The saturation number of C_6

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Abstract

A graph G is called C_k-saturated if G is C_k-free but $G+e$ is not for any $e \in E(\overline{G})$. The saturation number of C_k, denoted $sat(n, C_k)$, is the minimum number of edges in a C_k-saturated graph on n vertices. Finding the exact values of $sat(n, C_k)$ has been one of the most intriguing open problems in extremal graph theory. In this paper, we study the saturation number of C_6. We prove that $4n/3 - 2 \le sat(n, C_6) \le (4n+1)/3$ for all $n \ge 9$, which significantly improves the existing lower and upper bounds for $sat(n, C_6)$.

Keywords: saturation number; cycles; saturated graph

1 Introduction

All graphs considered in this paper are finite and simple. Throughout the paper we use the terminology and notation of [27]. For a graph G, we use $e(G)$ to denote the number of edges, $|G|$ the number of vertices, $\delta(G)$ the minimum degree and $\overline{G}$ the complement of G. For $A, B \subseteq V(G)$, let $B \setminus A := B - A$ and $N_G(A)$ denote the subset of $V(G) \setminus A$ in which each vertex is adjacent to some vertex of A in G. If $A = \{v\}$, then we write as $B \setminus v$ and $N_G(v)$. The degree of vertex v in G, denoted $d_G(v)$, is the size of $N_G(v)$. Let $N_G[v] = N_G(v) \cup \{v\}$. For a graph G and $u, v \in V(G)$, let $d_G(u, v)$ denote the length of a shortest path between u and v. Without confusion, we abbreviate as $N(A)$, $N(v)$, $N[v]$, $d(v)$ and $d(u, v)$, respectively. We use k-path (resp. k-cycle) to denote the path (resp. k-cycle) of length k. For positive integer k, let $[k] := \{1, 2, \ldots, k\}$.

Given a family of graphs $\mathcal{F}$, a graph is $\mathcal{F}$-free if it contains no any member in $\mathcal{F}$ as a subgraph. A graph G is called *$\mathcal{F}$-saturated* if G is $\mathcal{F}$-free but $G+e$ is not for any $e \in E(\overline{G})$. The *saturation number* of $\mathcal{F}$, denoted $sat(n, \mathcal{F})$, is the minimum number of edges in an $\mathcal{F}$-saturated graph on n vertices. If $\mathcal{F} = \{F\}$, then we write as $sat(n, F)$. Erdős, Hajnal and Moon [11] initiated the study of the saturation numbers of graphs and proved that $sat(n, K_{k+1}) = (k-1)n - \binom{k}{2}$ with $K_{k-1} \vee \overline{K_{n-k+1}}$ as the unique extremal graph. Later, Kászonyi and Tuza [20] extended this to the case when K_k is replaced by any family $\mathcal{F}$ of graphs, showing that $sat(n, \mathcal{F}) = O(n)$. Since then, a large quantity of work in this area has been carried out in determining the saturation numbers of complete multipartite graphs, cycles, trees, forests and hypergraphs, see e.g. [2, 4, 7, 13, 18, 19, 22–24, 26]. Surveys on the saturation problem of graphs and hypergraphs could be found in [12, 16, 25]. It is worth noting that the proofs of almost all results are quite technically involved and require significant efforts.

Finding the exact values of $sat(n, C_k)$ has been one of the most intriguing open problems in extremal graph theory. Until now, the exact values of $sat(n, C_k)$ are known for a few k. The result of Erdős, Hajnal and Moon [11] pointed out the star S_n is the unique extremal graph for C_3. Ollmann [23] determined all extremal graphs for C_4. Later, Tuza [26] gave a shorter proof for the exact value of $sat(n, C_4)$. Fisher, Fraughnaugh and Langley [14] derived an upper bound for $sat(n, C_5)$ by constructing a class of C_5-saturated graphs. Chen finally [5, 6] confirmed the upper bound obtained in [14] is the exact value for $sat(n, C_5)$ and also characterized all extremal graphs. For the case of Hamilton cycle, Bondy [3] first showed that $sat(n, C_n) \geq \lceil \frac{3n}{2} \rceil$ for all $n > 6$, which was later improved to $sat(n, C_n) \geq \lfloor \frac{3n+1}{2} \rfloor$ for all $n \geq 20$ in [8–10, 21]. We summarize some of these results as follows.

Theorem 1.1 *Let n be a positive integer.*

(a) **[23]** $sat(n, C_3) = n - 1$ *for any* $n \geq 3$.

(b) **[23, 26]** $sat(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$ *for any* $n \geq 5$.

(c) **[5, 6, 14]** $sat(n, C_5) = \left\lceil \frac{10(n-1)}{7} \right\rceil$ *for any* $n \geq 21$.

For the general case $k \geq 6$, many researchers devoted to the study of the saturation numbers of C_k, see [1, 3, 8–10, 15, 21, 28]. We present the previous best results as follows.

Theorem 1.2 ([17, 28]) *For any $n \geq 9$, we have* $\lceil \frac{7n}{6} \rceil - 2 \leq sat(n, C_6) \leq \lfloor \frac{3n-3}{2} \rfloor$.

Theorem 1.3 ([15]) *For all $k \geq 7$ and $n \geq 2k - 5$, we have* $\left(1 + \frac{1}{k+2}\right) n - 1 \leq sat(n, C_k) \leq \left(1 + \frac{1}{k-4}\right) n + \binom{k-4}{2}$.

Based on the constructions that yield the upper bounds, Füredi and Kim believed the upper bounds are approximately optimal and proposed the following conjecture, which is widely open.

Conjecture 1.4 ([15]) *There exists a constant k_0 such that* $sat(n, C_k) = \left(1 + \frac{1}{k-4}\right) n + O(k^2)$ *for all* $k \geq k_0$.

It seems quite difficult to determine the exact values of $sat(n, C_k)$ for any $k \geq 6$ as mentioned by Faudree, Faudree and Schmitt [12]. In this paper, we prove the following result, which improves Theorem 1.2.

Theorem 1.5 *Let $n \geq 9$ be a positive integer and $n \equiv \varepsilon$ (mod 3) with $\varepsilon \in \{0, 1, 2\}$. Then*

(a) $sat(n, C_6) \leq \frac{4n}{3} + \frac{\varepsilon^2}{2} - \frac{5\varepsilon}{6}$;

(b) $sat(n, C_6) \geq \frac{4n}{3} - 2$.

From Theorem 1.5, one can see $sat(n, C_6) = 4n/3 + O(1)$ for any $n \geq 9$, which confirms that the constant k_0 in Conjecture 1.4 should be at least 7.

We organize our paper as follows. In Section 2, we prove Theorem 1.5(a) by giving a new construction. In Section 3, we prove Theorem 1.5(b) by assuming Theorem 3.4 holds. To complete the proof of Theorem 3.4, we investigate that we just need to prove Theorems 4.7, 4.8 and 4.9 in Section 4. Several lemmas are provided in Section 5 but their proofs are contained in Appendix. We then prove Theorems 4.7, 4.8 and 4.9 in Sections 6-8. We need to introduce more notations. Given a graph G, let $\mathcal{F}_G$ denote the family of all graphs $G + e$ containing a 6-cycle as a subgraph, where $e \in E(\overline{G})$, and $C_6(e)$ denote the family of 6-cycles containing e in $G + e$. By abusing notation, we also use $C_6(e)$ to denote one member in $C_6(e)$. Denote by $P(uv)$ the path with ends u and v. Given a graph G with vertex partition $V(G) = V_1 \cup \cdots \cup V_s$ and $x \in V(G)$, let $N_i(x) = N(x) \cap V_i$ and $n_i(x) = |N_i(x)|$.

2 Proof of Theorem 1.5(a)

Let n and ε be given as in the statement. Let H be the graph depicted in Figure 1(a). Let P^i be a path with vertices a_i, b_i, c_i in order. For $n = 3t$ with $t \geq 3$, let G_t^0 be a graph on $n = 3t$ vertices obtained from $H \cup P^1 \cup \cdots \cup P^{t-3}$ by adding edges x_1a_i and x_2c_i for any $i \in [t-3]$. Then G_t^0 is C_6-free and

$$e(G_t^0) = e(H) + e(P^1) + \cdots + e(P^{t-3}) + 2(t-3) = 12 + 2(t-3) + 2(t-3) = 4t = \frac{4n}{3}.$$

For $n = 3t + \varepsilon$ with $t \geq 3$ and $\varepsilon \in [2]$, let G_t^ε be a graph on $n = 3t + \varepsilon$ vertices obtained from $G_t^0 \cup K_\varepsilon$ by joining y_4 to all vertices of K_ε. The graph G_t^1 is depicted in Figure 1(b). Then for any $\varepsilon \in [2]$, G_t^ε is C_6-free and

$$e(G_t^\varepsilon) = e(G_t^0) + e(K_\varepsilon) + \varepsilon = 4t + \frac{\varepsilon(\varepsilon-1)}{2} + \varepsilon = \frac{4n-\varepsilon}{3} + \frac{\varepsilon(\varepsilon-1)}{2} = \frac{4n}{3} + \frac{\varepsilon^2}{2} - \frac{5\varepsilon}{6}.$$

We next prove that G_t^ε is C_6-saturated for any $t \geq 3$ and $\varepsilon \in \{0,1,2\}$. Observe that if G_t^1 is C_6-saturated, then both G_t^0 and G_t^2 are C_6-saturated. Thus, we shall prove that G_t^1 is C_6-saturated for any $t \geq 3$. It has been proved in [28] that H is C_6-saturated. Thus we just need to show that $G_t^1 + z_1v \in \mathcal{F}_{G_t^1}$ for any $v \in V(H) \setminus y_4$, and $\{G_t^1 + a_ia_j, G_t^1 + a_ib_j, G_t^1 + a_ic_j, G_t^1 + b_ib_j, G_t^1 + b_ic_j, G_t^1 + c_ic_j\} \subseteq \mathcal{F}_{G_t^1}$ for any $0 \leq i < j \leq t-3$. It is clear that the latter holds since $C_6(a_ia_j) = a_ib_ic_ix_2x_1a_j$, $C_6(a_ib_j) = a_ib_ic_ix_2c_jb_j$, $C_6(a_ic_j) = a_ix_1y_2y_3x_2c_j$, $C_6(b_ib_j) = b_ia_ix_1x_2c_jb_j$, $C_6(b_ic_j) = b_ia_ix_1a_jb_jc_j$ and $C_6(c_ic_j) = c_ib_ia_ix_1x_2c_j$. For $v \in \{x_1, y_1, c_0, y_3\}$, $C_6(z_1v) = z_1P^1v$ where $P^1 = y_4x_2y_3y_2$ or $P^1 = y_4y_2x_1x_2$. For $v \in \{x_2, a_0\}$, $C_6(z_1v) = z_1P^2v$, where $P^2 = y_4y_2y_1x_1$. For $v \in \{y_2, b_0\}$, $C_6(z_1v) = z_1P^3v'v$, where $P^3 = y_4x_2x_1$ and $v' \in N(x_1) \cap N(v)$. Hence, for any $t \geq 3$ and $\varepsilon \in \{0,1,2\}$, G_t^ε is C_6-saturated. Thus, $sat(n, C_6) \leq e(G_t^\varepsilon) \leq 4n/3 + \varepsilon^2/2 - 5\varepsilon/6$, as desired. □

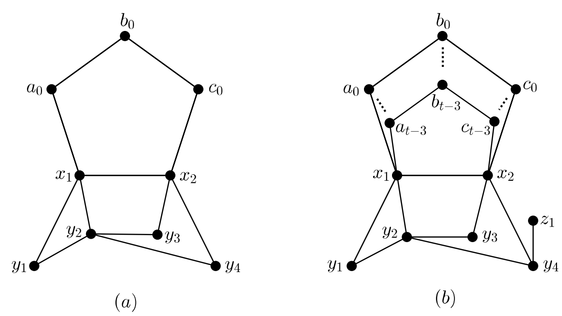


Figure 1: (a) H and (b) G_t^1.

3 Proof of Theorem 1.5(b)

We call a vertex of degree two in a graph a **root**. A root is **good** if it does not lie in any triangle, otherwise it is **bad**. For any graph G, let $B(G)$ denote the set of all bad roots in G, $B_1(G) = \{v \in B(G) | d(v') \geq 3 \text{ for every } v' \in N(v)\}$ and $B_2(G) = B(G) \backslash B_1(G)$.

Lemma 3.1 ([28]) *Let G be a C_6-saturated graph with $\delta(G) = 2$. Then at least one of the following is satisfied:*

(1) There exists a root $\alpha \in B_1(G)$ with $N(\alpha) = \{\alpha_1, \alpha_2\}$ such that $N(\alpha_i) = \{\alpha, \alpha_{3-i}, x_i\}$ for any $i \in [2]$ and $x_1 \neq x_2$.

(2) $D(X) \geq 3|X|$ holds for $X = V(G) \setminus (\{v \in V(G) | d(v) = 2\} \backslash B_1(G))$, where $D(X) = \sum_{v \in X} d(v)$.

In fact, we can derive a stronger result than Lemma 3.1.

Lemma 3.2 *Let G be a C_6-saturated graph with $\delta(G) = 2$. Then we must have $D(X) \geq 3|X|$ for $X = V(G)\backslash(\{v \in V(G)|d(v) = 2\}\backslash B_1(G))$, where $D(X) = \sum_{v\in X} d(v)$.*

Proof. Suppose not. By Lemma 3.1, we see Lemma 3.1(1) holds. Let α be such a root. Let $N(\alpha)$, $N(\alpha_1)$ and $N(\alpha_2)$ be defined as in the statement of the Lemma 3.1. Then G has a 5-path $P(\alpha x_1)$ because $G + \alpha x_1 \in \mathcal{F}_G$. Let $P(\alpha x_1) = \alpha a_1 a_2 a_3 a_4 x_1$. Note that $d(\alpha_i) = 3$ for any $i \in [2]$. If $a_1 = \alpha_2$, then $\alpha_1 = a_i$ for some $i \in \{2, 3, 4\}$ because G is C_6-free. But then $d(\alpha_1) \geq 4$, a contradiction. If $a_1 = \alpha_1$, then $a_2 = \alpha_2$ and $a_3 = x_2$. But then G has a copy of C_6 with vertices $x_1, \alpha_1, \alpha, \alpha_2, x_2, a_4$ in order, a contradiction. □

This immediately leads to the following theorem.

Theorem 3.3 *Let $n \geq 6$ be a positive integer. Let G be a C_6-saturated graph on n vertices with $B_2(G) = \emptyset$. If $\delta(G) = 2$ and G has no good root, then $e(G) \geq 4n/3 - 2$.*

Proof. Since G has no good root, we see $\{v \in V(G)|d(v) = 2\} = B_1(G) \cup B_2(G)$. Hence, $\{v \in V(G)|d(v) = 2\}\backslash B_1(G) = B_2(G) = \emptyset$. By Lemma 3.2, $D(V(G)) = \sum_{v\in V(G)} d(v) \geq 3|V(G)|$. Then $e(G) = (\sum_{v\in V(G)} d(v))/2 \geq 3n/2 \geq 4n/3 - 2$. □

To complete the proof of Theorem 1.5(b), we just need to prove the following Theorem.

Theorem 3.4 *Let $n \geq 6$ be a positive integer. Let G be a C_6-saturated graph on n vertices with $B_2(G) = \emptyset$. If $\delta(G) = 1$, or $\delta(G) = 2$ and G has at least one good root, then $e(G) \geq 4n/3 - 2$.*

Proof of Theorem 1.5(b): Let G be a C_6-saturated graph on $n \geq 9$ vertices with $e(G) = sat(n, C_6)$. If $B_2(G) = \emptyset$, then by Theorems 3.3 and 3.4, we have $e(G) \geq 4n/3 - 2$. So we may assume $B_2(G) \neq \emptyset$. Let G_1 be a graph obtained from G by deleting every vertex of $B_2(G)$. Clearly, G_1 is a C_6-saturated graph and $B_2(G_1) = \emptyset$. If $|G_1| \leq 5$, then G_1 is a complete graph and so $e(G_1) = |G_1|(|G_1| - 1)/2 > 4|G_1|/3 - 2$. If $|G_1| \geq 6$, then $e(G_1) \geq 4|G_1|/3 - 2$ by Theorem 3.4. It is easy to see that $e(G[B_2(G)]) + e(B_2(G), V(G) \setminus B_2(G)) = 3|B_2(G)|/2$. Thus, $e(G) = e(G_1) + e(B_2(G), V(G) \setminus B_2(G)) + e(G[B_2(G)]) \geq 4|G_1|/3 - 2 + 3|B_2(G)|/2 > 4n/3 - 2$. □

4 Proof of Theorem 3.4

Let G be a C_6-saturated graph on n vertices. Let $\alpha \in V(G)$ such that $d(\alpha) = \delta(G)$. Set $N[\alpha] = \{\alpha, \alpha_1, \ldots, \alpha_{\delta(G)}\}$. Denote by Θ_5 the graph obtained from a 5-cycle by joining two non-adjacent vertices. Let $S = \{v \in V(G)|d(v) = 2\}$, $\mathcal{C}_4 = \{v \in S|v \text{ is contained in a 4-cycle}\}$, $\mathcal{C}_5 = \{v \in S|v \text{ is contained in a 5-cycle}\}$, $S_5 = \{v \in S|v \text{ is contained in a } \Theta_5\}$, $S_4 = \{v \in S|v \text{ is not contained in a } \Theta_5\}$, $S_3 = \{v|v \in \mathcal{C}_4 - \mathcal{C}_5\}$, $S_2 = \{v|v \in \mathcal{C}_5 - \mathcal{C}_4\}$ and $S_1 = S - \mathcal{C}_4 - \mathcal{C}_5$. We choose such α satisfying: when $\delta(G) = 1$, the number of 4-cycles containing α_1 is minimum; when $\delta(G) = 2$, α is a good root and $\alpha \in S_i$ for some $i \in [5]$ such that i is as small as possible. Let $V_1 = N[\alpha]$ and $V_i = \{x \in V(G)|d(x, \alpha) = i\}$ for any $i \geq 2$. Clearly, $V_i = \emptyset$ for any $i \geq 6$ because G is C_6-saturated, that is, any two non-adjacent vertices are the ends of some 5-path in G. Thus, $V_1, \ldots, V_5$ form a partition of $V(G)$. We define a function as follows.

Definition 4.1 For any $x \in V_i$ and $i \geq 1$, let

$$g(x) = \begin{cases} \frac{1}{2}n_i(x) - \frac{4}{3} & \text{if } i = 1; \\ n_{i-1}(x) + \frac{1}{2}n_i(x) - \frac{4}{3} & \text{if } i \geq 2. \end{cases}$$

Remark 1. For any $x \in V_i$ with $i \geq 2$, $g(x) \geq n_{i-1}(x)/3 + n_i(x)/6 \geq \frac{2}{3}$ if $n_{i-1}(x) \geq 2$ or $n_i(x) \geq 2$; $g(x) = \frac{1}{6}$ if $n_{i-1}(x) = 1$ and $n_i(x) = 1$; and $g(x) = -\frac{1}{3}$ if $n_{i-1}(x) = 1$ and $n_i(x) = 0$.

We define $N_i^\star(x) := N_i(x) \cap V_i^\star$ and $n_i^\star(x) := |N_i^\star(x)|$, where $\star \in \{-, +, 1, 2, -1, -2\}$. By Remark 1, for $i \geq 2$, V_i can be partitioned into V_i^+, V_i^- such that

$$V_i^+ := \{x \in V_i | g(x) \geq 0\} \text{ and } V_i^- = V_i \backslash V_i^+ = \{x \in V_i | g(x) = -1/3\}.$$

Moreover, we partition V_i^+ into V_i^1, V_i^2 and V_i^- into V_i^{-1}, V_i^{-2} such that

$$V_i^2 := \{x \in V_i^+ | g(x) \geq 2/3\} \text{ and } V_i^1 := V_i^+ \backslash V_i^2 = \{x \in V_i^+ | g(x) = 1/6\};$$

$$V_i^{-1} := \{x \in V_i^- | n_{i+1}^2(x) \geq 2\} \text{ and } V_i^{-2} := V_i^- \backslash V_i^{-1}.$$

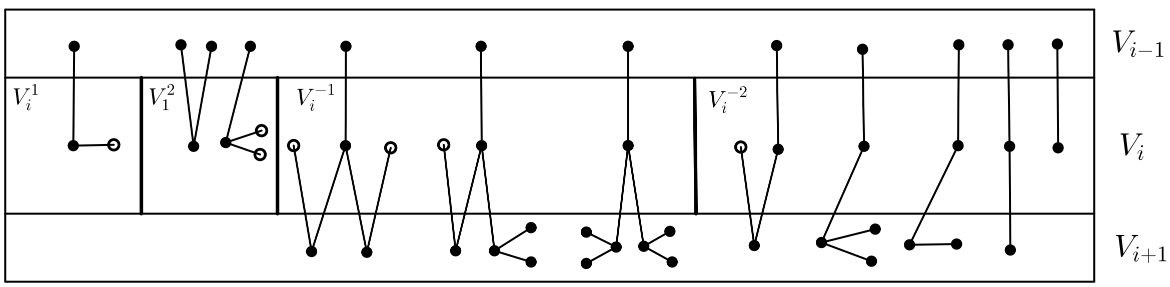


Figure 2: The partition of V_i.

For any $i \geq 2$, the partition of V_i is depicted in Figure 2, where the hollow vertices do not necessarily belong to the set shown in the Figure. By Definition 4.1,

$$\begin{aligned}
e(G) &= \sum_{i=1}^{5} e(G[V_i]) + \sum_{i=2}^{5} e(V_i, V_{i-1}) = \sum_{i=1}^{5} \frac{1}{2} \sum_{x \in V_i} n_i(x) + \sum_{i=2}^{5} \sum_{x \in V_i} n_{i-1}(x) \\
&= \sum_{i=1}^{5} \sum_{x \in V_i} \left(g(x) + \frac{4}{3} \right) = \sum_{x \in V_1} g(x) + \sum_{x \in V(G) \backslash V_1} g(x) + \frac{4}{3} n \\
&= \sum_{x \in V(G) \backslash V_1} g(x) + \frac{4}{3} n + e(G[V_1]) - \frac{4}{3} |V_1| \\
&= \sum_{x \in V(G) \backslash V_1} g(x) + \frac{4}{3} n + (|V_1| - 1) - \frac{4}{3} |V_1| \\
&= \sum_{x \in V(G) \backslash V_1} g(x) + \frac{4}{3} n - \frac{1}{3} |V_1| - 1 \\
&\geq \sum_{x \in V(G) \backslash V_1} g(x) + \frac{4}{3} n - 2, \qquad (1)
\end{aligned}$$

where $e(V_i, V_{i-1})$ is the number of edges with one end in V_i and the other end in V_{i-1}.

To obtain the desired lower bound in Theorem 3.4, we just need to show that $\sum_{x \in V(G) \backslash V_1} g(x) \geq 0$. The basic idea of our proof is as follows: we first allocate a charge to each vertex in G such that the initial charge function is the function g defined in Definition 4.1, and then reallocate the charge to each vertex in G such that the final charge at each vertex of $V(G) \setminus V_1$ is at least zero. To reallocate the charge to each vertex in G, we need to define a series of charge functions, which can be regarded as two stages: at the first stage, the charge function g_i for any $i \in [5]$ is defined; at the second stage, the charge function f_i for any $i \in [7]$ is defined, such that $\sum_{x\in V(G)} g_{i-1}(x) = \sum_{x\in V(G)} g_i(x)$ for any $i \in [5]$ and $\sum_{x\in V(G)} f_{j-1}(x) = \sum_{x\in V(G)} f_j(x)$ for any $j \in [7]$, where $g_0 = g$ and $f_0 = g_5$. Since the charges of vertices of V_1 are never altered, it suffices to prove that $\sum_{x\in V(G)\setminus V_1} f_7(x) \geq 0$.

In the following, we define the charge functions at the first stage. For any vertex x uninvolved in the definition of g_i, $g_i(x)$ is still equal to $g_{i-1}(x)$. In the following, for $x \in V(G)$, we call x as j-vertex if $g(x) = j$; j^+-vertex if $g(x) \geq j$.

Definition 4.2 (1) Several $\frac{2}{3}^+$-vertices send charges to some $-\frac{1}{3}$-vertices in the previous level and $\frac{1}{6}$-vertices in the same level; several $\frac{1}{6}$-vertices send charges to $-\frac{1}{3}$-vertices in the previous level. We define the concrete laws as follows:

(1.1) For any $u \in V_2^2$, let $g_1(u) = g(u) - \frac{1}{6}n_2^1(u)$, as depicted in Figure 3(a_1). For any $u \in V_i^2$ with $i \in \{3, 4\}$, let $g_1(u) = g(u) - \frac{1}{6}|A_1(u)| - \frac{1}{3}|B_1(u)| - \frac{1}{6}n_i^1(u)$, as depicted in Figure 3(a_2). For any $u \in V_i^2$ with $i = 5$, let $g_1(u) = g(u) - \frac{1}{6}|A_1(u)| - \frac{1}{3}|B_1(u)| - \frac{1}{6}(n_5^1(u) - |C_1(u)|)$, as depicted in Figure 3(a_3), where $A_1(u) = \{v \in N_{i-1}^-(u) | n_i^2(v) \geq 2\}$, $B_1(u) = \{v \in N_{i-1}^-(u) | n_i^2(v) = 1\}$ and

$$C_1(u) = \begin{cases} \{w \in N_5^1(u) | N_4(w) \subseteq N_4(w') \text{ for some } w' \in N_5(u) \setminus w\} & \text{if } d(u) = 3, n_4(u) = n_4^-(u) = 1; \\ \emptyset & \text{else.} \end{cases}$$

(1.2) For any $w \in V_2^1$, let $g_1(w) = g(w) + \frac{1}{6}n_i^2(w)$, as depicted in Figure 3(b_1). For any $w \in V_i^1$ with $i \in \{3, 4\}$, let $g_1(w) = g(w) + \frac{1}{6}n_i^2(w) - \frac{1}{6}|A_2(w)|$, as depicted in Figure 3(b_2), where $A_2(w) = \{v \in N_{i-1}^-(w) | n_i^2(v) = 0\}$. For any $w \in V_5^1$, let $g_1(w) = g(w) + \frac{1}{6}n_5^2(w)$ if $w \notin C_1(u)$ for any $u \in V_5^2$, as depicted in Figure 3(b_3), $g_1(w) = g(w)$ otherwise.

(1.3) For any $v \in V_i^-$ with $i \in \{2, 3, 4\}$, let $g_1(v) = g(v) + \frac{1}{6}n_{i+1}^2(v)$ if $n_{i+1}^2(v) \geq 2$, as depicted in Figure 3(c_1); $g_1(v) = g(v) + \frac{1}{3}$ if $n_{i+1}^2(v) = 1$, as depicted in Figure 3(c_2); $g_1(v) = g(v) + \frac{1}{6}n_{i+1}^1(v)$ when $i \in \{2, 3\}$ (as depicted in Figure 3(c_3)) and $g_1(v) = g(v)$ when $i = 4$ if $n_{i+1}^2(v) = 0$.

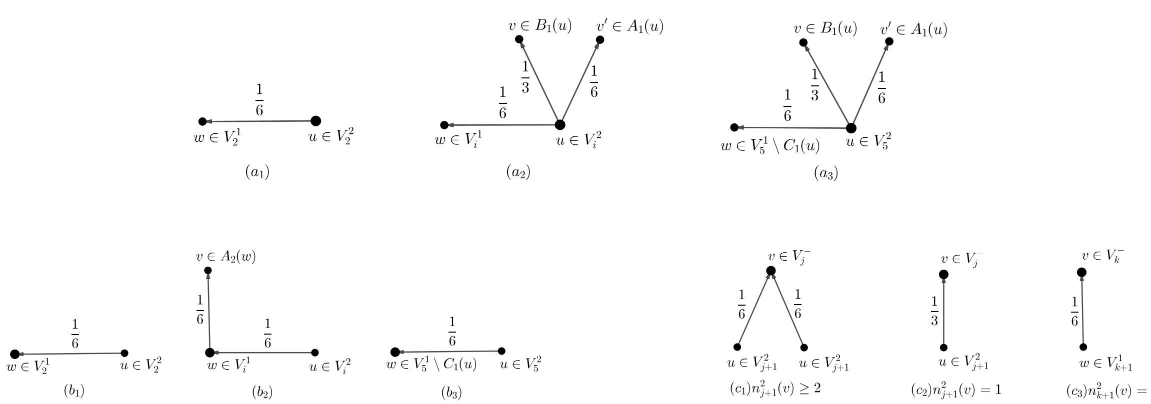


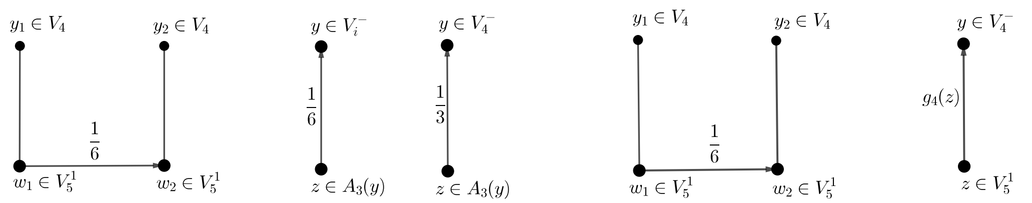
Figure 3: (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) are the explanations of Definition 4.2(1.1), (1.2), (1.3) respectively, where $i \in \{3, 4\}$, $j \in \{2, 3, 4\}$ and $k \in \{2, 3\}$.

(2) Several $\frac{1}{6}$-vertices of V_5 send charges to some $\frac{1}{6}$-vertices in V_5. Concretely, for any two adjacent vertices $w_1, w_2 \in V_5^1$ with $N_4(w_i) = \{y_i\}$, let $g_2(w_i) = g_1(w_i) - \frac{1}{6}$ and $g_2(w_{3-i}) = g_1(w_{3-i}) + \frac{1}{6}$ if $g_1(y_i) \geq 0$ and $g_1(y_{3-i}) < 0$, as depicted in Figure 4(a); $g_2(w_i) = g_1(w_i)$ otherwise.

(3) Several $\frac{1}{6}$-vertices send charges to their neighbours in the previous level. Concretely, for any $y \in V_i^-$ with $g_2(y) < 0$, where $i \in \{2, 3, 4\}$, let $g_3(y) = g_2(y) + t|A_3(y)|$ and $g_3(z) = g_2(z) - g_2(z) = 0$ for $z \in A_3(y)$, where $A_3(y) = \{z \in N_{i+1}^1(y) | g_2(z) = t\}$, and $t = \frac{1}{6}$ when $i \in \{2, 3\}$ and $t = \frac{1}{3}$ when $i = 4$, as depicted in Figure 4(b).

(4) Several $\frac{1}{6}$-vertices of V_5 send charges to some $\frac{1}{6}$-vertices in V_5. Concretely, for any two adjacent vertices $w_1, w_2 \in V_5^1$ with $N_4(w_i) = \{y_i\}$ and $g_3(w_1) = g_3(w_2) = \frac{1}{6}$, let $g_4(w_i) = g_3(w_i) - \frac{1}{6}$ and $g_4(w_{3-i}) = g_3(w_{3-i}) + \frac{1}{6}$ if $g_3(y_i) \geq 0$ and $g_3(y_{3-i}) < 0$, as depicted in Figure 4(c); $g_4(w_i) = g_3(w_i)$ otherwise.

(5) Several $\frac{1}{6}$-vertices of V_5 send charges to their neighbours in the previous level. Concretely, for any $y \in V_4^-$ with $g_4(y) < 0$, let $g_5(y) = g_4(y) + \sum_{z \in N_5^1(y)} g_4(z)$ and $g_5(z) = g_4(z) - g_4(z) = 0$ for any $z \in N_5^1(y)$, as depicted in Figure 4(d).



(a) $g_1(y_1) \geq 0$, $g_1(y_2) < 0$ (b) $g_2(y) < 0$, $i \in \{2, 3\}$ (c) $g_3(y_1) \geq 0$, $g_3(y_2) < 0$ (d) $g_4(y) < 0$

Figure 4: (a), (b), (c) and (d) are the explanations of Definition 4.2(2), (3), (4) and (5), respectively.

For any $x \in V(G)$, let $g^*(x) = g_5(x)$ and $g_0(x) = g(x)$. Clearly,

$$\sum_{x \in V(G) \setminus V_1} g^*(x) = \sum_{x \in V(G) \setminus V_1} g(x). \tag{2}$$

Moreover, for any $0 \leq i < j \leq 5$, if $g_i(x) \geq 0$, then $g_j(x) \geq 0$; and if $g_i(x) < 0$, then $g_0(x) \leq \cdots \leq g_i(x) < 0$. For any $i \in [5]$, let $V_i^{*-} := \{x \in V_i | g^*(x) < 0\}$, $N_i^{*-}(x) := N_i(x) \cap V_i^{*-}$ and $n_i^{*-}(x) := |N_i^{*-}(x)|$. By Definitions 4.1 and 4.2, we see that for any $x \in V_i^2$ and $i \in \{3, 4, 5\}$,

$$\begin{aligned} g^*(x) &\geq g(x) - \frac{1}{3} n_{i-1}^{-2}(x) - \frac{1}{6} n_{i-1}^{-1}(x) - \frac{1}{6} n_i^1(x) \\ &\geq g(x) - \frac{1}{3}(n_{i-1}(x) - n_{i-1}^+(x)) + \frac{1}{6} n_{i-1}^{-1}(x) - \frac{1}{6}(n_i(x) - n_i^2(x)) \\ &= \frac{2}{3} n_{i-1}(x) + \frac{1}{3} n_{i-1}^+(x) + \frac{1}{6} n_{i-1}^{-1}(x) + \frac{1}{3} n_i(x) + \frac{1}{6} n_i^2(x) - \frac{4}{3}. \end{aligned} \tag{3}$$

By Definition 4.2 and Ineq. (3), we have the following observations.

Observation 4.3 Let $i \in \{2, 3, 4\}$. For $x \in V_i^{*-}$, we have
(1) $n_{i+1}^2(x) = 0$ and $n_{i+1}^1(x) \leq 1$; (2) if $y \in N_{i+1}^1(x)$, then $n_{i+1}^1(y) = 1$.

Observation 4.4 Let $i \in \{3, 4, 5\}$. For any $x \in V_i^2$, $g^*(x) \geq \frac{1}{3} n_{i-1}(x) + \frac{1}{6} n_i(x)$ if one of the following holds.
(1) $n_{i-1}^+(x) \geq 2$ or $n_{i-1}(x) \geq 3$ and $n_{i-1}(x) + n_i(x) \geq 5$;
(2) $n_{i-1}^+(x) + n_{i-1}(x) \geq 4$;
(3) $n_{i-1}^+(x) + n_{i-1}(x) = 3$ and $n_i^2(x) \geq 1$;
(4) $n_{i-1}^+(x) + n_{i-1}(x) = 3$ and $n_{i-1}^{-1}(x) + n_i(x) \geq 2$.

Observation 4.5 Let $i \in \{3, 4, 5\}$. For any $x \in V_i^2$, $g^*(x) \geq \frac{1}{6} n_{i-1}(x) + \frac{1}{6} n_i(x)$ if one of the following holds.
(1) $n_{i-1}^+(x) \geq 2$ or $n_{i-1}(x) \geq 2$ and $n_{i-1}(x) + n_i(x) \geq 4$;
(2) $n_{i-1}^+(x) + n_{i-1}(x) \geq 3$;

(3) $n^{+}_{i-1}(x) + n_{i-1} = 2$ and $n^{2}_{i}(x) \geq 1$;

(4) $n^{+}_{i-1}(x) + n_{i-1} = 2$ and $n^{-1}_{i-1}(x) + n_i(x) \geq 3$;

(5) $n^{+}_{i-1}(x) + n_{i-1} = 1$ and $n^{-1}_{i-1}(x) + n^{2}_{i}(x) \geq 3$.

Now we are ready to define the functions at the second stage. Here, the "initial charge" of x can be regarded as $g^*(x)$. For any vertex uninvolved in the following definition of f_i, the value of f_i remains unchanged. For any $x \in V(G)$, let $N^{j-}_{i}(x) := \{v \in N_i(x) | f_j(v) < 0\}$ and $N^{j+}_{i}(x) = N_i(x) \setminus N^{j-}_{i}(x)$. We define $n^{j-}_{i}(x) = |N^{j-}_{i}(x)|$ and $n^{j+}_{i}(x) = |N^{j+}_{i}(x)|$. Let $L(v) \subseteq N(v)$ denote the set of vertices with degree one.

Definition 4.6 (1) Every vertex of V_5 sends charge equally to its neighbours in V_4. Concretely, for any $z \in V_4$ and $w \in V_5$, $f_1(z) = g^*(z) + \sum_{v \in N_5(z)} g^*(v)/n_4(v)$ and $f_1(w) = g^*(w) - \sum_{u \in N_4(w)} g^*(w)/n_4(w) = 0$, as depicted in Figure 5(a). With the following two exceptional cases:

(1.1) Suppose $w \in V_5$ with $N(w) = \{z_1, z_2, w_1\}$ such that $n^{-}_{5}(z_1) \neq 0$, $n^{-}_{5}(z_2) = 0$ and $N^{1}_{5}(w_1) \cap N^{1}_{5}(z_2) \neq \emptyset$, where $z_1, z_2 \in V^{-}_{4}$ and $w_1 \in V_5$. Then w will send $2g^*(w)/3$ charge to z_1, and $g^*(w)/3$ charge to z_2, as depicted in Figure 5(b).

(1.2) Suppose $w \in V_5$ with $N(w) = N^{-}_{4}(w) = \{z_1, z_2, z_3\}$ such that $n^{-}_{5}(z_1) \neq 0$, $n^{-}_{5}(z_2) = n^{-}_{5}(z_2) = 0$ and $N_3(z_2) = N_3(z_3)$. Then w will send $g^*(w)/2$ charge to z_1, and $g^*(w)/4$ charge to z_2, and $g^*(w)/4$ charge to z_3, as depicted in Figure 5(c).

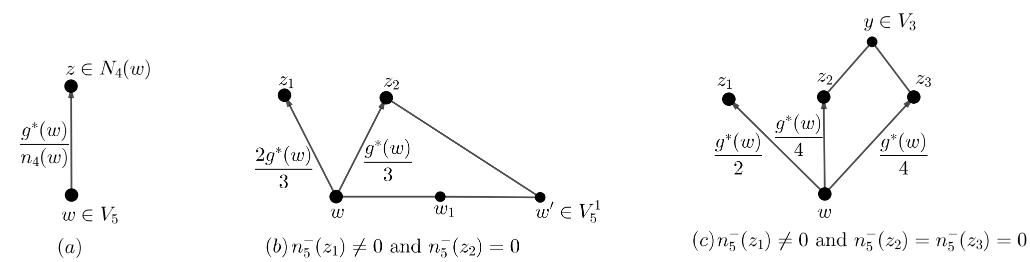


Figure 5: (a), (b) and (c) are the explanations of Definition 4.6 (1), (1.1) and (1.2), respectively.

(2) Several $\frac{1}{6}^{+}$-vertices of V_4 send charges to their neighbours in V_4, as depicted in Figure 6. Concretely, for any $z \in V_4$, if $f_1(z) \geq \frac{1}{6} n^{1-}_{4}(z)$, then $f_2(z) = f_1(z) - \frac{1}{6} n^{1-}_{4}(z)$, and $f_2(z') = f_1(z') + \frac{1}{6}$ for any $z' \in N^{1-}_{4}(z)$. Moreover, for any two adjacent vertices $z_1, z_2 \in V^{1}_{4}$ with $N_3(z_i) = \{y_i\}$ and $f_1(z_j) = 0$ and $f_1(y_j) < 0$, if $f_1(z_{3-j}) \geq \frac{1}{3}$ and $f_1(y_{3-j}) < 0$, or $f_1(z_{3-j}) \geq \frac{1}{6}$ and $f_1(y_{3-j}) \geq 0$, then let $f_2(z_{3-j}) = f_1(z_{3-j}) - \frac{1}{6}$ and $f_2(z_j) = f_1(z_j) + \frac{1}{6}$.

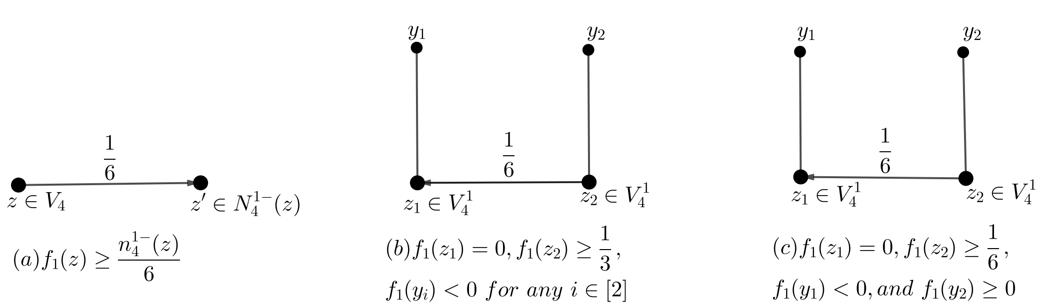


Figure 6: (a), (b) and (c) are the explanations of Definition 4.6(2).

(3) Several vertices of V_4 send charges equally to their neighbours in V_3. Concretely, for any $y \in V_3$, then $f_3(y) = f_2(y) + \sum_{z \in N_4^{2+}(y)} f_2(z)/n_3(z)$ and $f_3(z) = f_2(z) - f_2(z) = 0$ for any $z \in N_4^{2+}(y)$, as depicted in Figure 7(a). With the exception of the following case:

(3.1) Suppose $z \in V_4$ with $N_3(z) = \{y_1, y_2, y_3\}$, $n_4(z) \le 1$ and $n_5^-(z) = 0$ such that $L(y_1) \neq \emptyset$ and $L(y_2) = L(y_3) = \emptyset$. Then z will send $f_2(z)/2$ charge to y_1, and $f_2(z)/4$ charge to y_2, and $f_2(z)/4$ charge to y_3, as depicted in Figure 7(b).

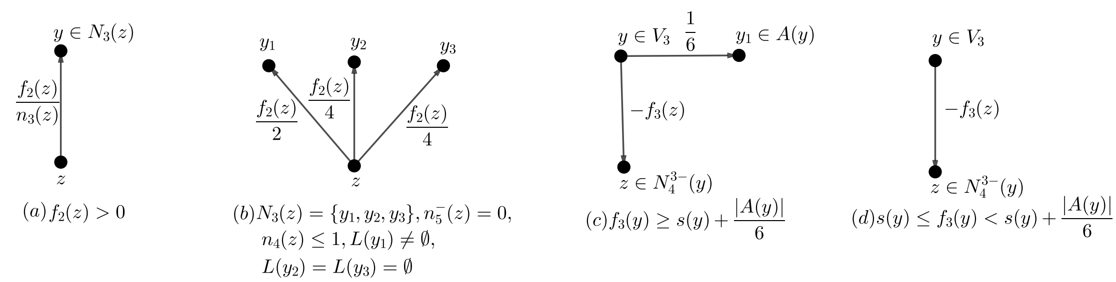


Figure 7: (a, b) and (c, d) are the explanations of Definition 4.6(3) and (4), respectively.

(4) Several vertices of V_3 send charges to their neighbours in $V_3 \cup V_4$. Concretely, for any $y \in V_3$, let $s(y) := \sum_{z \in N_4^{3-}(y)} -f_3(z)$ and $A(y) = \{y_1 \in N_3(y) | f_3(y_1) - s(y_1) < 0\}$, if $f_3(y) \ge s(y) + \frac{1}{6}|A(y)|$, then $f_4(y) = f_3(y) - s(y) - \frac{1}{6}|A(y)|$, $f_4(z) = f_3(z) + (-f_3(z))$ for any $z \in N_4^{3-}(y)$ and $f_4(y_1) = f_3(y_1) + \frac{1}{6}$ for any $y_1 \in A(y)$, as depicted in Figure 7(c); if $s(y) + \frac{1}{6}|A(y)| > f_3(y) \ge s(y)$, then $f_4(y) = f_3(y) - s(y)$ and $f_4(z) = f_3(z) + (-f_3(z))$ for any $z \in N_4^{3-}(y)$, as depicted in Figure 7(d).

(5) Several vertices of V_3 send charges to their neighbours in V_4. Concretely, for $y \in V_3$, if $f_4(y) \ge \sum_{z \in N_4^{4-}(y)} -f_4(z)$, then $f_5(y) = f_4(y) - \sum_{z \in N_4^{4-}(y)} -f_4(z)$ and $f_5(z) = f_4(z) + (-f_4(z)) = 0$ for any $z \in N_4^{4-}(y)$, as depicted in Figure 8(a).

(6) Several $\frac{1}{6}$-vertices of V_3 send charges to their neighbours in V_3. Concretely, for any two adjacent vertices $y_1, y_2 \in V_3^1$ with $N_2(y_i) = \{x_i\}$, if $f_5(x_1) \ge 0$, $f_5(x_2) < 0$, $f_5(y_1) \ge \frac{1}{6}$ and $f_5(y_2) = 0$, then let $f_6(y_1) = f_5(y_1) - \frac{1}{6}$, $f_6(y_2) = f_5(y_2) + \frac{1}{6}$, as depicted in Figure 8(b).

(7) Every vertex of $V_3 \cup V_4$ sends charge to some vertices in V_2. Concretely, for $x \in V_2$, if $f_6(x) + \sum_{y \in N_3^{6+}(x)} f_6(y)/n_2(y) \ge \sum_{y \in A_4(x) \cup N_3^{6-}(x)} -f_6(y)$, then $f_7(x) = f_6(x) + \sum_{y \in N_3^{6+}(x)} f_6(y)/n_2(y) + \sum_{y \in A_4(x) \cup N_3^{6-}(x)} f_6(y)$, $f_7(y) = f_6(y) - f_6(y) = 0$ for any $y \in N_3(x) \cup A_4(x)$, where $A_4(x) = \{y \in N_4(N_3(x)) | f_6(y) < 0\}$, as depicted in Figure 8(c).

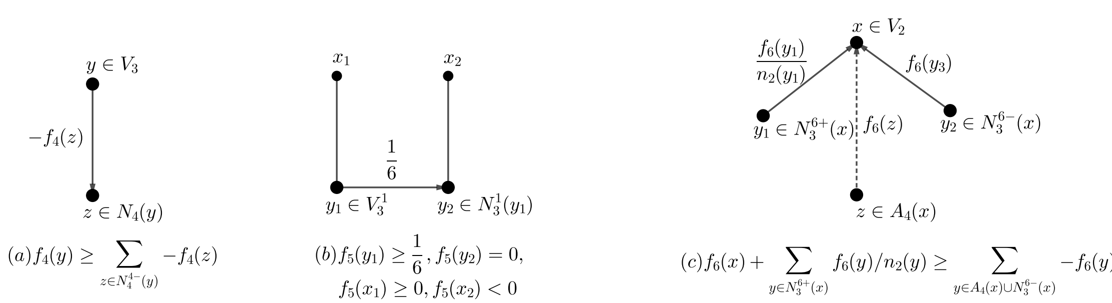


Figure 8: (a), (b) and (c) are the explanations of Definition 4.6(5), (6) and (7), respectively.

Clearly,

$$\sum_{x\in V(G)\setminus V_1} f_7(x) = \sum_{x\in V(G)\setminus V_1} g^*(x). \tag{4}$$

Observe that for any $x \in V(G)$ and $1 \le i < j \le 6$, we have

($\circledast$) if $f_i(x) \ge 0$, then $f_j(x) \ge 0$; if $f_i(x) < 0$, then $f_1(x) \le \cdots \le f_i(x) < 0$.

In order to prove Theorem 3.4, it is enough to prove Theorem 4.9 by inequalities (2) and (4), which will be proved in Section 8. Before we do this, we firstly prove the following two results in Sections 6 and 7, which indicate that the number of vertices in V_i^{5-} for any $i \in \{3,4\}$ are few.

Theorem 4.7 *Let $x \in V_2$. Then $\sum_{y\in N_3(x)} n_4^{5-}(y) \le 1$.*

Theorem 4.8 *Let $x \in V_2$. Then $n_3^{5-}(x) \le 1$. Moreover, if $n_3^{5-}(x) = 1$, then $n_4^{5-}(y) = 0$ for any $y \in N_3^{5+}(x)$.*

Theorem 4.9 *For all $x \notin V_1$, we have $f_7(x) \ge 0$.*

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4: If $\delta(G) \ge 3$, then $e(G) \ge 3n/2 > 4n/3-2$. So we may assume that $\delta(G) \le 2$. If $\delta(G) = 1$, then $\sum_{v\in V_1} g(v) = 2\times(\frac{1}{2}-\frac{4}{3}) = -\frac{5}{3}$. If $\delta(G) = 2$, then $\sum_{v\in V_1} g(v) = 2\times(\frac{1}{2}-\frac{4}{3})+(1-\frac{4}{3}) = -2$. By Theorem 4.9 and inequalities (2) and (4), $\sum_{v\in V(G)\setminus V_1} g(v) \ge 0$. Therefore, by Ineq. (1), $e(G) \ge \sum_{v\in V_1} g(v)+\sum_{v\in V(G)\setminus V_1} g(v)+\frac{4}{3}n \ge \frac{4n}{3} - 2$, as desired. □

5 Several Lemmas

Lemma 5.1 *Let $y \in V_3$. For $z \in N_4^{*-}(y)$ with $d(z) \ge 2$, we have*
(a) $n_5^-(z) = 0$ and $d(z) = 2$,
(b) G has a 3-path $P = zww_1z_1$ such that $d(w) = d(w_1) = d(z_1) = 2$ and $yz_1 \notin E(G)$.

Lemma 5.2 *For any $y \in V_3$, we have $n_4^{*-}(y) \le 1$.*

Remark 2. It is easy to check that $n_5^-(v) \le 1$ for any $v \in V_4$. From the above Definition 4.6, we see that $f_6(v) = \cdots = f_1(v) = g^*(v)$ for any $v \in V_2$, $f_2(v) = f_1(v) = g^*(v)$ for any $v \in V_3$, $f_1(v) \ge g^*(v)-\frac{1}{3}n_5^-(v) \ge g^*(v)-\frac{1}{3}$ for any $v \in V_4$ and $f_2(v) \ge f_1(v) - \frac{1}{6}n_4^{1-}(v) \ge f_1(v) - \frac{1}{6}n_4(v)$ for any $v \in V_4^2$.

Let $t_v^* := \frac{g^*(v)}{n_{i-1}(v)}$ for any $v \in V_i$ and $i \in \{3,4,5\}$, $t_v^2 := \frac{f_2(v)}{n_3(v)}$ for any $v \in V_4$ and $t_v^j := \frac{f_j(v)}{n_2(v)}$ for any $v \in V_3$ and $j \in \{5,6\}$.

Lemma 5.3 *For any $z \in V_4$ with $n_5^-(z) \ge 1$, we have $f_2(z) \ge 0$; moreover, $f_3(y) \ge 0$ for any $y \in N_3(z)$.*

By ($\circledast$), Lemmas 5.2 and 5.3, one can easily see the following corollary.

Corollary 5.4 *For any $x \in V_3$, $N_4^{i-}(x) \subseteq N_4^{*-}(x)$ and $n_4^{i-}(x) \le 1$ for any $2 \le i \le 5$.*

Lemma 5.5 *For any $y \in V_3$ with $N_5^-(N_4(y)) \ne \emptyset$, we have $n_4^{5-}(y) = 0$.*

Lemma 5.6 *Let $z \in V_4$ and $N_3(z) = \{y_1, y_2, y_3\}$. If $N_2(y_1) = N_2(y_2) = N_2(y_3)$ and $n_2(y_1) = 1$, then $n_4^{5-}(y_i) = 0$ for any $i \in [3]$.*

Lemma 5.7 *For any $y \in V_3$ with $L(y) \ne \emptyset$, we have $n_4^{5-}(y) = 0$, moreover $f_5(y) \ge 0$.*

Lemma 5.8 *Let* $y \in V_3^{5-}$.
(a) If $z \in N_4^{5-}(y)$, *then* $n_4^+(y) = 0$.
(b) If $z \in N_4^1(y)$ *and* $N_4(z) = \{z_1\}$, *then* $n_5^-(z_1) = 0$.

For readability, we have removed all proofs of these lemmas to Appendix.

6 Proof of Theorem 4.7

Suppose not. Let $z_i \in N_4^{5-}(y_i)$ for any $i \in [2]$, where $y_i \in N_3(x)$. By (⊛) and Corollary 5.4, $N_4^{5-}(y_i) = N_4^{3-}(y_i) = N_4^{*-}(y_i) = \{z_i\}$ and $y_1 \neq y_2$. By Lemmas 5.7 and 5.1(a, b), for any $i \in [2]$, G has a 3-path $P^i = z_i w_i w_i' z_i'$ such that $d(v) = 2$ for any $v \in V(P^i)$ and $z_i' y_i \notin E(G)$. Clearly, $f_3(z_i) \geq g^*(z_i) \geq -\frac{1}{6}$. Hence, $f_3(y_i) < \frac{1}{6}$ for any $i \in [2]$, else $f_4(z_i) \geq f_3(z_i) + \frac{1}{6} \geq 0$ because $N_4^{3-}(y_i) = \{z_i\}$. Note that either $P_1 = P_2$ or $V(P_1) \cap V(P_2) = \emptyset$. Because $d(z_i') = d(w_i') = 2$, we see $z_i' \in V_4^{*-}$ and so $z_i' y_{3-i} \notin E(G)$ if $z_i' \neq z_{3-i}$ for any $i \in [2]$. Since $G + z_1 z_2 \in \mathcal{F}_G$, we see G has a 3-path $P(y_1 y_2)$. Let $P(y_1 y_2) = y_1 a_1 a_2 y_2$. We next prove two claims.

Claim 6.1 $x \notin \{a_1, a_2\}$.

Proof. W.l.o.g., suppose $x = a_1$. Then $N_2^+(y_2) \cup (N_3^2(x) \setminus y_2) = \emptyset$, else $f_3(y_2) \geq g^*(y_2) \geq \frac{1}{6}$. Hence, $a_2 \in V_3^1$ and $g(x) < 0$. By the choice of α, we see $\alpha_1 \alpha_2 \notin E(G)$ if $\delta(G) = 2$. This implies $g(y_1) = \frac{1}{6}$ because $G + z_1 \alpha \in \mathcal{F}_G$, $d(z_1, \alpha) = 4$ and $g(x) < 0$. Clearly, $g(y_2) \leq \frac{1}{6}$, else $f_3(y_1) \geq g^*(y_1) \geq \frac{1}{6}$. Note that G has a 5-path $P(y_2 \alpha)$. Let $P(y_2 \alpha) = y_2 b_1 b_2 b_3 \alpha_i \alpha$. By the choice of α, α belongs to no 4-cycle because $d(z_i) = 2$ and z_i belongs to no 4-cycle. Then $b_1 \in V_4$ and $b_2 \in V_3$ because $d(y_2, \alpha) = 3$, $N_3^2(x) \setminus y_2 = \emptyset$ and $g(x) < 0$. By Lemma 5.5, $n_5^-(b_1) = 0$. By Observation 4.5(2), $g^*(b_1) \geq \frac{1}{6} n_3(b_1) + \frac{1}{6} n_4(b_1)$ because $y_2 \in N_3^+(b_1)$ and $n_3(b_1) \geq 2$, which yields $t_{b_1}^2 \geq \frac{1}{6}$. But then $f_3(y_2) \geq f_2(y_2) + t_{b_1}^2 \geq \frac{1}{6}$, a contradiction. This proves Claim 6.1. □

Claim 6.2 $|\{a_1, a_2\} \cap V_4| = 1$.

Proof. Obviously, $|\{a_1, a_2\} \cap V_4| \geq 1$, else $f_3(y_i) \geq g^*(y_i) \geq \frac{1}{6}$ because $n_3^2(x) + n_3^2(y_i) \geq 1$ for some $i \in [2]$. Suppose $|\{a_1, a_2\} \cap V_4| = 2$. By Lemma 5.5, for any $i \in [2]$, $n_5^-(a_i) = 0$ and $f_1(a_i) \geq 0$, which means $n_4^{1-}(a_i) \leq n_4(a_i) - 1$. Then $g(y_i) < 0$ for any $i \in [2]$, else let $g(y_1) \geq 0$, we see $f_3(y_1) \geq f_2(y_1) + t_{a_1}^2 \geq \frac{1}{6}$ because $f_1(a_1) \geq g^*(a_1) \geq \frac{1}{6} n_3(a_1) + \frac{1}{6}(n_4(a_1) - 1)$ and $t_{a_1}^2 \geq \frac{1}{6}$. Hence, G has a 3-path $P(x y_i)$ for some $i \in [2]$ as $G + z_1 y_2 \in \mathcal{F}_G$. Let $P(x y_i) = x b_1 b_2 y_i$. Then $b_1 \in V_3$ and $b_2 \in V_4$. Since $G + z_i b_1 \in \mathcal{F}_G$, we have $g(b_1) \geq \frac{1}{6}$ or $G - y_i - b_1$ has a 2-path with ends x and b_2. By Observation 4.5(2), $f_1(b_2) \geq g^*(b_2) \geq \frac{1}{6} n_3(b_2) + \frac{1}{6} n_4(b_2)$ because $n_5^-(b_2) = 0$ and $n_3^+(b_2) + n_3(b_2) \geq 3$, which implies $t_{b_2}^2 \geq \frac{1}{6}$. But then $f_3(y_i) \geq f_2(y_i) + t_{b_2}^2 \geq \frac{1}{6}$, a contradiction. □

By Claims 6.1 and 6.2, w.l.o.g., let $a_1 \in V_4$. Then $a_2 \in V_3$ and so $t_{a_1}^2 \geq \frac{1}{6}$ because $n_3^+(a_1) + n_3(a_1) \geq 3$ and $n_5^-(a_1) = 0$. But then $f_3(y_1) \geq f_2(y_1) + t_{a_1}^2 \geq \frac{1}{6}$, a contradiction. This completes the proof of Theorem 4.7. □

7 Proof of Theorem 4.8

We first show that $n_3^{5-}(x) \leq 1$. Suppose not. Let $y_1, y_2 \in N_3^{5-}(x)$. By (⊛) and Corollary 5.4, $f_j(y_i) < 0$ for any $j \in [5]$ and $i \in [2]$, and $y_1, y_2 \in N_3^{*-}(x)$.

Claim 7.1 $n_4^2(y_i) = 0$, $n_4^1(y_i) \leq 1$, $N_5^-(N_4(y_i)) = \emptyset$ and there is no s-path $P(x y_i)$ for any $s \in \{2, 3\}$.

Proof. The desired results hold by Observation 4.3(1) and Lemma 5.3. □

Claim 7.2 Suppose $N_4^1(y_i) = \{z_i\}$, $N_4(z_i) = \{z'_i\}$, $w'_i \in N_5(z'_i)$ and $w_i \in N_5(z_i)$. Then $n_4^2(z_i) = n_5^2(z_i) = n_5^2(w_i) = n_5^-(z'_i) = n_5^2(w'_i) = 0$. Furthermore, $f_1(z'_i) < \frac{1}{3}$ and $f_1(v) < \frac{1}{6}$ for any $v \in N_4(y_i)$.

Proof. By Observation 4.3(2), $n_4^2(z_i) = 0$. By Lemma 5.8(2), $n_5^-(z'_i) = 0$. Clearly, $f_2(y_i) = g^*(y_i) \geq -\frac{1}{6}$. Hence, $f_1(v) < \frac{1}{6}$ for any $v \in N_4(y_i)$, otherwise let $f_1(v_0) \geq \frac{1}{6}$ for some $v_0 \in N_4(y_i)$, $t_{v_0}^2 \geq \frac{1}{6}$ as $n_4^{1-}(v_0) = 0$, which yields $f_3(y_i) \geq f_2(y_i) + t_{v_0}^2 \geq 0$. Moreover, $f_1(z'_i) < \frac{1}{3}$, otherwise by Definition 4.6(2), $f_2(z_i) \geq \frac{1}{6}$ and so $f_3(y_i) \geq 0$. Next we prove $n_5^2(z_i) = n_5^2(w_i) = 0$. Suppose $w_i \in N_5^2(z_i)$ or $n_5^2(w_i) \neq 0$. By Ineq. (3), $g^*(w_i) \geq \frac{1}{6}n_4(w_i)$ because $z_i \in N_4^+(w_i)$, which means $t_{w_i}^* \geq \frac{1}{6}$. But then $f_1(z_i) \geq g^*(z_i) + t_{w_i}^* \geq \frac{1}{6}$, a contradiction, as desired. It remains to show $n_5^2(w'_i) = 0$. Suppose not. Then $w'_i \in V_5^1$ and $g^*(w'_i) < \frac{1}{3}$, else, $t_{w'_i}^* \geq \frac{1}{3}$ and so $f_1(z'_i) \geq t_{w'_i}^* \geq \frac{1}{3}$. By Definition 4.2(1.1), we see there exists a vertex $w''_i \in N_5(z'_i) \setminus w'_i$ such that $t_{w''_i}^* \geq \frac{1}{6}$. Note that $g^*(w'_i) \geq \frac{1}{6}$ and so $t_{w'_i}^* \geq \frac{1}{6}$, which follows that $f_1(z'_i) \geq t_{w'_i}^* + t_{w''_i}^* \geq \frac{1}{3}$, a contradiction. □

Claim 7.3 Suppose $z_i \in N_4^-(y_i)$ and $z'_i \in N_5^2(z_i)$. Then $d(z'_i) \leq 3$. Moreover, $f_1(z_i) < \frac{1}{3}$.

Proof. Clearly, $f_1(z_i) < \frac{1}{3}$, else $t_{z_i}^2 \geq \frac{1}{3}$ and so $f_3(y_i) \geq f_2(y_i) + t_{z_i}^2 \geq 0$. Suppose $d(z'_i) \geq 4$. By Ineq. (3), $g^*(z'_i) \geq \frac{1}{3}n_4(z'_i)$, that is, $t_{z'_i}^* \geq \frac{1}{3}$. By Claim 7.1, $n_5^-(z_i) = 0$. Hence, $f_1(z_i) \geq g^*(z_i) + t_{z'_i}^* \geq \frac{1}{3}$, a contradiction. □

Claim 7.4 G has no 4-cycle containing y_i.

Proof. W.l.o.g., suppose G has a 4-cycle with vertices y_1, u_1, u_2, u_3 in order. By Claim 7.1, we have $u_1, u_3 \in V_4$ and $u_2 \in V_5$ and $u_1u_3 \notin E(G)$. By Claims 7.1, 7.2 and 7.3, $n_5^-(u_i) = 0$, $g(u_i) < 0$ for any $i \in \{1, 3\}$ and $d(u_2) \leq 3$. We assert $d(u_2) = 2$. Suppose not. By Ineq. (3), $g^*(u_2) \geq \frac{1}{6}n_4(u_2)$ because $n_4(u_2) + n_5(u_2) = d(u_2) \geq 3$ and $n_4(u_2) \geq 2$, which means $t_{u_2}^* \geq \frac{1}{6}$. Hence, $f_2(u_i) = f_1(u_i) \geq g^*(u_i) + t_{u_2}^* \geq \frac{1}{6}$ for any $i \in \{1, 3\}$, which means $t_{u_i}^2 \geq \frac{1}{6}$. But then $f_3(y_1) \geq f_2(y_1) + t_{u_1}^2 + t_{u_3}^2 \geq 0$, a contradiction, as asserted. By Claim 7.1, $G - y_1 - u_2$ has a 2-path with vertices u_1, u_4, u_3 in order such that $u_4 \in V_5$ as $G + u_2x \in \mathcal{F}_G$. By Ineq. (3), $g^*(u_2) \geq \frac{1}{6}n_4(u_2)$ since $n_4(u_2) \geq 2$ and $u_1, u_3 \in N_4^{-1}(u_2)$. Similarly, we have $f_3(y_1) \geq 0$, a contradiction. □

Clearly, $d(y_i) \geq 2$ for some $i \in [2]$, else $G + y_1y_2 \notin \mathcal{F}_G$. W.l.o.g., assume $d(y_1) \geq 2$. Let $z_1 \in N_4(y_1)$. Note that $d(y_2) \geq 2$, otherwise by Claims 7.1 and 7.4, G has no 4-path with ends x and z_1 containing y_1, which yields $G + z_1y_2 \notin \mathcal{F}_G$. By Claim 7.1, $z_1y_2 \notin E(G)$, which means G has a 5-path with vertices $z_1, a_1, a_2, a_3, z_2, y_2$ in order. By Claims 7.1 and 7.4, $a_1 \notin V_3$ and $z_2 \neq x$ which implies $z_2 \in N_4(y_2)$. Note that $z_1z_2 \notin E(G)$, otherwise $a_1, a_3 \in V_5$ and $a_2 \in V_4 \cup V_5$ which violates Claim 7.2. This yields G has a 5-path with vertices $z_1, b_1, b_2, b_3, b_4, z_2$ in order. Note that $\{b_1, b_4\} \subseteq V_3 \cup V_4 \cup V_5$. Clearly, by Lemma 5.3, $n_5^-(z_i) = 0$ for any $i \in [2]$, which yields $f_1(z_i) \geq g^*(z_i)$.

Claim 7.5 $|\{b_1, b_4\} \cap V_3| \leq 1$.

Proof. Suppose not. By Claim 7.1, $b_1 = y_1$, $b_4 = y_2$, $b_2, b_3 \in V_4^1$ and $b_1b_3 \notin E(G)$. This implies G has a 5-path with vertices $b_1, c_1, c_2, c_3, c_4, b_3$ in order. By Claims 7.1 and 7.4, $c_4 \neq y_2$ and $c_1 \neq x$. Then $c_1 \in V_4$ as $g(y_1) < 0$. Moreover, $c_4 \notin V_5$, otherwise by Claim 7.2, $c_3 \in V_5$, $c_2 \in V_4$, which yields $g(c_1) \geq \frac{1}{6}$ and so $n_4^2(y_1) \geq 1$ or $n_4^1(y_1) \geq 2$ violating Claim 7.1. Thus, $c_4 = b_2$. By Claims 7.1 and 7.2, $c_3, c_2 \in V_5$ and $d(c_2) = d(c_3) = 2$. But then $G + c_1c_3 \notin \mathcal{F}_G$ by Claims 7.1 and 7.4, a contradiction. □

Claim 7.6 $|\{b_1, b_4\} \cap V_3| = 0$.

Proof. Suppose not. By Claim 7.5, $|\{b_1, b_4\} \cap V_3| = 1$. We may assume $b_1 \in V_3$. Then $b_1 = y_1$ and $b_4 \in V_4 \cup V_5$.

Case A: $b_4 \in V_4$**.** Then $a_3 = b_4$, otherwise $a_2, a_3 \in V_5^1$ and $a_1 \in V_4$ by Claim 7.2, we see $g^*(a_3) \geq \frac{1}{6}$ and so $t_{a_3}^* \geq \frac{1}{6}$, which yields $f_1(z_2) \geq g^*(z_2) + t_{a_3}^* \geq \frac{1}{6}$ violating Claim 7.2. Note that $a_2 \in V_3 \cup V_5$. We next prove

(a) $a_2 \in V_5$.

To see why (a) is true, suppose $a_2 \in V_3$. By Claim 7.1, $a_1 \in V_4$. Observe that $f_1(a_i) < \frac{1}{6}$ for any $i \in \{1,3\}$, otherwise let $f_1(a_1) \geq \frac{1}{6}$, by Definition 4.6(2), $f_2(z_1) \geq \frac{1}{6}$ and so $t^2_{z_1} \geq \frac{1}{6}$ as $g^*(a_2) \geq 0$, which implies $f_3(y_1) \geq f_2(y_1) + t^2_{z_1} \geq 0$. It follows that $b_3 = a_2$, otherwise $b_3 \in V_5$ and $b_2 \in V_4$ by Claim 7.2, we see $g^*(b_3) \geq \frac{1}{6}n_4(b_3)$ and so $t^*_{b_3} \geq \frac{1}{6}$, which yields $f_1(a_3) \geq g^*(a_3) + t^*_{b_3} \geq \frac{1}{6}$. Furthermore, $b_2 = x$ by Claim 7.1. Since $G + z_1a_2 \in \mathcal{F}_G$, we set $C_6(z_1a_2) = z_1c_1c_2c_3c_4a_2$. Note that $g(a_2) < 0$, else $f_1(a_3) \geq g^*(a_3) \geq \frac{1}{6}$. By Claims 7.1 and 7.4, $c_4 \in V_4$. We assert $c_1 \notin V_3$. Suppose not. Then $c_1 = y_1$. Note that $c_2 = x$, otherwise by Claim 7.1 and $g(y_1) < 0$, $c_2 \in V_4$ and $c_3 \in V_5$, which yields G has a copy of C_6 with vertices $y_1, x, a_2, c_4, c_3, c_2$ in order. So $c_3 \in V_3$, which means $g(c_4) \geq \frac{2}{3}$. But then $f_1(a_1) \geq g^*(a_1) \geq \frac{1}{6}$ as $g(c_4) \geq \frac{2}{3}$, a contradiction. Clearly, $c_1 \in V_4$, otherwise $c_1, c_2 \in V_5^1$ and $c_3 \in V_4$ by Claim 7.2, which implies $g^*(c_1) \geq \frac{1}{6}$ and so $f_1(z_1) \geq \frac{1}{6}$. Then $c_1 = a_1$ and $c_2 \in V_5$. Moreover, $c_2, c_3 \in V_5^1$, otherwise $g^*(c_2) \geq \frac{1}{6}n_4(c_2)$ and so $t^*_{c_2} \geq \frac{1}{6}$, which implies $f_1(a_1) \geq g^*(a_1) + t^*_{c_2} \geq \frac{1}{6}$. Then G has a 3-path $P(a_2a_1)$ or $P(a_2c_4)$ as $G + c_2c_4 \in \mathcal{F}_G$. We have $f_1(a_1) \geq \frac{1}{6}$, a contradiction. This proves (a).

By (a), $a_1 \in V_5$ and $d(a_2) = 2$, otherwise $g^*(a_2) \geq \frac{1}{3}n_4(a_2)$ and so $t^*_{a_2} \geq \frac{1}{3}$, which implies $f_1(b_4) \geq g^*(b_4) + t^*_{a_2} \geq \frac{1}{3}$ violating Claim 7.2. It is easy to check that $d(a_1) = 2$ by Claim 7.2. Moreover, $g(z_1) < 0$, otherwise $g^*(a_1) = \frac{1}{6}$ and so $t^*_{a_1} \geq \frac{1}{6}$, which yields $f_1(z_1) \geq g^*(z_1) + t^*_{a_1} \geq \frac{1}{6}$ violating Claim 7.2. Note that $b_2 \in V_2 \cup V_4$ as $b_1 = y_1$. We assert $b_2 \in V_4$. Suppose not. Then $b_2 = x$ and $b_3 \in V_3$. Since $G + a_2x \in \mathcal{F}_G$ and $d(a_1) = d(a_2) = 2$, $a_2b_4 \in E(C_6(a_2x))$, otherwise $a_2a_1 \in E(C_6(a_2x))$ and G has a 3-path with ends x and z_1, which violates $g(z_1) < 0$ and $g(y_1) < 0$. Then G has a 4-path with ends x and b_4 containing b_3, which means G has a 3-path with ends v and b_3, where $v \in \{x, b_4\}$ as $g(z_2) = g(b_4) = \frac{1}{6}$. By Definition 4.6(2), we have $n^2_4(b_3) = 0$ and $g(b_3) < 0$, otherwise $f_2(z_2) \geq \frac{1}{6}$ and so $t^2_{z_2} \geq \frac{1}{6}$, which means $f_3(y_2) \geq f_2(y_2) + t^2_{z_2} \geq 0$. It follows that G has no 3-path with ends x and b_3. Then G must have a 3-path $P(b_3b_4)$. Let $P(b_3b_4) = b_3b'_3b'_4b_4$. Then $b'_4 \in V_5$ and $b'_3 \in V_4$. We see $d(b'_4) = 2$, otherwise $g^*(b'_4) \geq \frac{1}{3}n_4(b'_4)$ and so $t^*_{b'_4} \geq \frac{1}{3}$, which means $f_1(b_4) \geq g^*(b_4) + t^*_{b'_4} \geq \frac{1}{3}$. Since $G + xb'_4 \in \mathcal{F}_G$, there exists one vertex of $\{b_4, b'_3\}$, say u, such that $G - b'_4$ has a 4-path with ends x and u containing b_3. It follows $G - b'_4$ has a 3-path with ends b_3 and u containing $\{b_4, b'_3\} \setminus \{u\}$. That is, there is a vertex $b''_4 \in (N_5(b_4) \cap N_5(b'_3)) \setminus b'_4$. Then $g^*(v) \geq \frac{1}{6}n_4(v)$ for any $v \in \{b'_4, b''_4\}$ and so $t^*_v \geq \frac{1}{6}$. But then $f_1(b_4) \geq g^*(b_4) + t^*_{b'_4} + t^*_{b''_4} \geq \frac{1}{3}$, a contradiction, as asserted. Thus, $b_2 \in V_4$. Then $b_3 \in V_5$. We have $d(b_3) = 2$ and $g(b_2) < 0$, otherwise $t^*_{b_3} \geq \frac{1}{3}$ and so $f_1(b_4) \geq g^*(b_4) + t^*_{b_3} \geq \frac{1}{3}$. Since $G + a_1b_3 \in \mathcal{F}_G$, we see $b_3b_2 \in E(C_6(a_1b_3))$ as $d(a_2) = 2$, which implies $G - b_3$ has a 3-path $P(b_2z_1)$ or a 2-path $P(b_2b_4)$. Note that $g^*(b_3) \geq \frac{1}{6}n_4(b_3)$ and so $t^*_{b_3} \geq \frac{1}{6}$. Since $g(b_2) < 0$ and $n^-_5(b_2) = 0$, we see $t^2_{b_2} \geq \frac{1}{6}$. If the 3-path $P(b_2z_1)$ exists, let $P(b_2z_1) = b_2b'_2z'_1z_1$, then $\{b'_2, z'_1\} \subseteq V_5$. If $n^2_5(z_1) \neq 0$, then $g^*(a_1) \geq \frac{1}{6}$ and so $t^*_{a_1} \geq \frac{1}{6}$. If $n^2_5(z_1) = 0$, then $g_2(a_1) = \frac{1}{3}$, which follows $g^*(z'_1) \geq \frac{1}{6}$ and so $t^*_{z'_1} \geq \frac{1}{6}$. In both cases, we see $t^2_{z_1} \geq \frac{1}{6}$ because $n^-_5(z_1) = 0$ and $g(z_1) < 0$. But then $f_3(y_1) \geq f_2(y_1) + t^2_{z_1} + t^2_{b_2} \geq 0$, a contradiction. If the 2-path $P(b_2b_4)$ exists, let $P(b_2b_4) = b_2b'_2b_4$. Then $b'_2 \in V_5$, otherwise $b'_2 = y_1$, we see $f_2(y_1) \geq g^*(y_1) \geq -\frac{1}{6}$, which yields $f_3(y_1) \geq f_2(y_1) + t^2_{b_2} \geq 0$. Similarly, $t^*_{b'_2} \geq \frac{1}{6}$ and so $f_1(b_4) \geq g^*(b_4) + t^*_{b_3} + t^*_{b'_2} \geq \frac{1}{3}$ violating Claim 7.2.

Case B: $b_4 \in V_5$. Then $b_2 \in V_4$ and $b_3 \in V_4 \cup V_5$. If $b_3 \in V_4$, then $g(z_2) < 0$ and $d(b_4) = 2$ otherwise $t^*_{b_4} \geq \frac{1}{3}$ and $f_1(b_3) \geq g^*(b_3) + t^*_{b_4} \geq \frac{1}{3}$. Clearly, $a_3 \in V_5$. Note that $a_3 \neq b_4$, otherwise $a_2 = b_3$ and $a_1 \in V_5$, which means $f_1(b_3) \geq g^*(b_3) + t^*_{b_4} + t^*_{a_1} \geq \frac{1}{3}$ because $t^*_v \geq \frac{1}{6}$ for any $v \in \{a_1, b_4\}$. It is easy to check $g^*(v) \geq \frac{1}{6}n_4(v)$ for any $v \in \{b_4, a_3\}$, which means $t^*_v \geq \frac{1}{6}$. But then $f_1(z_2) \geq g^*(z_2) + t^*_{b_4} + t^*_{a_3} \geq \frac{1}{3}$, which contradicts to Claim 7.3. Next we assume that $b_3 \in V_5$. By Claim 7.4, $b_3 \neq a_1$. Moreover, $b_3 \neq a_3$ and $b_4 \notin \{a_1, a_2\}$, otherwise there is a 6-cycle in G, that is, $C_6 = z_2y_2xy_1b_2b_3$ when $b_3 = a_3$, or $C_6 = z_1y_1xy_2z_2b_4$ when $b_4 = a_1$, or $C_6 = z_1y_1b_2b_3b_4a_1$ when $b_4 = a_2$. We next prove that

(b) $|\{b_4, b_3\} \cap \{a_2, a_3\}| \leq 1$.

To prove (b), suppose $\{b_4, b_3\} = \{a_2, a_3\}$. Then $b_4 = a_3$ and $b_3 = a_2$. Then $a_1 \in V_5$, else $t^*_{b_3} \geq \frac{1}{3}$ and so $f_1(b_2) \geq g^*(b_2) + t^*_{b_3} \geq \frac{1}{3}$, which violates Claim 7.2. Moreover, $d(a_1) = 2$, else $d(a_1) \geq 3$, $g^*(v) \geq \frac{1}{6}n_4(v)$ as $n^2_5(v) \geq 1$ for any $v \in \{a_1, b_3\}$, which implies $t^2_u \geq \frac{1}{6}$ for any $u \in \{z_1, b_2\}$ and so $f_3(y_1) \geq f_2(y_1) + t^2_{z_1} + t^2_{b_2} \geq 0$. By Claim 7.3, $d(b_3) = 3$. Since $G + a_1b_2 \in \mathcal{F}_G$ and by Claims 7.1 and 7.4, $a_1a_2 \in E(C_6(a_1b_2))$ and $G - a_1 - a_2$ has a 3-path $P(b_2b_4)$. Let $P(b_2b_4) = b_2b'_2b'_4b_4$. By Claim 7.2, $g(b_2) < 0$ and $g(z_2) < 0$. Then $b'_2 \in V_5$, else $b'_2 = y_1$, we see G has a copy of C_6, that is, $C_6 = y_1xy_2z_2b_4b'_4$ when $b'_4 = z_1$, or $C_6 = y_1z_1a_1b_3b_4b'_4$ when $b'_4 \neq z_1$. We assert $b'_4 \in V_5$. Suppose $b'_4 \in V_4$. Then $b'_4 \neq z_2$, else there is a $C_6 = b'_2b_2y_1xy_2z_2$ in G. By Ineq. (3), $g^*(v) \geq \frac{1}{6}n_4(v)$ for any $v \in \{b'_2, b_3\}$ as $n^{-1}_4(b'_2) + n^+_4(b'_2) \geq 2$, $n^{-1}_4(b_3) + n^+_4(b_3) \geq 1$ and $n_4(b_3) + n_5(b_3) \geq 3$, that is, $t^*_v \geq \frac{1}{6}$. But then $f_1(b_2) \geq g^*(b_2) + t^*_{b_3} + t^*_{b'_2} \geq \frac{1}{3}$, which contradicts to Claim 7.3, as asserted. So $b'_4 \in V^2_5$. Since $n^2_5(b_4) \geq 2$, $g^*(b_4) \geq \frac{1}{3}n_4(b_4)$ and so $t^*_{b_4} \geq \frac{1}{3}$. But $f_1(z_2) \geq g^*(z_2) + t^*_{b_4} \geq \frac{1}{3}$, which contradicts to Claim 7.3. This proves (b).

(c) $|\{b_4, b_3\} \cap \{a_2, a_3\}| = 0$.

To prove (c), suppose not. By (b), $|\{b_4, b_3\} \cap \{a_2, a_3\}| = 1$. Clearly, $b_4 \in \{a_2, a_3\}$, otherwise $b_3 = a_2$ and $b_4 \neq a_3$, which yields $d(b_3) \geq 4$ violating Claims 7.2 and 7.3. Then $b_4 = a_3$ and $b_3 \neq a_2$. By Claim 7.2, $g(z_2) < 0$. By Claim 7.3, $d(b_4) = 3$. We assert $d(b_3) = 2$. Suppose not. Then $g^*(b_3) \geq \frac{1}{6}n_4(b_3)$ because $b_4 \in N^2_5(b_3)$ and $d(b_3) = n_4(b_3) + n_5(b_3) \geq 3$ and so $t^*_{b_3} \geq \frac{1}{6}$. By Claim 7.2, $a_1 \in V_5$, otherwise $f_1(b_2) \geq g^*(b_2) + t^*_{b_3} \geq \frac{1}{6}$. It follows that $g^*(b_4) \geq \frac{1}{3}n_4(b_4)$ because $n_4(b_4) + n_5(b_4) = 3$ and $n^2_5(b_4) + n^{-1}_4(b_4) + n^+_4(b_4) \geq 2$. That is, $t^*_{b_4} \geq \frac{1}{3}$. But then $f_1(z_2) \geq g^*(z_2) + t^*_{b_4} \geq \frac{1}{3}$, which contradicts to Claim 7.3, as asserted. Since $G + z_1b_3 \in \mathcal{F}_G$, let $C_6(z_1b_3) = z_1c_1c_2c_3c_4b_3$. By Claims 7.1 and 7.4, $c_4 \neq b_2$. Hence, $c_4 = b_4$ as $d(b_3) = 2$. We assert $c_1 \notin V_3$. Suppose not. Then $b_2 = c_i$ for some $i \in \{2, 3\}$, otherwise G has a copy of $C_6 = c_1c_2c_3c_4b_3b_2$. By Claim 7.1, $b_2 = c_2$. Since $d(b_4) = 3$, we have $c_3 = a_2$. Note that $a_2 \in V^+_4 \cup V^2_5$. Hence, $g^*(b_3) \geq \frac{1}{3}$ or $g^*(b_4) \geq \frac{1}{3}n_4(b_4)$. That is, $t^*_{b_3} \geq \frac{1}{3}$ or $t^*_{b_4} \geq \frac{1}{3}$. But then $f_1(b_2) \geq g^*(b_2) + t^*_{b_3} \geq \frac{1}{3}$ or $f_1(z_2) \geq g^*(z_2) + t^*_{b_4} \geq \frac{1}{3}$, a contradiction, as asserted. Since $d(b_4) = 3$, we have $c_3 \in \{z_2, a_2\}$.

We next show that $c_3 \neq a_2$. Suppose not. Then $c_2 \neq a_1$, otherwise $c_2, c_1 \subseteq V_5$, we see $g^*(v) \geq \frac{1}{6}n_4(v)$ for any $v \in \{c_1, c_2\}$ as $n^2_5(v) \geq 1$, which implies $t^*_v \geq \frac{1}{6}$ and so $f_1(z_1) \geq g^*(z_1) + t^*_{c_1} + t^*_{c_2} \geq \frac{1}{3}$. We assert $c_1 \neq a_1$. Suppose not. By Claim 7.2, $c_1 \in V_5$. Moreover, $a_2 \in V_4$, otherwise $a_2 \in V_5$, we see $t^*_{c_1} \geq \frac{1}{3}$ because $d(c_1) \geq 3$ and $n^+_4(c_1) + n^{-1}_4(c_1) + n^2_5(c_1) \geq 2$, which yields $f_1(z_1) \geq g^*(z_1) + t^*_{c_1} \geq \frac{1}{3}$ violating Claims 7.2 and 7.3. Clearly, $c_2 \in V_5$ and $d(c_2) = 2$, otherwise $t^*_{c_1} \geq \frac{1}{3}$ and so $f_1(z_1) \geq g^*(z_1) + t^*_{c_1} \geq \frac{1}{3}$. But then $G + z_1c_2 \notin \mathcal{F}_G$ because $d(c_1) = 3$ and $d(c_2) = 2$, a contradiction, as asserted. By Claim 7.2, $c_1 \in V_5$. Furthermore, $c_2 \in V_5$, otherwise $a_1 \in V_5$ by Claim 7.2, we see $t^*_v \geq \frac{1}{6}$ for any $v \in \{c_1, a_1\}$ and so $f_1(z_1) \geq g^*(z_1) + t^*_{a_1} + t^*_{c_1} \geq \frac{1}{3}$, which violates Claims 7.2 and 7.3. By Claim 7.2, $a_1 \in V_5$. Note that $a_2 \in V_4 \cup V_5$. Then $d(v) = 2$ for any $v \in \{a_1, c_1\}$, otherwise we see $t^*_v \geq \frac{1}{6}$ and so $f_1(z_1) \geq g^*(z_1) + t^*_{a_1} + t^*_{c_1} \geq \frac{1}{3}$. By Claim 7.2, $g(z_1) < 0$. If $a_2 \in V_5$, then $f_1(z_1) \geq g^*(z_1) \geq \frac{1}{3}$, which violates Claim 7.3. If $a_2 \in V_4$, then $d(c_2) \geq 3$, otherwise, $G + z_1c_2 \notin \mathcal{F}_G$. By Claim 7.3, $g^*(c_1) < \frac{1}{3}$. By Definition 4.2(1), there exists a vertex $c'_1 \in N_5(z_1) \cap N_5(c_2)$. It is easy to check that for any $v \in \{c_1, c'_1\}$, $t^*_v \geq \frac{1}{6}$ and so $f_1(z_1) \geq g^*(z_1) + t^*_{c_1} + t^*_{c'_1} \geq \frac{1}{3}$, which violates Claim 7.3, as desired.

Thus, $c_3 \neq a_2$ and so $c_3 = z_2$. By Claim 7.2, $a_1, c_1 \in V_5$. Recall that $g(z_2) < 0$. Hence, $c_2 \in V_5$. Note that $a_2 \in V_4 \cup V_5$. By Ineq. (3), $t^*_{b_4} \geq \frac{1}{6}$ because $n_4(b_4) + n_5(b_4) \geq 3$ and $n^{-1}_4(b_4) + n^2_5(b_4) \geq 1$. If $n^2_5(z_1) \neq 0$ or $a_1 = c_1$, then $t^*_{c_2} \geq \frac{1}{6}$, which follows that $f_1(z_2) \geq t^*_{c_2} + t^*_{b_4} \geq \frac{1}{3}$ violating Claim 7.3. If $n^2_5(z_1) = 0$ and $a_1 \neq c_1$, then $g_1(a_1) = \frac{1}{3}$ as $d(b_4) = 3$. That is, $z_1b_4 \notin E(G)$. It follows that $g_2(c_1) = \frac{1}{3}$, or $g_2(c_1) = \frac{1}{6}$ and there exists a vertex $c'_1 \in N_5(z_1) \setminus c_1$ such that $g^*(c'_1) \geq \frac{1}{6}$. That is, $g^*(z_1) \geq \frac{1}{3}$, or $t^*_v \geq \frac{1}{6}$ for any $v \in \{c_1, c'_1\}$. But then $f_1(z_1) \geq g^*(z_1) \geq \frac{1}{3}$ or $f_1(z_1) \geq g^*(z_1) + t^*_{c_1} + t^*_{c'_1} \geq \frac{1}{3}$, which violates Claim 7.3. This proves (c).

By (c), $\{b_3, b_4\} \cap \{a_1, a_2, a_3\} = \emptyset$. We first prove $d(b_4) = 2$. Suppose not. By Claim 7.2, $g(z_2) < 0$ and so $a_3 \in V_5$. By Claim 7.3, $d(b_4) = 3$. Then $a_2 \in V_5$, otherwise $a_2 \in V_4$ and so $a_1 \in V_5$, we see for any $v \in \{a_3, b_4\}$,

$t^*_v \geq \frac{1}{6}$ because $d(b_4) = 3$, $z_2 \in N_4^{-1}(v)$ and $a_2 \in N_4^{-1}(a_3) \cup N_4^+(a_3)$, which means $f_1(z_2) \geq g^*(z_2) + t^*_{a_3} + t^*_{b_4} \geq \frac{1}{3}$ violating Claim 7.2. Clearly, $a_3 \in V_5^1$ and $n_5^2(b_4) = 0$, otherwise $t^*_v \geq \frac{1}{6}$ for any $v \in \{a_3, b_4\}$, which means $f_1(z_2) \geq \frac{1}{3}$. Moreover, $a_2 \in V_5^1$, otherwise since $b_4 \in N_5^2(z_2)$, we see $g^*(a_3) \geq \frac{1}{3}$, or there exists $a_3' \in N_5(z_2) \setminus a_3$ such that $t^*_v \geq \frac{1}{6}$ for any $v \in \{a_3, a_3'\}$, which means $f_1(z_2) \geq \frac{1}{3}$. Hence, $a_1 \in V_4$. Since $G + a_3b_3 \in \mathcal{F}_G$, we see G has a 5-path $P(a_3b_3) = a_3c_1c_2c_3c_4b_3$. Note that $c_1 \in \{a_2, z_2\}$ and $c_4 \in \{b_2, b_4\}$ because $d(a_3) = d(b_3) = 2$. We assert $c_4 \neq b_4$. Suppose not. Then $z_2 = c_i$ for some $i \in [3]$ because G is C_6-free. Since $d(a_2) = d(a_3) = 2$, we see $z_2 = c_1$. By Claim 7.4, $\{c_2, c_3\} \subseteq V_5$. But then $c_3 \in N_5^2(b_4)$, a contradiction, as asserted. Thus, $c_4 = b_2$. Then $c_1 = z_2$, otherwise $c_1 = a_2$, $c_2 = a_1$ and $c_3 \in V_5$, we see $t^*_v \geq \frac{1}{6}$ for any $v \in \{c_3, a_2\}$ because $n_4^+(c_3) + n_4(c_3) \geq 3$, which means $f_1(a_1) \geq g^*(a_1) + t^*_{c_3} + t^*_{a_2} \geq \frac{1}{3}$. Since G is C_6-free, $b_4 = c_i$ for some $i \in \{2, 3, 4\}$. Since $d(b_4) = 3$ and $b_4 \neq c_4$, we see $c_2 = b_4$. By Claim 7.1, $c_3 \in V_5^1$ because $n_5^2(b_4) = 0$. By Definition 4.2(1), $g^*(b_4) = \frac{1}{3}$ and so $t^*_{b_4} = \frac{1}{3}$. But then $f_1(z_2) \geq g^*(z_2) + t^*_{b_4} \geq \frac{1}{3}$, a contradiction, as desired. Thus, $d(b_4) = 2$.

We next prove $d(b_3) = 2$. Suppose not. By Claim 7.2, $g(v) < 0$ for any $v \in \{b_2, z_2\}$, which means $a_3 \in V_5$. By Claim 7.3, $d(b_3) = 3$. Then $g_1(b_4) = \frac{1}{3}$, otherwise $g_1(b_4) = \frac{1}{6}$ which implies $g^*(b_3) \geq g_1(b_3) \geq \frac{1}{3}$, that is, $t^*_{b_3} \geq \frac{1}{3}$, and so $f_1(b_2) \geq t^*_{b_3} \geq \frac{1}{3}$. Moreover, $a_3, a_2 \in V_5^1$, otherwise $f_1(z_2) \geq \frac{1}{3}$. It follows that $a_1 \in V_4$. Then $n_4(b_3) = 0$ and $n_5^2(b_3) + n_4^{-1}(b_3) = 0$, otherwise by Ineq. (3), $t^*_{b_3} \geq \frac{1}{6}$ and so $f_1(b_2) \geq t^*_{b_3} \geq \frac{1}{6}$ violating Claim 7.2. Since $G + a_2b_4 \in \mathcal{F}_G$, we see G has a 5-path $P(a_2b_4) = a_2c_1c_2c_3c_4b_4$. Then $c_4 \in \{z_2, b_3\}$ and $c_1 \in \{a_1, a_3\}$ since $a_3, a_2 \in V_5^1$. Since G is C_6-free, $c_4 \neq z_2$ as $a_3 \in V_5^1$. Hence, $c_4 = b_3$. Since $g_1(b_4) = \frac{1}{3}$ and $g(z_2) < 0$, we see $G - b_4$ has no 2-path, which implies $c_1 \neq a_3$. Hence, $c_1 = a_1$. By Claims 7.1 and 7.2, $c_3 = b_2$. By Claim 7.1 and $g(b_2) < 0$, we have $c_2 \in V_5$. But then $b_2 \in N_4^{-1}(b_3)$, a contradiction, as desired. Thus, $d(b_3) = 2$.

Since $G + b_4x \in \mathcal{F}_G$, G has a 5-path $P(xb_4) = xc_1c_2c_3c_4b_4$. Then $c_4 \in \{z_2, b_3\}$ as $d(b_4) = 2$. By Claims 7.1 and 7.4, $c_4 \neq z_2$. Hence, $c_4 = b_3$ and so $c_3 = b_2$ as $d(b_3) = 2$. By Claim 7.1, $c_2 \in V_4$. Then $g(z_2) < 0$ and $n_5^2(z_2) = 0$, otherwise $t^*_{b_3} \geq \frac{1}{6}$ so $f_1(b_2) \geq t^*_{b_3} \geq \frac{1}{6}$ violating Claim 7.2. It follows that $a_1, a_2, a_3 \in V_5$. Then $g_1(a_3) = \frac{1}{3}$, which implies $t^*_{b_3} \geq \frac{1}{6}$. But then $f_1(b_2) \geq t^*_{b_3} \geq \frac{1}{6}$ violating Claim 7.2, a contradiction. □

Claim 7.7 $b_4 \neq a_3$ or $b_1 \neq a_1$.

Proof. Suppose not. Then $b_4 = a_3$ and $b_1 = a_1$. Note that $\{a_1, a_3\} \subseteq V_4 \cup V_5$. We first assert $a_2 \notin \{b_2, b_3\}$. Suppose not. By symmetry, assume $a_2 = b_2$. Then $a_3 \in V_5$, otherwise $\{a_2, b_3\} \subseteq V_5$ and $a_2 \in V_5^2$, we see $t^*_{a_2} \geq \frac{1}{3}$ because $n_4(a_2) + n_5(a_2) + n_4^+(a_2) \geq 4$, which follows $f_1(a_3) \geq g^*(a_3) + t^*_{a_2} \geq \frac{1}{3}$ violating Claim 7.2. By Claim 7.2, $g(z_2) < 0$. Moreover, $a_2 \in V_4$, otherwise $a_2 \in V_5$, we see $t^*_{a_3} \geq \frac{1}{3}$ because $n_4(a_3) + n_5(a_3) \geq 3$ and $n_4^+(a_3) + n_4^{-1}(a_3) + n_5^2(a_3) \geq 2$, which yields $f_1(z_2) \geq g^*(z_2) + t^*_{a_3} \geq \frac{1}{3}$ violating Claim 7.3. By Claim 7.3, $d(a_3) = 3$. Clearly, $d(b_3) = 2$, otherwise $t^*_{a_3} \geq \frac{1}{3}$ and so $f_1(z_2) \geq t^*_{a_3} \geq \frac{1}{3}$. Since $G + z_2b_3 \in \mathcal{F}_G$, $G - b_3 - a_3$ has a 3-path with vertices a_2, a_2', z_2', z_2 in order. But then G has a copy of $C_6 = a_2b_3a_3z_2z_2'a_2'$, a contradiction, as asserted.

We next assert $\{a_1, a_3\} \subseteq V_5$. W.l.o.g., assume $a_3 \in V_4$. By Claims 7.1 and 7.2, $a_1 \in V_4$. By Claim 7.1, $\{b_2, b_3\} \subseteq V_5$, otherwise $\{b_2, b_3\} \subseteq V_3$, by Definition 4.6(2), $f_2(z_1) \geq \frac{1}{6}$ and so $t^2_{z_1} \geq \frac{1}{6}$, which yields $f_3(y_1) \geq f_2(y_1) + t^2_{z_1} \geq 0$ as $f_2(y_1) \geq g^*(y_1) \geq -\frac{1}{6}$. We see $a_2 \in V_3$ and $d(b_2) = d(b_3) = 2$, otherwise $f_1(a_i) \geq \frac{1}{3}$ for some $i \in \{1, 3\}$. Then G has a 3-path $P(a_2a_i)$ for some $i \in \{1, 3\}$ since $G + b_2a_3 \in \mathcal{F}_G$. W.l.o.g., let $i = 1$. Let $P(a_2a_1) = a_2a_2'a_1'a_1$. Then $a_1' \in V_5$ and so $a_2' \in V_4$. Thus for any $v \in \{b_2, a_1'\}$, $g^*(v) \geq \frac{1}{6}n_4(v)$ as $a_1 \in N_4^+(v)$ and so $t^*_v \geq \frac{1}{6}$. But then $f_1(a_1) \geq g^*(a_1) + t^*_{b_2} + t^*_{a_1'} \geq \frac{1}{3}$, violating Claim 7.2, as asserted.

By Claim 7.2, $g(z_i) < 0$ since $d(a_1) \geq 3$ and $d(a_3) \geq 3$. Note that $\{b_2, b_3\} \cap V_4 \neq \emptyset$ and $\{b_2, b_3\} \cap V_5 \neq \emptyset$, otherwise $t^*_{a_1} \geq \frac{1}{3}$ because $d(a_1) = n_4(a_1) + n_5(a_1) \geq 3$ and $n_4^{-1}(a_1) + n_4^+(a_1) + n_5^2(a_1) \geq 2$, which yields $f_1(z_1) \geq t^*_{a_1} \geq \frac{1}{3}$ violating Claim 7.3. W.l.o.g., assume $b_2 \in V_4$ and $b_3 \in V_5$. By Claim 7.3, $d(a_1) = d(a_3) = 3$. Clearly, $d(b_3) = 2$, otherwise $t^*_{a_3} \geq \frac{1}{3}$ and so $f_1(z_2) \geq t^*_{a_3} \geq \frac{1}{3}$. Then $G - a_1$ has a 2-path $P(z_1v)$ for some $v \in \{a_2, b_2\}$, or $G - a_3 - b_3$ has a 3-path $P(z_2a_2)$ as $G+b_3a_2\in\mathcal{F}_G$. We assert G has no 3-path $P(z_2a_2)$. Suppose not. Let $P(z_2a_2)=z_2z_2'a_2'a_2$. Then $n_4^1(y_2)=0$, else $g^*(y_2)\geq -\frac{1}{6}$ and so $f_3(y_2)\geq 0$ because $t^*_{a_3}\geq\frac{1}{6}$ and so $t^2_{z_2}\geq\frac{1}{6}$. Hence, $z_2'\in V_3\cup V_5$. Moreover, $z_2'\in V_5$, otherwise $z_2'=y_2$, $a_2'\in V_4$ and so $a_2\in V_5$, which follows that $f_1(a_2')\geq\frac{1}{3}$ as $t^*_{a_2}\geq\frac{1}{3}$ violating Claim 7.3. It is easy to verify that $t^*_v\geq\frac{1}{6}$ for any $v\in\{a_3,z_2'\}$. But then $f_1(z_2)\geq t^*_{a_3}+t^*_{z_2'}\geq\frac{1}{3}$ violating Claim 7.3, as asserted. Thus, $G-a_1$ has a 2-path $P(z_1v)$ for some $v\in\{a_2,b_2\}$. Let $P(z_1v)=z_1v'v$. By Claim 7.4, $v'\notin V_3$. Hence, $v'\in V_5$ as $g(z_1)<0$. Furthermore, one can easily check $t^*_u\geq\frac{1}{6}$ for any $u\in\{v',a_1\}$. But then $f_1(z_2)\geq t^*_{a_1}+t^*_{v'}\geq\frac{1}{3}$ violating Claim 7.3. This completes the proof of Claim 7.7. □

Note that $\{a_1,a_3\}\subseteq V_4\cup V_5$. By Claim 7.7 and symmetry, we may assume $b_1\neq a_1$. We first show $g(z_1)<0$. Suppose not. Note that $a_1\notin V_5$, otherwise $\{a_1,a_2\}\in V_5^1$ and so $a_3\in V_4$ by Claim 7.2, we see $g^*(a_1)\geq\frac{1}{6}$ and so $t^*_{a_1}\geq\frac{1}{6}$, which yields $f_1(z_1)\geq t^*_{a_1}\geq\frac{1}{6}$ violating Claim 7.2. By Claim 7.1, $|\{b_1,a_1\}\cap V_5|\geq 1$. Thus, $b_1\in V_5$. By Claim 7.2, $b_1\in V_5^1$, $\{b_2,b_4\}\subseteq V_5$ and $b_3\in V_4$. Then $g^*(b_1)\geq\frac{1}{6}$ because $n_5^2(b_3)\geq 1$, which means $t^*_{b_1}\geq\frac{1}{6}$. But then $f_1(z_1)\geq t^*_{b_1}\geq\frac{1}{6}$, which contradicts to Claim 7.2, as desired. Thus, $\{a_1,b_1\}\subseteq V_5$. Clearly, $g^*(v)<\frac{n_4(v)}{6}$ for some $v\in\{a_1,b_1\}$, otherwise $f_1(z_1)\geq t^*_{a_1}+t^*_{b_1}\geq\frac{1}{3}$, which contradicts to Claim 7.2.

Assume first that $g^*(a_1)<\frac{1}{6}n_4(a_1)$. We first assert $d(a_1)=2$. Suppose $d(a_1)\geq 3$. Since $g^*(a_1)<\frac{1}{6}n_4(a_1)$, $n_4(a_1)=1$ and $n_5^2(a_1)+n_4^{-1}(a_1)+n_4^+(a_1)=0$. Hence, $a_2\in V_5^1$, which means $a_3\in V_4$. But then $a_1\in N_5^2(a_2)$ violating Claim 7.2, as asserted. We next assert $a_2\in V_5$. Suppose not. Then $a_2\in V_4$. By Claim 7.1, $a_3\in V_5$. Since $g^*(a_1)<\frac{1}{6}n_4(a_1)$, we see $b_1\in V_5^1$, which implies $b_2\in V_5$. Moreover, $b_2\in V_5^1$, otherwise $g^*(b_1)=\frac{1}{3}$ or G has a vertex $z_1'\in(N_5(z_1)\cap N_5(b_2))\setminus b_1$ such that $g^*(v)\geq\frac{1}{6}n_4(v)$ for any $v\in\{z_1',b_1\}$, which implies $f_1(z_1)\geq\frac{1}{3}$ violating Claim 7.3. Hence, $b_3\in V_4$. By Claim 7.1, $b_4\in V_5$. Clearly, $b_3\neq a_2$, otherwise $G+b_1a_2\notin\mathcal{F}_G$. If $b_4\neq a_3$, then $d(b_4)=2$, else $t^*_u\geq\frac{1}{6}$ for any $u\in\{a_3,b_4\}$, which yields $f_1(z_2)\geq t^*_{a_3}+t^*_{b_4}\geq\frac{1}{3}$. Clearly, $d(a_3)=2$, else $t^*_{a_3}\geq\frac{1}{3}$ and so $f_1(z_2)\geq t^*_{a_3}\geq\frac{1}{3}$. Since $G+b_1b_4\in\mathcal{F}_G$, $G-b_4$ has a 2-path $P(b_3z_2)$ or G has a 3-path $P(z_1z_2)$. In both cases, $b_3y_2\in E(G)$ or there exists $z_2'\in N_5(z_2)\setminus a_3$ such that $t^*_v\geq\frac{1}{6}$ for any $v\in\{z_2',a_3\}$. But then $f_3(y_1)+t^2_{b_3}+t^2_{z_2}\geq 0$ because $t^2_v\geq\frac{1}{6}$ for any $v\in\{b_3,z_2\}$, or $f_1(z_2)\geq t^*_{z_2'}+t^*_{a_3}\geq\frac{1}{3}$, a contradiction. If $b_4=a_3$, then $d(a_3)=3$ by Claim 7.3. Since $G+a_2b_2\in\mathcal{F}_G$, $G-b_1-b_2$ has a 3-path with vertices z_1,z_1',a_2',a_2 in order, or $G-b_4$ has a 2-path with vertices $z_2u'u$ for some $u\in\{b_3,a_2\}$. For the former, we have $f_1(z_1)\geq t^*_{z_1'}+t^*_{b_1}\geq\frac{1}{3}$ since $t^*_v\geq\frac{1}{6}$ for any $v\in\{z_1',b_1\}$. For the latter, we have $f_1(z_2)\geq t^*_{b_4}\geq\frac{1}{3}$ because $n_4^+(b_4)+n_4^{-1}(b_4)\geq 2$ and $n_4(b_4)=3$ when $u'\in V_5$, or $u'\in N_4^2(z_2)$ when $u'\in V_4$, or $f_3(y_2)\geq f_2(y_2)+t^2_{z_2}+t^2_u\geq 0$ since $t^2_v\geq\frac{1}{6}$ for any $v\in\{z_2,u\}$ when $u'\in V_3$, a contradiction, as asserted. Thus, $a_2\in V_5$. Then $a_1\in V_5^1$, which yields $n_5^2(z_1)=0$ since $g^*(a_1)<\frac{1}{6}n_4(a_1)$. Hence, $b_1\in V_5^1$ and so $b_2\in V_5$. By Claim 7.2, $a_3\in V_5$. Then $b_3\in V_4$, otherwise $g_1(v)=\frac{1}{3}$ for any $v\in\{a_1,b_1\}$, which implies $g^*(z_1)\geq\frac{1}{3}$. Hence, $b_4\in V_5$. But then $f_1(z_2)\geq t^*_{a_3}\geq\frac{1}{3}$, violating Claim 7.3.

Assume next that $g^*(b_1)<\frac{1}{6}n_4(b_1)$. We assert that $d(b_1)=2$. Suppose $d(b_1)\geq 3$. Since $g^*(b_1)<\frac{1}{6}n_4(b_1)$, we see $n_4(b_1)=1$ and $n_5^2(b_1)+n_4^{-1}(b_1)+n_4^+(b_1)=0$. Hence, $a_1,b_2\in V_5^1$, which means $b_3\in V_4$ and $a_2,b_4\in V_5$. By Claim 7.2, $a_3\in V_5$. But then $t^*_{a_1}\geq\frac{1}{3}$ and so $f_1(z_1)\geq t^*_{a_1}\geq\frac{1}{3}$, which contradicts to Claim 7.3, as asserted. Note that $b_2\in V_4$, otherwise $n_5^2(z_1)=0$ and $g(z_1)<0$ as $g^*(b_1)<\frac{1}{6}n_4(b_1)$, which implies $a_2,b_2\in V_5^1$, that is, $a_3,b_3\in V_4$ violating Claims 7.1 and 7.2. By Claim 7.2, $g(z_1)<0$. Since $g^*(b_1)<\frac{1}{6}n_4(b_1)$, we have $n_4^+(b_1)=0$ and $n_4^{-1}(b_1)\leq 1$. Hence, $b_3\in V_5$. Note that $b_4\in V_4\cup V_5$. If $b_4\in V_4$, then $a_1\in V_5^1$ as $n_4^{-1}(b_1)\leq 1$. Hence, $a_2\in V_5$. By Claims 7.1 and 7.2, $a_3=b_4$. Note that $a_2\neq b_3$, otherwise $f_1(b_4)\geq t^*_{b_3}\geq\frac{1}{3}$ violating Claim 7.2. But then $t^*_v\geq\frac{1}{6}$ for any $v\in\{a_2,b_3\}$ and so $f_1(b_4)\geq\frac{1}{3}$, a contradiction. If $b_4\in V_5$, then $g(z_2)<0$, otherwise $f_1(z_2)\geq t^*_{b_4}\geq\frac{1}{6}$ violating Claim 7.2. Hence, $a_3\in V_5$. Clearly $a_2\in V_4$, else $t^*_{a_1}\geq\frac{1}{3}$, which implies $f_1(z_1)\geq\frac{1}{3}$ violating Claim 7.3. Since $n_4^{-1}(b_1)\leq 1$, we see $d(b_3)=2$. As $G+z_1b_3\in\mathcal{F}_G$ and $d(b_1)=d(b_3)=2$, G has a 4-path with vertices z_1,c_1,c_2,c_3,b_4 in order. Obviously, $c_3\in V_4$, otherwise $a_3\neq b_4$ and $t^*_v\geq\frac{1}{6}$ for any $v\in\{a_3,b_4\}$, which implies $f_1(z_2)\geq\frac{1}{3}$. Moreover, $g(c_3)<0$, otherwise $c_3\neq z_2$ yields $t^*_{b_4}\geq\frac{1}{3}$ and so $f_1(z_2)\geq\frac{1}{3}$. It follows that $c_1,c_2\in V_5$.

Note that $c_1 \neq a_1$, otherwise $f_1(a_1) \geq t^*_{a_1} \geq \frac{1}{3}$. It is easy to check that $t^*_v \geq \frac{1}{6}$ for any $v \in \{c_1, a_1\}$. But then $f_1(z_1) \geq t^*_{a_1} + t^*_{c_1} \geq \frac{1}{3}$ violating Claim 7.3.

Finally, let $N_3^{5-}(x) = \{y_1\}$. It remains to prove that $n_4^{5-}(v) = 0$ for any $v \in N_3(x) \setminus y_1$. Suppose not. Let $z \in N_4^{5-}(y)$, where $y \in N_3(x) \setminus y_1$. By Corollary 5.4 and Lemmas 5.7 and 5.1(a,b), $g^*(z) < 0$ and G has a 3-path $P = zww_2z_2$ such that $d(v) = 2$ for any $v \in V(P)$ and $z_2y \notin E(G)$. Let $N_3(z_2) = \{y_2\}$. Note that $g^*(y) \leq f_3(y) < \frac{1}{6}$, otherwise $f_4(z) \geq 0$. Then G has a 3-path $P(xy)$ since $G + y_1z \in \mathcal{F}_G$ and $f_5(y_1) < 0$. Let $P(xy) = xx_1x_2y$. Then $x_1y_2 \in E(G)$ or $G - x_1 - y$ has a path of length at most two with ends x and x_2 as $G + zx_1 \in \mathcal{F}_G$. Clearly, $x_2 \in V_4$, otherwise $g^*(y) \geq \frac{1}{6}$. Then $x_1 \in V_3$. If $x_1y_2 \notin E(G)$, then $G - x_1 - y$ has a 2-path with ends x and x_2, that is, $n_3(x_2) \geq 3$. If $x_1y_2 \in E(G)$, then $n_3^+(x_2) + n_3(x_2) \geq 3$. By Lemma 5.5, $n_5^-(x_2) = 0$. Hence, in both cases, $f_1(x_2) \geq g^*(x_2) \geq \frac{1}{6}n_3(x_2) + \frac{1}{6}n_4(x_2)$ and so $t^2_{x_2} \geq \frac{1}{6}$. But then $f_3(y) \geq f_2(y) + t^2_{x_2} \geq \frac{1}{6}$, a contradiction. This completes the proof of Theorem 4.8. □

8 Proof of Theorem 4.9

By Definition 4.6(1), for all $x \in V_5$, $f_7(x) \geq 0$. To prove $f_7(x) \geq 0$ for all $x \in V_2 \cup V_3 \cup V_4$, we just need to prove that for any $x \in V_2$, $f_7(x) \geq 0$, $n_3^{7+}(x) = n_3(x)$ and $n_4^{7+}(v) = n_4(v)$ for any $v \in N_3(x)$. Let $\alpha_1 \in N_2(x)$.

Case 1. $n_3^{5-}(x) \neq 0$.

Let $y \in N_3^{5-}(x)$. By Theorem 4.8, $n_3^{7+}(x) \geq n_3^{5+}(x) \geq n_3(x) - 1$ and $n_4^{7+}(v) = n_4^{5+}(v) = n_4(v)$ for any $v \in N_3(x) \setminus y$. It remains to prove $f_7(x) \geq 0$ and $f_7(y) \geq 0$ and $n_4^{7+}(y) = n_4(y)$. If $d(y) = 1$, then $V_1 = \{\alpha, \alpha_1\}$, which yields G has a 3-path $P(x\alpha_1)$ because $G + y\alpha \in \mathcal{F}_G$. Let $P(x\alpha_1) = xx_1\alpha_1'\alpha_1$. Then $G - x - \alpha_1'$ has a t-path with vertices α_1, x_t, x_1 in order for some $t \in [2]$ because $G + y\alpha_1' \in \mathcal{F}_G$. Hence, $G[\{x, x_t, \alpha_1'\}]$ contains at least two edges because $G + \alpha x_1 \in \mathcal{F}_G$, which implies $g(x) \geq \frac{2}{3}$ or $n_2^2(x) \geq 1$. Thus, $f_6(x) \geq g^*(x) \geq \frac{1}{3}$, which means $f_7(x) \geq 0$ and $f_7(y) \geq 0$. So we next assume $d(y) \geq 2$.

Case 1.1 $n_4^{5-}(y) \neq 0$.

Let $z \in N_4^{5-}(y)$. By Theorem 4.7, $n_4^{7+}(y) \geq n_4^{5+}(y) \geq n_4(y) - 1$. So we shall prove $f_7(v) \geq 0$ for any $v \in \{x, y, z\}$. Note that $n_2(x) \neq 0$ as $G + \alpha z \in \mathcal{F}_G$, which yields $f_6(x) \geq g^*(x) \geq \frac{1}{6}$. By Corollary 5.4, Lemma 5.1(a,b), $g^*(y) < 0$, $g^*(z) < 0$ and G has a 3-path $P = zww_1z_1$ such that $d(v) = 2$ for any $v \in V(P)$, and $yz_1 \notin E(G)$, where $z_1 \in V_4$ and $w, w_1 \in V_5$. Let $N_3(z_1) = \{y_1\}$. Clearly, $f_5(y) \geq -\frac{1}{3}$ and $f_5(z) \geq -\frac{1}{6}$. Thus, we need to prove $f_6(x) + \sum_{i\geq 1} t^6_{v_i} \geq \frac{1}{2}$ or $\sum_{i \geq 1} t^6_{v_i} \geq \frac{1}{3}$, where $v_i \in N_3(x) \setminus \{y\}$. Since $g^*(y) < 0$, we have $y_1x \in E(G)$ and G has a 3-path $P(yy_1)$ as $\{G + wz_1, G + zz_1\} \subseteq \mathcal{F}_G$. Let $P(y_1y) = y_1a_1a_2y$. Since $g^*(y) < 0$, we see $a_2 = x$, otherwise $a_2 \in V_4$ violating Lemma 5.8(a). Thus, $a_1 \in V_2 \cup V_3$. For any $v \in V_3$, let $A(v)$ be defined as in Definition 4.6(4), and $n_4^{3-}(v) \leq n_4^{*-}(v) \leq 1$ by Corollary 5.4. Since $d(z_1) = d(w_1) = 2$, $f_3(z_1) = -\frac{1}{6}$, which yields $f_5(y_1) \geq f_3(y_1) - (-f_3(z_1)) - \frac{1}{6}|A(y_1)| \geq g^*(y_1) - \frac{1}{6} - \frac{1}{6}n_3(y_1)$.

Suppose $a_1 \in V_2$. Then $a_1, x \in N_2^+(y_1)$. By Observation 4.4(1), $g^*(y_1) \geq \frac{1}{3}n_2(y_1) + \frac{1}{6}n_3(y_1) \geq \frac{1}{6}n_2(y_1) + \frac{1}{6}(n_3(y_1) + 1)$ which means $t^6_{y_1} \geq t^5_{y_1} \geq \frac{1}{6}$. If $n_2(y_1) + n_3(y_1) \geq 3$, then $t^6_{y_1} \geq t^5_{y_1} \geq \frac{1}{3}$ as $g^*(y_1) \geq \frac{1}{3}n_2(y_1) + \frac{1}{6}(n_3(y_1) + 1)$. If $g(v) \geq \frac{2}{3}$ for some $v \in \{a_1, x\}$, then $f_6(x) \geq g^*(x) \geq \frac{1}{3}$ and so $f_6(x) + t^6_{y_1} \geq \frac{1}{2}$. So we next assume $n_2(y_1) + n_3(y_1) = 2$ and $g(x) = g(a_1) = \frac{1}{6}$. Note that $N_1(a_1) = \{\alpha_1\}$, else $\alpha_2a_1 \in E(G)$ yields G has a copy of $C_6 = \alpha_2\alpha\alpha_1a_2y_1a_1$. Then G has a 4-path $P(\alpha_1y_1)$ containing a_1 and x or 4-path $P(\alpha_2y_1)$ because $G + \alpha y_1 \in \mathcal{F}_G$. Suppose $P(\alpha_1y_1)$ exists. Then $G - \alpha_1 - y_1$ has a 2-path with vertices a_1, a_3, x in order such that $a_3 \in V_3$. Similarly, $t^6_{a_3} \geq \frac{1}{6}$. Hence, $t^6_{y_1} + t^6_{a_3} \geq \frac{1}{3}$. Suppose $P(\alpha_2y_1)$ exists. Then $\delta(G) = 2$ and let $P(\alpha_2y_1) = \alpha_2b_1b_2b_3y_1$. Then $b_1 \in V_2 \setminus \{a_1, x\}$ since $N_1(a_1) = \{\alpha_1\}$ and $g(x) = \frac{1}{6}$. Note that $b_2 \notin V_1$, otherwise $b_2 = \alpha_1$, which violates the choice of α because α is contained in a 4-cycle but z_1 is a 2-vertex that is not contained in a 4-cycle. Since $n_2(y_1)+n_3(y_1)=2$, we see $b_3 \in V_2 \cup V_4$. In fact, $b_3 \in V_4$, otherwise $b_3 \in V_2$ and so G has a copy of $C_6 = b_3b_2b_1\alpha_2\alpha\alpha_1$ because $b_2 \notin V_1$. Then $b_2 \in V_3$, which implies $n_3(b_3) \geq 2$ and $y_1 \in N_3^+(b_3)$. Note that $n_5^-(b_3) = 0$ as $\delta(G) = 2$. By Observation 4.5(2), $g^*(b_3) \geq \frac{1}{6}n_3(b_3) + \frac{1}{6}n_4(b_3)$, which implies $t_{b_3}^2 \geq \frac{1}{6}$. Hence, $f_3(y_1) \geq f_2(y_1) + t_{b_3}^2 \geq g^*(y_1) + \frac{1}{6}$ and so $t_{y_1}^6 \geq \frac{1}{3}$.

Suppose $a_1 \in V_3$. By contradiction, assume $t_{y_1}^5 + t_{a_1}^5 < \frac{1}{3}$ as $t_v^6 = t_v^5$ for any $v \in \{a_1, y_1\}$. Then G has a 3-path $P(y_1x)$ as $G + y_1z \in \mathcal{F}_G$ and $g^*(y) < 0$. Let $P(y_1x) = y_1b_1b_2x$. Note that $f_5(a_1) \geq g^*(a_1) - (-f_3(a_1')) - \frac{1}{6}|A(a_1)| \geq g^*(a_1) - \frac{1}{3} - \frac{1}{6}|A(a_1)|$, where $a_1' \in N_4^{3-}(a_1)$. We next prove that

(a) $a_1 \notin \{b_1, b_2\}$.

To prove (a), suppose first $a_1 = b_1$. Then $b_2 \in V_2 \cup V_3$. In fact $b_2 \in V_3$, otherwise $g^*(a_1) \geq \frac{1}{3}n_2(a_1) + \frac{1}{6}(n_3(a_1)+1)$ because $b_2, x \in N_2^+(a_1)$ and $n_2(a_1) + n_3(a_1) \geq 3$, which implies $t_{a_1}^5 \geq \frac{1}{3}$ as $y_1 \notin A(a_1)$. Moreover, $y_1 \in V_3^1$, otherwise for any $v \in \{a_1, y_1\}$, $g^*(v) \geq \frac{1}{6}n_2(v) + \frac{1}{6}n_3(v)$ since $x \in N_2^+(v)$ and $n_3^2(v) \geq 1$ which implies $t_v^5 \geq \frac{1}{6}$ as $b_2, y_1 \notin A(a_1)$ and $a_1 \notin A(y_1)$. Note that $g^*(y_1) \geq \frac{1}{3}$. Clearly, $t_{y_1}^5 \geq \frac{1}{6}$ since $a_1 \notin A(y_1)$. Then $t_{a_1}^5 < \frac{1}{6}$. By Ineq. (3), $g^*(a_1) \geq \frac{1}{6}n_2(a_1) + \frac{1}{6}(n_3(a_1) - 1)$ since $x \in N_2^+(a_1)$ and $n_3(a_1) \geq 2$. Then $n_4^{3-}(a_1) = 1$, otherwise $t_{a_1}^5 \geq \frac{1}{6}$ as $b_1, y_1 \notin A(a_1)$. Let $N_4^{3-}(a_1) = \{z_2\}$. Then $d(z_2) = 1$, otherwise $f_3(z_2) \geq -\frac{1}{6}$, which implies $t_{a_1}^5 \geq \frac{1}{6}$ as $b_1, y_1 \notin A(a_1)$. Then $G - a_1 - x$ has a 2-path with vertices y_1, b_3, b_2 in order such that $b_3 \in V_4$ as $G + z_2x \in \mathcal{F}_G$ and $y_1 \in V_3^1$. Hence, $g^*(b_3) \geq \frac{1}{6}n_3(b_3) + \frac{1}{3} + \frac{1}{6}n_4(b_3)$ since $b_2, y_1 \in N_3^+(b_3)$, which yields $t_{b_3}^2 \geq \frac{1}{6}$. Hence, $f_3(y_1) \geq f_2(y_1) + t_{b_3}^2 \geq \frac{1}{2}$. But then $t_{y_1}^5 \geq \frac{1}{3}$ as $a_1 \notin A(y_1)$, a contradiction. Suppose now $a_1 = b_2$. Then $b_1 \in V_2 \cup V_3 \cup V_4$. In fact $b_1 \in V_4$, otherwise for any $v \in \{a_1, y_1\}$, $g^*(v) \geq \frac{1}{6}n_2(v) + \frac{1}{6}(n_3(v)+1)$ because $x \in N_2^+(v)$ and $n_3^2(v) + n_2(v) \geq 3$, which implies $t_v^5 \geq \frac{1}{6}$ since $y_1 \notin A(a_1)$ and $a_1 \notin A(y_1)$. For any $v \in \{a_1, y_1\}$, then $g^*(v) \geq \frac{1}{6}n_2(v) + \frac{1}{6}n_3(v) - \frac{1}{6}$ as $x \in N_2^+(v)$. Then $G - a_1 - y_1$ has a 2-path with ends x and b_1 because $G + z_1a_1 \in \mathcal{F}_G$. By Ineq. (3), $g^*(b_1) \geq \frac{1}{3}n_3(b_1) + \frac{1}{6}(n_4(b_1)+2)$ since $a_1, y_1 \in N_3^+(b_1)$ and $n_3(b_1) \geq 3$, which implies $t_{b_1}^2 \geq \frac{1}{3}$ and so $f_3(v) \geq f_2(v) + t_{b_1}^2$. But then $t_v^5 \geq \frac{1}{6}$ for any $v \in \{a_1, y_1\}$ as $y_1 \notin A(a_1)$ and $a_1 \notin A(y_1)$, that is, $t_{y_1}^5 + t_{a_1}^5 \geq \frac{1}{3}$, a contradiction. This proves (a).

Then $G - b_2 - y_1$ has an s-path with ends x and b_1 as $G + z_1b_2 \in \mathcal{F}_G$ for some $s \in [2]$. Note that $b_1 \notin V_2$, otherwise by Observation 4.4(2,4), $g^*(y_1) \geq \frac{1}{3}n_2(y_1) + \frac{1}{6}n_3(y_1)$ because $x \in N_2^+(y_1)$, $n_2(y_1) \geq 2$, $b_1 \in N_2^+(y_1) \cup N_2^{-1}(y_1)$ and $n_3(y_1) \geq 1$, which yields $t_{y_1}^5 \geq \frac{1}{3}$ as $a_1 \notin A(y_1)$. We assert $b_1 \in V_4$. Suppose not. Then $b_1 \in V_3$. Hence, $n_2^+(b_1) \geq 1$ or $n_2(b_1) + n_3(b_1) \geq 4$ which implies $f_3(b_1) \geq g^*(b_1) \geq \frac{1}{3}$ and so $b_1 \notin A(y_1)$. Since $x \in N_2^+(a_1)$ and $y_1 \in N_3^2(a_1)$, we have $a_1 \notin A(y_1)$. Hence, $|A(y_1)| \leq n_3(y_1) - 2$. By Ineq. (3), $g^*(y_1) \geq \frac{1}{3}n_2(y_1) + \frac{1}{6}(n_3(y_1) - 1)$ because $x \in N_2^+(y_1)$ and $b_1 \in N_3^2(y_1)$. But then $t_{y_1}^5 \geq \frac{1}{3}$, a contradiction, as asserted. Then $y_1 \in N_3^+(b_1)$ and $n_3(b_1) \geq 3$. Note that when $N_5^-(b_1) = \{w_2\}$, we have $n_3^+(b_1) \geq 2$ as $G + w_2x \in \mathcal{F}_G$. By Ineq. (3), $g^*(b_1) \geq \frac{1}{3}n_3(b_1) + \frac{1}{6}n_4(b_1) + \frac{1}{3}n_5^-(b_1)$ and so $t_{b_1}^2 \geq \frac{1}{3}$. Hence, $f_3(y_1) \geq f_2(y_1) + t_{b_1}^2$. By Ineq. (3), $g^*(y_1) \geq \frac{1}{3}n_2(y_1) + \frac{1}{6}(n_3(y_1) - 2)$ as $x \in N_2^+(y_1)$. Thus, $a_1 \in A(y_1)$, otherwise $t_{y_1}^5 \geq \frac{1}{3}$. Hence, $n_4^{3-}(a_1) = 1$. Let $N_4^{3-}(a_1) = \{z_2\}$. Clearly, $d(z_2) = 1$ and $g(a_1) = \frac{1}{6}$. Then G has a 3-path with vertices a_1, c_1, c_2, x in order such that $c_1 \in V_4$ as $G + z_2y \in \mathcal{F}_G$. Hence, $a_1 \in N_3^+(c_1)$ and $n_3(c_1) \geq 3$ since $G + z_2c_2 \in \mathcal{F}_G$, which yields $g^*(c_1) \geq \frac{1}{6}n_3(c_1) + \frac{1}{6}n_4(c_1) + \frac{1}{3}$ and so $t_{c_1}^2 \geq \frac{1}{6}$. It follows $f_3(a_1) \geq f_2(a_1) + t_{c_1}^2 \geq \frac{1}{3}$. But then $a_1 \notin A(y_1)$, a contradiction.

Case 1.2 $n_4^{5-}(y) = 0$.

Note that $n_4^{7+}(y) = n_4^{5+}(y) = n_4(y)$. So we just need to prove $f_7(x) \geq 0$ and $f_7(y) \geq 0$.

Case 1.2.1 $n_4^-(y) \neq 0$.

Let $z \in N_4^-(y)$. Note that $n_2(x) \neq 0$ because $G + \alpha z \in \mathcal{F}_G$ and $d(\alpha, z) = 4$ and $\alpha_1\alpha_2 \notin E(G)$, which yields $f_6(x) \geq \frac{1}{6}$. If $f_5(y) \geq -\frac{1}{6}$, then $f_6(x) + f_6(y) = f_6(x) + f_5(y) \geq 0$, which yields $f_7(x) \geq 0$ and $f_7(y) \geq 0$. So we may assume $f_5(y) < -\frac{1}{6}$. By (⁎), $f_i(y) < -\frac{1}{6}$ for any $i \in [4]$. Then we just need to prove $f_6(x) + \sum_{i\geq 1} t^6_{v_i} \geq \frac{1}{3}$, where $v_i \in N_3(x) \setminus y$ and $t^6_{v_i} \geq 0$. If $g(x) \geq \frac{2}{3}$ or $n^2_2(x) \geq 1$, then $f_6(x) \geq g^*(x) \geq \frac{1}{3}$. So we next assume $g(x) = \frac{1}{6}$ and $n^2_2(x) = 0$. We claim $G - \alpha$ has a 3-path $P(x\alpha_1)$. Suppose not. Then G has a 4-path with ends α and x containing α_1 because $G + \alpha y \in \mathcal{F}_G$ and $g^*(y) < 0$. It follows $G - x$ has a 3-path $P(\alpha\alpha_1)$. Let $P(\alpha\alpha_1) = \alpha\alpha_2 x_2 \alpha_1$, that is $\alpha \in \mathcal{C}_4$. Then $x_2 \in V_2$. Since $G + \alpha_1 z \in \mathcal{F}_G$, we see G has a 5-path with vertices $\alpha_1, x_3, y_3, z_3, w, z$ in order. Then $w \in V_5$, otherwise $w = y$ and $z_3 = x$ because $g^*(y) < 0$ and $g(z) < 0$, which implies $G - \alpha$ has a 3-path $P(x\alpha_1)$. Hence, $z_3 \in V_4$. By Lemma 5.3, $n^-_5(z) = 0$. Moreover, $d(w) = 2$ and $N_5(z) \cap N_5(z_3) = \{w\}$, otherwise $t^*_w \geq \frac{1}{6}$, which yields $f_2(z) \geq g^*(z) + t^*_w \geq \frac{1}{6}$ and so $f_3(y) \geq f_2(y) + t^2_z \geq -\frac{1}{6}$. By the choice of α, $w \in \mathcal{C}_4$ and $|N(z) \cap N(z_3)| \geq 2$. Let $w_1 \in N(z) \cap N(z_3)$ and $w_1 \neq w$. Then $w_1 = y$. Since $G + wx \in \mathcal{F}_G$ and $g^*(y) < 0$, G has a 3-path with ends y and $v \in \{z, z_3\}$ containing $v' \in \{z, z_3\}$ such that $v \neq v'$. But then $|N_5(z) \cap N_5(z_3)| \geq 2$ as $g(z) < 0$, a contradiction, as claimed. Let $P(x\alpha_1) = xy_1x_1\alpha_1$. Then $y_1 \in V_3$ and $x_1 \in V_2$ as $g(x) = \frac{1}{6}$ and $n^2_2(x) = 0$. If $n^{3-}_4(y_1) = 0$ or $g(x_1) \geq 0$, then $f_3(y_1) \geq g^*(y_1) \geq \frac{1}{6}n_2(y_1) + \frac{1}{6}n_3(y_1) + \frac{1}{3}(n^+_2(y_1) - 1)$ since $x \in N^+_2(y_1)$ and $n_2(y_1) \geq 2$, which implies $t^6_{y_1} \geq \frac{1}{6}$ and so $f_6(x) + t^6_{y_1} \geq \frac{1}{3}$. So we next assume $n^{3-}_4(y_1) = 1$ and $g(x_1) < 0$. Let $N^{3-}_4(y_1) = \{z_1\}$. Then G has a 4-path with ends α_1 and y_1 containing x and x_1 since $G + z_1\alpha_1 \in \mathcal{F}_G$ and $d(z_1, \alpha_1) = 3$. Then $G - \alpha_1 - y_1$ has a 2-path with vertices x, y_2, x_1 in order. Then $y_2 \in V_3$ as $g(x_1) < 0$. Hence, for any $i \in [2]$, $x \in N^+_2(y_i)$ and $x_1 \in N^{-1}_2(y_i)$, which yields $f_3(y_i) \geq g^*(y_i) \geq \frac{1}{12}n_2(y_i) + \frac{1}{6}(n_3(y_i) + 2)$. Thus, $t^6_{y_i} \geq \frac{1}{12}$ and so $f_6(x) + t^6_{y_1} + t^6_{y_2} \geq \frac{1}{3}$.

Case 1.2.2 $n^-_4(y) = 0$.

By Observation 4.3(1), $N_4(y) = N^1_4(y)$ and $d(y) = 2$ which yields no 4-cycle contains y. By the choice of α, $\alpha \notin \mathcal{C}_4$. Clearly, $f_6(y) \geq -\frac{1}{6}$. If $g(x) \geq \frac{1}{6}$, then $f_6(x) + f_6(y) \geq 0$, which yields $f_7(x) \geq 0$ and $f_7(y) \geq 0$. So we next assume $g(x) < 0$. Let $N_4(y) = \{z\}$ and $N_4(z) = \{z_1\}$. By Observation 4.3(2), $g(z_1) = \frac{1}{6}$. Let $N_3(z_1) = \{y_3\}$. Then G has a 3-path $P(x\alpha_1)$ because $G + \alpha y \in \mathcal{F}_G$, $g(x) < 0$ and no 4-cycle contains α. Let $P(x\alpha_1) = xy_1x_1\alpha_1$. Then $y_1 \in V_3$ and $x_1 \in V_2$, which means $f_5(x) \geq g^*(x) \geq 0$. So we just need to prove $\sum_{i\geq 1} t^6_{v_i} \geq \frac{1}{6}$, where $v_i \in N_3(x) \setminus y$. Note that if $n^{3-}_4(y_1) = 1$, then $G - \alpha_1 - y_1$ has a 2-path with vertices x, y_2, x_1 in order such that $y_2 \in V_4$ as $G + z_2\alpha_1 \in \mathcal{F}_G$, where $N^{3-}_4(y_1) = \{z_2\}$.

Assume $g(x_1) \geq 0$. If $n^{3-}_4(y_1) = 0$, then $g^*(y_1) \geq \frac{1}{6}n_2(y_1) + \frac{1}{6}n_3(y_1)$ as $n^+_2(y_1) + n_2(y_1) \geq 3$, which yields $t^6_{y_1} \geq \frac{1}{6}$. If $n^{3-}_4(y_1) = 1$, then for any $i \in [2]$, $g^*(y_i) \geq \frac{1}{12}n_2(y_i) + \frac{1}{6}(n_3(y_i) + 2)$ since $n^+_2(y_i) + n_2(y_i) \geq 3$ and $x \in N^{-1}_2(y_i)$, which means $t^6_{y_i} \geq \frac{1}{12}$ and so $t^6_{y_1} + t^6_{y_2} \geq \frac{1}{6}$.

So we further assume $g(x_1) < 0$. Suppose $n^{3-}_4(y_1) = 1$. Then $x, x_1 \in N^{-1}_2(y_i)$ for any $i \in [2]$. Note that $n_2(y_1) + n_3(y_1) \geq 3$ as $G + z_2\alpha \in \mathcal{F}_G$ and $d(\alpha, z_2) = 4$. Similarly, $n_2(y_2) + n_3(y_2) \geq 3$ when $n^{3-}_4(y_2) \neq 0$. Hence, $g^*(y_i) \geq \frac{1}{12}n_2(y_i) + \frac{1}{6}n_3(y_i) + \frac{1}{3}n^{3-}_4(y_i)$, which implies $t^6_{y_i} \geq \frac{1}{12}$ and so $t^6_{y_1} + t^6_{y_2} \geq \frac{1}{6}$. Suppose $n^{3-}_4(y_1) = 0$. If $\delta(G) = 1$, then $x, x_1 \in N^{-1}_2(y_1)$ as $G + \alpha y_1 \in \mathcal{F}_G$. Hence, $g^*(y_1) \geq \frac{1}{6}n_2(y_1) + \frac{1}{6}n_3(y_1)$, which yields $t^6_{y_1} \geq \frac{1}{6}$. If $\delta(G) = 2$, then G has a 4-path $P(zx_1)$ or $G - x - x_1$ has a 2-path with ends α_1 and y_1 as $G + yx_1 \in \mathcal{F}_G$. We claim the later holds. Suppose not. Let $P(zx_1) = za_1a_2a_3x_1$. Then $a_1 \in V_4 \cup V_5$. If $a_1 \in V_4$, then $a_1 = z_1$, $a_2 = y_3$ and $a_3 \in V_3$ as $g(x_1) < 0$. Note that $n^-_5(z_1) = 0$ as $\delta(G) = 2$. Hence, $f_1(z_1) \geq g^*(z_1) \geq \frac{1}{6}$ since $y_3 \in N^+_3(z_1)$, which means $z_1 \in N^{1+}_4(z)$. Thus, $t^2_z \geq \frac{1}{6}$. If $a_1 \in V_5$, then $a_2 \in V_4$, which implies $g^*(a_1) \geq \frac{1}{6}n_4(a_1)$ since $z \in N^+_4(a_1)$ and $n_4(a_1) \geq 2$. Thus, $t^2_z \geq \frac{1}{6}$ as $z_1 \in N^{1+}_4(z)$. But then in both cases, $f_3(y) \geq f_2(y) + t^2_z \geq \frac{1}{6} \geq 0$, a contradiction, as claimed. Thus, $n_2(y_1) \geq 3$, which implies $g^*(y_1) \geq \frac{1}{6}n_2(y_1) + \frac{1}{6}n_3(y_1)$. Hence, $t^6_{y_1} \geq \frac{1}{6}$.

Case 2. $n^{5-}_3(x) = 0$.

Note that $n^{7+}_3(x) = n^{5+}_3(x) = n_3(x)$. By Theorem 4.7, $\sum_{v\in N_3(x)} n^{5-}_4(v) \leq 1$.

Case 2.1 $\sum_{v\in N_3(x)} n^{5-}_4(v) = 1$.

Let $z \in N_4^{5-}(y)$, where $y \in N_3(x)$. So we just need to prove that $f_7(x) \geq 0$ and $f_7(z) \geq 0$. By Corollary 5.4, Lemmas 5.7 and 5.1(a,b), $g^*(z) < 0$ and G has a 3-path $P = zww_1z_1$ such that $d(v) = 2$ for any $v \in V(P)$, and $yz_1 \notin E(G)$, where $z_1 \in V_4$ and $w, w_1 \in V_5$. Let $N_3(z_1) = \{y_1\}$. Note that $f_6(z) \geq -\frac{1}{6}$. Then $f_3(y) < \frac{1}{6}$, else $f_5(z) \geq 0$. We claim $g^*(x) \geq \frac{1}{6}$. Suppose not. Then $n_2(x) = 0$, otherwise $g^*(x) \geq \frac{1}{6}$. Since $z \notin \mathcal{C}_4$, we see $n_1(x) = 1$ by the choice of α, $\alpha \notin \mathcal{C}_4$. It follows $g(x) < 0$. Clearly, $n_3(y) \neq 0$ and G has a 4-path with ends y and α_1 containing x because $\{G + \alpha z, G + \alpha_1 z\} \subseteq \mathcal{F}_G$, $g(x) < 0$ and $g^*(z) < 0$. Note that G has no 3-path $P(x\alpha_1)$, otherwise let $P(x\alpha_1) = xy_2x_1\alpha_1$, we see $x_1 \in V_2$ and $y_2 \in V_3$, which yields $f_3(y) \geq g^*(y) \geq \frac{1}{6}$ as $y_2 \in N_3^2(x)$. Hence, $G - \alpha_1$ has a 3-path $P(xy)$. Let $P(xy) = xb_1b_2y$. Then $b_1 \in V_3$. Note that $b_2 \in V_4$, otherwise $f_3(y) \geq g^*(y) \geq \frac{1}{6}$. By Lemma 5.5, $n_5^-(b_2) = 0$. Then $g^*(b_2) \geq \frac{1}{6}n_3(b_2) + \frac{1}{6}n_4(b_2)$ as $y \in N_3^+(b_2)$ and $n_3(b_2) \geq 2$, which yields $t_{b_2}^2 \geq \frac{1}{6}$ and so $f_3(y) \geq f_2(y) + t_{b_2}^2 \geq \frac{1}{6}$, a contradiction. Then $f_6(x) + f_6(z) \geq 0$, which yields $f_7(x) \geq 0$ and $f_7(z) \geq 0$.

Case 2.2 $\sum_{v \in N_3(x)} n_4^{5-}(v) = 0$.

Note that $n_4^{7+}(v) = n_4^{5+}(v) = n_4(v)$ for any $v \in N_3(x)$. So we shall prove $f_7(x) \geq 0$. We assume $f_7(x) < 0$, then $g^*(x) < 0$. Since $G + \alpha x \in \mathcal{F}_G$, G has a 3-path $P(\alpha_1\alpha_2)$ where $N(\alpha) = \{\alpha_1, \alpha_2\}$ or a 3-path $P(xx_1)$ for some $x_1 \in V_2$. Suppose $P(\alpha_1\alpha_2)$ exists, that is $\alpha \in \mathcal{C}_5$. Let $P(\alpha_1\alpha_2) = \alpha_1b_1b_2\alpha_2$ where $b_1, b_2 \in V_2$. In this case, $\delta(G) = 2$ and $v \in \mathcal{C}_4 \cup \mathcal{C}_5$ for any vertex $v \in V(G)$ with $d(v) = 2$ by the choice of α. By Lemma 5.1, $V_4^{*-} = V_5^{*-} = \emptyset$. Since $G + x\alpha_2 \in \mathcal{F}_G$, G has a 5-path $P(x\alpha_2)$. Let $P(x\alpha_2) = xc_1c_2c_3c_4\alpha_2$. We have $c_1 \in V_3$, $c_4 \in V_2$, $c_2 \in V_3 \cup V_4$ and $c_3 \in V_3 \cup V_2$. When $c_2 \in V_3$, then $c_3 \in V_2$ as $g^*(x) < 0$ and so $g(c_i) = \frac{1}{6}$ for any $i \in [2]$ and $g(c_3) \geq \frac{1}{6}$. It follows that $f_5(c_3) \geq 0$ and $f_5(c_2) \geq \frac{1}{6}$, which yields $f_7(x) \geq 0$ by Definition 4.6(6). When $c_2 \in V_4$, then $c_3 \in V_3$. Observe that $g(c_1) < 0$, otherwise $g^*(x) \geq -\frac{1}{6}$ and $t_{c_2}^2 \geq t_{c_2}^* \geq \frac{1}{6}$ and so $t_{c_1}^6 \geq \frac{1}{6}$ yielding $f_7(x) \geq 0$. It follows that $d(x) \geq 3$ otherwise $x \in \mathcal{C}_4 \cup \mathcal{C}_5$. We assert $d(c_1) \geq 3$. Suppose not. Then $c_1 \notin \mathcal{C}_4$, otherwise there is another vertex $c_0 \in (N(x) \cap N(c_2)) \setminus \{c_1, c_3\}$ and so $t_{c_2}^2 \geq t_{c_2}^* \geq \frac{1}{6}$, $t_{c_i}^6 \geq \frac{1}{6}$ for $i \in \{0,1\}$ yielding $f_7(x) \geq 0$. Then $N_2(\alpha_1) \cap N_2(\alpha_2) = \emptyset$, $c_1 \in \mathcal{C}_5$ and $G - c_1$ has a 3-path $P(xc_2)$. Let $P(xc_2) = xc_5c_6c_2$ where $c_5 \in V_3$. Then $c_6 \in V_4$ otherwise $g^*(x) \geq -\frac{1}{6}$ and $t_{c_2}^* \geq \frac{1}{6}$ and so $t_{c_1}^6 \geq \frac{1}{6}$ yielding $f_7(x) \geq 0$. Because $G + c_5\alpha \in \mathcal{F}_G$ and $n_2(x) = 0$, there is a 5-path $P(\alpha c_5)$. Let $P(\alpha c_5) = \alpha w_1w_2w_3w_4c_5$ where $w_1 \in V_1$ and $w_2 \in V_2$. When $w_4 \in V_3$, then $g(w_4) = \frac{1}{6}$, $w_3 \in V_2$ and $g(w_3) \geq 0$, which follows that $f_6(w_4) \geq \frac{1}{6}$ and so $f_7(x) \geq 0$ by Definition 4.6(6). When $w_4 \in V_4$, we see $w_3 \in V_3$ and $n_3(w_4) \geq 2$, which follows that $t_{c_6}^* \geq \frac{1}{6}$ and $t_{c_2}^* \geq \frac{1}{6}$ yielding $t_{c_i}^6 \geq \frac{1}{6}$ for $i \in \{1,5\}$, which follows that $f_7(x) \geq 0$. Thus $d(c_1) \geq 3$. There is another vertex $c_{11} \in N_4(c_1) \setminus \{c_2\}$ and $n_4(c_{11}) \geq 1$ because $G + c_{11}\alpha \in \mathcal{F}_G$ and $n_3(c_1) = n_2(x) = 0$. We see $t_{c_{11}}^2 \geq \frac{1}{6}$. When $N_2(\alpha_1) \cap N_2(\alpha_2) \neq \emptyset$, then $d(c_2) \geq 3$, otherwise $c_2 \in \mathcal{C}_4$ and there is a vertex $v \in (N_4(c_1) \cap N_4(c_3)) \setminus c_2$, which follows that $t_{c_1}^6 \geq \frac{1}{3}$ yielding $f_7(x) \geq 0$. We claim $f_2(c_2) \geq \frac{1}{6}n_3(c_2)$. Suppose not. Then $n_3(c_2) + n_4(c_2) = 2$ and there is a vertex $c_{21} \in N_5(c_2)$. When $d(c_{21}) \geq 3$ or $g(c_{21}) = \frac{1}{6}$, we see $f_2(c_2) \geq \frac{1}{6}n_3(c_2)$. Then we just consider the case $N(c_{21}) = N_4(c_{21}) = \{v_1, c_2\}$. By the choice of α, we see $c_{21} \in \mathcal{C}_4 \cap \mathcal{C}_5$. If there is a vertex $c_{22} \in (N_5(c_2) \cap N_5(v_1)) \setminus c_{21}$ or $g(v_1) \geq 0$, then $f_2(c_2) \geq \frac{1}{6}n_3(c_2)$. Thus $v_1c_3 \in E(G)$ otherwise $c_1v_1 \in E(G)$ but then $G + v_1\alpha \notin \mathcal{F}_G$. Note that there is a 3-path $P(c_2v_1)$ with $g(c_2) = \frac{2}{3}$ and $g(v_1) < 0$. Let $P(c_2v_1) = c_2u_1u_2v_1$ where $u_i \in V_5$ for $i \in [2]$, which follows $f_2(c_2) \geq \frac{1}{6}n_3(c_2)$, as claimed. Note that $t_{c_{11}}^2 \geq \frac{1}{6}$, we see $t_{c_1}^6 \geq \frac{1}{3}$ and so $f_7(x) \geq 0$. When $N_2(\alpha_1) \cap N_2(\alpha_2) = \emptyset$, we see there is a vertex $d_2 \in N_4(d_1)$ with $n_3(d_2) \geq 2$ because $G + d_1\alpha \in \mathcal{F}_G$ for $d_1 \in N_3(x) \setminus c_1$. Similarly we have $g(d_1) < 0$ and there is a vertex $d_{11} \in N_4(d_1) \setminus d_2$ with $t_{d_{11}}^2 \geq \frac{1}{6}$, together with $t_{c_{11}}^2 \geq \frac{1}{6}$ we see $f_7(x) \geq 0$.

Suppose $P(xx_1)$ exists and let $P(xx_1) = xy_1y_2x_1$. By Observation 4.3(1,2), $y_i \in V_3^1$ for any $i \in [2]$. Hence, $f_6(x) \geq g^*(x) \geq -\frac{1}{6}$. So we just need to prove $\sum_{i \geq 1} t_{v_i}^6 \geq \frac{1}{6}$, where $v_i \in N_3(x)$. We shall proceed it by contradiction. Then, $f_6(y_1) < \frac{1}{6}$ and so $f_5(y_1) < \frac{1}{6}$.

Note that $g^*(x) < 0$, we see G has no 3-path with ends x and α_1. We assert that $G - \alpha_1$ has no 3-path with ends x and y_1. Suppose not. Let $P(xy_1) = xy_3z_1y_1$. Then $y_3 \in V_3$. By Observation 4.3(1), $g(y_3) < 0$ and so

$z_1 \in V_4$. Note that $n_4^{3-}(y_3) = 0$, otherwise let $z_3 \in N_4^{3-}(y_3)$, $G + z_3\alpha \in \mathcal{F}_G$ yields $n_3(y_3) + n_2(x) \neq 0$ and so $g(y_3) \geq 0$ or $g(x) \geq 0$. Furthermore, $n_5^-(z_1) \neq 0$, otherwise $f_2(z_1) \geq g^*(z_1) - \frac{1}{6}n_4(z_1) \geq \frac{1}{6}n_3(z_1)$ as $y_1 \in N_3^+(z_1)$ and $n_3(z_1) \geq 2$, which implies $f_6(y_3) \geq f_3(y_3) \geq f_2(y_3) + t_{z_1}^2 \geq \frac{1}{6}$ and so $t_{y_3}^6 \geq \frac{1}{6}$. Let $w_1 \in N_5^-(z_1)$. Then $G - x - z_1$ has a 2-path with vertices y_1, z_2, y_3 in order such that $z_2 \in V_4$ because $G + w_1x \in \mathcal{F}_G$. Hence, for any $i \in [2]$, $f_2(z_i) \geq g^*(z_i) - \frac{1}{3}n_5^-(z_i) - \frac{1}{6}n_4(z_i) \geq \frac{1}{12}n_3(z_i)$ because $y_1 \in N_3^+(z_i)$, $y_3 \in N_3^{-1}(z_i)$ and $n_5^-(z_i) \leq 1$, which means $t_{z_1}^2 \geq \frac{1}{12}$. But then $f_6(y_3) \geq f_3(y_3) \geq g^*(y_3) + t_{z_1}^2 + t_{z_2}^2 \geq \frac{1}{6}$ and so $t_{y_3}^6 \geq \frac{1}{6}$, a contradiction, as asserted. Thus, G has no 4-path with ends y_1 and α_1.

Since $G + y_1\alpha \in \mathcal{F}_G$, G has a 4-path $P(y_1\alpha_2)$. Let $P(y_1\alpha_2) = y_1a_1a_2a_3\alpha_2$. Note that $\delta(G) = 2$. We claim $n_4^{3-}(v) = 0$ for any $v \in N_3(x)$. Suppose not. Let $z_1 \in N_4^{3-}(y)$ for some $y \in N_3(x)$. Then $y = y_1$, else $G + \alpha z_1 \notin \mathcal{F}_G$ because $d(z_1, \alpha) = 4$, $g(y) < 0$ and $g(x) < 0$. By Corollary 5.4 and Lemma 5.1(a, b), G has a 3-path consisting of vertices of degree 2 with one end z_1. But then G has a 4-path with ends α_1 and y_1 containing x because $G + z_1\alpha_1 \in \mathcal{F}_G$, a contradiction, as claimed. Note that $a_1 \in V_2 \cup V_3 \cup V_4$.

Assume $a_1 \in V_4$. Then $a_2 \in V_3$ and $a_3 \in V_2$. Then $f_2(a_1) \geq g^*(a_1) - \frac{1}{6}n_3(a_1) \geq \frac{1}{6}n_4(a_1)$ since $y_1 \in N_3^+(a_1)$ and $n_3(a_1) \geq 2$, which means $t_{a_1}^2 \geq \frac{1}{6}$. Hence, $f_3(y_1) \geq f_2(y_1) + t_{a_1}^2 \geq \frac{1}{6}$. Then $n_4^{3-}(y_2) \neq 0$, otherwise $y_2 \notin A(y_1)$ yields $f_6(y_1) \geq f_3(y_1) \geq \frac{1}{6}$ and so $t_{y_1}^6 \geq \frac{1}{6}$. Let $z_2 \in N_4^{3-}(y_2)$. Similarly, G has a 3-path $P = z_2w_2w_3z_3$ such that $d(v) = 2$ for any $v \in V(P)$ and $z_3y_2 \notin E(G)$, where $w_2, w_3 \in V_5$ and $z_3 \in V_4$. Since $G + \alpha_iz_2 \in \mathcal{F}_G$ for some $i \in [2]$ such that $\alpha_i \in N_1(x_1)$, G has a 4-path with ends y_2 and α_i containing x_1. Hence, $g(x_1) > 0$ or $n_3^2(x_1) \geq 1$ or G has a 3-path $P(x_1y_2)$. If $g(x_1) > 0$ or $n_3^2(x_1) \geq 1$, then $f_3(y_2) \geq g^*(y_2) \geq \frac{1}{6}$. If 3-path $P(x_1y_2)$ exists, then let $P(x_1y_2) = x_1x_1'y_2'y_2$. Since $y_2 \in V_3^1$, we see $y_2' \in V_4$ and $x_1' \in V_3$. Hence, $f_2(y_2') \geq \frac{1}{6}n_3(y_2')$ as $y_2 \in N_3^+(y_2')$, $n_3(y_2') \geq 2$ and $n_5^-(y_2') = 0$, which yields $t_{y_2'}^2 \geq \frac{1}{6}$ and so $f_3(y_2) \geq f_2(y_2) + t_{y_2'}^2 \geq \frac{1}{6}$. Thus, in both cases, $y_2 \notin A(y_1)$ as $f_3(z_2) \geq -\frac{1}{6}$ and $f_3(y_2) + f_3(z_2) \geq 0$. But then $f_6(y_1) \geq f_3(y_1) \geq \frac{1}{6}$ and so $t_{y_1}^6 \geq \frac{1}{6}$, a contradiction.

Assume $a_1 \in V_3$. Then $a_1 = y_2$, $a_2 = x_1$ and $a_3 \in V_2$ as $y_1, y_2 \in V_3^1$. Then $f_5(y_1) \geq g^*(y_1) \geq 0$ and $g^*(y_2) \geq \frac{1}{6}$ as $g(x_1) > 0$. Note that $n_4^{3-}(y_2) \neq 0$, otherwise $f_5(y_2) \geq g^*(y_2) \geq \frac{1}{6}$, which means $f_6(y_1) \geq f_5(y_1) + \frac{1}{6} \geq \frac{1}{6}$ by Definition 4.6(6). Let $z_2 \in N_4^{3-}(y_2)$. Similarly, G has a 3-path $P = z_2w_2w_3z_3$ such that $d(v) = 2$ for any $v \in V(P)$ and $z_3y_2 \notin E(G)$, where $w_2, w_3 \in V_5$ and $z_3 \in V_4$. Let $y_3 \in N_3(z_3)$. We claim $y_3x \notin E(G)$. Suppose not. Then G has a 2-path with vertices y_2, z_4, y_3 in order such that $z_4 \in V_4$ because $G + z_2w_3 \in \mathcal{F}_G$. Then $f_2(z_4) \geq g^*(z_4) - \frac{1}{6}n_4(z_4) \geq \frac{1}{6}n_3(z_4)$ because $y_2 \in N_3^+(z_4)$ and $n_3(z_4) \geq 2$, which means $t_{z_4}^2 \geq \frac{1}{6}$. Hence, $f_3(y_2) \geq f_2(y_2) + t_{z_4}^2 \geq \frac{1}{3}$ which means $f_5(y_2) \geq f_3(y_2) - (-f_3(z_2)) \geq \frac{1}{6}$ because $f_3(z_2) = -\frac{1}{6}$. But then $f_6(y_1) \geq f_5(y_1) + \frac{1}{6} \geq \frac{1}{6}$, a contradiction, as claimed. Hence, $G - x$ has a 3-path $P(y_1y_2)$ because $G + xz_2 \in \mathcal{F}_G$ and $G - \alpha_1$ has no 3-path with ends x and y_1. Let $P(y_1y_2) = y_1z_5z_6y_2$. Clearly, $z_5, z_6 \in V_4$ and $f_2(z_6) \geq g^*(z_6) - \frac{1}{6}(n_4(z_6) - 1) \geq \frac{1}{6}n_3(z_6)$ because $y_2 \in N_3^+(z_6)$ and $z_5 \in N_4^{1+}(z_6)$, which means $t_{z_6}^2 \geq \frac{1}{6}$. Similarly, we have $f_3(y_2) \geq f_2(y_2) + t_{z_6}^2 \geq \frac{1}{3}$ and $f_5(y_2) \geq \frac{1}{6}$. But then $f_6(y_1) \geq f_5(y_1) + \frac{1}{6} \geq \frac{1}{6}$, a contradiction.

Assume $a_1 \in V_2$. Then $a_1 = x$, $a_2 = \alpha_1$ and $a_3 \in V_2$ as $g^*(x) < 0$. Since $\alpha \in \mathcal{C}_4$, then $v \in \mathcal{C}_4$ for any $v \in V(G)$ with degree two. Then G has a 5-path with vertices $x, b_1, b_2, b_3, x_2, \alpha_2$ in order as $G + x\alpha_2 \in \mathcal{F}_G$. Since G has no 3-path with ends x and α_1, we see $x_2 \in V_2$. Note that $b_1 \in V_3$ because $g^*(x) < 0$ and $d(\alpha) = 2$. Then $b_1 \in V_3^-$, otherwise $b_1 = y_1$ and we get a contradiction by similar analysis with the case $a_1 \in V_3 \cup V_4$. Hence, $b_2 \in V_4$ and $b_3 \in V_3$. We claim $t_{b_2}^2 \geq \frac{1}{6}$. Suppose not. Then $n_3^+(b_2) = 0$ and $n_3(b_2) + n_4(b_2) = 2$ and $n_3^{-1}(b_2) \leq 1$, otherwise $f_2(b_2) \geq g^*(b_2) \geq \frac{1}{6}n_3(b_2)$ since $n_5^-(b_2) + n_4^{1-}(b_2) = 0$. By the choice of α, $n_5(b_2) \neq 0$. Let $w \in N_5(b_2)$. Note that $g^*(b_2) \geq \frac{1}{6}n_3(b_2) - \frac{1}{3}$. Then $n_4^+(w) \leq 1$ and $d(w) = 2$, otherwise $t_w^* \geq \frac{1}{3}$ as $b_2 \in N_4^+(w)$ which yields $f_2(b_2) \geq g^*(b_2) + t_w^* \geq \frac{1}{6}n_3(b_2)$. Let $b_4 \in N(w) \setminus b_2$. Since $d(w) = 2$, we have w belongs a 4-cycle, which means $G - w$ has a 2-path $P(b_2b_4)$. Let $P(b_2b_4) = b_2b_5b_4$. Note that $b_5 \in V_3 \cup V_5$. In fact, $b_5 \in V_3$, that is, $b_5 \in \{b_1, b_3\}$, otherwise $t_v^* \geq \frac{1}{6}$ for any $v \in \{w, b_5\}$ which implies $f_2(b_2) \geq g^*(b_2) + t_w^* + t_{b_5}^* \geq \frac{1}{6}n_3(b_2)$. W.l.o.g., let $b_5 = b_3$. Then

$b_4 \in V_4^-$ as $n_4^+(w) \le 1$. We further may assume $N(b_2) \cap N(b_4) = \{w, b_5\}$. Then for any $i \in [2]$, G has a 2-path with vertices α_i, α_i', x_2 in order because $G + \alpha_i w \in \mathcal{F}_G$. Note that $\alpha_1' = \alpha_2'$, else G has a copy of C_6 with vertices $x_2, \alpha_1', \alpha_1, \alpha, \alpha_2, \alpha_2'$ in order. This means α belongs to a graph Θ_5. By the choice of α, each vertex of degree two belongs to a graph Θ_5 which yields G has a 2-path with ends b_3 and b_i for some $i \in \{2, 4\}$. But then $b_4 \in V_4^+$ or $n_3(b_2) + n_4(b_2) \ge 3$, a contradiction, as claimed. Thus, $t_{b_2}^2 \ge \frac{1}{6}$. We see $n_4^-(b_1) = 0$, otherwise $G + b_1'\alpha \notin \mathcal{F}_G$ because $g(x) < 0$ and $g(b_1) < 0$ for any $b_1' \in N_4^-(b_1)$. Hence, $f_6(b_1) \ge f_3(b_1) \ge f_2(b_1) + t_{b_2}^2 \ge \frac{1}{6}$ and so $t_{b_1}^6 \ge \frac{1}{6}$, a contradiction. □

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Appendix

Proof of Lemma 5.1 Suppose $n_5^-(z) \neq 0$. Let $w \in N_5^-(z)$. Then $d(w) = 1$. Hence, G has a 4-path with vertices y, z_1, w_2, w_1, z in order since $G + yw \in \mathcal{F}_G$. By Observation 4.3(1), we see that $w_1 \in V_5^1$ as $z \in V_4^{*-}$. That is, $d(w_1) = 2$. Then G has a 4-path with ends z and w_2 containing w_1 because $G + ww_2 \in \mathcal{F}_G$. But this is impossible since $d(w_1) = 2$. So we derive that $n_5^-(z) = 0$. Note that $n_3(z) + n_4(z) = 1$. By Observation 4.3(1), $n_5^1(z) = n_5(z) \leq 1$. Thus, $d(z) = 2$, since otherwise $n_5^1(z) = n_5(z) = d(z) - n_3(z) - n_4(z) \geq 2$. This proves (a).

To prove (b), let $P = zww_1z_1$ be a 3-path of G such that $w \in V_5$. By Observation 4.3, $w, w_1 \in V_5^1$. This implies $z_1 \in V_4$ and $d(w) = d(w_1) = 2$. Note that $g(z_1) < 0$ and $n_5^2(z_1) = 0$, else $g_2(w) = \frac{1}{3}$ which yields $g^*(z) \geq g_3(z) \geq 0$.

Now we shall show that $yz_1 \notin E(G)$. Suppose not. Then G has a 3-path with vertices y, z_2, w_2, z_1 in order because $G + wz_1 \in \mathcal{F}_G$ and $d(z) = 2$. Then $w_2 \in V_5$ and $z_2 \in V_4$ since $g(z_1) < 0$. But then $w_2 \in N_5^2(z_1)$, a contradiction. Hence $yz_1 \notin E(G)$.

It remains to show that $d(z_1) = 2$. Let $y_1 \in N_3(z_1)$. Then $y_1 \neq y$. Suppose $d(z_1) \geq 3$. Let $w_2 \in N(z_1) \setminus \{y_1, w_1\}$. Then, $w_2 \in V_5$ and $d(w_2) \leq 2$ because $g(z_1) < 0$ and $n_5^2(z_1) = 0$. Moreover, $d(w_2) = 2$, otherwise G has a 4-path with ends w and z_1 containing w_1 because $G + ww_2 \in \mathcal{F}_G$ which means $d(w_1) \geq 3$. Let $w_3 \in N(w_2) \setminus z_1$.

Since $n_5^2(z_1) = 0$, we see $w_3 \in V_5$. Note that $n_4(w_3) = 1$, else $n_4(w_3) \geq 2$, which means $g_1(w_2) \geq \frac{1}{3}$, $g_3(z_1) \geq 0$, $g_4(w) \geq \frac{1}{3}$ and $g^*(z) = g_5(z) \geq 0$. Let $N_4(w_3) = \{z_2\}$. We claim that $yz_2 \in E(G)$. Suppose not. Note that G has a 5-path $P(w_1w_3)$ since $G + w_1w_3 \in \mathcal{F}_G$. Let $P(w_1w_3) = w_1a_1a_2a_3a_4w_3$. Then $a_1 \in \{w, z_1\}$. If $a_1 = w$, then $a_2 = z$, $a_3 = y$. And so $a_4 = z_2$ since $n_4(w_3) = 1$, which contradicts to $yz_2 \notin E(G)$. Hence, $a_1 = z_1$. Then G has a 4-path with ends z_1 and w_3 containing w_2. But this is impossible because $d(w_2) = 2$, as claimed. Since $yz_1 \notin E(G)$, we see $z_1 \neq z_2$. Clearly, $g(z_2) < 0$, else $g_2(w_2) \geq \frac{1}{3}$, $g_3(z_1) \geq 0$, $g_4(w) \geq \frac{1}{3}$ and so $g^*(z) = g_5(z) \geq 0$. Note that G has a 5-path $P(w_1w_2)$ because $G + w_1w_2 \in \mathcal{F}_G$. Let $P(w_1w_2) = w_1b_1b_2b_3b_4w_2$. Then $b_1 \in \{w, z_1\}$. If $b_1 = w$, then $b_2 = z$,

$b_3 = y$. And so $b_4 = z_1$ since $d(w_2) = 2$, which contradicts to $yz_1 \notin E(G)$. Thus $b_1 = z_1$ and so $b_4 = w_3$. Since $g(z_1) < 0$, we see that $b_2 \in V_3 \cup V_5$. In fact, $b_2 \in V_5$, else $b_2 = y_1$ and $b_3 = z_2$ as $n_4(w_3) = 1$, which contradicts to $g(z_2) < 0$. Then $b_3 \in V_5$ and $b_3 z_1 \notin E(G)$ because $n_5^2(z_1) = 0$. But then $g_1(w_2) \geq \frac{1}{3}$, $g_3(z_1) \geq 0$, $g_4(w) \geq \frac{1}{3}$ and so $g^*(z) = g_5(z) \geq 0$, a contradiction, as desired. □

Proof of Lemma 5.2: Suppose not. Let $z_1, z_2 \in N_4^{*-}(y)$. We claim that $d(z_i) \geq 2$ for any $i \in [2]$. W.l.o.g., suppose $d(z_1) = 1$. Then $d(z_2) \geq 2$, else $G + z_1 z_2 \notin \mathcal{F}_G$. By Lemma 5.1($a$), $d(z_2) = 2$. Let $w \in N(z_2) \backslash y$. Then G has a 4-path with ends y and w containing z_2 because $G + z_1 w \in \mathcal{F}_G$. But this is impossible since $d(z_2) = 2$, By Lemma 5.1, for any $i \in [2]$, we see $d(z_i) = 2$ and G has a 3-path $P^i = z_i w_i w_{i+2} z_{i+2}$ such that $d(w_i) = d(w_{i+2}) = d(z_{i+2}) = 2$ and $yz_{i+2} \notin E(G)$. This implies $V(P^1) \cap V(P^2) = \emptyset$. But then $G + z_1 z_2 \notin \mathcal{F}_G$, a contradiction. □

Proof of Lemma 5.3: Note that $n_5^-(z) = 1$. Let $N_5^-(z) = \{w\}$ and $y_1 \in N_3(z)$. Clearly, $d(z) \geq 3$, else $G + y_1 w \notin \mathcal{F}_G$.

Firstly, we shall show that $f_2(z) \geq 0$. Suppose $f_2(z) < 0$. By (⊛), we have $f_1(z) < 0$. Clearly, $g^*(z) < \frac{1}{3}$, else $f_1(z) \geq g^*(z) - \frac{1}{3} n_5^-(z) \geq 0$. We next prove several claims.

Claim 1. $n_3(z) = 1$.

Proof. Suppose not. Let $y_2 \in N_3(z) \backslash y_1$. Hence, for any $i \in [2]$, $n_2(y_i) = 1$ and $N_2(y_1) = N_2(y_2)$, else let $x_i \in N_2(y_i)$, G has a copy of C_6 with vertices $z, y_1, x_1, \alpha_1, x_2, y_2$ since $\delta(G) = 1$, where $\alpha_1 \in V_1$. Let $N_2(y_1) = N_2(y_2) = \{x\}$. Then G has a 4-path with ends x and z containing y_1 and y_2 since $G + wx \in \mathcal{F}_G$. Then $G - x - z$ has an s-path with ends y_1 and y_2 for some $s \in [2]$. Hence, $n_3^+(z) \geq 1$ or $n_3^{-1}(z) \geq 2$. But then by Ineq. (3), $g^*(z) \geq \frac{1}{3}$, a contradiction. □

Claim 2. $n_5(z) = 1$.

Proof. Suppose not. We first assert $d(v) = 3$ for any $v \in N_5(z) \setminus w$. If $d(u) = 2$ for some $u \in N_5(z)$, then let $N(u) = \{z, u'\}$, $G + wu' \notin \mathcal{F}_G$ because G has no 4-path with ends z and u' containing u, a contradiction. If $d(u) \geq 4$ for some $u \in N_5(z)$, then $g^*(u) \geq \frac{1}{3} n_4(u)$ as $d(u) = n_4(u) + n_5(u)$, which implies $t_u^* \geq \frac{1}{3}$ and $f_1(z) \geq g^*(z) - \frac{1}{3} n_5^-(z) + t_u^* \geq 0$, a contradiction, as asserted. Hence, $N_5(z) \setminus w = N_5^2(z)$. Note that $n_5(z) = 2$, otherwise let $w_1, w_2 \in N_5(z) \setminus w$, $g^*(w_i) \geq \frac{1}{6} n_4(w_i)$ because $n_4^{-1}(w_i) + n_4^+(w_i) \geq 1$ and $n_4(w_i) + n_5(w_i) = 3$ which yields $t_{w_i}^* \geq \frac{1}{6}$ and $f_1(z) \geq g^*(z) - \frac{1}{3} n_5^-(z) + t_{w_1}^* + t_{w_2}^* \geq 0$. Let $N_5(z) = \{w, w_1\}$. Note that $n_4^+(w_1) = 0$, otherwise $t_{w_1}^* \geq \frac{1}{3}$ and so $f_1(z) \geq 0$. We next prove that

(a) $n_4(w_1) = 1$.

To see why (a) is true, suppose $n_4(w_1) \geq 2$. Let $z_1 \in N_4(w_1) \setminus z$. Then $g(z_1) < 0$ as $n_4^+(w_1) = 0$. Then G has a 4-path with ends z and z_1 containing w_1 because $G + wz_1 \in \mathcal{F}_G$. Since $d(w_1) = 3$, $G - z_1$ has a 3-path $P(zw_1)$ or $G - z$ has a 3-path $P(z_1 w_1)$. We assert that $G - z_1$ has no 3-path $P(zw_1)$. Suppose not. Let $P(zw_1) = za_1 b_1 w_1$. Since $g(z) < 0$ and $n_5(z) = 2$, we see that $a_1 = y_1$ and $b_1 \in V_4 \setminus z_1$. Then G has a 4-path with ends z and b_1 containing y_1 and w_1 because $G + wb_1 \in \mathcal{F}_G$, which implies $G - z - b_1$ has a 2-path between y_1 and w_1. Hence, $y_1 z_1 \in E(G)$ as $d(w_1) = 3$. Since $n_5(z) = 2$, $n_3(z) = 1$ and $g(z) < 0$, G has a 2-path with ends y_1 and v because $G + ww_1 \in \mathcal{F}_G$, where $v \in \{b_1, z_1\}$. But then $g(v) > 0$ and so $n_4^+(w_1) \neq 0$, a contradiction, as asserted. Thus, $G - z$ has a 3-path $P(z_1 w_1) = z_1 a_1 b_1 w_1$. We see that $a_1 \in V_3 \cup V_5$ because $g(z_1) < 0$. Assume first $a_1 \in V_3$. Then $b_1 \in V_4$. Moreover, $n_5^-(z_1) + n_5^-(b_1) = 0$, otherwise let $w_2 \in N_5^-(z_1)$, we see $y_1 = a_1$ because $G + w_2 b_1 \in \mathcal{F}_G$ which yields $G - z_1$ has a 3-path $P(zw_1)$ violating the above assert. But then $f_1(z) \geq g^*(z) - \frac{1}{3} n_5^-(z) + \frac{1}{3} \geq 0$ by Definition 4.6(1.2), a contradiction. Assume now $a_1 \in V_5$. Then $b_1 \in V_4 \cup V_5$. Note that $b_1 \in V_5$, otherwise by Ineq. (3), we see $g^*(w_1) \geq \frac{1}{3} n_4(w_1)$ because $z_1, b_1 \in N_4^{-1}(w_1)$, which implies $t_{w_1}^* \geq \frac{1}{3}$ and $f_1(z) \geq g^*(z) - \frac{1}{3} n_5^-(z) + t_{w_1}^* \geq 0$. We see $d(a_1) \geq 3$, else $f_1(z) \geq g^*(z) - \frac{1}{3}n_5^-(z) + \frac{1}{3} \geq 0$ by Definition 4.6(1.1). By Ineq. (3), $g^*(w_1) \geq \frac{1}{3}n_4(w_1)$ because $z_1 \in N_4^{-1}(w_1)$ and $b_1 \in N_5^2(w_1)$. That is, $t^*_{w_1} \geq \frac{1}{3}$. But then $f_1(z) \geq g^*(z) - \frac{1}{3}n_5^-(z) + t^*_{w_1} \geq 0$, a contradiction. This proves (a).

By (a), $N(w_1) \setminus z = N_5(w_1)$. Let $w_2 \in N(w_1) \setminus z$. Then G has a 4-path with ends z and w_2 containing w_1 because $G + ww_2 \in \mathcal{F}_G$. Since $g(z) < 0$, $n_4(w_1) = 1$ and $n_5(z) = 2$, we see $G - w_2$ has no s-path with ends z and w_1 for any $s \in \{2, 3\}$. Hence, $G - z$ has a 3-path $P(w_1w_2)$. Let $P(w_1w_2) = w_1a_2b_2w_2$. Then $a_2 \in V_5$. Note that $\{w_2, a_2\} \cap N_5^1(w_1) \neq \emptyset$, otherwise $g^*(w_1) \geq \frac{1}{3}n_4(w_1)$ because $w_2, a_2 \in N_5^2(w_1)$ which yields $f_1(z) \geq 0$. Hence, $b_2 \in V_4$. By Definition 4.2(1.1), we see $g^*(w_1) = \frac{1}{3}$. But then $f_1(z) \geq 0$, a contradiction. □

By Claims 1 and 2, $N(z) \setminus \{y_1, w\} = N_4(z) \neq \emptyset$ since $d(z) \geq 3$. Let $z_1 \in N_4(z)$ and $y_2 \in N_3(z_1)$. We assert $G - y_2$ has no 2-path with ends z and z_1. Suppose not. Let $P(zz_1) = za_3z_1$. Then $a_3 \in V_3 \cup V_4$. Clearly, $a_3 \in V_3$, otherwise $a_3 \in V_4$, $\{z_1, a_3\} \subseteq N_4^2(z)$ yields $g^*(z) \geq \frac{1}{3}$. Then $a_3 = y_1$. Hence, $z_1 \in N_4^2(z) \cap N_4^2(y_1)$. But then $g^*(z) \geq \frac{1}{3}$, a contradiction, as asserted. Then G has a 4-path with ends z and y_2 containing z_1 because $G + wy_2 \in \mathcal{F}_G$. Hence, $G - y_2$ has a 3-path $P(zz_1)$ or $G - z$ has a 3-path $P(y_2z_1)$.

Claim 3. $G - z$ has no 3-path $P(y_2z_1)$.

Proof. Suppose not. Let $P(y_2z_1) = y_2a_4b_4z_1$. We assert $g(y_1) < 0$. Suppose $g(y_1) \geq 0$. Then $g(z) = g(z_1) = \frac{1}{6}$, otherwise $g^*(z) \geq \frac{1}{3}$. Hence, $d(z) = 3$, $b_4 \in V_5$ and $a_4 \in V_4$. Then $y_1y_2 \notin E(G)$(if $y_1 \neq y_2$), else G has a copy of C_6 with vertices $y_1, z, z_1, b_4, a_4, y_2$ in order. Note that $n_5^-(z_1) = 0$, otherwise let $w_1 \in N_5^-(z_1)$, we see $G - z$ has a 2-path with ends y_1 and z_1 because $G + ww_1 \in \mathcal{F}_G$ and $d(z) = 3$ which yields $g(z_1) \geq \frac{2}{3}$. By Observation 4.5(2), $g^*(b_4) \geq \frac{1}{6}n_4(b_4)$ because $z_1 \in N_4^+(b_4)$ and $n_4(b_4) \geq 2$. Hence, $t^*_{b_4} \geq \frac{1}{6}$ and $f_1(z_1) \geq g^*(z_1) + t^*_{b_4} \geq \frac{1}{6}$. Note that $g^*(z) = \frac{1}{6}$ and $f_1(z) = g^*(z) - \frac{1}{3} = -\frac{1}{6}$. But then $f_2(z) \geq f_1(z) + \frac{1}{6} \geq 0$, a contradiction, as asserted. Let $x_1 \in N_2(y_1)$. Then G has a 4-path with ends z and x_1 containing y_1 because $G + wx_1 \in \mathcal{F}_G$. Since $g(y_1) < 0$, we see $G - z$ has a 3-path $P(x_1y_1)$ or $G - x_1$ has a 3-path $P(y_1z)$. We next prove that

(b) $G - z$ has no 3-path $P(x_1y_1)$.

To see why (b) is true, suppose $P(x_1y_1)$ exists and let $P(x_1y_1) = x_1c_4d_4y_1$. Then $d_4 \in V_4$ and $c_4 \in V_3$ as $g(y_1) < 0$. Hence, $d_4 \in N_4^2(y_1)$, which means $g^*(z) \geq \frac{1}{6}$. Hence, $f_1(z) \geq g^*(z) - \frac{1}{3}n_5^-(z) \geq -\frac{1}{6}$. Note that $g(z_1) = \frac{1}{6}$, otherwise $g(z_1) \geq \frac{2}{3}$ yields $g^*(z) \geq \frac{1}{3}$. Then $b_4 \in V_5$ and $a_4 \in V_4$. Similarly, we have $t^*_{b_4} \geq \frac{1}{6}$ and $y_1y_2 \notin E(G)$(if $y_1 \neq y_2$). Then $n_5^-(z_1) \neq 0$, otherwise $f_1(z_1) \geq g^*(z_1) + t^*_{b_4} \geq \frac{1}{6}$ yields $f_2(z) \geq f_1(z) + \frac{1}{6} \geq 0$. Let $w_1 \in N_5^-(z_1)$. Then $G - z_1 - a_4$ has a 2-path $P(y_2b_4) = y_2z_2b_4$ and G has a 3-path $P(zz_1) = zz'z_1'z_1$ because $\{G + w_1a_4, G + ww_1\} \in \mathcal{F}_G$. Then $z' \in V_4$ because $n_3(z) = n_5(z) = 1$, $g(z_1) = \frac{1}{6}$ and $y_1y_2 \notin E(G)$(if $y_1 \neq y_2$). Moreover, $z' \notin \{z_2, a_4\}$, else let $z' = z_2$, we see G has a copy of $C_6 = b_4z_1zz'y_2a_4$. Clearly, $z_1' \in V_5$ because $g(z_1) = \frac{1}{6}$. Then $z_1' \neq b_4$, else G has a copy of $C_6 = zz'z_1'z_2y_2z_1$. By Observation 4.4(1), $g^*(z_1') \geq \frac{1}{3}n_4(z_1')$ because $z', z_1 \in N_4^+(z_1')$. Then $t^*_{z_1'} \geq \frac{1}{3}$. Hence, $f_1(z_1) \geq g^*(z_1) - \frac{1}{3}n_5^-(z_1) + t^*_{z_1'} + t^*_{b_4} \geq \frac{1}{6}$. But then $f_2(z) \geq f_1(z) + \frac{1}{6} \geq 0$, a contradiction. This proves (b).

By (b), $G - x_1$ has a 3-path $P(y_1z)$. Let $P(y_1z) = y_1c_4d_4z$. Then $c_4, d_4 \in V_4$ because $g(y_1) < 0$. We assert $d_4 = z_1$. Suppose not. Note that $d_4 \in N_4^2(z)$. Then $g(z_1) = \frac{1}{6}$, $N_4^2(y_1) = \{z\}$ and $d(z) = 4$, otherwise $n_3^{-1}(z) + n_4^2(z) \geq 2$ or $n_3(z) + n_4(z) = d(z) - 1 \geq 4$ which yields $g^*(z) \geq \frac{1}{3}$. This means $b_4 \in V_5$ and $a_4 \in V_4$. Similarly, $t^*_{b_4} \geq \frac{1}{6}$. Note that $n_5^-(z_1) = 0$, otherwise let $w_1 \in N_5^-(z_1)$, we see G has a 3-path $P(zz_1) = zz'z_1'z_1$ such that $z' = d_4$ and $z_1' \notin \{y_1, c_4, z\}$ because $G + ww_1 \in \mathcal{F}_G$, which yields G has a copy of $C_6 = y_1c_4d_4z_1'z_1z$. Hence, $f_1(z_1) \geq g^*(z_1) + t^*_{b_4} \geq \frac{1}{6}$. Note that $g^*(z) \geq \frac{1}{6}$ and so $f_1(z) \geq g^*(z) - \frac{1}{3} \geq -\frac{1}{6}$. But then $f_2(z) \geq f_1(z) + \frac{1}{6} \geq 0$, a contradiction, as asserted. Note that $z_1 \in N_4^2(z)$. Moreover, $N_4^2(y_1) \setminus z = \emptyset$, else $g^*(z) \geq \frac{1}{3}$ because $z_1 \in N_4^2(z)$. Hence, $y_1z_1 \notin E(G)$. Then $G - z - c_4$ has a 2-path $P(y_1z_1) = y_1y_1'z_1$ because $G + wc_4 \in \mathcal{F}_G$. We see $c_4, y_1' \in N_4^1(y_1)$ because $g(y_1) < 0$.

Note that $n_5^-(z_1) = 0$, otherwise let $w_1 \in N_5^-(z_1)$, we see G has a 4-path with ends y_1 and z_1 containing each of $\{z, c_4, y_1'\}$ because $G + y_1w_1 \in \mathcal{F}_G$, which yields $zv \in E(G)$ for some $v \in \{c_4, y_1'\}$ and so $v \in N_4^2(y_1)$. If $b_4 \in V_5$, then $t_{b_4}^* \geq \frac{1}{6}$. By Ineq. (3), $f_1(z_1) \geq g^*(z_1) + t_{b_4}^* \geq \frac{1}{6}n_4(z_1)$ because $n_4(z_1) \geq 3$. If $b_4 \in V_3 \cup V_4$, then $b_4 \notin \{y_1', c_4\}$ because $y_1', c_4 \in V_4^1$ and $y_1y_2 \notin E(G)$. Note that $b_4 \neq z$. Thus $n_4(z_1) \geq 4$ and by Ineq. (3), $f_1(z_1) \geq g^*(z_1) \geq \frac{1}{6}n_4(z_1)$. Since $n_4^2(z) \geq 1$ and $n_4^1(y_1) \geq 2$, we have $g^*(z) \geq \frac{1}{6}$ and so $f_1(z) \geq g^*(z) - \frac{1}{3} \geq -\frac{1}{6}$. But then $f_2(z) \geq f_1(z) + \frac{1}{6} \geq 0$, a contradiction. $\square$

By Claim 3, $G - y_2$ has a 3-path $P(zz_1)$. Let $P(zz_1) = za_5b_5z_1$. Then $a_5 \in V_3 \cup V_4$. We next prove that

(c) $a_5 \in V_4$.

To prove (c), suppose $a_5 \in V_3$. Then $a_5 = y_1$ because $n_3(z) = 1$. Moreover, $z_1 \in N_4^2(z)$ as $b_5 \neq y_2$. Hence, $y_1z_1 \notin E(G)$, $g(y_1) < 0$ and $N_4^2(y_1) \setminus z = \emptyset$, else $g^*(z) \geq \frac{1}{3}$. Then $G - z - b_5$ has a 2-path $P(y_1z_1)$ because $G + wb_5 \in \mathcal{F}_G$. Let $P(y_1z_1) = y_1c_5z_1$. We see $b_5, c_5 \in V_4^1$. It is easy to see $g^*(z) \geq \frac{1}{6}$ and $f_1(z) \geq g^*(z) - \frac{1}{3}n_5^-(z) \geq -\frac{1}{6}$. Then $n_5^-(z_1) = 0$, otherwise let $w_1 \in N_5^-(z_1)$, we see G has a 4-path with ends y_1 and z_1 containing each of $\{z, b_5, c_5\}$ because $G + y_1w_1 \in \mathcal{F}_G$, which yields $zv \in E(G)$ for some $v \in \{b_5, c_5\}$ and so $v \in N_4^2(y_1)$. Note that $n_4^2(z_1) = 0$, else $f_1(z_1) \geq g^*(z_1) \geq \frac{1}{6}n_4(z_1)$ because $n_3(z_1) + n_4(z_1) \geq 4$ and $n_4^2(z_1) \geq 1$, which implies $f_2(z) \geq f_1(z) + \frac{1}{6} \geq 0$. This follows $d(z) = 3$. Moreover, $n_5^-(b_5) \neq 0$ or $n_5^-(c_5) \neq 0$, else $f_1(z_1) \geq g^*(z_1) \geq \frac{1}{6}(n_4(z_1) - 1) \geq \frac{1}{6}n_4^{1-}(z_1)$ because $n_3(z_1) + n_4(z_1) \geq 4$ and $b_5 \in N_4^{1+}(z_1)$, which means $f_2(z) \geq f_1(z) + \frac{1}{6} \geq 0$. W.l.og, let $w_2 \in N_5^-(b_5)$. Then G has a 3-path $P(zb_5)$ because $G + ww_2 \in \mathcal{F}_G$. Let $P(zb_5) = zz'b_5'b_5$. We see $z' = z_1$, else $z' = y_1$ because $d(z) = 3$, which means $b_5 \in N_4^2(z_1)$ or $y_1z_1 \in E(G)$. Moreover, $b_5' \in V_5$. By Observation 4.4(1), we have $t_{b_5'}^* \geq \frac{1}{3}$ because $z_1, b_5 \in N_4^+(b_5')$. This implies $b_5 \in N_4^{1+}(z_1)$. Similarly, we have $f_1(z_1) \geq \frac{1}{6}n_4^{1-}(z_1)$ and so $f_2(z) \geq 0$, a contradiction. This proves (c).

By (c), $a_5 \in V_4$. Then $\{z_1, a_5\} \cap N_4^1(z) \neq \emptyset$, else $g^*(z) \geq \frac{1}{3}$. W.l.o.g., let $z_1 \in N_4^1(z)$. Then $b_5 \in V_5$ because $b_5 \neq y_2$. Note that $z_1a_5 \notin E(G)$. Hence, $G - z - b_5$ has a 2-path $P(z_1a_5) = z_1d_5a_5$ because $G + wb_5 \in \mathcal{F}_G$. Clearly, $n_5^-(z_1) = 0$, otherwise let $w_1 \in N_5^-(z_1)$ we see G has a 4-path with ends z_1 and a_5 containing each of $\{z, b_5, d_5\}$ because $G + w_1a_5 \in \mathcal{F}_G$, which implies $d_5 \in V_4$ and so $z_1 \notin N_4^1(z)$. By Observation 4.4(1), $g^*(b_5) \geq \frac{1}{3}n_4(b_5)$ and so $t_{b_5}^* \geq \frac{1}{3}$ because $z_1, a_5 \in N_4^+(b_5)$, which yields $f_1(z_1) \geq g^*(z_1) + t_{b_5}^* \geq \frac{1}{3}$. If $a_5 \notin N_4^1(z)$, then $g^*(z) \geq \frac{1}{6}$, which yields $f_2(z) \geq f_1(z) + \frac{1}{6} \geq 0$, a contradiction. If $a_5 \in N_4^1(z)$, then we have $f_1(a_5) \geq \frac{1}{3}$ by similar analysis with z_1, which implies $f_2(z) \geq f_1(z) + \frac{1}{6} \times 2 \geq 0$, a contradiction. This completes the proof of $f_2(z) \geq 0$.

Finally, we shall show that $f_3(y) \geq 0$ for any $y \in N_3(z)$. Suppose not. Let $f_3(y_1) < 0$ for some $y_1 \in N_3(z)$. By ($\circledast$), $g^*(y_1) < 0$. By Observation 4.3(1), $n_4^2(y_1) = 0$ and $n_4^1(y_1) \leq 1$. Let $x_1 \in N_2(y_1)$. Then G has a 4-path with ends x_1 and z containing y_1 because $G + wx_1 \in \mathcal{F}_G$. This implies that G has a 3-path $P(y_1z) = y_1z_1w_1z$ such that $w_1 \in V_5$ and $z_1 \in V_4$ because $g^*(y_1) < 0$. Since $G + wz_1 \in \mathcal{F}_G$, we have $G - z - z_1$ has a 2-path $P(y_1w_1) = y_1z_2w_1$. Clearly, $g(z_i) < 0$ for some $i \in [2]$. W.l.o.g., let $g(z_1) < 0$. We claim that $n_5^-(z_1) + n_5^-(z_2) \neq 0$. Suppose not. We say $g(v) < 0$ for any $v \in \{z, z_2\}$, otherwise by Observation 4.4(2), $t_{w_1}^* \geq \frac{1}{3}$ because $n_4^+(w_1) \geq 1$ and $n_4(w_1) \geq 3$, which implies $f_2(z_1) \geq f_1(z_1) \geq t_{w_1}^*$, that is, $t_{z_1}^2 \geq \frac{1}{3}$, and so $f_3(y_1) \geq f_2(y_1) + t_{z_1}^2 \geq 0$. By Observation 4.5(2), $t_{w_1}^* \geq \frac{1}{6}$ because $n_4(w_1) \geq 3$. Hence, $f_2(z_i) \geq f_1(z_i) \geq t_{w_1}^* \geq \frac{1}{6}$ for any $i \in [2]$. That is, $t_{z_i}^2 \geq \frac{1}{6}$. But then $f_3(y_1) \geq f_2(y_1) + t_{z_1}^2 + t_{z_2}^2 \geq 0$, a contradiction, as claimed. Thus, $n_5^-(z_1) + n_5^-(z_2) \neq 0$. W.l.o.g., let $w_2 \in N_5^-(z_2)$. Then G has a 3-path $P(zz_2) = za_6b_6z_2$ because $G + ww_2 \in \mathcal{F}_G$. We claim that $a_6, b_6 \in V_5$. Suppose not. W.l.o.g., let $a_6 \notin V_5$. Then $a_6 \in V_4$ and $b_6 \in V_5$ because $n_4^2(y_1) = 0$ and $n_4^1(y_1) \leq 1$. Clearly, $f_2(y_1) \geq g^*(y_1) \geq -\frac{1}{6}$. Moreover, $g(z_2) < 0$. We see $b_6 \neq w_1$, otherwise $t_{w_1}^* \geq \frac{1}{2}$ because $n_4^+(w_1) \geq 2$ and $n_4(w_1) \geq 4$, which implies $t_{z_2}^2 \geq \frac{1}{6}$ and $f_3(y_1) \geq f_2(y_1) + t_{z_2}^2 \geq 0$. By Observations 4.4(2) and 4.5(2), $t_{w_1}^* \geq \frac{1}{3}$ because $n_4(w_1) \geq 3$ and $z \in N_4^+(w_1)$, and $t_{b_6}^* \geq \frac{1}{6}$ because $n_4(b_6) \geq 2$ and $a_6 \in N_5^+(b_6)$. Hence, $f_2(z_2) \geq f_1(z_2) \geq g^*(z_2) - \frac{1}{3}n_5^-(z_2) + t_{w_1}^* + t_{b_6}^* \geq \frac{1}{6}$, that is, $t^2_{z_2} \geq \frac{1}{6}$. But then $f_3(y_1) \geq f_2(y_1) + t^2_{z_2} \geq 0$, a contradiction, as claimed. Thus, $a_6, b_6 \in V_5$. Obviously, $d(a_6) \geq 3$ and $d(b_6) \geq 3$, else $G + wb_6 \notin \mathcal{F}_G$ or $G + w_2a_6 \notin \mathcal{F}_G$. Hence, $n_4(v) + n_5(v) = d(v) \geq 3$ for any $v \in \{a_6, b_6\}$. By Ineq. (3), $t^*_v \geq \frac{1}{3}$ because $n^2_5(v) \geq 1$ and $n^+_4(v) + n^{-1}_4(v) + d(v) \geq 4$. We assert $g(u) < 0$ for any $u \in \{z, z_2\}$. Suppose not. Then $f_2(y_1) \geq g^*(y_1) \geq -\frac{1}{6}$. Similarly, $t^*_{w_1} \geq \frac{1}{3}$. W.l.o.g., we assume $w_1 \neq a_6$. Then $f_1(z) \geq g^*(z) - \frac{1}{3}n^-_5(z) + t^*_{a_6} + t^*_{w_1} \geq \frac{1}{3}$, which means $f_2(z) \geq \frac{1}{6}$ because $g(z) \leq \frac{1}{6}$. That is, $t^2_z \geq \frac{1}{6}$. But then $f_3(y_1) \geq f_2(y_1) + t^2_z \geq 0$, a contradiction, as asserted. If $w_1 \in \{a_6, b_6\}$, say $w_1 = b_6$, then $t^*_{w_1} \geq \frac{1}{3}$ because $n_4(w_1) + n_5(w_1) \geq 4$. Hence, $f_2(z) \geq f_1(z) \geq g^*(z) - \frac{1}{3}n^-_5(z) + t^*_{w_1} + t^*_{a_6} \geq \frac{1}{3}$. That is, $t^2_z \geq \frac{1}{3}$. But then $f_3(y_1) \geq f_2(y_1) + t^2_z \geq 0$, a contradiction. If $w_1 \notin \{a_6, b_6\}$, then $t^*_{w_1} \geq \frac{1}{6}$ because $n_4(w_1) \geq 3$. Hence, $f_2(u) \geq f_1(u) \geq g^*(u) - \frac{1}{3}n^-_5(u) + t^*_{u'} + t^*_{w_1} \geq \frac{1}{6}$ for any $u \in \{z, z_2\}$, where $u' \in \{a_6, b_6\} \cap N(u)$. That is, $t^2_u \geq \frac{1}{6}$. But then $f_3(y_1) \geq f_2(y_1) + t^2_z + t^2_{z_2} \geq 0$, a contradiction. Thus, $f_3(y) \geq 0$ for any $y \in N_3(z)$. □

Proof of Lemma 5.5: Let $w \in N^-_5(z_1)$ and $z_1 \in N_4(y)$. Note that $d(w) = 1$. By Lemma 5.3 and (⊛), $f_5(y) \geq 0$ and $f_2(z_1) \geq 0$. To establish the desired result, suppose $f_5(z) < 0$ for some $z \in N_4(y)\backslash z_1$. Then $g^*(y) \leq f_3(y) < \frac{1}{3}$. By (⊛), $f_i(z) < 0$ for any $i \in [5]$. By Corollary 5.4, $g^*(z) < 0$. By Lemma 5.1(a, b), $d(z) \leq 2$ and G has a 3-path $P = zw_1w_2z_2$ such that $d(u) = 2$ for each $u \in V(P)$ and $yz_2 \notin E(G)$ when $d(z) = 2$, where $w_1, w_2 \in V_5$, $z_2 \in V_4$. When $d(z) = 2$, let $N_3(z_2) = \{y_2\}$. We first prove that

(a) for any $v \in N^2_4(y)$, $n^+_3(v) \leq 1$, or $n^+_3(v) = 2$ and $n^2_4(v) = 0$, or $n^+_3(v) = 2$ and $N^1_4(v) \cap N_4(N_3(v)) = \emptyset$.

To see why (a) is true, suppose first $n^+_3(v) \geq 3$, or $n^+_3(v) = 2$ and $n^2_4(v) \geq 1$. By Ineq. (3), $g^*(v) \geq \frac{1}{3}n_3(v) + \frac{1}{6}n_4(v) + \frac{1}{3}$. Thus, $f_2(v) \geq \frac{1}{3}n_3(v)$. Suppose $n^+_3(v) = 2$ and $N^1_4(v) \cap N_4(N_3(v)) \neq \emptyset$. Let $y_1 \in N_3(v)$ and $v_1 \in N^1_4(v) \cap N_4(y_1)$. By Ineq. (3), $g^*(v) \geq \frac{1}{3}n_3(v) + \frac{1}{6}n_4(v) + \frac{1}{6}$ and $g^*(v_1) = \frac{1}{3}$, which means $f_1(v_1) \geq 0$ and so $n^{1-}_4(v) \leq n_4(v) - 1$. Hence, $f_2(v) \geq f_1(v) - \frac{1}{6}n^{1-}_4(v) \geq \frac{1}{3}n_3(v)$. In both cases, we see $t^2_v \geq \frac{1}{3}$. But then $f_3(y) \geq f_2(y) + t^2_v \geq \frac{1}{3}$, a contradiction. This proves (a).

Then G has a 3-path $P(yz_1)$ because $G + zw \in \mathcal{F}_G$ and $g^*(z) < 0$. Let $P(yz_1) = ya_1b_1z_1$. We then prove that

(b) $a_1z_1 \notin E(G)$.

To prove (b), suppose $a_1z_1 \in E(G)$. We claim that $b_1 \in V_5$. Suppose not. Then $b_1 \in V_3 \cup V_4$. Note that $b_1 \in V_4$, otherwise by (a) and Ineq. (3), we see $a_1 \in V_4$ and so $g^*(v) \geq \frac{1}{6}n_3(v) + \frac{1}{6}n_4(v) + \frac{1}{3}$ for any $v \in \{z_1, a_1\}$ because $n^{-1}_3(v) + n^+_3(v) \geq 2$ and $n^2_4(v) \geq 1$, which means $t^2_v \geq \frac{1}{6}$ and so $f_3(y) \geq f_2(y) + t^2_{z_1} + t^2_{a_1} \geq \frac{1}{3}$. By (a), $a_1 \in V_4$. Hence, $\{a_1, z_1, b_1\} \subseteq V^2_4$, which means that $g^*(v') \geq \frac{1}{3}$ and $f_1(v') \geq 0$ for any $v' \in \{a_1, z_1, b_1\}$. By Observation 4.5(2,3,5), $g^*(v) \geq \frac{1}{6}n_3(v) + \frac{1}{6}n_4(v)$ for any $v \in \{z_1, a_1\}$ because $n^{-1}_3(v) + n^+_3(v) \geq 1$ and $n^2_4(v) \geq 2$. This implies $t^2_v \geq \frac{1}{6}$ as $n^{1+}_4(v) \geq 2$. But then $f_3(y) \geq f_2(y) + t^2_{z_1} + t^2_{a_1} \geq \frac{1}{3}$, a contradiction, as claimed. Then $G - z_1 - a_1$ has a 2-path $P(yb_1)$ because $G + a_1w \in \mathcal{F}_G$. Let $P(yb_1) = yc_1b_1$. Then G has a 4-path with ends y and b_1 containing each of $\{z_1, c_1, a_1\}$ since $G + zb_1 \in \mathcal{F}_G$, which means $c_1a_1 \in E(G)$ or $c_1z_1 \in E(G)$. W.l.o.g., let $c_1a_1 \in E(G)$. Then $a_1 \in V^2_4$. Note that $g^*(b_1) \geq \frac{1}{2}n_4(b_1)$ as $n^+_4(b_1) \geq 3$, which means $t^*_{b_1} \geq \frac{1}{2}$. Then $v \in V^2_4$ for any $v \in \{z_1, c_1\}$, otherwise let $z_1 \in V^1_4$, we see $g^*(z_1) = \frac{1}{3}$ and so $f_2(z_1) \geq g^*(z_1) - \frac{1}{3} - \frac{1}{6} + t^*_{b_1} \geq \frac{1}{3}$, which yields $t^2_{z_1} \geq \frac{1}{3}$ and $f_3(y) \geq f_2(y) + t^2_{z_1} \geq \frac{1}{3}$. Hence, for any $v \in \{a_1, z_1, c_1\}$, $g^*(v) \geq \frac{1}{6}n_3(v) + \frac{1}{6}(n_4(v) - 1)$ because $n^+_3(v) + n^{-1}_3(v) \geq 1$ and $n^2_4(v) \geq 1$, which implies $t^2_v \geq \frac{1}{6}$ because $f_1(v) \geq g^*(v) - \frac{1}{3} + t^*_{b_1}$. But then $f_3(y) \geq f_2(y) + t^2_{z_1} + t^2_{a_1} + t^2_{c_1} \geq \frac{1}{2}$, a contradiction. This proves (b).

(c) $yb_1 \notin E(G)$.

To see why (c) is true, suppose $yb_1 \in E(G)$. Then $b_1 \in V_3 \cup V_4$. We assert $b_1 \in V_4$. Suppose not. Then $b_1 \in V_3$. Then $a_1 \notin V_4$, otherwise $g^*(v) \geq \frac{1}{6}n_3(v) + \frac{1}{6}n_4(v) + \frac{1}{3}$ for any $v \in \{z_1, a_1\}$ because $n^+_3(v) \geq 2$, which implies $t^2_v \geq \frac{1}{6}$ and so $f_3(y) \geq f_2(y) + t^2_{z_1} + t^2_{a_1} \geq \frac{1}{3}$. Since $g^*(y) < \frac{1}{3}$, we see $a_1 \notin V_3$. Hence, $a_1 \in V_2$. Note that $y_2b_1 \notin E(G)$ when

$d(z)=2$, otherwise $g^*(y)\geq\frac{1}{3}$. Then $G-y-b_1$ has a 2-path with ends a_1 and z_1 because $G+zb_1\in\mathcal{F}_G$. This implies $n_3(z_1)\geq 3$. By Ineq. (3), $g^*(z_1)\geq\frac{1}{3}n_3(z_1)+\frac{1}{6}n_4(v)+\frac{1}{3}$, which yields $t^2_{z_1}\geq\frac{1}{3}$. But then $f_3(y)\geq f_2(y)+t^2_{z_1}\geq 0$, a contradiction, as asserted. By (a), $a_1\in V_4$. We assert $y_2b_1\notin E(G)$. Suppose not. Then $d(z)=2$ and $g^*(z)\geq-\frac{1}{6}$. Hence, $f_3(y)<\frac{1}{6}$, otherwise $f_4(y)\geq 0$ and $f_4(z)\geq 0$. By Observation 4.5(1), $g^*(b_1)\geq\frac{1}{6}n_3(b_1)+\frac{1}{6}n_4(b_1)$ because $n_3(b_1)\geq 2$ and $n_3(b_1)+n_4(b_1)\geq 4$. Clearly, $g^*(v)\geq\frac{1}{3}$ for any $v\in\{z_1,a_1\}$ because $b_1\in V_4^2$, which implies $\{z_1,a_1\}\subseteq N_4^{1+}(b_1)$. Thus $t^2_{b_1}\geq\frac{1}{6}$ as $n_4^{1-}(b_1)\leq n_4(b_1)-2$, which yields that $f_3(y)\geq f_2(y)+t^2_{b_1}\geq\frac{1}{6}$, a contradiction, as asserted. By (b), $G-y-b_1$ has a 2-path $P(z_1a_1)$ because $G+zb_1\in\mathcal{F}_G$. Let $P(z_1a_1)=z_1c_1a_1$. We assert $c_1\in V_5$. Suppose not. Then $c_1\in V_4$, otherwise $g^*(v)\geq\frac{1}{6}n_3(v)+\frac{1}{6}(n_4(v)+2)$ because $n_3^{-1}(v)+n_3^{+}(v)\geq 2$ and $n_4^2(v)\geq 1$ for any $v\in\{z_1,a_1\}$, which means that $t^2_v\geq\frac{1}{6}$ and so $f_3(y)\geq f_2(y)+t^2_{z_1}+t^2_{a_1}\geq\frac{1}{3}$. By Observation 4.5(2,3,5), $g^*(v)\geq\frac{1}{6}n_3(v)+\frac{1}{6}n_4(v)$ for any $v\in\{z_1,a_1,b_1\}$ and $g^*(c_1)\geq\frac{1}{3}$ because $n_4^2(c_1)\geq 2$, which implies $n_4^{1+}(v)\geq 2$. Hence, $t^2_v\geq\frac{1}{6}$ because $n_4^{1-}(v)\leq n_4(v)-2$ for any $v\in\{z_1,a_1,b_1\}$. But then $f_3(y)\geq f_2(y)+t^2_{z_1}+t^2_{a_1}+t^2_{b_1}\geq\frac{1}{2}$, a contradiction, as asserted. By Observation 4.4(1), $g^*(c_1)\geq\frac{1}{3}n_4(c_1)+\frac{1}{6}n_5(c_1)$ because $n_4^+(c_1)\geq 2$, which means $t^*_{c_1}\geq\frac{1}{3}$. We assert $t^2_v\geq\frac{1}{6}$ for any $v\in\{z_1,a_1\}$. W.l.o.g., suppose $t^2_{z_1}<\frac{1}{6}$. Then $z_1\in V_4^2$, otherwise $g^*(z_1)\geq\frac{1}{3}$ and so $f_1(z_1)\geq g^*(z_1)-\frac{1}{3}+t^*_{c_1}\geq\frac{1}{3}$, which means $t^2_{z_1}\geq\frac{1}{6}$. By Ineq.(3), $g^*(z_1)\geq\frac{1}{6}n_3(z_1)+\frac{1}{6}(n_4(z_1)-1)\geq\frac{1}{3}$ because $n_3^+(z_1)+n_3^{-1}(z_1)\geq 1$ and $b_1\in N_4^2(z_1)$. Similarly, $g^*(b_1)\geq\frac{1}{3}$, which means $b_1\in N_4^{1+}(z_1)$ and $n_4^{1-}(z_1)\leq n_4(z_1)-1$. But then $f_1(z_1)\geq g^*(z_1)-\frac{1}{3}n_5^-(z_1)+t^*_{c_1}\geq g^*(z_1)$, which yields that $t^2_{z_1}\geq\frac{1}{6}$, a contradiction, as asserted. Hence, $f_3(y)\geq f_2(y)+t^2_{z_1}+t^2_{a_1}\geq\frac{1}{3}$, a contradiction. This proves (c).

By (b) and (c), we may assume any 4-cycle containing the edge yz_1 is an induced cycle in $G-z$. By (c), $G-z_1-a_1$ has a 2-path $P(yb_1)$ since $G+wa_1\in\mathcal{F}_G$. Let $P(yb_1)=yc_1b_1$. We assert $y_2b_1\notin E(G)$. Suppose not. Then $d(z)=2$ and $b_1\in V_3\cup V_4$. Hence, $f_3(y)<\frac{1}{6}$ as $f_3(z)\geq g^*(z)\geq-\frac{1}{6}$. If $b_1\in V_3$, then $g(y)<0$, otherwise by Observation 4.4(1), $g^*(z_1)\geq\frac{1}{3}n_3(z_1)+\frac{1}{6}n_4(z_1)\geq\frac{1}{6}n_3(z_1)+\frac{1}{6}n_4(z_1)+\frac{1}{3}$ because $\{y,b_1\}\subseteq N_3^+(z_1)$, which means that $t^2_{z_1}\geq\frac{1}{6}$ and so $f_3(y)\geq f_2(y)+t^2_{z_1}\geq\frac{1}{6}$. Then $\{a_1,c_1\}\cap V_4\neq\emptyset$. W.l.o.g., let $c_1\in V_4$. By Ineq. (3), for $v\in\{z_1,c_1\}$, $g^*(v)\geq\frac{1}{12}n_3(v)+\frac{1}{6}(n_4(v)+2)$ because $b_1\in N_3^+(v)$ and $y\in N_4^{-1}(v)$, which yields $t^2_v\geq\frac{1}{12}$. But then $f_3(y)\geq f_2(y)+t^2_{z_1}+t^2_{c_1}\geq\frac{1}{6}$, a contradiction. If $b_1\in V_4$, then $y_2z_1\notin E(G)$, otherwise by Ineq. (3), $b_1\in N_4^{1+}(z_1)$ and $g^*(z_1)\geq\frac{1}{6}n_3(z_1)+\frac{1}{6}(n_4(z_1)-1)+\frac{1}{3}$ because $n_3(z_1)\geq 2$, $n_3^{-1}(z_1)+n_3^+(z_1)\geq 1$ and $b_1\in N_4^2(z_1)$, which implies $t^2_{z_1}\geq\frac{1}{6}$ and so $f_3(y)\geq\frac{1}{6}$.

Hence, G has 2-path $P(yz_1)$ and $P(yy_2)$ because $\{G+ww_1,G+w_2z\}\subseteq\mathcal{F}_G$ and $y_2z_1\notin E(G)$. Let $P(yz_1)=yz_{11}z_1$ and $P(yy_2)=yy_1y_2$. By (c), $z_{11}\neq b_1$. Then $y_1\in\{z_1,z_{11}\}$, else there is a $C_6=yy_1y_2b_1z_1z_{11}$ in G. But then G has a copy of $C_6=yz_{11}z_1y_2b_1a_1$ when $y_1=z_1$, or $C_6=yz_1z_{11}y_2b_1a_1$ when $y_1=z_{11}$, a contradiction, as asserted. Because $G+zb_1\in\mathcal{F}_G$ and $y_2b_1\notin E(G)$, we see there are at least two edges in $G[\{z_1,a_1,c_1\}]$. By (b), $z_1c_1\in E(G)$. But this is impossible because 4-cycle $C_4=yc_1b_1z_1$ is not an induced cycle in G, a contradiction. This completes the proof of Lemma 5.5. □

Proof of Lemma 5.6: By Corollary 5.4, $N_4^{5-}(y_i)\subseteq N_4^{*-}(y_i)$ and $n_4^{5-}(y_i)\leq 1$ for any $i\in[3]$. We proceed it by contradiction. W.l.o.g., suppose $n_4^{5-}(y_1)\neq 0$. Let $z_1\in N_4^{5-}(y_1)$. By ($\circledast$), $f_j(z_1)<0$ for any $j\in[5]$. By Lemma 5.5, $N_5^-(N(y_1))=\emptyset$. By Observation 4.5(2), $f_1(z)\geq g^*(z)\geq\frac{1}{6}n_3(z)+\frac{1}{6}n_4(z)$ because $n_3(z)\geq 3$, which implies $t^2_z\geq\frac{1}{6}$. We claim $d(z_1)=1$. Suppose not. By Lemma 5.1(a), $d(z_1)=2$. One can easily check $f_3(z_1)\geq g^*(z_1)\geq-\frac{1}{6}$ as $n_5^-(z_1)=0$. Note that $f_3(y_1)\geq f_2(y_1)+t^2_z\geq\frac{1}{6}$. But then $f_4(z_1)\geq f_3(z_1)+\frac{1}{6}\geq 0$, a contradiction, as claimed. Hence, $f_3(z_1)=g(z_1)=-\frac{1}{3}$ which means $f_3(y_1)<\frac{1}{3}$. Clearly, $g(y_i)<0$ for any $i\in[3]$ and $n_3^{-1}(z)+n_4(z)\leq 1$, otherwise by Observation 4.4(2,4), we see $g^*(z)\geq\frac{1}{3}n_3(z)+\frac{1}{6}n_4(z)$ which implies $t^2_z\geq\frac{1}{3}$ and so $f_3(y_1)\geq f_2(y_1)+t^2_z\geq\frac{1}{3}$. Note that $L(y_i)\neq\emptyset$ for some $i\in\{2,3\}$, otherwise by Definition 4.6(3.1), $f_3(y_1)\geq f_2(y_1)+\frac{1}{2}f_2(z)\geq\frac{1}{3}$ because $f_2(z)\geq\frac{2}{3}n_3(z)-\frac{4}{3}\geq\frac{2}{3}$. W.l.o.g., let $L(y_2)=\{z_2\}$. Then G has a 3-path $P(y_1y_2)$ because $G+z_1z_2\in\mathcal{F}_G$. Let $P(y_1y_2)=y_1a_1a_2y_2$. We see $a_1,a_2\in V_4$ because $g(y_i)<0$. Moreover, $\{a_1,a_2\}\cap V_4^1\neq\emptyset$ as $n_3^{-1}(z)+n_4^2(z)\leq 1$. We next prove several claims.

Claim 1. $a_1 \neq z$.

Proof. Suppose not. Then $a_2 \in V_4^1$. Then $g^*(a_2) = \frac{1}{3}$ as $z \in N_4^2(y_2) \cap N_4^2(a_2)$, which means $f_1(a_2) \geq 0$ and so $a_2 \in N_4^{1+}(z)$. Hence, $n_4^{1-}(z) \leq n_4(z) - 1$. By Ineq. (3), $g^*(z) \geq \frac{1}{3}n_3(z) + \frac{1}{6}(n_4(z) - 1)$ since $n_3(z) \geq 3$ and $n_4(z) \geq 1$. Hence, $t_z^2 \geq \frac{1}{3}$. Then $f_3(y_1) \geq f_2(y_1) + t_z^2 \geq \frac{1}{3}$, a contradiction. □

By Claim 1, we see $t_{a_1}^2 < \frac{1}{6}$, otherwise $f_3(y_1) \geq f_2(y_1) + t_z^2 + t_{a_1}^2 \geq \frac{1}{3}$.

Claim 2. $a_2 \in V_4^1$ and $a_2 \in N_4^{1+}(a_1)$.

Proof. Suppose $a_2 \notin V_4^1$. Then $a_1 \in V_4^1$. Hence, $g^*(a_1) \geq \frac{1}{3}$ because $a_2 \in N_4^2(a_1)$ and $z \in N_4^2(y_1)$. But then $f_2(a_1) \geq \frac{1}{6}$ and so $t_{a_1}^2 \geq \frac{1}{6}$, a contradiction. Thus, $a_2 \in V_4^1$. Clearly, $g^*(a_2) \geq \frac{1}{6}$. Now we prove $a_2 \in N_4^{1+}(a_1)$. Suppose not. Clearly, $g(a_1) = \frac{1}{6}$ and $n_5^-(a_2) \neq 0$. Let $w_2 \in N_5^-(a_2)$. Then G has a 3-path $P(y_2a_2)$ because $G + z_2w_2 \in \mathcal{F}_G$. Let $P(y_2a_2) = y_2b_1b_2a_2$. Then $b_2 \in V_5$ and $b_1 \in V_4$ as $g(a_2) = g(a_1) = \frac{1}{6}$ and $g(y_i) < 0$ for any $i \in [2]$. By Observation 4.5(2), $g^*(b_2) \geq \frac{1}{6}n_4(b_2) + \frac{1}{6}n_5(b_2)$ as $n_4^+(b_2) + n_4(b_2) \geq 3$, which means $t_{b_2}^* \geq \frac{1}{6}$. But then $f_1(a_2) \geq g^*(a_2) - \frac{1}{3}n_5^-(a_2) + t_{b_2}^* \geq 0$, a contradiction. □

By Claims 1 and 2, we see $n_3(a_1) = 1$, otherwise $g^*(a_1) \geq \frac{1}{6}n_3(a_1) + \frac{1}{6}(n_4(a_1) - 1)$ because $n_3(a_1) \geq 2$ and $y_1 \in N_3^{-1}(a_1)$, which implies $t_{a_1}^2 \geq \frac{1}{6}$. Then G has a 4-path with ends y_1 and a_2 containing a_1 because $G + z_1a_2 \in \mathcal{F}_G$. Note that $N(y_1) \cap N(a_1) = \emptyset$, else let $b_1 \in N(y_1) \cap N(a_1)$, we see G has a copy of $C_6 = y_1b_1a_1a_2y_2x$ when $b_1 \neq a_2$ or $C_6 = y_1b_1y_2xy_3z$ when $b_1 = a_2$. Thus, $G - a_2$ has a 3-path $P(y_1a_1)$ or $G - y_1$ has a 3-path $P(a_1a_2)$. We next prove that

(a) $G - a_2$ has no 3-path $P(y_1a_1)$.

To prove (a), suppose 3-path $P(y_1a_1)$ exists and let $P(y_1a_1) = y_1c_1c_2a_1$. Then $c_2 \in V_4 \cup V_5$ as $n_3(a_1) = 1$. We assert $c_2 \in V_5$. Suppose not. Then $c_1 \in V_4$ because $g(y_1) < 0$. By Ineq. (3), $g^*(a_1) \geq \frac{1}{6}n_3(a_1) + \frac{1}{6}(n_4(a_1) - 1)$ because $c_2 \in N_4^2(a_1)$ and $y_1 \in N_3^{-1}(a_1)$. By Claim 2, $n_4^{1-}(a_1) \leq n_4(a_1) - 1$. But then $t_{a_1}^2 \geq \frac{1}{6}$, a contradiction, as asserted. Then $c_1 \in V_4$. By Observation 4.5(2), $g^*(c_2) \geq \frac{1}{6}n_4(c_2) + \frac{1}{6}n_5(c_2)$ because $n_4^+(c_2) + n_4(c_2) \geq 3$, which means $t_{c_2}^* \geq \frac{1}{6}$. Thus, $f_1(a_1) \geq g^*(a_1) + t_{c_2}^*$. One can easily check $g^*(a_1) \geq \frac{1}{6}n_3(a_1) + \frac{1}{6}(n_4(a_1) - 2)$. But then $t_{a_1}^2 \geq \frac{1}{6}$ because $n_4^{1-}(a_1) \leq n_4(a_1) - 1$, a contradiction. This proves (a).

By (a), $G - y_1$ has a 3-path $P(a_1a_2)$ and let $P(a_1a_2) = a_1c_1c_2a_2$. Then $c_1 \in V_4 \cup V_5$ as $n_3(a_1) = 1$. We assert $c_1 \in V_5$. Suppose not. Then $c_2 \in V_3 \cup V_5$ since $a_2 \in V_4^1$. If $c_2 \in V_3$, then $c_2 = y_2$, which implies $g^*(c_1) \geq \frac{1}{3}$ as $a_1 \in N_4^2(c_1)$ and $z \in N_4^2(y_2)$. Hence, $f_1(c_1) \geq 0$. If $c_2 \in V_5$, then by Observation 4.4(2), $g^*(c_2) \geq \frac{1}{3}n_4(c_2) + \frac{1}{6}n_5(c_2)$ as $\{c_1, a_2\} \subseteq N_4^+(c_2)$, which means $t_{c_2}^* \geq \frac{1}{3}$. Hence, $f_1(c_1) \geq 0$. By Claim 2, $\{c_1, a_2\} \subseteq N_4^{1+}(a_1)$, which yields $n_4^{1-}(a_1) \leq n_4(a_1) - 2$. But then $t_{a_1}^2 \geq \frac{1}{6}$ because $g^*(a_1) \geq \frac{1}{6}n_3(a_1) + \frac{1}{6}(n_4(a_1) - 2)$, a contradiction, as asserted. Then $c_2 \in V_5$ as $a_2 \in V_4^1$. By Ineq. (3), $g^*(c_1) \geq \frac{1}{6}n_4(c_1)$ since $a_1 \in N_4^+(c_1)$, which means $t_{c_1}^* \geq \frac{1}{6}$. Hence, $f_1(a_1) \geq g^*(a_1) + t_{c_1}^*$. But then $t_{a_1}^2 \geq \frac{1}{6}$ because $g^*(a_1) \geq \frac{1}{6}n_3(a_1) + \frac{1}{6}(n_4(a_1) - 2)$, a contradiction. This completes the proof of Proposition 5.6. □

Proof of Lemma 5.7: Clearly, we only need to prove $n_4^{5-}(y) = 0$. Suppose not. Then $f_3(y) < \frac{1}{3}$ and $f_4(y) < \frac{1}{3}$. By Lemma 5.5, $N_5^-(N_4(y)) = \emptyset$. Let $z \in L(y)$, $x \in N_2(y)$ and $N_1(x) = \{\alpha_1\}$. For any $v \in V_3$, $N_4^{3-}(v) \subseteq N_4^{*-}(v)$ and $n_4^{3-}(v) \leq 1$ by Corollary 5.4. We first prove several Claims.

Claim 1. If $g(y) = \frac{1}{6}$, then $n_3^2(y) = 0$, or $g(x) < 0$ and $n_3^2(x) = 0$. If $g(y) \geq \frac{2}{3}$, then $n_2^+(y) = 0$ and $n_3^2(y) + n_2^{-1}(y) \leq 1$.

Proof. Obviously, the results hold since otherwise $f_3(y) \geq g^*(y) \geq \frac{1}{3}$. □

Claim 2. G contains no 2-path $P(yv)$ for some $v \in \{x, \alpha_1\}$.

Proof. Suppose G contains 2-path $P(yv)$ for any $v \in \{x, \alpha_1\}$. Then $g(x) \geq 0$ and $g(y) \geq 0$. Let $P(xy) = xa_1y$. By Claim 1, $y, a_1 \in V_3^1$. Hence, $f_3(y) \geq f_2(y) = g^*(y) = \frac{1}{6}$. Then $G - a_1$ has a 3-path $P(xy)$ or $G - y$ has a 3-path $P(xa_1)$ because $G + za_1 \in \mathcal{F}_G$. We assert $G - a_1$ has no 3-path $P(xy)$. Suppose not. Let $P(xy) = xb_1b_2y$. Then $b_1 \in V_3$ and $b_2 \in V_4$ because $y \in V_3^1$. By Observation 4.5(2), $g^*(b_2) \geq \frac{1}{6}n_3(b_2) + \frac{1}{6}n_4(b_2)$ because $n_3(b_2) \geq 2$ and $y \in N_3^+(b_2)$. Since $n_5^-(b_2) = 0$, we have $t_{b_2}^2 \geq \frac{1}{6}$. But then $f_3(y) \geq f_2(y) + t_{b_2}^2 \geq \frac{1}{3}$, a contradiction, as asserted. Hence, $G - y$ has a 3-path $P(xa_1)$. Let $P(xa_1) = xb_1b_2a_1$. Similarly, $b_1 \in V_3$, $b_2 \in V_4$. Then $n_4^{3-}(a_1) \neq 0$, else $f_4(y) \geq f_3(y) + \frac{1}{6} \geq \frac{1}{3}$ since $f_3(a_1) \geq g^*(a_1) = \frac{1}{6}$. Let $z_1 \in N_4^{3-}(a_1)$. Then G has a 4-path with ends y and z_1 containing a_1 as $G + zz_1 \in \mathcal{F}_G$. Then G has a 3-path $P(ya_1)$ because $g^*(z_1) < 0$. Let $P(ya_1) = yc_1c_2a_1$. Then $c_1, c_2 \in V_4$. Clearly, $G - y - c_2$ has an s-path with ends c_1 and a_1 for some $s \in [2]$ since $G + zc_2 \in \mathcal{F}_G$. Then $n_3(c_1) + n_4(c_1) \geq 3$ and $y \in N_3^+(c_1)$. By Ineq. (3), $f_1(c_1) \geq g^*(c_1) \geq \frac{1}{6}n_3(c_1) + \frac{1}{6}(n_4(c_1) - 1)$ as $n_5^-(c_1) = 0$. Note that $g^*(c_2) \geq \frac{1}{3}$ because $a_1 \in N_3^+(c_2)$ and $c_1 \in N_4^2(c_2)$, which means $f_1(c_2) \geq 0$ and so $c_2 \in N_4^{1+}(c_1)$. Hence, $n_4^{1-}(c_1) \leq n_4(c_1) - 1$. Then, $t_{c_1}^2 \geq \frac{1}{6}$. But then $f_3(y) \geq f_2(y) + t_{c_1}^2 \geq \frac{1}{3}$, a contradiction. This proves Claim 2. $\square$

Note that G has a 4-path with ends α_1 and y containing x since $G + z\alpha_1 \in \mathcal{F}_G$. By Claim 2, $G - y$ has a 3-path with ends x and α_1 or $G - \alpha_1$ has a 3-path with ends x and y. We next prove the following claim.

Claim 3. $G - \alpha_1$ has no 3-path with ends x and y.

Proof. Suppose not. Let $P(xy) = xx_1x_2y$. By Claim 1, $xx_2 \notin E(G)$. Then G has a 4-path with ends y and x_1 containing x and x_2 as $G + zx_1 \in \mathcal{F}_G$. We thus see that $G - y - x_1$ has a 2-path $P(xx_2)$. Let $P(xx_2) = xx_3x_2$. By Claim 1, $x_1, x_3 \in V_3$ and $x_2 \in V_3 \cup V_4$. We next prove that

(a) $x_2 \in V_3$ and $L(x_2) = \emptyset$.

To see why (a) is true, suppose first $x_2 \in V_4$. Clearly, $N_3(x_2) = \{y, x_1, x_3\}$, else by Observation 4.4(2), $g^*(x_2) \geq \frac{1}{3}n_3(x_2) + \frac{1}{6}n_4(x_2)$ which implies $t_{x_2}^2 \geq \frac{1}{3}$ and so $f_3(y) \geq f_2(y) + t_{x_2}^2 \geq \frac{1}{3}$. Moreover, we see $N_2(y) = N_2(x_1) = N_2(x_3) = \{x\}$, else we assume $x' \in N_2(x_1) \setminus x$, we see G has a copy of $C_6 = \alpha_1xx_3x_2x_1x'$. But then $n_4^{5-}(y) = 0$ by Lemma 5.6, a contradiction. Thus, $x_2 \in V_3$. We now show that $L(x_2) = \emptyset$. Suppose not. Let $z_1 \in L(x_2)$. Then $z_1 \in V_4$. Then G has a 4-path with ends x and x_2 containing each of $\{y, x_1, x_3\}$ because $G + xz_1 \in \mathcal{F}_G$. Hence, $G[\{y, x_1, x_3\}]$ has at least two edges. W.l.o.g., let $yx_3 \in E(G)$. Then $y \in V_3^2$ and $\{x_2, x_3\} \subseteq N_3^2(y)$, which violates Claim 1. This proves (a).

By (a) and Claim 1, $g(x) < 0$ and $x_1, x_3 \in V_3^1$. For any $v \in V_3$, let $A(v)$ be defined as in Definition 4.6(4). By (a) and Lemma 5.1(a, b), $f_3(v) \geq -\frac{1}{6}$ for any $v \in N_4^{3-}(x_2)$. Hence, $f_3(x_2) < \frac{1}{6}|A(x_2)| + \frac{1}{6}n_4^{3-}(x_2)$, else $f_4(y) \geq f_3(y) + \frac{1}{6} \geq \frac{1}{3}$. Since $g(x_2) \geq \frac{2}{3}$, we see $f_3(v) \geq g^*(v) \geq \frac{1}{6}$ for any $v \in \{y, x_1, x_3\}$. By Ineq. (3), $g^*(x_2) \geq \frac{1}{6}(n_3(x_2) - 1)$ because $n_2(x_2) + n_3(x_2) \geq 4$. We assert $g(y) = \frac{1}{6}$. Suppose not. For any $i \in \{1, 3\}$, $f_3(x_i) \geq g^*(x_i) = \frac{1}{3}$ because $y \in N_3^2(x)$ and $x_2 \in N_3^2(x_i)$, which yields $f_3(x_i) + f_3(x_i') \geq 0$ for any $x_i' \in N_4(x_i)$. Hence, $x_1, x_3 \notin A(x_2)$, which means $|A(x_2)| \leq n_3(x_2) - 2$. But then $f_3(x_2) \geq g^*(x_2) \geq \frac{1}{6}|A(x_2)| + \frac{1}{6} \geq \frac{1}{6}|A(x_2)| + \frac{1}{6}n_4^{3-}(x_2)$, a contradiction, as asserted. We further assert $L(x_1) \cup L(x_3) \neq \emptyset$. Suppose not. By Lemma 5.1(a, b), $f_3(v) \geq -\frac{1}{6}$ for any $v \in N_4^{3-}(x_i)$, which yields $f_3(x_i) + f_3(x_i') \geq 0$ for any $x_i' \in N_4(x_i)$ and $i \in \{1, 3\}$. Hence, $x_1, x_3 \notin A(x_2)$. Similarly, we also get a contradiction, as asserted. W.l.o.g., let $z_1 \in L(x_1)$. Because $G + zz_1 \in \mathcal{F}_G$, we see G has a 3-path $P(yx_1)$. Let $P(yx_1) = ya_1a_2x_1$. We next show that

(b) $a_1, a_2 \in V_4^1$.

To prove (b), suppose first $a_1 \notin V_4^1$. We assert $a_1 \in V_4^2$. Suppose not. Then $a_1 = x_2$ and $a_2 \in V_4$ because $g(y) = g(x_1) = \frac{1}{6}$ and $xx_2 \notin E(G)$. By Ineq. (3), $g^*(a_2) \geq \frac{1}{6}n_3(a_2) + \frac{1}{6}(n_4(a_2) + 2)$ because $x_1, x_2 \in N_3^+(a_2)$, which means $t^2_{a_2} \geq \frac{1}{6}$. Hence, $f_3(x_1) \geq f_2(x_1) + t^2_{a_2} \geq \frac{1}{3}$. Then $x_1 \notin A(x_2)$, which means $|A(x_2)| \leq n_3(x_2) - 1$. But then $f_3(x_2) \geq f_2(x_2) + t^2_{a_2} \geq g^*(x_2) + \frac{1}{6} \geq \frac{1}{6}|A(x_2)| + \frac{1}{6}n_4^{3-}(x_2)$, a contradiction, as asserted. Thus, $a_1 \in V_4^2$. Note that $a_2 \in V_4$, else $a_2 = x_2$ yields $t^2_{a_1} \geq \frac{1}{6}$ and so $f_3(y) \geq f_2(y) + t^2_{a_1} \geq \frac{1}{3}$. Then $g^*(a_2) \geq \frac{1}{3}$ because $x_1 \in N_3^+(a_2)$ and $a_1 \in N_4^2(a_2)$, which means $a_2 \in N_4^{1+}(a_1)$ and so $n_4^{1-}(a_1) \leq n_4(a_1) - 1$. By Ineq. (3), $g^*(a_1) \geq \frac{1}{6}n_3(a_1) + \frac{1}{6}(n_4(a_1) - 1)$ because $y \in N_3^+(a_1)$. Then, $t^2_{a_1} \geq \frac{1}{6}$ because $n_5^-(a_1) = 0$. But then $f_3(y) \geq f_2(y) + t^2_{a_1} \geq \frac{1}{3}$, a contradiction. Thus $a_1 \in V_4^1$. Then $a_2 \in V_4$. Moreover, $a_2 \in V_4^1$, otherwise $f_1(a_1) \geq g^*(a_1) \geq \frac{1}{3}$ because $y \in N_3^+(a_1)$ and $n_5^-(a_1) = 0$, which means $t^2_{a_1} \geq \frac{1}{6}$ and so $f_3(y) \geq f_2(y) + t^2_{a_1} \geq \frac{1}{3}$. This proves (b).

By (b), $g^*(a_1) = \frac{1}{6}$ as $y \in N_3^+(a_1)$. Then G has a 4-path with ends y and a_2 containing a_1 because $G + za_2 \in \mathcal{F}_G$. Since $g(a_1) = \frac{1}{6}$, we see $G - a_2$ has a 3-path $P(a_1y)$ or $G - y$ has a 3-path $P(a_1a_2)$. Let $P(a_1v) = a_1b_1b_2v$, where $v \in \{y, a_2\}$. Then $b_1 \in V_5$ because $g(a_1) = \frac{1}{6}$. Clearly, $g^*(b_1) \geq \frac{1}{6}n_4(b_1)$ since $a_1 \in N_4^+(b_1)$, which means $t^*_{b_1} \geq \frac{1}{6}$. Thus, $f_1(a_1) \geq g^*(a_1) + t^*_{b_1} \geq \frac{1}{3}$ as $n_5^-(a_1) = 0$, which means $t^2_{a_1} \geq \frac{1}{6}$. But then $f_3(y) \geq f_2(y) + t^2_{a_1} \geq \frac{1}{3}$, a contradiction. □

By Claim 3, $G - y$ has a 3-path $P(x\alpha_1)$ and let $P(x\alpha_1) = xa_2a_1\alpha_1$. Then $a_1 \in V_2$ since $\delta(G) = 1$. We assert $n_2(y) = 1$. Suppose not. Let $x' \in N_2(y) \setminus x$. By Claim 1, $x' \neq a_2$. Hence, $x' = a_1$, else G has a copy of $C_6 = \alpha_1a_1a_2xyx'$. But then $n_2^+(y) + n_2^{-1}(y) \geq 2$, which violates Claim 1, as asserted. We next prove that

(c) $n_3(y) = 0$.

To see why (c) is true, suppose $n_3(y) \neq 0$. By Claim 1, $n_3^2(y) = 0$. So $N_3(y) = N_3^1(y)$. Let $y_1 \in N_3(y)$ and $x_1 \in N_2(y_1)$. Clearly, $f_3(y) \geq g^*(y) \geq \frac{1}{6}$ because $x \in N_2^+(y)$ or $N_3^2(x) \setminus y \neq \emptyset$.

We first show that $G - x_1$ has no 3-path with ends y and y_1. Suppose not. Let $P(yy_1) = yc_1c_2y_1$. Then $c_2 \in V_4$ because $g(y_1) = \frac{1}{6}$. We claim $c_1 \in V_4$. Suppose not. Then $c_1 \in V_3$. Clearly, $N_2(c_1) = \{x_1\}$. Hence, $f_2(y_1) \geq g^*(y_1) \geq \frac{1}{6}$ because $y \in N_3^2(y_1)$. By Ineq. (3), $g^*(c_2) \geq \frac{1}{6}n_3(c_2) + \frac{1}{6}(n_4(c_2) + 2)$ because $c_1, y_1 \subseteq N_3^+(c_2)$, which implies $t^2_{c_2} \geq \frac{1}{6}$. Hence, $f_3(y_1) \geq f_2(y_1) + t^2_{c_2} \geq \frac{1}{3}$. Moreover, $L(y_1) \neq \emptyset$, otherwise by Lemma 5.1(a, b), $f_3(v) \geq -\frac{1}{6}$ for any $v \in N_4^{3-}(y_1)$, which yields $f_4(y) \geq f_3(y) + \frac{1}{6} \geq \frac{1}{3}$. Let $z' \in L(y_1)$. We see $G - c_1 - y_1$ has a path of length at most two with ends y and c_2 because $G + z'c_1 \in \mathcal{F}_G$, which implies $t^2_{c_2} \geq \frac{1}{3}$ and so $f_3(y_1) \geq f_2(y_1) + t^2_{c_2} \geq \frac{1}{2}$. But then $f_4(y) \geq f_3(y) + \frac{1}{6} \geq \frac{1}{3}$, a contradiction, as claimed. Note that $y_1c_1 \notin E(G)$, otherwise $f_1(c_1) \geq g^*(c_1) \geq \frac{1}{3}n_3(c_1) + \frac{1}{6}n_4(c_1)$ because $y, y_1 \subseteq N_3^+(c_1)$ and $n_5^-(c_1) = 0$, which means $t^2_{c_1} \geq \frac{1}{3}$ and so $f_3(y) \geq f_2(y) + t^2_{c_1} \geq \frac{1}{3}$. Then $G - y - c_2$ has a 2-path $P(y_1c_1)$ as $G + zc_2 \in \mathcal{F}_G$. Let $P(y_1c_1) = y_1d_1c_1$. Then $d_1 \in V_4$ as $g(y_1) = \frac{1}{6}$. By Ineq. (3), $f_1(c_1) \geq g^*(c_1) \geq \frac{1}{6}n_3(c_1) + \frac{1}{6}(n_4(c_1) - 1)$ since $y \in N_3^+(c_1)$ and $n_5^-(c_1) = 0$. Clearly, $f_1(c_2) \geq 0$, which means $c_2 \in N_4^{1+}(c_1)$ and so $n_4^{1-}(c_1) \leq n_4(c_1) - 1$. Hence, $t^2_{c_1} \geq \frac{1}{6}$. But then $f_3(y) \geq f_2(y) + t^2_{c_1} \geq \frac{1}{3}$, a contradiction.

Thus, G has no 3-path with ends y and y_1. Clearly, $n_4^{3-}(y_1) = 0$, else let $v \in N_4^{3-}(y_1)$, $G + zv \notin \mathcal{F}_G$ by Lemma 5.1(a, b). Since $G + zx_1 \in \mathcal{F}_G$ and $y_1 \in N_3^1(y)$, we see G has a 3-path $P(y_1x_1) = y_1c_1c_2x_1$ such that $c_1 \in V_4$ and $c_2 \in V_3$. If $n_5^-(c_1) = 0$, then by Observation 4.5(2), $g^*(c_1) \geq \frac{1}{6}n_3(c_1) + \frac{1}{6}n_4(c_1)$ because $y_1 \in N_3^+(c_1)$ and $n_3(c_1) \geq 2$, which means $t^2_{c_1} \geq \frac{1}{6}$ and so $f_3(y_1) \geq f_2(y_1) + t^2_{c_1} \geq \frac{1}{6}$. If $n_5^-(c_1) \neq 0$, then let $w \in N_5^-(c_1)$. Then $G - x_1 - c_1$ has a 2-path $P(y_1c_2) = y_1c_3c_2$ such that $c_3 \in V_4$ because $G + wx_1 \in \mathcal{F}_G$ and $y_1 \in N_3^1(y)$. Hence, for $i \in \{1, 3\}$, $g^*(c_i) \geq \frac{1}{12}n_3(c_i) + \frac{1}{6}n_4(c_i) + \frac{1}{3}$ because $y_1 \in N_3^+(c_i)$, $n_3(c_i) \geq 2$ and $c_2 \in N_3^+(c_i) \cup N_3^{-1}(c_i)$. This implies $t^2_{c_i} \geq \frac{1}{12}$. Thus, $f_3(y_1) \geq f_2(y_1) + t^2_{c_1} + t^2_{c_3} \geq \frac{1}{6}$. But then in both cases $f_4(y) \geq f_3(y) + \frac{1}{6} \geq \frac{1}{3}$, a contradiction. This proves (c).

Let $N_1(\alpha_1) = \{\alpha\}$. By the choice of α, y belongs to some 4-cycle with vertices y, z_1, z_2, z_3 in order. By (c) and Claim 3, $z_1, z_3 \in V_4$. Note that $n_5^-(z_i) = 0$ for any $i \in \{1, 3\}$, which means $f_1(z_i) \geq g^*(z_i)$. Since $G + zz_2 \in \mathcal{F}_G$, we have $z_1z_3 \in E(G)$ or $G - y - z_2$ has a 2-path $P(z_1z_3) = z_1z_4z_3$. We assert $g(z_i) \geq 0$ for any $i \in \{1, 3\}$. W.l.o.g., suppose $g(z_1) < 0$. Then $z_2, z_4 \in V_5$. By Ineq. (3), for any $i \in \{2,4\}$, $g^*(z_i) \geq \frac{1}{6}n_4(z_i)$ because $n_4(z_i) \geq 2$ and $n_4^+(z_i) + n_4^{-1}(z_i) \geq 2$, which means $t^*_{z_i} \geq \frac{1}{6}$. Hence, $f_2(z_1) \geq g^*(z_1) + t^*_{z_2} + t^*_{z_4} \geq \frac{1}{3}$, which means $t^2_{z_1} \geq \frac{1}{3}$. But then $f_3(y) \geq f_2(y) + t^2_{z_1} \geq \frac{1}{3}$, a contradiction, as asserted. Then $g^*(y) \geq 0$. Obviously, $t^2_{z_i} < \frac{1}{6}$ for some $i \in \{1,3\}$, else $f_3(y) \geq f_2(y) + t^2_{z_1} + t^2_{z_3} \geq \frac{1}{3}$. We further assert $N_5(z_1) \cap N_5(z_3) \neq \emptyset$. Suppose not. Then $z_1, z_3 \in V_4^2$. By Ineq. (3), for any $i \in \{1,3\}$, $g^*(z_i) \geq \frac{1}{6}n_3(z_i) + \frac{1}{6}n_4(z_i)$ because $n_3^+(z_i) + n_3^{-1}(z_i) + n_4^2(z_i) \geq 3$, which yields $t^2_{z_i} \geq \frac{1}{6}$, a contradiction, as asserted. W.l.o.g., let $z_2 \in N_5(z_1) \cap N_5(z_3)$. By Ineq. (3), $g^*(z_2) \geq \frac{1}{3}n_4(z_2)$ because $\{z_1, z_3\} \subseteq N_4^+(z_2)$, which means $t^*_{z_2} \geq \frac{1}{3}$. Note that $t^2_{z_i} < \frac{1}{6}$ for some $i \in \{1,3\}$. W.l.o.g., suppose $t^2_{z_1} < \frac{1}{6}$. Then $z_1 \in V_4^2$, otherwise $g^*(z_1) \geq 0$ and $f_1(z_1) \geq g^*(z_1) + t^*_{z_2} \geq \frac{1}{3}$, which implies $t^2_{z_1} \geq \frac{1}{6}$. Moreover, $z_3 \in V_4^2$, otherwise $g^*(z_3) \geq \frac{1}{6}$ because $z_1 \in N_4^2(y)$, which implies $f_2(z_3) \geq g^*(z_3) + t^*_{z_2} - \frac{1}{6} \geq \frac{1}{3}$ and so $f_3(y) \geq f_2(y) + t^2_{z_3} \geq \frac{1}{3}$. It is easy to see that if z_4 exists, then $z_4 \in V_3 \cup V_4$ otherwise $t^*_{z_4} \geq \frac{1}{3}$ and $t^2_{z_1} \geq \frac{1}{6}$. By Ineq. (3), $g^*(z_1) \geq \frac{1}{6}n_3(z_1) + \frac{1}{6}(n_4(z_1) - 2)$ because $n_3^+(z_1) + n_3^{-1}(z_1) + n_4^2(z_1) \geq 2$ and $n_3(z_1) + n_4(z_1) \geq 2$. Hence, $f_1(z_1) \geq g^*(z_1) + t^*_{z_2} \geq \frac{1}{6}n_3(z_1) + \frac{1}{6}n_4(z_1)$. But then $t^2_{z_1} \geq \frac{1}{6}$, a contradiction. This completes the proof of Lemma 5.7. □

Proof of Lemma 5.8: To prove (a), suppose $n_4^+(y) \neq 0$. Let $z_0 \in N_4^+(y)$. By (⁎) and Corollary 5.4, $g^*(y) \leq f_3(y) < 0$ and $g^*(z) < 0$. By Lemmas 5.7 and 5.1(a, b), G has a 3-path $P = zww_1z_1$ such that $d(v) = 2$ for any $v \in V(P)$, and $yz_1 \notin E(G)$, where $z_1 \in V_4$, $w, w_1 \in V_5$. By Observation 4.3(1), $N_4^+(y) = N_4^1(y) = \{z_0\}$, which yields $f_2(y) = g^*(y) \geq -\frac{1}{6}$. Let $N_3(z_1) = \{y_1\}$, $N_4(z_0) = \{z_2\}$ and $N_2(y) = \{x\}$. Then G has a 2-path with vertices x, x', y_1 in order because $G + wx \in \mathcal{F}_G$ and $g^*(y) < 0$. Then $y_1z_2 \notin E(G)$, else $y_1x'xyz_0z_2$ is a 6-cycle. This means G has no 2-path with ends y_1 and z_0. Then G has a 3-path $P(yz_0)$ because $G + wz_0 \in \mathcal{F}_G$. Let $P(yz_0) = ya_1a_2z_0$. Then $a_1 \in N_4^-(y)$ and $a_2 \in V_5$. Note that $z_0 \in N_4^+(a_2)$ and $n_4(a_2) \geq 2$. By Observation 4.5(2), $g^*(a_2) \geq \frac{1}{6}n_4(a_2) + \frac{1}{6}n_5(a_2)$ and so $t^*_{a_2} \geq \frac{1}{6}$. By Lemma 5.5, $n_5^-(a_1) = 0$. Hence, $f_2(a_1) \geq g^*(a_1) + t^*_{a_2} \geq \frac{1}{6}$ and so $t^2_{a_1} \geq \frac{1}{6}$. But then $f_3(y) \geq f_2(y) + t^2_{a_1} \geq 0$, a contradiction. This proves (a).

Now we shall prove (b). Suppose $n_5^-(z_1) \neq 0$. Let $w_1 \in N_5^-(z_1)$. By (⁎) and $g(z) = \frac{1}{6}$, $-\frac{1}{6} \leq f_2(y) \leq f_3(y) < 0$. Then G has a 4-path with ends y and z_1 containing z as $G + yw_1 \in \mathcal{F}_G$. Clearly, $G - z_1$ has no 3-path with ends y and z, otherwise $f_3(y) \geq 0$ by similar analysis with the case $P(yz_0)$ exists in the proof of Lemma 5.8(a). Then $G - y$ has a 3-path $P(zz_1)$. Let $P(zz_1) = za_1a_2z_1$. By Observation 4.3(2), $g(z_1) = \frac{1}{6}$. Hence, $a_1, a_2 \in V_5$. Then $G - z_1 - a_1$ has a path of length at most two with ends z and a_2 because $G + w_1a_1 \in \mathcal{F}_G$. Hence, $a_2 \in N_5^2(a_1)$, which means $g^*(a_1) \geq \frac{1}{3}n_4(a_1)$ because $z \in N_4^+(a_1)$ and so $t^*_{a_1} \geq \frac{1}{3}$. By Lemma 5.3, $n_5^-(z) = 0$. Hence, $f_1(z) \geq g^*(z) + t^*_{a_1} \geq \frac{1}{3}$ and so $t^2_z \geq \frac{1}{6}$. But then $f_3(y) \geq f_2(y) + t^2_z \geq 0$, a contradiction. □