for r-regular Moore graphs H, we cannot reduce from $\rho^2(G)$ to $\rho(G)$ in (2). However, we may be wondering whether $\rho^2(G)$ can be replaced with $\rho^{3/2}(G)$ in (2). In fact, we provide graphs F such that $\chi_r(F) = \Theta(\rho^2(F))$ and both of $\chi_r(F)$ and $\rho(F)$ are independent of r (see Section 4), which means that we cannot reduce the exponent of $\rho(G)$ in (2). Thus, we can say that the bound in (2) is almost sharp. When $r = 2$, we improve the bound; $\chi_r(G) \leq 1 + \rho^2(G)$. For $r = 3$ and $\Delta(G) \geq 5$, we have $\chi_3(G) \leq 1 + 2\rho^2(G)$ by using the inequality $\chi_3(G) \leq 1 + 2\Delta(G)$. Note that for $r = 3$ and $\Delta(G) \leq 4$, the bound may not be true and in fact, we prove $\chi_3(G) \leq 10$; equality holds when G is the Petersen graph.

2 Tools

In this section, we introduce tools to prove the main results.

To prove Theorem 3.1, we use the bound on $\chi_r(G)$ in terms of $\Delta(G)$ given by Jahanbekam, Kim, O, and West [11].

Theorem 2.1 ([11])**.** *For a graph G, $\chi_r(G) \leq 1 + r\Delta(G)$, with equality if and only if G is r-regular with diameter 2 and girth 5.*

For $r = 2$, to improve the bound in Theorem 3.1, the bound for the 2-dynamic chromatic number in Theorem 2.2 will be used.

Theorem 2.2 ([13])**.** *For a graph G with $\Delta(G) \geq 3$, we have $\chi_2(G) \leq 1 + \Delta(G)$. Furthermore, if $\Delta(G) \leq 2$, then $\chi_2(G) \leq 5$; equality holds only when $G \cong C_5$.*

For $r = 3$, the bound in Theorem 3.1 will be ameliorated by using induction on n.

To prove Theorem 3.1, we compare the spectral radii of a graph G and its certain subgraph by using Theorem 2.3.

Theorem 2.3 ([4, 8])**.** *If H is a subgraph of a connected graph G, then $\rho(H) \leq \rho(G)$; equality holds only when $H = G$.*

Let $P = \{V_1, \ldots, V_s\}$ be a partition of the vertex set of a graph G into s non-empty subsets. The *quotient matrix* Q corresponding to P is the $s \times s$ matrix whose (i, j)-entry is the average number of incident edges in V_j of the vertices in V_i. More precisely, $Q_{i,j} = \frac{|[V_i,V_j]|}{|V_i|}$ if $i \neq j$, and $Q_{i,i} = \frac{2|E(G[V_i])|}{|V_i|}$. A partition P is *equitable* if for each $1 \leq i, j \leq s$, any vertex $v \in V_i$ has exactly $Q_{i,j}$ neighbors in V_j.

Theorem 2.4 ([4, 8])**.** *For a graph G, if P is an equitable partition of $V(G)$ and Q is its corresponding quotient matrix, then we have $\rho(G) = \rho(Q)$.*

Theorem 2.4 is used to compute the spectral radii of graphs in Section 4.

3 Bounds in terms of $\Delta(G)$ and $\rho(G)$

In this section, we first give an upper bound on $\chi_r(G)$ in terms of the spectral radius of a graph G. Then for $r = 2, 3$, the bound will be ameliorated.

For simplicity, we use Δ and δ instead of $\Delta(G)$ and $\delta(G)$ in this section.

Theorem 3.1. *For a graph G and a positive integer $r \geq 2$, we have*

$$\chi_r(G) \leq 1 + r\rho^2(G). \tag{2}$$

Proof. By Theorem 2.1, we have $\Delta \geq \frac{\chi_r(G)-1}{r}$. Thus G contains the star $S_{1,\lceil \frac{\chi_r(G)-1}{r} \rceil}$ as a subgraph. By Theorem 2.3, we have

$$\sqrt{\lceil \frac{\chi_r(G)-1}{r} \rceil} = \rho(S_{1,\lceil \frac{\chi_r(G)-1}{r} \rceil}) \leq \rho(G),$$

which implies that $\lceil \frac{\chi_r(G)-1}{r} \rceil \leq \rho^2(G)$. Thus we have the desired result. $\square$

Even though the upper bound in Theorem 3.1 is easily achieved, it is almost tight in the sense that there are graphs H with $\chi_r(H) = \Theta(\rho^2(H))$ (see Section 4.)

Now, we improve the bound in (2) for $r = 2$.

Theorem 3.2. *For a connected graph G, we have $\chi_2(G) \leq 1 + \rho^2(G)$; equality holds only when $G = P_1, P_2, P_3$ or C_5.*

Proof. We consider two cases depending on the maximum degree of G.

Case 1: $\Delta \geq 3$. By Theorem 2.2, we have $\Delta \geq \chi_2(G) - 1$. Thus G contains the star $S_{1,\chi_2(G)-1}$ as a subgraph. By Theorem 2.3, we have

$$\sqrt{\chi_2(G) - 1} = \rho(S_{1,\chi_2(G)-1}) \leq \rho(G), \tag{3}$$

which implies that $\chi_2(G) \leq 1 + \rho^2(G)$.

To have equality in (3), G must be the star by Theorem 2.3 with at least 4 vertices (since $\Delta \geq 3$). Thus we have

$$\chi_2(S_{1,n-1}) = 3 < n = 1 + (\sqrt{n-1})^2.$$

Now, we see that for $\Delta \geq 3$, equality does not hold in (3).

Case 2: $\Delta \leq 2$. In this case, G must be a path or a cycle. If G is the n-vertex path, then

$$\chi_2(P_n) = 3 < 1+\rho^2(P_n) = 1+4\cos^2(\frac{\pi}{n+1}) \text{ for } n \geq 4 \text{ or } \chi_2(P_n) = 1+\rho^2(P_n) \text{ for } n = 1, 2, 3.$$

If G is the n-vertex cycle, then we have

$$\chi_2(C_n) \leq 5 = 1 + 2^2 = 1 + \rho^2(C_n);$$

equality holds only when $G = C_5$. $\square$

For each $k > 2$, we construct the graph $H_{k,2}$ with $\chi_2(H_{k,2}) = k$ and $\rho(H_{k,2}) = \sqrt{2(k-1)}$ in Section 4. Thus there are infinitely many graphs H such that $\chi_r(H) = \Theta(\rho^2(H))$.

For positive integers a and b, a graph G is (a, b)-*bireguar* if it is a bipartite graph with the vertices of one part all having degree a and the others all having degree b. For vertices in a graph G, we use the following notations for their neighbors in the proofs of Theorem 3.3 and Theorem 3.4. For a vertex $v \in V(G)$, $N(v) = \{v_1, \ldots, v_{d(v)}\}$ and for a vertex v_i, $N(v_i) =$

$\{v_{i,1}, \ldots, v_{i,d(v_i)}\}$. Similarly, we can define for vertices $v_{i,j}$ and $v_{i,j,k}$. Also, for a set $A \subseteq V(G)$ and for a coloring f on $V(G)$, $f(A) := \bigcup_{v \in A} f(v)$.

In addition to those, if a construction of a graph in the proof of Theorem 3.3 and Theorem 3.4 creates multiple edges between two vertices, then we remove one edge between the two vertices in order to assume that the resulting graph is simple. (#)

Note that the existence of multiple edges does not affect a vertex coloring.

Theorem 3.3. *For a connected graph G, if $\Delta \leq 4$, then we have $\chi_3(G) \leq 10$.*

Proof. We prove this by induction on $|V(G)|$. For $\Delta \leq 3$, we have $\chi_3(G) \leq 3\Delta + 1 \leq 10$. Equality holds only when G is the Petersen graph by Theorem 2.1. Thus we may assume that $\Delta = 4$ and $|V(G)| \geq 11$. Then we consider the following four cases depending on the minimum degree.

Case 1: $\delta = 1$. Then there are two vertices v and v_1 such that $N(v) = \{v_1\}$. By applying the induction hypothesis to $H := G - v$, we have $\chi_3(H) \leq 10$. Let f be a 3-dynamic 10-coloring of H. Note that $d_H(v_1) = d_G(v_1) - 1 \leq 3$. Then by assigning v to a color in $S - f(N_H[v_1])$, we can have a 3-dynamic 10-coloring of G.

Case 2: $\delta = 2$. Then there are three vertices v, v_1 and v_2 such that $N(v) = \{v_1, v_2\}$. By applying the induction hypothesis to to $H : G - v + v_1v_2$, (we may assume that H is simple by (#).) then we have $\chi_3(H) \leq 10$. Let f be a 3-dynamic 10-coloring of H. Since $\Delta \leq 4$, we can assign v to a color in $S - f(N_H[v_1]) - f(N_H[v_2])$. Thus we can have a 3-dynamic 10-coloring of G.

Case 3: $\delta = 3$. In this case, we first show that G is $(3, 4)$-biregular by the folloiwng claims.

Claim 1. *If there exist two vertices v and v_1 such that $d(v) = d(v_1) = 3$, then $\chi_3(G) \leq 10$.*

Proof. Suppose that $H := G - \{v, v_1\} + v_2v_3 + v_{1,1}v_{1,2}$. Note that v, $v_{1,1}$, and $v_{1,2}$ are all different, but $v_{1,1}, v_{1,2}$ are possibly equal to v_2 or v_3. By applying the induction hypothesis to H, we have $\chi_3(H) \leq 10$. Let f be a 3-dynamic 10-coloring of H. By the definition of f, there exist two vertices in each neighborhood of $v_2, v_3, v_{1,1}$, and $v_{1,2}$ with two different colors. Thus we may assume that $f(v_{2,1}) \neq f(v_{2,2})$, $f(v_{3,1}) \neq f(v_{3,2})$, $f(v_{1,1,1}) \neq f(v_{1,1,2})$ and $f(v_{1,2,1}) \neq f(v_{1,2,2})$. Now, by assigning v_1 to a color in $S - \{f(N_G(v_1)), f(v_2), f(v_3), f(v_{1,1,1}), f(v_{1,1,2}), f(v_{1,2,1}), f(v_{1,2,2,})\}$ and by assigning v to a color in $S - \{f(N_G(v)), f(v_{1,1}), f(v_{1,2}), f(v_{2,1}), f(v_{2,2}), f(v_{3,1}), f(v_{3,2})\}$, we can have a 3-dynamic 10-coloring of G. □

Claim 2. *If there exist three vertices v, v_1 and $v_{1,3}$ such that $d(v) = 3$ and $d(v_1) = d(v_{1,3}) = 4$, then $\chi_3(G) \leq 10$.*

Proof. By setting H in the same way in Claim 1, the proof is similar to that of Claim 1. The difference between Claim 1 and Claim 2 is the size of $N_H(v_1)$. □

By Claim 1 and Claim 2, G must be $(3,4)$-biregular since $\delta = 3$.

Since $\delta \geq 3$, there exists a cycle in G. Let C_m be a shortest cycle in G.

If there exists a vertex v in $V(G) - V(C_m)$ such that $|N(v) \cap V(C_m)| \geq 2$, then we have $m = 4$ since G is biregular and C_m is a shortest cycle. Let $C_4 := a_1a_2a_3a_4$ be a 4-cycle. Without loss of generality, we may assume that the vertex v is adjacent to a_1 and a_3 so that $a_1a_2a_3v$ is also a 4-cycle. Since G is $(3,4)$-biregular, we have

(i) $d(a_1) = d(a_3) = 3$, $d(a_2) = d(a_4) = d(v) = 4$ or

(ii) $d(a_1) = d(a_3) = 4$, $d(a_2) = d(a_4) = d(v) = 3$.

For (i), we also have $d(a_{2,1}) = d(a_{2,2}) = 3$. Suppose that $H := G - a_1 - a_2 + a_{2,1}a_3 + a_4v$. By applying the induction hypothesis to H, we have $\chi_3(H) \leq 10$. Let f be a 3-dynamic 10-coloring of H. By the definition of f, we may assume that $f(a_{2,1,1}) \neq f(a_{2,1,2})$, $f(a_{2,2,1}) \neq f(a_{2,2,2})$, $f(a_{4,1}) \neq f(a_3)$ and $f(v_1) \neq f(a_3)$. Then by assigning a_2 to a color in $S - \{f(a_{2,1}), f(a_{2,1,1}), f(a_{2,1,2}), f(a_{2,2}), f(a_{2,2,1}), f(a_{2,2,2}), f(a_3), f(a_4), f(v)\}$ and by assigning a_1 to a color in $S - \{f(a_2), f(a_{2,1}), f(a_3), f(a_4), f(a_{4,1}), f(v), f(v_1)\}$, we can have a 3-dynamic 10-coloring of G.

The proof for (ii) is similar to that for (i).

Now, we may assume that for every vertex $v \in V(G) - V(C_m)$, we have $|N(v) \cap V(C_m)| \leq 1$. Let $C_m := a_1 \dots a_m$ be an m-cycle. Without loss of generality, assume that $d(a_1) = 4$ and $d(a_m) = 3$. Let $H := G - V(C_m) - \{a_{1,1}\} + a_{1,1,1}a_{1,1,2}$. By applying the induction hypothesis to H, we have $\chi_3(H) \leq 10$. Let f be a 3-dynamic 10-coloring of H. Then we assign the vertices $a_1, \dots, a_m, a_{1,1}$ to colors in the following order; we assign
(i) a_3 to a color in $S - \{f(a_{3,1}), f(a_{3,2}), f(a_{3,1,1}), f(a_{3,1,2}), f(a_{3,2,1}), f(a_{3,2,2}), f(a_{4,1}), f(a_{2,1})\}$,
(ii) for $i \in [2, \frac{m-2}{2}]$ in an increasing order, a_{2i+1} to a color in $S - \{f(a_{2i+1,1}), f(a_{2i+1,2}), f(a_{2i+1,1,1}), f(a_{2i+1,1,2}), f(a_{2i+1,2,1}), f(a_{2i+1,2,2}), f(a_{2i+2,1}), f(a_{2i,1}), f(a_{2i-1})\}$,
(iii) a_1 to a color in $S - \{f(a_{1,2}), f(a_{1,2,1}), f(a_{1,2,2}), f(a_{1,1,1}), f(a_{1,1,2}), f(a_3), f(a_{m-1}), f(a_{2,1}), f(a_{m,1})\}$,
(iv) $a_{1,1}$ to a color in $S - \{f(a_{1,2}), f(a_{1,1,1}), f(a_{1,1,2}), f(a_{1,1,1,1}), f(a_{1,1,1,2}), f(a_{1,1,2,1}), f(a_{1,1,2,2}), f(a_1)\}$,
(v) a_2 to a color in $S - \{f(a_1), f(a_{1,1}), f(a_{2,1}), f(a_3), f(a_{3,1})\}$,
(vi) for $i \in [2, \frac{m-2}{2}]$ in increasing order, a_{2i} to a color in $S - \{f(a_{2i-2}), f(a_{2i-1}), f(a_{2i-1,1}), f(a_{2i,1}), f(a_{2i+1}), f(a_{2i+1,1})\}$
(vii) a_m to a color in $S - \{f(a_1), f(a_2), f(a_{1,1}), f(a_{m,1}), f(a_{m-1}), f(a_{m-1,1}), f(a_{m-2})\}$.

Then we can have a 3-dynamic 10-coloring of G.

Case 4: $\delta = 4$. Thus G is 4-regular, and suppose that C_m is a shortest cycle in G.

Subcase 4-1: For each vertex $v \in V(G) - V(C_m)$, $|N(v) \cap V(C_m)| \leq 1$. Let $C_m := a_1 \dots a_m$ be an m-cycle. Suppose that $H := G - V(C_m) + \sum_{i \in [m]} a_{i,1}a_{i,2}$. By applying the induction hypothesis to H, we have $\chi_3(H) \leq 10$. Let f be a 3-dynamic 10-coloring of H. Then we extend f from $V(H)$ to $V(G)$ to have a 3-dynamic 10-coloring in the following way.

First, we assign a_1 to a color in $S - \{f(a_{1,1}), f(a_{1,2}), f(a_{1,1,1}), f(a_{1,1,2}), f(a_{1,2,1}), f(a_{1,2,2}), f(a_{2,1}), f(a_{2,2})\}$. Then for $i \in [2, m-2]$ in an increasing order, we assign a_i to a color in $S - \{f(a_{i,1}), f(a_{i,2}), f(a_{i,1,1}), f(a_{i,1,2}), f(a_{i,2,1}), f(a_{i,2,2}), f(a_{i+1,1}), f(a_{i+1,2}), f(a_{i-1})\}$. Now, if $f(a_1) \in \{f(a_{m,1}), f(a_{m,2})\}$, then without loss of generality, assume that $f(a_1) = f(a_{m,1})$. Then we can assign a_{m-1} to a color in $S - \{f(a_{m-1,1}), f(a_{m-1,2}), f(a_{m-1,1,1}), f(a_{m-1,1,2}), f(a_{m-1,2,1}), f(a_{m-1,2,2}), f(a_{m,1}), f(a_{m,2}), f(a_{m-2})\}$ and a_m to a color in $S - \{f(a_{m,1}), f(a_{m,2}), f(a_{m,1,1}), f(a_{m,1,2}), f(a_{m,2,1}), f(a_{m,2,2}), f(a_{1,1}), f(a_{1,2}), f(a_{m-1})\}$. If $f(a_1) \notin \{f(a_{m,1}), f(a_{m,2})\}$, then we can assign a_m to a color in $S - \{f(a_{m,1}), f(a_{m,2}), f(a_{m,1,1}), f(a_{m,1,2}), f(a_{m,2,1}), f(a_{m,2,2}), f(a_{1,1}), f(a_{1,2}), f(a_1)\}$ and a_{m-1} to a color in $S - \{f(a_{m-1,1}), f(a_{m-1,2}), f(a_{m-1,1,1}), f(a_{m-1,1,2}), f(a_{m-1,2,1}), f(a_{m-1,2,2}), f(a_m), f(a_{m-2})\}$.

Subcase 4-2: For some vertex v in $V(G) - V(C_m)$, $|N(v) \cap V(C_m)| \geq 2$. Then, we must have $m = 3$ or $m = 4$.

For $m = 3$, let $C_3 := a_1a_2a_3$ be a triangle. Suppose that for a vertex $v \in V(G) - V(C_3)$, we have $\{a_1, a_2\} \subseteq N(v) \cap V(C_3)$. Suppose that $H := G - a_1 - a_2 + va_3$. By applying the induction hypothesis to H, we have $\chi_3(H) \leq 10$. Let f be a 3-dynamic 10-coloring of H. Now, by assigning a_1 to a color in $S - \{f(v), f(v_1), f(a_3), f(a_{3,1}), f(a_{1,1})\}$ and by assigning a_2 to a color in $S - \{f(a_1), f(a_3), f(a_{3,1}), f(a_{2,1}), f(v), f(v_1)\}$, we can have a 3-dynamic 10-coloring of G.

For $m = 4$, suppose that $C_4 := a_1a_2a_3a_4$ is a 4-cycle. Without loss of generality, assume that there is a vertex $v \in V(G) - V(C_4)$ such that v is adjacent to a_1 and a_3. Note that $a_1a_2a_3v$ is also a 4-cycle. Suppose that $H := G - a_1 - a_2 + a_{2,1}a_3 + a_4v$. By applying the induction hypothesis to H, we have $\chi_3(H) \leq 10$. Let f be a 3-dynamic 10-coloring of H. By the definition of f, we may assume that $f(a_{2,1,1}) \neq f(a_{2,1,2})$, $f(a_{4,1}) \neq f(a_3)$ and $f(v_1) \neq f(a_3)$. Then by assigning a_1 to a color in $S - \{f(N_G(a_1)), f(a_{2,1}), f(a_3), f(a_{4,1}), f(v_1)\}$ and by assigning a_2 to a color in $S - \{f(N_G(a_2)), f(a_{2,1,1}), f(a_{2,1,2}), f(a_4), f(v)\}$, we can have a 3-dynamic 10-coloring of G. □

Note that the Petersen graph holds equality in the bound in Theorem 3.3.

Observation 1. *If P is the Petersen graph, then $\chi_3(P) = 10$.*

Proof. For any pair of two distinct vertices u and v, either u and v are adjacent or $d(u, v) = 2$. Thus we must assign the vertices to all different colors. □

Also, there is a graph whose 3-dynamic chromatic number is 9, close to 10.

Observation 2. *If P' is the graph obtained from the Petersen graph P by adding a single pendant vertex, then $\chi_3(P') = 9$.*

Proof. Note that there is the vertex with degree 4 (say u), the vertex with degree 1 (say v), and $uv \in E(P')$. Except the vertices u, v, and another neighbor of u, we must assign the other vertices to all different colors with the same reason in the proof of Observation 1. If we assign the vertex u to a used color for a vertex, which is already colored, then there is a vertex with only two colors including the used color on the neighborhood, which is not a 3-dynamic coloring. Thus we need to assign the vertex u to another color, which implies that $\chi_3(P') \geq 9$. By assigning the remaining neighbor of u with degree 3 to one of the colors assigned the other two neighbors with degree 3, and by assigning v to any color, which is not used for the neighbors of u with degree 3, we can have a 3-dynamic 9-coloring. □

We handle the case $\Delta \geq 5$ separately from $\Delta \leq 4$. Note that the graph $H_{k,3}$ in Section 4 guarantees that the bound in Theorem 3.4 is sharp.

Theorem 3.4. *For a connected graph G, if $\Delta \geq 5$, we have $\chi_3(G) \leq 2\Delta + 1$.*

Proof. We prove this by induction on $|V(G)|$. Since $\Delta \geq 5$, we have $|V(G)| \geq 6$.

Now, we consider the following four cases depending on the minimum degree. Note that the graph H that we consider in the following cases may have the maximum degree smaller than 5 so that we may not apply the induction hypothesis to H. However, even if $\Delta(H) \leq 4$, then by Theorem 3.3, we have $\chi_3(H) \leq 10 < 2\Delta + 1$ since $\Delta \geq 5$.

Case 1: $\delta = 1$. Then there are vertices v and v_1 such that $N(v) = \{v_1\}$. By applying the induction hypothesis to $H := G - v$, we have $\chi_3(H) \leq 2\Delta + 1$. Let f be a 3-dynamic $(2\Delta + 1)$-coloring of H. Then by assigning v to a color in $S - \{f(N[v_1])\}$, we can have a 3-dynamic $(2\Delta + 1)$-coloring of G.

Case 2: $\delta = 2$. Then there are vertices v, v_1, v_2 such that $N(v) = \{v_1, v_2\}$. Suppose that $H := G - v + v_1v_2$. By applying the induction hypothesis to H, we have $\chi_3(H) \leq 2\Delta + 1$. Let f be a 3-dynamic $(2\Delta + 1)$-coloring of H. If each neighborhood of v_1 and v_2 has two distinct vertices with different colors, then without loss of generality, we assume that $f(v_{1,1}) \neq f(v_{1,2})$ and $f(v_{2,1}) \neq f(v_{2,2})$. Now, we obtain a 3-dynamic $(2\Delta + 1)$-coloring of G by assigning v to a color in $S - \{f(v_1), f(v_2), f(v_{1,1}), f(v_{1,2}), f(v_{2,1}), f(v_{2,2})\}$. If a neighborhood of v_1 or v_2 has no such vertices, then we have more options to color v.

Case 3: $\delta = 3$. There exists vertices v, v_1, v_2, v_3 such that $N(v) = \{v_1, v_2, v_3\}$. Suppose that $A = N(v_1) \cap \{u \in V(G) | u \neq v, d(u) = 3\}$ and $B = N(v_1) - A - \{v\}$. Let $A = \{a_1, \ldots, a_t\}$ and $B = \{b_1, \ldots, b_{d(v_1)-t-1}\}$.

First, assume that $t > 0$. By setting $H := G - \{v, v_1\} + v_2v_3$ and then applying the induction hypothesis to H, we have $\chi_3(H) \leq 2\Delta + 1$. Let f be a 3-dynamic $(2\Delta + 1)$-coloring of H. By the definition of f, we may assume that for $i \in [t]$ and $j = 1, 2$, $f(a_{i,1}) \neq f(a_{i,2})$ and let $f(a_{i,j,1}) \neq f(a_{i,j,2})$. Let S' be a subset of S such that $|S'| = 2(\Delta - 1)$ and $f(B) \cup f(N_H(A)) \subseteq S'$. Note that we can reassign the vertices of A to colors such that $f(N_G(v_1) \setminus \{v\}) \cup f(N_H(A)) \subseteq S'$ and $|f(N_G(v_1) \setminus \{v\})| \geq 2$ since for each vertex $a_i \in A$, we can reassign to a color in $S' - \{f(a_{i,1}), f(a_{i,2}), f(a_{i,1,1}), f(a_{i,1,2}), f(a_{i,2,1}), f(a_{i,2,2})\}$. For $t \geq 2$, when we recolor a_2, we can allow $f(a_2) \neq f(a_1)$. For $t = 1$, when we recolor a_1, we can allow $f(a_1) \neq f(b_1)$. Now, we obtain a 3-dynamic 10-coloring of G by assigning v_1 to a color in $S - \{f(N_G(v_1) \setminus \{v\}) \cup f(N_H(N(A))), f(v_2), f(v_3)\}$ and v to a color in $S - \{f(v_1), f(v_2), f(v_3), f(a_1), f(a_2), f(v_{2,1}), f(v_{2,2}), f(v_{3,1}), f(v_{3,2})\}$.

Now, assume that $t = 0$, that is, $A = \emptyset$. By setting $H := G - \{v, v_1\} + v_2v_3 + b_1b_2$ and then applying the induction hypothesis to H, we have $\chi_3(H) \leq 2\Delta + 1$. Let f be a 3-dynamic $(2\Delta + 1)$-coloring of H. By the definition of f, we may assume that for $i = 1, 2$, $f(b_{i,1}) \neq f(b_{i,2})$. Now, we obtain a 3-dynamic $(2\Delta + 1)$-coloring of G by assigning v_1 to a color in $S - \{f(N_G(v_1) \setminus \{v\}), f(b_{1,1}), f(b_{1,2}), f(b_{2,1}), f(b_{2,2})\}$ and v to a color in $S - \{f(v_1), f(v_2), f(v_3), f(b_1), f(b_2), f(v_{2,1}), f(v_{2,2}), f(v_{3,1}), f(v_{3,2})\}$.

Case 4: $\delta \geq 4$. Let $C_m := a_1 \dots a_m$ be a shortest cycle in G, and for each $i \in [m]$, let $a_{i,j}$ be the neighbors of a_i in $V(G) - V(C_m)$. We consider the following three cases depending on m.

Subcase 4.1: $m = 3$. By setting $H := G - \{a_1, a_{1,1}\} + a_{1,1,1}a_{1,1,2}$ and then applying the induction hypothesis to H, we have $\chi_3(H) \leq 2\Delta + 1$. Let f be a 3-dynamic $(2\Delta + 1)$-coloring of H. By the definition of f, we may assume that for $i = 1, 2$, $f(a_{1,1,i,1}) \neq f(a_{1,1,i,2})$. Now, we obtain a 3-dynamic $(2\Delta + 1)$-coloring of G by assigning $a_{1,1}$ to a color in $S - \{f(N_G(a_{1,1}) \setminus \{a_1\}), f(a_{1,1,1,1}), f(a_{1,1,1,2}), f(a_{1,1,2,1}), f(a_{1,1,2,2}), f(a_2), f(a_3)\}$ and a_1 to a color in $S - \{f(N_G(a_1)), f(a_{1,1}), f(a_{1,1,1}), f(a_{1,1,2})\}$.

Subcase 4.2: $m = 4$.By setting $H := G - \{a_1, a_2\} + a_{1,1}a_4 + a_{2,1}a_3$ and then applying the induction hypothesis to H, we have $\chi_3(H) \leq 2\Delta + 1$ and let f be a 3-dynamic $(2\Delta + 1)$-coloring of H. By the definition of f, we can assume that for $i = 1, 2$, $f(a_{4,1}) \neq f(a_3)$, $f(a_{3,1}) \neq f(a_4)$ and $f(a_{i,1,1}) \neq f(a_{i,1,2})$. Then by assigning a_1 to a color in $S - \{f(N_G(a_1) \setminus \{a_2\}), f(a_{1,1,1}), f(a_{1,1,2}), f(a_{41}), f(a_3), f(a_{2,1})\}$ and a_2 to a color in $S - \{f(N_G(a_2)), f(a_{2,1,1}), f(a_{2,1,2}), f(a_{31}), f(a_4), f(a_{1,1})\}$, we obtain a 3-dynamic $(2\Delta + 1)$-coloring of G.

Subcase 4.3: $m \geq 5$. By setting $H := G - V(C_m) + \sum_{i \in [m]} a_{i,1}a_{i,2}$ and then applying the induction hypothesis to H, we have $\chi_3(H) \leq 2\Delta + 1$. Let f be a 3-dynamic $(2\Delta + 1)$-coloring of H. Note that for any $v \in V(G) - V(C_m)$, $|N(v) \cap V(C_m)| \leq 1$ since C_m is a shortest cycle in G. By the definition of f, we can assume that for $i \in [m]$ and $j = 1, 2$, $f(a_{i,j,1}) \neq f(a_{i,j,2})$. If we have $|f(N_G(a_i) \setminus V(C_m))| = 2$ for all $i \in [m]$, then we obtain a 3-dynamic $(2\Delta + 1)$-coloring of G by assigning a_1 to a color in $S - \{f(a_{m,1}), f(a_{m,2}), f(a_{1,1}), f(a_{1,2}), f(a_{1,1,1}), f(a_{1,1,2}), f(a_{1,2,1}), f(a_{1,2,2})\}$, for $i = [2, m-1]$ in an increasing order, a_i to a color in $S - \{f(a_{i-1}), f(a_{i-1,1}), f(a_{i-1,2}), f(a_{i,1}), f(a_{i,2}), f(a_{i,1,1}), f(a_{i,1,2}), f(a_{i,2,1}), f(a_{i,2,2})\}$, and then a_m to a color in $S - \{f(a_1), f(a_{m-1}), f(a_{m-1,1}), f(a_{m-1,2}), f(a_{m,1}), f(a_{m,2}), f(a_{m,1,1}), f(a_{m,1,2}), f(a_{m,2,1}), f(a_{m,2,2})\}$. Now, we assume that there exists $i \in [m]$ (say $i = m - 1$) such that $|f(N_G(a_i) \setminus V(C_m))| \geq 3$. Then we obtain a 3-dynamic $(2\Delta + 1)$-coloring of G by assigning a_1 to a color in $S - \{f(N_G(a_1) \setminus V(C_m)), f(a_{m,1}), f(a_{m,2}), f(a_{1,1,1}), f(a_{1,1,2}), f(a_{1,2,1}), f(a_{1,2,2})\}$, for $i = [2, m - 1]$ in an increasing order, a_i to a color in $S - \{f(N_G(a_i) \setminus V(C_m)), f(a_{i-1}), f(a_{i-1,1}), f(a_{i-1,2}), f(a_{i,1,1}), f(a_{i,1,2}), f(a_{i,2,1}), f(a_{i,2,2})\}$, and a_m to a color in $S - \{f(N_G(a_m) \setminus V(C_m)), f(a_1), f(a_{m-1}), f(a_{m,1,1}), f(a_{m,1,2}), f(a_{m,2,1}), f(a_{m,2,2})\}$. $\square$

By Theorem 3.4 and by following the idea of Theorem 3.1, for $r = 3$ and $\Delta \geq 5$, we improve the bound in (2).

Corollary 3.5. *For a connected graph G and $\Delta \geq 5$, we have $\chi_3(G) \leq 1 + 2\rho^2(G)$.*

4 Example

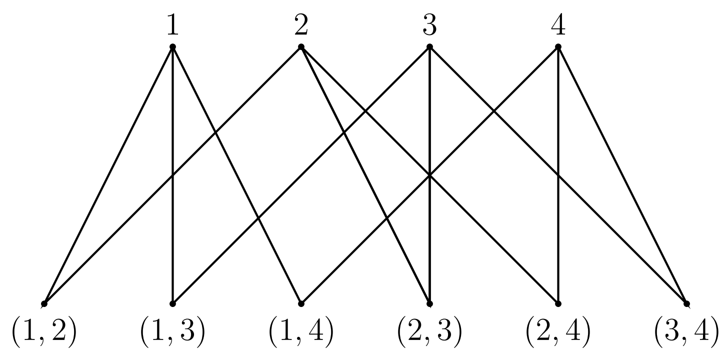

Figure 1: $H_{4,2}$

In this section, for given positive integers $k > r$ and $r \geq 2$, we provide graphs $H_{k,r}$ such that $\chi_r(H_{k,r}) = k$ and $\rho(H_{k,r}) = \sqrt{r(\frac{k-1}{r-1})}$. These examples show that $\chi_r(H_{k,r}) = \Theta(\rho^2(H_{k,r}))$, which implies that the bound in Theorem 3.1 is almost sharp. Thus for $r \geq 2$, we cannot replace $\rho^2(G)$ by $\rho(G)$ in Theorem 3.1.

First, we provide such a family of graphs for $r = 2$.

Definition 4.1. *For a positive integer $k > 2$, let $H_{k,2}$ be the bipartite graph with partite sets A and B such that*
(i) $A = \{v_1, \ldots, v_k\}$ and $B = \{x_{ij} : 1 \leq i < j \leq k\}$
(ii) For $1 \leq i < j \leq k$, $N(x_{ij}) = \{v_i, v_j\}$.

Now, we determine the 2-dynamic chromatic number and the spectral radius of $H_{k,2}$.

Theorem 4.2. *For a positive integer $k > 2$, we have*

$$\chi_2(H_{k,2}) = k \text{ and } \rho(H_{k,2}) = \sqrt{2(k-1)}.$$

Proof. Suppose that a 2-dynamic coloring f of $H_{k,2}$. If there exist $u, v \in A$ such that $f(u) = f(v)$, then the vertex x_{uv} in B (note that $d(x_{uv}) = 2$) have the neighbors with the same color violating the property of 2-dynamic coloring, which implies that $\chi_2(H_{k,2}) \geq k$.

Since $\Delta(H_{k,2}) = k - 1$, we have $\chi_2(H_{k,2}) \leq k$, which determines the 2-dynamic chromatic number of $H_{k,2}$.

The quotient matrix corresponding to the vertex partition $\{A, B\}$ is

$$Q = \begin{pmatrix} 0 & k-1 \\ 2 & 0 \end{pmatrix}.$$

Thus the characteristic polynomial of Q is $p(x) = x^2 - 2(k-1)$. Since the vertex partition is equitable, we have $\rho(H_{k,2}) = \rho(Q) = \sqrt{2(k-1)}$ by Theorem 2.4. $\square$

Thus for $r = 2$, we cannot reduce the exponent of ρ in the bound of Theorem 3.2 since $\chi_2(H_{k,2}) = \Theta(\rho^2(H_{k,2}))$. This is different from the Wilf's bound that the usual chromatic number of G is linearly bounded above by its spectral radius. In fact, for $r \geq 3$, we can provide more graphs F such that $\chi_2(F) = \Theta(\rho^2(F))$. This guarantees that for even $r \geq 3$ (at least up to 9), the bound in Theorem 3.1 is almost sharp.

Now, we construct such graphs from the graph $H_{k,2}$ with partite sets A and B in Definition 4.1.

Definition 4.3. *For $r \geq 2$, let $P = \{B_1, \ldots, B_s\}$ be a partition of B such that for each $1 \leq i \leq s$, $|B_i| = \binom{r}{2}$ and $|\cup_{v \in B_i} N_{H_{k,2}}(v)| = r$. The graph $H_{k,r}$ is a bipartite graph with partite sets A and P obtained from $H_{k,2}$ such that for each $1 \leq i \leq s$, $N_{H_{k,r}}(B_i) = \cup_{v \in B_i} N_{H_{k,2}}(v)$.*

Similarly to the proof of Theorem 4.2, we can also determine the r-dynamic chromatic number and the spectral radius of $H_{k,r}$.

Theorem 4.4. *For positive integers r and k with $r \geq 2$ and $k \geq r^2 + 1$, we have*

$$\chi_r(H_{k,r}) = k \text{ and } \rho(H_{k,r}) = \sqrt{\frac{r(k-1)}{r-1}}.$$

Proof. Let f be an r-dynamic coloring of $H_{k,r}$. If there exist $u, v \in A$ such that $f(u) = f(v)$, then a vertex in P (note that every vertex in P has degree r) has two neighbors with the same color violating the property of r-dynamic coloring, which implies that $\chi_r(H_{k,r}) \geq k$.

Now, we prove that $\chi_r(H_{k,r}) \leq k$ by showing the existence of an r-dynamic k-coloiring f. First, for each $i \in [k]$, we set $f(v_i) = i$. Second, for a vertex $v \in P$ not assigned to a color yet, we assign a color to v by following the rules: (i) $f(v) \notin \cup_{w \in N(v)} f(w)$ and (ii) we avoid $\min\{|f(N(w))|, r-1\}$ colors in $N(w)$ for each $w \in N(v)$. Since we have $k - r(r-1) - r \geq 1$, we can do (i) and (ii).

The quotient matrix corresponding to the vertex partition $\{A, B\}$ is

$$Q = \begin{pmatrix} 0 & \frac{k-1}{r-1} \\ r & 0 \end{pmatrix}.$$

Thus the characteristic polynomial of Q is $p(x) = x^2 - r(\frac{k-1}{r-1})$. Since the vertex partition is equitable, we have the desired result by Theorem 2.4. $\square$

Now, we say that the bound in Theorem 3.1 is almost sharp by Theorem 4.4 because $\chi_r(H_{k,r}) = \Theta(\rho^2(H_{k,r}))$. However, by looking at Definition 4.3, how can we guarantee the existence of a partition of B? Note that the existence of such a partition of B is equivalent to the existence of a Steiner system $S(2, r, k)$. For $r = 2$, see Definition 4.1. In fact, for each $3 \leq r \leq 9$, a Steiner system $S(2, r, k)$ exists (see [1, 2, 6, 9, 10, 17]). Thus we can guarantee the existence of $H_{k,r}$ for $2 \leq r \leq 9$. The existence of $H_{k,3}$ says that the bound in Theorem 3.4 is also sharp.

Definition 4.5. *A Steiner system $S(a, b, c)$ is a c-set C together with a family $\mathcal{B}$ of b-subsets of C with the property that every a-subset of C is contained in exactly one b-subset.*

We mention that very recently, Jendrol and Onderko [12] improved the bound in Theorem 2.1; $\chi_r(G) \leq (r-1)(\Delta(G)+1)+2$. They also provided graphs whose r-dynamic chromatic number equals $(r-1)\Delta(G)+1$, and the graphs are related to the existence of Steiner systems $S(2, r, k)$. In fact, they are the same as $H_{k,r}$.

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