The Briggs inequality for partitions and overpartitions

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Abstract. A sequence of $\{a_n\}_{n\geq 0}$ satisfies the Briggs inequality if

$$a_n^2(a_n^2 - a_{n-1}a_{n+1}) > a_{n-1}^2(a_{n+1}^2 - a_n a_{n+2})$$

holds for any $n \geq 1$. In this paper we show that both the partition function $\{p(n + N_0)\}_{n\geq 0}$ and the overpartition function $\{\overline{p}(n + \overline{N}_0)\}_{n\geq 0}$ satisfy the Briggs inequality for some N_0 and $\overline{N}_0$. Based on Chern's formula for η-quotients, we further prove that the k-regular partition function $\{p_k(n + N_k)\}_{n\geq 0}$ and the k-regular overpartition function $\{\overline{p}_k(n + \overline{N}_k)\}_{n\geq 0}$ also satisfy the Briggs inequality for $2 \leq k \leq 9$ and some $N_k, \overline{N}_k$.

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1 Introduction

In the study of binding polynomials, Briggs [5] proposed the following conjecture.

Conjecture 1.1. *Suppose that $f(x) = \sum_{l=0}^{n} a_l x^l$ is a polynomial with nonnegative coefficients. If $f(x)$ has only negative zeros, then, for $1 \leq l \leq n-1$,*

$$a_{l-1}a_{l+2}^2 + a_l^2 a_{l+3} + a_{l+1}^3 > a_{l+1}(a_{l-1}a_{l+3} + 2a_l a_{l+2}), \quad (1)$$

$$a_l^2(a_l^2 - a_{l-1}a_{l+1}) > a_{l-1}^2(a_{l+1}^2 - a_l a_{l+2}). \quad (2)$$

As pointed out by Zhang and Zhao [35], the first inequality (1) can be deduced from a result due to Bränden [4]. This paper is mainly concerned with the second inequality (2), which has been proved by Fan and Wang [17] recently. Although Briggs' original conjecture is stated for a finite sequence, it is natural to study such inequalities for an infinite sequence. Following Zhang and Zhao [35], we say that a sequence of $\{a_n\}_{n\geq 0}$ satisfies the Briggs inequality if

$$a_n^2(a_n^2 - a_{n-1}a_{n+1}) > a_{n-1}^2(a_{n+1}^2 - a_n a_{n+2})$$

holds for any $n \geq 1$. Zhang and Zhao [35] proved that both the Boros-Moll sequence and two of its variations satisfy the Briggs inequality.

In this paper we aim to show that the Briggs inequality is satisfied by some partition function, such as the partition function, the overpartition function, the k-regular partition function, or the k-regular overpartition function. Recall that a partition of n is a weakly decreasing sequence of positive numbers whose sum is n. The partition function $p(n)$ counts the number of partitions of n. As a broad generalization of partitions, Corteel and Lovejoy [11] introduced the concept of overpartitions. By an overpartition of n we mean a partition of n such that the frist occurrence of a number may be overlined. The overpartition function $\overline{p}(n)$ counts the number of overpartitions of n. For example, there are three partitions of 3, namely $(3), (2,1), (1,1,1)$, and eight overpartitions of 3, namely $(3), (\overline{3}), (2,1), (\overline{2},1), (2,\overline{1}), (\overline{2},\overline{1}), (1,1,1), (\overline{1},1,1)$. Thus $p(3) = 3$ and $\overline{p}(3) = 8$. For $k \geq 2$, by a k-regular partition of n we mean a partition of n with no part divisible by k. A k-regular overpartition of n can be defined in the same manner. As usual, we use $p_k(n)$ and $\overline{p}_k(n)$ to denote the k-regular partition function and the k-regular overpartition function respectively.

Various interesting inequalities have been established for the partition function, the overpartition function, the k-regular partition function and the k-regular overpartition function. The log-concavity of $\{p(n)\}_{n\geq 26}$ was independently proved by Nicolas [29] and by DeSalvo and Pak [12]. Chen, Jia and Wang [9] showed that $\{p(n)\}_{n\geq 95}$ also satisfies higher order Turán inequalities. For more information on higher order Turán inequalities, see [13, 30, 22]. Furthermore, Chen, Jia and Wang conjectured that for $d \geq 4$ there is a positive number N_d such that the order d Turán inequalities are valid for $p(n)$ when $n \geq N_d$. Later, this conjecture was solved by Griffin, Ono, Rolen, and Zagier [15]. Hou and Zhang [20] proved the asymptotic r-log-concavity of $\{p(n)\}_{n\geq 1}$ for any $r \geq 1$. In particular, they established the 2-log-concavity of $\{p(n)\}_{n\geq 221}$, which was independently proved by Jia and Wang [23]. For the overpartition function, Engel [16] proved that $\{\overline{p}(n)\}_{n\geq 2}$ is log-concave. Liu and Zhang [25] showed that the higher order Turán inequalities are satisfied by $\{\overline{p}(n)\}_{n\geq 16}$. Following the work in [23], Mukherjee [27] showed that $\{\overline{p}(n)\}_{n\geq 42}$ satisfies the double Turán inequalities. Later, Mukherjee, Zhang and Zhong [28] proved the asymptotic r-log-concavity of $\{\overline{p}(n)\}_{n\geq 1}$. By employing the result in [15], Craig and Pun [10] showed that $\{p_2(n)\}$ satisfies the order d Turán inequalities for sufficiently large n. Furthermore, they conjectured that $\{p_2(n)\}_{n\geq 33}$ is log-concave and $\{p_2(n)\}_{n\geq 121}$ satisfies the higher order Turán inequalities. Based on Chern's asymptotic formula [8], Dong and Ji [14] showed that $\{p_k(n)\}_{n\geq N_k}$ is log-concave and satisfies higher order inequalities for $2 \leq k \leq 5$ and some N_k, thus particularly confirming the conjectures of Craig and Pun. Wang and Yang [34] showed that $\{p_2(n)\}_{n\geq 271}$ satisfies double Turán inequalities, as conjectured by Dong and Ji [14]. Peng [10], Zhang and Zhong [31] proved that $\{\overline{p}_k(n)\}_{n\geq \overline{N}_k}$ is log-concave and satisfies higher order inequalities for $2 \leq k \leq 9$ and some $\overline{N}_k$.

We would like to point out that, for a given positive sequence $\{a_n\}_{n\geq 0}$, the inequality (1) is equivalent to the double Turán inequality, while the Briggs inequality (2) is closely related to the log-concavity. Recall that a sequence $\{a_n\}_{n\geq 0}$ is said to be log-concave if

$a_n^2 - a_{n+1}a_{n-2} \geq 0$ for any $n \geq 1$. Note that if a log-concave sequence $\{a_n\}_{n\geq 0}$ satisfies

$$a_{n+1}(a_n^2 - a_{n-1}a_{n+1}) > a_{n-1}(a_{n+1}^2 - a_n a_{n+2}), \tag{3}$$

then it also satisfies the Briggs inequality. This is clear since the log-concavity of $\{a_n\}_{n\geq 0}$ tells that $a_n^2 \geq a_{n+1}a_{n-1}$, and hence

$$a_n^2(a_n^2 - a_{n-1}a_{n+1}) \geq a_{n+1}a_{n-1}(a_n^2 - a_{n-1}a_{n+1}) > a_{n-1}^2(a_{n+1}^2 - a_n a_{n+2}).$$

In order to show that $p(n), \overline{p}(n), p_k(n)$ and $\overline{p}_k(n)$ (after ignoring some initial terms) satisfy the Briggs inequality, it suffices to show that they satisfy the stronger inequality (3) in view of the aforementioned log-concavity of these partition functions.

The remainder of this paper is organized as follows. In Section 2 we prove the Briggs inequality of the partition function and the overpartition function by using the bounds of $p(n)$ and $\overline{p}(n)$ given by Wang and Yang. In Section 3 we show that, for $2 \leq k \leq 9$, the k-regular partition function $p_k(n)$ and the k-regular overpartition function $\overline{p}_k(n)$ satsify the Briggs inequality by using some explicit bounds of $p_k(n)$ and $\overline{p}_k(n)$, which can be obtained from Chern's formula of η-quotients.

2 Partition functions

The main objective of this section is to prove that both the partition function and the overpartition function satisfy the Briggs inequality. Taking a_n to be the partition function $p(n)$ or $\overline{p}(n)$, we only need to prove (3), as discussed earlier. Note that (3) can rewritten as

$$a_{n+1}a_n^2 - 2a_{n-1}a_{n+1}^2 + a_{n-1}a_n a_{n+2} > 0. \tag{4}$$

2.1 Partitions

For the partition function $p(n)$, we have the following result.

Theorem 2.1. *For all $n \geq 114$, we have*

$$p(n+1)p(n)^2 - 2p(n-1)p(n+1)^2 + p(n-1)p(n)p(n+2) > 0. \tag{5}$$

To prove Theorem 2.1, we need the following upper and lower bounds for $p(n)$ given by Wang and Yang [33]. Let $\mu(n) = \dfrac{\pi\sqrt{24n-1}}{6}$ and

$$f(t) = \frac{1}{t^2}\left(1 - \frac{1}{t} - \frac{1}{t^{10}}\right), \quad g(t) = \frac{1}{t^2}\left(1 - \frac{1}{t} + \frac{1}{t^{10}}\right). \tag{6}$$

Wang and Yang [33] obtained the following result.

Lemma 2.2 ([33], Lemma 2.1)**.** *Let $\mu(n), f(t), g(t)$ be defined as above. Then for all $n \geq 1520$, i.e., $\mu(n-1) \geq 100$, we have*

$$\frac{\sqrt{12}\pi^2 e^{\mu(n)}}{36} f(\mu(n)) < p(n) < \frac{\sqrt{12}\pi^2 e^{\mu(n)}}{36} g(\mu(n)).$$

Note that $p(n-1), p(n), p(n+1)$ and $p(n+2)$ appear in (5). In order to use the bounds of these values given by Lemma 2.2, for notational convenience, we set

$$x = \mu(n-1), \quad x_1 = \mu(n), \quad x_2 = \mu(n+1), \quad x_3 = \mu(n+2) \tag{7}$$

throughout this subsection. In our proof of (5), the value of x will be used to estimate x_1, x_2 and x_3. We have the following result.

Lemma 2.3. *Let x, x_1, x_2 and x_3 be functions of n as defined in (7), and let*

$$x_{11} = \check{h}_x(\frac{2\pi^2}{3}), \quad x_{21} = \check{h}_x(\frac{4\pi^2}{3}), \quad x_{31} = \check{h}_x(2\pi^2), \tag{8}$$

$$x_{12} = \hat{h}_x(\frac{2\pi^2}{3}), \quad x_{22} = \hat{h}_x(\frac{4\pi^2}{3}), \quad x_{32} = \hat{h}_x(2\pi^2), \tag{9}$$

where

$$\check{h}_x(a) := x + \frac{a}{2x} - \frac{a^2}{8x^3} + \frac{a^3}{16x^5} - \frac{5a^4}{64x^7}, \tag{10}$$

$$\hat{h}_x(a) := x + \frac{a}{2x} - \frac{a^2}{8x^3} + \frac{a^3}{16x^5}. \tag{11}$$

Then, for $n \geq 5$ and hence $x > \sqrt{2}\pi$,

$$x_{i1} < x_i < x_{i2} \tag{12}$$

holds for $1 \leq i \leq 3$.

Proof. One can directly verify that

$$x_1 = \sqrt{x^2 + \frac{2\pi^2}{3}}, \quad x_2 = \sqrt{x^2 + \frac{4\pi^2}{3}}, \quad x_3 = \sqrt{x^2 + 2\pi^2}. \tag{13}$$

Since each of x_i is of the form $\sqrt{x^2+a}$ for some positive number a, it suffices to show that $\check{h}_x(a) < \sqrt{x^2+a} < \hat{h}_x(a)$ for $x^2 > a$. Keeping in mind that x is always positive, Newton's binomial theorem tells that

$$\sqrt{x^2+a} = x\left(1+\frac{a}{x^2}\right)^{\frac{1}{2}} = x\left(\sum_{k\geq 0}\binom{\frac{1}{2}}{k}\left(\frac{a}{x^2}\right)^k\right)$$

$$= x + \frac{a}{2x} - \frac{a^2}{8x^3} + \frac{a^3}{16x^5} - \frac{5a^4}{128x^7} + \frac{7a^5}{256x^9} - \frac{21a^6}{1024x^{11}} + O(\frac{1}{x^{13}}).$$

By considering the difference of two adjacent terms in the above expansion, one can show that if $x^2 > a$ then $\check{h}_x(a) < \sqrt{x^2+a} < \hat{h}_x(a)$. This completes the proof. □

Now we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1. For $114 \le n \le 1519$ one can directly verify (5). From now on we assume that $n \ge 1520$, whence $x \ge 100$.

Recalling the bounds of $p(n)$ given in Lemma 2.2, we obtain that

$$p(n+1)p(n)^2 - 2p(n-1)p(n+1)^2 + p(n-1)p(n)p(n+2) > \left(\frac{\sqrt{12}\pi^2}{36}\right)^3 F_1(x),$$

where

$$F_1(x) = e^{x_{21}+2x_{11}} f(x_2)f(x_1)^2 - 2e^{2x_{22}+x} g(x_2)^2 g(x) + e^{x+x_{11}+x_{31}} f(x)f(x_1)f(x_3), \qquad (14)$$

the symbols $x, x_1, x_2, x_3, x_{11}, x_{21}, x_{31}, x_{22}$ are defined as in (7), (8) and (9), and the functions $f(t), g(t)$ are given by (6).

It remains to show that $F_1(x) > 0$. Let $z_2 = \hat{h}_x(\frac{8\pi^2}{9})$. Note that the sign of $F_1(x)$ coincides with that of $F_1(x)e^{-3z_2}$. We find that it is more convenient to deal with the latter. It is routine to verify that, for $x \ge 4$,

$$x_{21} + 2x_{11} - 3z_2 = \frac{-\pi^4(18x^4 - 26\pi^2x^2 + 135\pi^4)}{486x^7} < 0,$$

$$2x_{22} + x - 3z_2 = \frac{-4\pi^4\,(9x^2 - 10\pi^2)}{243x^5} < 0,$$

$$x + x_{11} + x_{31} - 3z_2 = \frac{-\pi^4(126x^4 - 188\pi^2x^2 + 615\pi^4)}{486x^7} < 0.$$

While, for $t < 0$, we have

$$E_1(t) < e^t < E_2(t), \qquad (15)$$

where

$$E_1(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}, \quad E_2(t) = 1 + t + \frac{t^2}{2}. \qquad (16)$$

Thus, for $x \ge 4$, there holds

$$e^{x_{21}+2x_{11}-3z_2} > E_1(x_{21} + 2x_{11} - 3z_2), \qquad (17)$$

$$e^{2x_{22}+x-3z_2} < E_2(2x_{22} + x - 3z_2), \qquad (18)$$

$$e^{x+x_{11}+x_{31}-3z_2} > E_1(x + x_{11} + x_{31} - 3z_2). \qquad (19)$$

To prove that $F_1(x)e^{-3z_2} > 0$, we also need to estimate the values of $f(x_i)$ and $g(x_i)$. For $i = 1, 2, 3$ letting

$$f_i(t) = \frac{t^{10} - t^8 x_{i2} - 1}{t^{12}} \qquad \text{and} \qquad g_i(t) = \frac{t^{10} - t^8 x_{i1} + 1}{t^{12}},$$

one can check that for $x \geq 1$,

$$f_i(x_i) < f(x_i) \qquad \text{and} \qquad g(x_i) < g_i(x_i). \tag{20}$$

Combining (14), (17), (18), (19) and (20), we see that

$$F_1(x)e^{-3z_2} > F_2(x)$$

holds for $x \geq 4$, where

$$\begin{aligned} F_2(x) = &\big(E_1(x_{21} + 2x_{11} - 3z_2)f_2(x_2)f_1(x_1)^2 - 2E_2(2x_{22} + x - 3z_2)g_2(x_2)^2 g(x) \\ &+ E_1(x + x_{11} + x_{31} - 3z_2)f(x)f_1(x_1)f_3(x_3)\big). \end{aligned}$$

It turns out that $F_2(x)$, after simplification, can be written as the following form:

$$F_2(x) = \frac{\sum_{l=0}^{92} a_l x^l}{2^5 3^{19} x^{43} (x^2 + 2\pi^2)^6 (3x^2 + 2\pi^2)^{12} (3x^2 + 4\pi^2)^{12}}, \tag{21}$$

where a_l are the known numbers, and the values of a_{92}, a_{91}, a_{90} are given below

$$a_{92} = 2^5 3^{41} \pi^6, \quad a_{91} = -2^5 3^{37}(225\pi^6 + 7\pi^8), \quad a_{90} = 2^7 3^{38}(162\pi^6 + 127\pi^8).$$

With the help of mathematical software, we find that the largest real zero of $F_2(x)$ is less than 15. Thus, for $x \geq 100$ we have $F_2(x) > 0$. This completes the proof. □

Based on Theorem 2.1 we obtain the following result.

Corollary 2.4. *The sequence* $\{p(n)\}_{n \geq 114}$ *satisfies the Briggs inequality.*

Proof. From Theorem 2.1 it follows that

$$p(n+1)(p(n)^2 - p(n-1)p(n+1)) > p(n-1)(p(n+1)^2 - p(n)p(n+2))$$

for $n \geq 114$. By the log-concavity of $\{p(n)\}_{n \geq 26}$ proved in [29] and [12], we immediately obtain the desired result. □

2.2 Overpartitions

For the overpartition function we have the following result.

Theorem 2.5. *For all* $n \geq 18$, *we have*

$$\overline{p}(n+1)\overline{p}(n)^2 - 2\overline{p}(n-1)\overline{p}(n+1)^2 + \overline{p}(n-1)\overline{p}(n)\overline{p}(n+2) > 0. \tag{22}$$

It's worth noting that our approach to the partition function can be carried over verbatimly to the overpartition function. Along the lines of the proof of Theorem 2.1, we first recall the upper and lower bounds for $\overline{p}(n)$ given by Wang and Yang [33].

Lemma 2.6 ([33], Lemma 5.1). *Let $\overline{\mu}(n) = \pi\sqrt{n}$. Then for any $n \geq 821$, i.e., $\overline{\mu}(n-1) \geq 90$, we have*

$$\frac{\pi^2 e^{\overline{\mu}(n)}}{8} f(\overline{\mu}(n)) < \overline{p}(n) < \frac{\pi^2 e^{\overline{\mu}(n)}}{8} g(\overline{\mu}(n)).$$

Now the proof of Theorem 2.5 can be given in the same manner as that of Theorem 2.1.

Proof of Theorem 2.5. Set $\overline{x} = \overline{\mu}(n-1), \overline{x}_1 = \overline{\mu}(n), \overline{x}_2 = \overline{\mu}(n+1), \overline{x}_3 = \overline{\mu}(n+2)$. Then

$$\overline{x}_1 = \sqrt{\overline{x}^2 + \pi^2}, \quad \overline{x}_2 = \sqrt{\overline{x}^2 + 2\pi^2}, \quad \overline{x}_3 = \sqrt{\overline{x}^2 + 3\pi^2}.$$

One can directly check that (22) holds for $18 \leq \overline{n} \leq 820$. Now we may assume that $n \geq 821$, and hence $\overline{x} \geq 90$.

Using the symbols given by (10) and (11), we set

$$\overline{x}_{11} = \check{h}_{\overline{x}}(\pi^2), \quad \overline{x}_{21} = \check{h}_{\overline{x}}(2\pi^2), \quad \overline{x}_{31} = \check{h}_{\overline{x}}(3\pi^2),$$
$$\overline{x}_{12} = \hat{h}_{\overline{x}}(\pi^2), \quad \overline{x}_{22} = \hat{h}_{\overline{x}}(2\pi^2), \quad \overline{x}_{32} = \hat{h}_{\overline{x}}(3\pi^2).$$

and

$$\overline{f}_i(t) = \frac{t^{10} - t^8\overline{x}_{i2} - 1}{t^{12}}, \quad \overline{g}_i(t) = \frac{t^{10} - t^8\overline{x}_{i1} + 1}{t^{12}}, \quad \overline{z}_2 = \hat{h}_{\overline{x}}(\frac{4\pi^2}{3}).$$

Similar to the proof of Lemma 2.3, we can show that if $\overline{x} \geq 6$ then $\overline{x}_{i1} < \overline{x}_i < \overline{x}_{i2}$ for $1 \leq i \leq 3$. One can also show that if $\overline{x} \geq 1$ then $\overline{f}_i(\overline{x}_i) < f(\overline{x}_i)$ and $g(\overline{x}_i) < \overline{g}_i(\overline{x}_i)$, where $f(t)$ and $g(t)$ are defined by (6). A little computation shows that $\overline{x}_{21} + 2\overline{x}_{11} - 3\overline{z}_2 < 0$, $2\overline{x}_{22} + \overline{x} - 3\overline{z}_2 < 0$ and $x + \overline{x}_{11} + \overline{x}_{31} - 3\overline{z}_2 < 0$ for $\overline{x} \geq 5$.

Let

$$\begin{aligned}\overline{F}_2(\overline{x}) =&\big(E_1(\overline{x}_{21} + 2\overline{x}_{11} - 3\overline{z}_2)\overline{f}_2(\overline{x}_2)\overline{f}_1(\overline{x}_1)^2 - 2E_2(2\overline{x}_{22} + \overline{x} - 3\overline{z}_2)\overline{g}_2(\overline{x}_2)^2\overline{g}(\overline{x}) \\ &+ E_1(\overline{x} + \overline{x}_{11} + \overline{x}_{31} - 3\overline{z}_2)f(\overline{x})\overline{f}_1(\overline{x}_1)\overline{f}_3(\overline{x}_3)\big),\end{aligned}$$

where $E_1(t)$ and $E_2(t)$ are defined as in (16).

By using the same arguments as in the proof of Theorem 2.1, we find that the inequality (22) is equivalent to the positivity of $\overline{F}_2(\overline{x})$. After some simplification, we see that

$$\overline{F}_2(\overline{x}) = \frac{\sum_{l=0}^{92} a_l \overline{x}^l}{2^{25}3^7 x^{43}\left(\overline{x}^2 + \pi^2\right)^{12}\left(\overline{x}^2 + 2\pi^2\right)^{12}\left(\overline{x}^2 + 3\pi^2\right)^{6}}, \tag{23}$$

where a_l are the known number, and the values of a_{92}, a_{91}, a_{90} are given below

$$a_{92} = 2^{22}3^8\pi^6, \quad a_{91} = -2^{21}3^5(450\pi^6 + 7\pi^8), \quad a_{90} = 2^{23}3^6(108\pi^6 + 127\pi^8).$$

By using mathematical software, one can check that the largest real zero of $\overline{F}_2(\overline{x})$ is less than 18. Thus $\overline{F}_2(\overline{x}) > 0$ for $\overline{x} \geq 90$, as desired. This completes the proof. □

Based on Theorem 2.5 and the log-concavity of $\{\overline{p}(n)\}_{n\geq 2}$ due to Engel [16], we immediately obtain the following result.

Corollary 2.7. *The sequence* $\{\overline{p}(n)\}_{n\geq 18}$ *satisfies the Briggs inequality.*

3 k-regular partition functions

In this section we aim to prove that, for $2 \leq k \leq 9$, both the k-regular partition and the k-regular overpartition satisfy the Briggs inequality. Here our approach is exactly the same as that for the partition function and the overpartition given in Section 2.

Fixing an integer $2 \leq k \leq 9$, let a_n to be $p_k(n)$ or $\overline{p}_k(n)$. In order to prove that the sequence $\{a_n\}_{n\geq 0}$ satisfies the Briggs inquality, it suffices to show that it is log-concave and moreover it satisfies (4). Dong and Ji [14] proved the log-concavity of $p_k(n)$ for 2, and their proof also works for $3 \leq k \leq 9$ provided that the appropriate upper and lower bounds of $p_k(n)$ are given. The log-concavity of $\overline{p}_k(n)$ has been established by Peng, Zhang and Zhong [31] when $2 \leq k \leq 9$. The proofs of Theorems 2.1 and 2.5 reveal that, to prove (4) for $a_n = p_k(n)$ and $a_n = \overline{p}_k(n)$, it is necessary to know some bounds of $p_k(n)$ and $\overline{p}_k(n)$. Dong and Ji [14] gave certain upper and lower bounds of $p_2(n)$. For our purpose, we shall present some explicit bounds of $p_k(n)$ when $3 \leq k \leq 9$. For the k-regular overpartitions where $2 \leq k \leq 9$, we shall use the bounds of $\overline{p}_k(n)$ given by Peng, Zhang and Zhong [31].

3.1 k-regular partitions

For the k-regular partition function $p(n)$, we obtain the following result.

Theorem 3.1. *For* $2 \leq k \leq 9$ *and* $n \geq N_k$*, we have*

$$p_k(n+1)p_k(n)^2 - 2p_k(n-1)p_k(n+1)^2 + p_k(n-1)p_k(n)p_k(n+2) > 0, \tag{24}$$

where

$$\begin{aligned} &N_2 = 150, \quad N_3 = 220, \quad N_4 = 75, \quad N_5 = 164, \\ &N_6 = 60, \quad N_7 = 148, \quad N_8 = 78, \quad N_9 = 138. \end{aligned}$$

To prove the above theorm, we need to use the following explicit bounds of $p_k(n)$. To maintain readability, we will provide the proofs for the explicit bounds in the appendix, as they are quite tedious.

Theorem 3.2. *For* $2 \leq k \leq 9$*, let*

$$\mu_k(n) = \frac{\pi}{6}\sqrt{(1-\frac{1}{k})(24n+k-1)}, \quad M_k(n) = \frac{(k-1)\pi^2}{3k\sqrt{k}\mu_k(n)} I_1(\mu_k(n)), \tag{25}$$

where $I_1(s)$ denotes the first modified Bessel function of the first kind. Then

$$M_k(n)\left(1-\frac{1}{\mu_k(n)^6}\right) < p_k(n) < M_k(n)\left(1+\frac{1}{\mu_k(n)^6}\right), \tag{26}$$

whenever $n \geq \hat{n}_k$, where

$$\hat{n}_2 = 1067, \quad \hat{n}_3 = 821, \quad \hat{n}_4 = 711, \quad \hat{n}_5 = 695,$$
$$\hat{n}_6 = 677, \quad \hat{n}_7 = 652, \quad \hat{n}_8 = 651, \quad \hat{n}_9 = 615.$$

We also need the following bounds of the first modified Bessel function of the first kind, due to Dong and Ji [14].

Lemma 3.3 ([14], Lemma 2.2 and equation(3.14))**.** *Let $I_1(s)$ denote the first modified Bessel function of the first kind, and let*

$$D_I(s) := 1-\frac{3}{8s}-\frac{15}{128s^2}-\frac{105}{1024s^3}-\frac{4725}{32768s^4}-\frac{72765}{262144s^5}. \tag{27}$$

Then for $s \geq 26$, we have

$$\frac{e^s}{\sqrt{2\pi s}}\left(D_I(s)-\frac{31}{s^6}\right) \leq I_1(s) \leq \frac{e^s}{\sqrt{2\pi s}}\left(D_I(s)+\frac{31}{s^6}\right). \tag{28}$$

Moreover,

$$I_1(s) \geq \frac{e^s}{\sqrt{2\pi s}}\left(1-\frac{1}{2s}\right). \tag{29}$$

For notational convenience, let

$$x = \mu_6(n-1), \quad y = \mu_6(n), \quad z = \mu_6(n+1), \quad w = \mu_6(n+2) \tag{30}$$

throughout this subsection, where $\mu_6(n)$ is given by (25). Before giving the proof of Theorem 3.1, let us estimate the values x, z and w in terms of y. By (25) we have

$$x = \sqrt{y^2-\frac{5\pi^2}{9}}, \quad z = \sqrt{y^2+\frac{5\pi^2}{9}}, \quad w = \sqrt{y^2+\frac{10\pi^2}{9}}.$$

The following result is analogous to Lemma 2.3, and its proof is ommited here.

Lemma 3.4. *Let x, y, z and w be defined as in (30), let $\check{h}_y(a)$ and $\hat{h}_y(a)$ be given by (10) and (11) respectively, and let*

$$x_1 = \check{h}_y\left(-\frac{5\pi}{9}\right), \quad z_1 = \check{h}_y\left(\frac{5\pi}{9}\right), \quad w_1 = \check{h}_y\left(\frac{10\pi}{9}\right),$$
$$x_2 = \hat{h}_y\left(-\frac{5\pi}{9}\right), \quad z_2 = \hat{h}_y\left(\frac{5\pi}{9}\right), \quad w_2 = \hat{h}_y\left(\frac{10\pi}{9}\right). \tag{31}$$

Then, for $y \geq 3$, we have

$$x_1 < x < x_2, \quad z_1 < z < z_2, \quad w_1 < w < w_2. \tag{32}$$

We proceed to give a proof of Theorem 3.1.

Proof of Theorem 3.1. We shall take $k = 6$ to illustrate our proof, and the proofs for other values of k can be given in the same manner. Let x, z and w be defined in (30). By Theorem 3.2, we find that for $n \geq 1067$, i.e., $y \geq 61$,

$$\frac{5\pi^2}{18\sqrt{6}y} I_1(y)\left(1 - \frac{1}{y^6}\right) < p_6(n) < \frac{5\pi^2}{18\sqrt{6}y} I_1(y)\left(1 + \frac{1}{y^6}\right).$$

By further applying Lemma 3.3, we get that

$$\frac{5\pi^{\frac{3}{2}} \cdot e^y}{36\sqrt{3}y^{\frac{3}{2}}} f(y) < p_6(n) < \frac{5\pi^{\frac{3}{2}} \cdot e^y}{36\sqrt{3}y^{\frac{3}{2}}} g(y) \tag{33}$$

whenever $y \geq 61$, where

$$f(y) = \left(1 - \frac{1}{y^6}\right)\left(D_I(y) - \frac{31}{y^6}\right), \qquad g(y) = \left(1 + \frac{1}{y^6}\right)\left(D_I(y) + \frac{31}{y^6}\right)$$

and $D_I(y)$ is given by (27).

For $t \geq 3$, it is routine to verify that

$$\begin{aligned} f(t) > \tilde{f}(t) &= 1 - \frac{3}{8t} - \frac{15}{128t^2} - \frac{105}{1024t^3} - \frac{4725}{32768t^4} - \frac{72765}{262144t^5} - \frac{32}{t^6}, \\ g(t) < \tilde{g}(t) &= 1 - \frac{3}{8t} - \frac{15}{128t^2} - \frac{105}{1024t^3} - \frac{4725}{32768t^4} - \frac{72765}{262144t^5} + \frac{32}{t^6}. \end{aligned} \tag{34}$$

Combining (33) and (34), we obtain that for $y \geq 61$,

$$p_6(n+1)p_6(n)^2 - 2p_6(n-1)p_6(n+1)^2 + p_6(n-1)p_6(n)p_6(n+2) \geq \left(\frac{5\pi^{\frac{3}{2}}}{36\sqrt{3}}\right)^3 F(y), \tag{35}$$

where

$$F(y) = \frac{e^{2y+z}}{y^3 z^{\frac{3}{2}}}\tilde{f}(z)\tilde{f}(y)^2 - \frac{2e^{x+2z}}{x^{\frac{3}{2}}z^3}\tilde{g}(x)\tilde{g}(z)^2 + \frac{e^{x+y+w}}{x^{\frac{3}{2}}y^{\frac{3}{2}}w^{\frac{3}{2}}}\tilde{f}(x)\tilde{f}(y)\tilde{f}(w).$$

To establish Theorem 3.1, it is sufficient to show $F(n) > 0$. Let

$$\theta_2 = \hat{h}_y\left(\frac{5\pi^2}{27}\right). \tag{36}$$

We find that the sign of $F(n)$ coincides with that of $F(n)e^{-3\theta_2}$. One can verify that for $y \geq 1$,

$$z_1 + 2y - 3\theta_2 = -\frac{25\left(432\pi^4 y^4 - 160\pi^6 y^2 + 125\pi^8\right)}{419904y^7} < 0$$

$$x_2 + 2z_2 - 3\theta_2 = -\frac{25\left(-5\pi^6 + 54\pi^4 y^2\right)}{13122 y^5} < 0$$
$$x_1 + y + w_1 - 3\theta_2 = -\frac{25\left(3024\pi^4 y^4 - 1240\pi^6 y^2 + 2125\pi^8\right)}{419904 y^7} < 0.$$

Then according to (15), we have

$$F(y) \geq e^{3\theta_2}\Bigg(E_1(z_1 + 2y - 3\theta_2)\frac{\tilde{f}(z)\tilde{f}(y)^2}{z^{\frac{3}{2}}y^3} - 2E_2(x_2 + 2z_2 - 3\theta_2)\frac{\tilde{g}(x)\tilde{g}(z)^2}{x^{\frac{3}{2}}z^3} + E_1(x_1 + y + w_1 - 3\theta_2)\frac{\tilde{f}(x)\tilde{f}(y)\tilde{f}(w)}{x^{\frac{3}{2}}y^{\frac{3}{2}}w^{\frac{3}{2}}}\Bigg). \tag{37}$$

where $E_1(y)$ and $E_2(y)$ are given by (16). Based on (34) and (32), one can verify that if $y \geq 3$ then

$$\begin{aligned}
\tilde{f}(x) > \lambda_1(y) &= 1 - \frac{3}{8x_1} - \frac{15}{128x^2} - \frac{105}{1024x^2x_1} - \frac{4725}{32768x^4} - \frac{72765}{262144x^4x_1} - \frac{32}{x^6},\\
\tilde{f}(z) > \lambda_2(y) &= 1 - \frac{3}{8z_1} - \frac{15}{128z^2} - \frac{105}{1024z^2z_1} - \frac{4725}{32768z^4} - \frac{72765}{262144z^4z_1} - \frac{32}{z^6},\\
\tilde{f}(w) > \lambda_3(y) &= 1 - \frac{3}{8w_1} - \frac{15}{128w^2} - \frac{105}{1024w^2w_1} - \frac{4725}{32768w^4} - \frac{72765}{262144w^4w_1} - \frac{32}{w^6},\\
\tilde{g}(x) < \lambda_4(y) &= 1 - \frac{3}{8x_2} - \frac{15}{128x^2} - \frac{105}{1024x^2x_2} - \frac{4725}{32768x^4} - \frac{72765}{262144x^4x_2} + \frac{32}{x^6},\\
\tilde{g}(z) < \lambda_5(y) &= 1 - \frac{3}{8z_2} - \frac{15}{128z^2} - \frac{105}{1024z^2z_2} - \frac{4725}{32768z^4} - \frac{72765}{262144z^4z_2} + \frac{32}{z^6},\\
\tilde{g}(w) < \lambda_6(y) &= 1 - \frac{3}{8w_2} - \frac{15}{128w^2} - \frac{105}{1024w^2w_2} - \frac{4725}{32768w^4} - \frac{72765}{262144w^4w_2} + \frac{32}{w^6}.
\end{aligned} \tag{38}$$

Thus, it is enough to show for $y \geq 61$,

$$E_1(z_1 + 2y - 3\theta_2)\frac{\lambda_2(y)\tilde{f}(y)^2}{z^{\frac{3}{2}}y^3} - 2E_2(x_2 + 2z_2 - 3\theta_2)\frac{\lambda_4(y)\lambda_5(y)^2}{x^{\frac{3}{2}}z^3} + E_1(x_1 + y + w_1 - 3\theta_2)\frac{\lambda_1(y)\tilde{f}(y)\lambda_3(y)}{x^{\frac{3}{2}}y^{\frac{3}{2}}w^{\frac{3}{2}}} > 0. \tag{39}$$

Notice that there exist the annoying terms $\sqrt{x}$, $\sqrt{z}$ and $\sqrt{w}$, we need to do a little change to estimate theses terms. Let

$$\theta = \sqrt{y^2 + \frac{5\pi^2}{27}}, \quad W_1 = \sqrt{\frac{\theta^9}{y^6z^3}}, \quad W_2 = \sqrt{\frac{\theta^9}{x^3z^6}}, \quad W_3 = \sqrt{\frac{\theta^9}{x^3y^3w^3}}. \tag{40}$$

Then it can be calculated using Taylor expansion that for $y \geq 5$,

$$W_1 > W_{11}, \quad W_2 < W_{22}, \quad W_3 > W_{31}, \tag{41}$$

where

$$W_{11} = 1 + \frac{25\pi^4}{324y^4} - \frac{250\pi^6}{6561y^6} + \frac{38125\pi^8}{1889568y^8} - \frac{34375\pi^{10}}{3188646y^{10}},$$
$$W_{22} = 1 + \frac{25\pi^4}{81y^4} - \frac{250\pi^6}{6561y^6} + \frac{11875\pi^8}{118098y^8},$$
$$W_{31} = 1 + \frac{175\pi^4}{324y^4} - \frac{3875\pi^6}{13122y^6} + \frac{848125\pi^8}{1889568y^8} - \frac{5171875\pi^{10}}{12754584y^{10}}.$$

We find that the left-hand side of (39) is greater than $\widetilde{F}(y)/\sqrt{\theta^9}$, where

$$\begin{aligned}\widetilde{F}(y) =& E_1(z_1 + 2y - 3\theta_2)W_{11}\lambda_2(y)\tilde{f}(y)^2 - 2E_2(x_2 + 2z_2 - 3\theta_2)W_{22}\lambda_4(y)\lambda_5(y)^2 \\ &+ E_1(x_1 + y + w_1 - 3\theta_2)W_{31}\lambda_1(y)\lambda_3(y)\tilde{f}(y).\end{aligned} \tag{42}$$

By substituting the expressions for x_1, x_2, z_1, z_2, w_1, w_2, and θ_2 into $\widetilde{F}(y)$, we can rewrite $\widetilde{F}(y)$ as

$$\widetilde{F}(y) = \frac{\sum_{k=0}^{104} a_k y^k}{2^{75}3^{38}y^{43}H_1(y)},$$

where

$$\begin{aligned}H_1(y) =& 2^{75}3^{38}y^{43}\left(9y^2 - 5\pi^2\right)^3\left(419904y^8 + 116640\pi^2y^6 - 16200\pi^4y^4 + 4500\pi^6y^2 - 3125\pi^8\right)\times \\ &\left(9y^2 + 5\pi^2\right)^6\left(419904y^8 - 116640\pi^2y^6 - 16200\pi^4y^4 - 4500\pi^6y^2 - 3125\pi^8\right)\times \\ &\left(9y^2 + 10\pi^2\right)^3\left(26244y^8 + 14580\pi^2y^6 - 4050\pi^4y^4 + 2250\pi^6y^2 - 3125\pi^8\right)\times \\ &\left(11664y^6 + 3240\pi^2y^4 - 450\pi^4y^2 + 125\pi^6\right)^2\left(11664y^6 - 3240\pi^2y^4 - 450\pi^4y^2 - 125\pi^6\right).\end{aligned}$$

and a_k are the known numbers. Specially, we give the values of a_{104} and a_{103} below:

$$a_{104} = 2^{98}3^{99}5^3\pi^6, \qquad a_{103} = 2^{95}3^{96}(2^25^4\pi^8 + 3^35^341\pi^6 - 2^{13}3^9).$$

One can check that $H_1(y)$ is positive for $y \geq 3$. It remains to show that

$$H(y) = \sum_{k=0}^{104} a_k y^k > 0. \tag{43}$$

It can be computed by mathematical software that the largest real zero of $H(y)$ is less than 12. Thus, for $y \geq 61$, i.e, $n \geq 1067$, we have $F(y) > 0$ along with the fact that $H(12) > 0$. Additionally, for $150 \leq n \leq 1066$ one can directly verify (24). This completes the proof. □

For the log-concavity of the k-regular partition function, we have the following result.

Theorem 3.5. *For $2 \leq k \leq 9$, let N_k be given as in Theorem 3.1. Then the sequence $\{p_k(n)\}_{n \geq N_k}$ is log-concave.*

Proof. The log-concavity of $\{p_k(n)\}_{n\geq 58}$ when $k = 2, 3, 4$ or 5 has been proved by Dong and Ji in [14, Theorem 1.4]. For each $6 \leq k \leq 9$, one can give a proof the log-concavity of $\{p_k(n)\}_{n\geq 36}$, exactly like that of Theorem 3.1. Again we take $k = 6$ to illustrate the idea. Following the approach to deduce (53), we obtain that for $y \geq 61$,

$$p_6(n)^2 - p_6(n-1)p_6(n+1) \geq \left(\frac{\pi^{\frac{3}{2}}}{12}\right)^2 e^{2y} J(y), \tag{44}$$

where

$$J(y) = \frac{1}{y^3}\tilde{f}(y)^2 - \frac{e^{x_2+z_2-2y}}{x^{\frac{3}{2}}z^{\frac{3}{2}}}\tilde{g}(x)\tilde{g}(z),$$

the symbols x, y, z, x_2, z_2 are defined as in (30) and (31), and the functions $\tilde{f}(t), \tilde{g}(t)$ are given by (34). It can be verified that $x_2 + z_2 - 2y < 0$ for $y \geq 1$. According to (15) and the above bounds of $\tilde{g}(x)$ and $\tilde{g}(z)$ (immediately before (39)), we have

$$J(y) \geq \frac{1}{y^3}\tilde{f}(y)^2 - \frac{E_2(x_2+z_2-2y)}{x^{\frac{3}{2}}z^{\frac{3}{2}}}\lambda_4(y)\lambda_5(y), \tag{45}$$

which $\lambda_4(y)$ and $\lambda_5(y)$ are defined in (38). Let

$$V = \sqrt{\frac{y^6}{x^3z^3}} \qquad \text{and} \qquad V_2 = 1 + \frac{25\pi^4}{108y^4} + \frac{4375\pi^8}{y^8} + \frac{240625\pi^{12}}{12754584}.$$

Using Taylor expansion, it can be checked that $V < V_2$ for $y \geq 4$. Thus the right-hand side of (45) is greater than $\widetilde{J}(y)/y^3$, where

$$\widetilde{J}(y) = \tilde{f}(y)^2 - E_2(x_2+z_2-2y)V_2\lambda_4(y)\lambda_5(y).$$

After some simplification, we see that

$$\widetilde{J}(y) = \frac{\sum_{l=0}^{39} a_l y^l}{J_1(y)}, \tag{46}$$

where

$$J_1(y) = 2^{40}3^{21}(9y^2-5\pi^2)^3(9y^2+5\pi^2)^3(11664y^6 - 3240\pi^2y^4 - 450\pi^4y^2 - 125\pi^6)$$
$$(11664y^6 + 3240\pi^2y^4 - 450\pi^4y^2 + 125\pi^6),$$

and a_l are the known number, and the values of a_{39} and a_{38} are given below

$$a_{39} = 2^{46}3^{41}5^2\pi^4, \quad a_{38} = -2^{44}3^{42}5^3\pi^4.$$

One can check that $J_1(y)$ is positive for $y \geq 3$. Then it is sufficient to show the denominator of $\widetilde{J}(y) > 0$. It can be computed by mathematical software that the largest real zero of $\widetilde{J}(y)$ is less than 5. Thus, for $y \geq 61$, i.e, $n \geq 1067$, we have $\widetilde{J}(y) > 0$. Additionally, for $36 \leq n \leq 1066$ one can directly verify that $p_6(n)^2 - p_6(n-1)p_6(n+1) \geq 0$. This completes the proof. ☐

Based on Theorems 3.1 and 3.5, we immediately obtain the following result.

Corollary 3.6. *For $2 \leq k \leq 9$, let N_k be given as in Theorem 3.1. Then the sequence $\{p_k(n)\}_{n\geq N_k}$ satisfies the Briggs inequality.*

3.2 k-regular overpartition

For the k-regular overpartition function $\overline{p}(n)$, we obtain the following result.

Theorem 3.7. *For $2 \leq k \leq 9$ and $n \geq N_k$, we have*

$$\overline{p}_k(n+1)\overline{p}_k(n)^2 - 2\overline{p}_k(n-1)\overline{p}_k(n+1)^2 + \overline{p}_k(n-1)\overline{p}_k(n)\overline{p}_k(n+2) > 0, \tag{47}$$

where

$$\begin{aligned} &\overline{N}_2 = 30, \quad \overline{N}_3 = 9, \quad \overline{N}_4 = 21, \quad \overline{N}_5 = 21, \\ &\overline{N}_6 = 15, \quad \overline{N}_7 = 18, \quad \overline{N}_8 = 18, \quad \overline{N}_9 = 15. \end{aligned}$$

To prove Theorem 3.7, we need the following upper and lower bounds of $\overline{p}_k(n)$ for $2 \leq k \leq 9$ given by Peng, Zhang and Zhong [31].

Theorem 3.8 ([31], Corollary 3.4)**.** *Let $I_1(s)$ denote the first modified Bessel function of the first kind and*

$$\overline{\mu}_k(n) = \sqrt{(1-\frac{1}{k})n\pi}.$$

For $2 \leq k \leq 9$ and $\overline{\mu}_k \geq \hat{n}_k$, we have

$$M_k(n)\left(1-\frac{1}{\overline{\mu}_k{}^6}\right) \leq \overline{p}_k(n) \leq M_k(n)\left(1+\frac{1}{\overline{\mu}_k{}^6}\right), \tag{48}$$

where $M_k(n) = C_k(n)I_1(\overline{\mu}_k)$, and the values of $\hat{n}_k$ and $C_k(n)$ are given in Table 1.

k	2	3	4	5	6	7	8	9
$C_k(n)$	$\frac{\pi^2}{\sqrt{8}\overline{\mu}_2}$	$\frac{2\sqrt{3}\pi^2}{9\overline{\mu}_3}$	$\frac{3\pi^3}{4\overline{\mu}_4}$	$\frac{8\sqrt{5}\pi^2}{25\overline{\mu}_5}$	$\frac{5\sqrt{6}\pi^2}{18\overline{\mu}_6}$	$\frac{18\sqrt{7}\pi^2}{49\overline{\mu}_7}$	$\frac{7\sqrt{2}\pi^2}{8\overline{\mu}_8}$	$\frac{8\pi^2}{9\overline{\mu}_9}$
$\hat{n}_k$	43	49	43	58	130	102	129	268

Table 1: Values of $\hat{n}_k$ and $C_k(n)$ for $2 \leq k \leq 9$.

Now we are in the position to prove Theorem 3.7.

Proof of Theorem 3.7. We shall take $k = 6$ to illustrate our proof, and the proofs for other values of k can be given in the same manner. Set

$$x = \overline{\mu}_6(n-1), \quad y = \overline{\mu}_6(n), \quad z = \overline{\mu}_6(n+1), \quad w = \overline{\mu}_6(n+2). \tag{49}$$

Then

$$x = \sqrt{y^2 - \frac{5\pi^2}{6}}, \quad z = \sqrt{y^2 + \frac{5\pi^2}{6}}, \quad w = \sqrt{y^2 + \frac{5\pi^2}{3}}.$$

By Theorem 3.8, we find that for $n \geq 225$, i.e., $y \geq 43$,

$$\frac{5\sqrt{6}\pi^2}{18y} I_1(y) \left(1 - \frac{1}{y^6}\right) < \overline{p}_6(n) < \frac{5\sqrt{6}\pi^2}{18y} I_1(y) \left(1 + \frac{1}{y^6}\right).$$

Applying Lemma 3.3, we have

$$\frac{5\pi^{\frac{3}{2}} \cdot e^y}{6\sqrt{3}y^{\frac{3}{2}}} f(y) < \overline{p}_6(n) < \frac{5\pi^{\frac{3}{2}} \cdot e^y}{6\sqrt{3}y^{\frac{3}{2}}} g(y) \tag{50}$$

whenever $y \geq 43$, where the functions $f(y), g(y)$ are given by (34). Using the symbols given by (31)and (36), we set

$$\begin{aligned} x_1 &= \check{h}_y\left(-\frac{5\pi^2}{6}\right), \quad z_1 = \check{h}_y\left(\frac{5\pi^2}{6}\right), \quad w_1 = \check{h}_y\left(\frac{5\pi^2}{3}\right), \\ x_2 &= \hat{h}_y\left(-\frac{5\pi^2}{6}\right), \quad z_2 = \hat{h}_y\left(\frac{5\pi^2}{6}\right), \quad w_2 = \hat{h}_y\left(\frac{5\pi^2}{3}\right), \end{aligned} \tag{51}$$

and $\theta_2 = \hat{h}_y\left(\frac{5\pi^2}{18}\right)$.

Similar to the proof of Lemma 2.3, we can show that if $y \geq 3$ then

$$x_1 < x < x_2, \quad z_1 < z < z_2, \quad w_1 < w < w_2. \tag{52}$$

Recall the expressions of $\tilde{f}(t)$ and $\tilde{g}(t)$ in (34). Combining (50) and (34), we obtain that for $y \geq 43$,

$$\overline{p}_6(n+1)\overline{p}_6(n)^2 - 2\overline{p}_6(n-1)\overline{p}_6(n+1)^2 + \overline{p}_6(n-1)\overline{p}_6(n)\overline{p}_6(n+2) \geq \left(\frac{5\pi^{\frac{3}{2}}}{6\sqrt{3}}\right)^3 F(y), \tag{53}$$

where

$$F(y) = \frac{e^{2y+z}}{y^3 z^{\frac{3}{2}}} \tilde{f}(z)\tilde{f}(y)^2 - \frac{2e^{x+2z}}{x^{\frac{3}{2}} z^3} \tilde{g}(x)\tilde{g}(z)^2 + \frac{e^{x+y+w}}{x^{\frac{3}{2}} y^{\frac{3}{2}} w^{\frac{3}{2}}} \tilde{f}(x)\tilde{f}(y)\tilde{f}(w).$$

A little computation shows that $z_1+2y-3\theta_2 < 0$, $x_2+2z_2-3\theta_2 < 0$ and $x_1+y+w_1-3\theta_2 < 0$ for $y \geq 1$. By (15), we have

$$\begin{aligned} F(y) \geq e^{3\theta_2} \Bigg(& E_1(z_1 + 2y - 3\theta_2) \frac{\tilde{f}(z)\tilde{f}(y)^2}{z^{\frac{3}{2}} y^3} - 2E_2(x_2 + 2z_2 - 3\theta_2) \frac{\tilde{g}(x)\tilde{g}(z)^2}{x^{\frac{3}{2}} z^3} \\ & + E_1(x_1 + y + w_1 - 3\theta_2) \frac{\tilde{f}(x)\tilde{f}(y)\tilde{f}(w)}{x^{\frac{3}{2}} y^{\frac{3}{2}} w^{\frac{3}{2}}} \Bigg). \end{aligned}$$

Using the same arguments as in the proof of Theorem 3.1, it is sufficient to show for $y \geq 43$,

$$E_1(z_1 + 2y - 3\theta_2)\frac{\lambda_2(y)\tilde{f}(y)^2}{z^{\frac{3}{2}}y^3} - 2E_2(x_2 + 2z_2 - 3\theta_2)\frac{\lambda_4(y)\lambda_5(y)^2}{x^{\frac{3}{2}}z^3} + E_1(x_1 + y + w_1 - 3\theta_2)\frac{\lambda_1(y)\tilde{f}(y)\lambda_3(y)}{x^{\frac{3}{2}}y^{\frac{3}{2}}w^{\frac{3}{2}}} > 0. \tag{54}$$

Using the symbols given by (40), we set

$$\theta = \sqrt{y^2 + \frac{5\pi^2}{18}}, \quad W_1 = \sqrt{\frac{\theta^9}{y^6 z^3}}, \quad W_2 = \sqrt{\frac{\theta^9}{x^3 z^6}}, \quad W_3 = \sqrt{\frac{\theta^9}{x^3 y^3 w^3}}.$$

and

$$\begin{aligned}
W_{11} &= 1 + \frac{25\pi^4}{144y^4} - \frac{125\pi^6}{972y^6} + \frac{38125\pi^8}{373248y^8} - \frac{34375\pi^{10}}{419904y^{10}}, \\
W_{22} &= 1 + \frac{25\pi^4}{36y^4} - \frac{125\pi^6}{972y^6} + \frac{11875\pi^8}{23328y^8}, \\
W_{31} &= 1 + \frac{175\pi^4}{144y^4} - \frac{3875\pi^6}{3888y^6} + \frac{848125\pi^8}{373248y^8} - \frac{5171875\pi^{10}}{1679616y^{10}}.
\end{aligned}$$

One can check that for $y \geq 1$, $W_1 > W_{11}, W_2 < W_{22}, W_3 > W_{31}$. We find that the left-hand side of (54) is greater than $\widetilde{F}(y)/\sqrt{\theta^9}$, where

$$\begin{aligned}
\widetilde{F}(y) =& E_1(z_1 + 2y - 3\theta_2)W_{11}\lambda_2(y)\tilde{f}(y)^2 - 2E_2(x_2 + 2z_2 - 3\theta_2)W_{22}\lambda_4(y)\lambda_5(y)^2 \\
&+ E_1(x_1 + y + w_1 - 3\theta_2)W_{31}\lambda_1(y)\lambda_3(y)\tilde{f}(y).
\end{aligned} \tag{55}$$

Note that for $1 \leq i \leq 4$, the function $\lambda_i(y)$ is defined in a similar way to (38) by substituting the variables in (38) with the variables in (49) and (51). After some simplification, we see that

$$\widetilde{F}(y) = \frac{\sum_{l=0}^{104} a_l y^l}{H_1(y)}, \tag{56}$$

where

$$\begin{aligned}
H_1(y) =& 2^{89}3^{24}y^{43}\left(5\pi^2 + 3y^2\right)^3\left(-5\pi^2 + 6y^2\right)^3\left(5\pi^2 + 6y^2\right)^6 \\
&\times \left(-125\pi^6 - 300\pi^4y^2 - 1440\pi^2y^4 + 3456y^6\right)\left(125\pi^6 - 300\pi^4y^2 + 1440\pi^2y^4 + 3456y^6\right)^2 \\
&\times \left(-3125\pi^8 + 1500\pi^6y^2 - 1800\pi^4y^4 + 4320\pi^2y^6 + 5184y^8\right) \\
&\times \left(-3125\pi^8 - 3000\pi^6y^2 - 7200\pi^4y^4 - 34560\pi^2y^6 + 82944y^8\right) \\
&\times \left(-3125\pi^8 + 3000\pi^6y^2 - 7200\pi^4y^4 + 34560\pi^2y^6 + 82944y^8\right),
\end{aligned}$$

and a_l are the known numbers, and the values of a_{104} and a_{103} are given below

$$a_{104} = 2^{139}3^{55}5^2\pi^6, \quad a_{103} = 2^{136}3^{53}(25^4\pi^8 - 3^25^341\pi^6 - 2^{16}3^5).$$

One can check that $H_1(y)$ is positive for $y \geq 3$. Then it is sufficient to show the denominator of $\widetilde{F}(y) > 0$ It can be computed by mathematical software that the largest real zero of $\widetilde{F}(y)$ is less than 9. Thus, for $y \geq 43$, i.e, $n \geq 225$, we have $\widetilde{F}(y) > 0$. Additionally, for $15 \leq n \leq 224$ one can directly verify that inequality (47). This completes the proof. $\square$

For $2 \leq k \leq 9$, Peng, Zhang and Zhong [31] proved that the log-concavity of $\{p_k(n)\}_{n\geq 1}$. Based on their result and Theorem 3.7, we have the following result.

Theorem 3.9. *For $2 \leq k \leq 9$, let $\overline{N}_k$ be given as in Theorem 3.7. Then the sequence $\{\overline{p}_k(n)\}_{n\geq \overline{N}_k}$ satisfies Briggs inequality.*

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4 Appendix: proof of Theorem 3.2

In this appendix, we will follow Dong and Ji [14] to establish upper and lower bounds of $p_k(n)$ for $3 \leq k \leq 9$ by using Chern's formula for η-quotients.

Firstly, let us review Chern's theorem. We adopt the notations in [14]. Let $\mathbf{m} = (m_1, \ldots, m_R)$ be a sequence of R distinct positive integers and $\delta = (\delta_1, \ldots, \delta_R)$ be a sequence of R non-zero integers. Assuming that h and j are positive integers with $\gcd(h, j) = 1$, set

$$C_1 = -\frac{1}{2}\sum_{r=1}^{R} \delta_r, \qquad C_2 = \sum_{r=1}^{R} m_r \delta_r,$$

$$C_3(l) = -\sum_{r=1}^{R} \frac{\delta_r \gcd^2(m_r, l)}{m_r}, \qquad C_4(l) = \prod_{r=1}^{R} \left(\frac{m_r}{\gcd(m_r, l)}\right)^{-\frac{\delta_r}{2}},$$

$$\hat{A}_l(n) = \sum_{\substack{0\leq h<l \\ \gcd(h,l)=1}} \exp\left(-\frac{2\pi n h i}{l} - \pi i \sum_{r=1}^{R} \delta_r s\left(\frac{m_r h}{\gcd(m_r, l)}, \frac{l}{\gcd(m_r, l)}\right)\right), \tag{57}$$

where $s(h, j)$ is the Dedekind sum. Take $L = \mathrm{lcm}(m_1, \ldots, m_R)$, the least common multiple of $m_1, \ldots, m_R$. We divide the set $\{1, 2 \cdots, L\}$ into the following two disjoint subsets:

$$\mathcal{L}_{>0} := \{1 \leq l \leq L \mid C_3(l) > 0\}, \qquad \mathcal{L}_{\leq 0} := \{1 \leq l \leq L \mid C_3(l) \leq 0\}. \tag{58}$$

Define

$$\mathbb{E}_{C_1}(s) := \begin{cases} 1, & C_1 = 0, \\ 2\sqrt{s}, & C_1 = -\frac{1}{2}, \\ s\log(s+1), & C_1 = -1, \\ s^{-2C_1-1}\zeta(-C_1), & \text{otherwise}, \end{cases} \tag{59}$$

where $\zeta(\cdot)$ is Riemann zeta-function. Now, we are able to give Chern's theorem.

Theorem 4.1 ([8], Theorem 1.1). *Let*

$$G(q) = \sum_{n\geq 0} g(n)q^n = \prod_{r=1}^{R} (q^{m_r}; q^{m_r})_\infty^{\delta_r}.$$

If $C_1 \leq 0$ and the inequality

$$\min_{1\leq r\leq R} \left(\frac{\gcd^2(m_r, l)}{m_r}\right) \geq \frac{C_3(l)}{24} \tag{60}$$

holds for all $1 \leq l \leq L$, then for positive integers N and $n > -\frac{c_2}{24}$, we have

$$g(n) = E(n) + \sum_{l\in\mathcal{L}_{>0}} 2\pi C_4(l) \left(\frac{24n + C_2}{C_3(l)}\right)^{-\frac{C_1+1}{2}} \sum_{\substack{1\leq t\leq N \\ t\equiv_L l}} I_{-C_1-1}\left(\frac{\pi}{6t}\sqrt{C_3(l)(24n + C_2)}\right) \frac{\hat{A}_t(n)}{t}, \tag{61}$$

where

$$\begin{aligned} |E(n)| \leq & \frac{2^{-C_1}\pi^{-1}N^{-C_1+2}}{n + \frac{C_2}{24}} \exp\left(\frac{2\pi}{N^2}\left(n + \frac{C_2}{24}\right)\right) \sum_{l\in\mathcal{L}_{>0}} \exp\left(\frac{C_3(l)\pi}{3}\right) \\ & + 2\exp\left(\frac{2\pi}{N^2}\left(n + \frac{C_2}{24}\right)\right) \mathbb{E}_{C_1}(N) \times \left(- \sum_{l\in\mathcal{L}_{>0}} C_4(l) \exp\left(\frac{\pi C_3(l)}{24}\right) \right. \\ & \left. + \sum_{1\leq l\leq L} C_4(l) \exp\left(\frac{\pi C_3(l)}{24} + \sum_{r=1}^{R} \frac{|\delta_r| \exp\left(-\pi \gcd^2(m_r, l)/m_r\right)}{\left(1 - \exp\left(-\pi \gcd^2(m_r, l)/m_r\right)\right)^2}\right)\right). \end{aligned} \tag{62}$$

and $I_\nu(s)$ is the ν-th modified Bessel function of the first kind.

Recall that, for $k \geq 2$, the generating function for the sequence $\{p_k(n)\}_n$ is as follows:

$$\sum_{n\geq 0} p_k(n)q^n = \prod_{n=1}^{\infty} \frac{1 - q^{kn}}{1 - q^n} = \frac{(q^k; q^k)_\infty}{(q; q)_\infty}, \tag{63}$$

where $(a; q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)$. In order to use Theorem 4.1 to get bounds of $p_k(n)$, we need to $g(n) = p_k(n)$ in Chern's theorem, in which case we have $\mathbf{m} = (1, k)$, $\delta = (-1, 1)$, $R = 2$, $L = k$ and

$$C_1 = 0, \qquad C_2 = k - 1, \qquad C_3(l) = 1 - \frac{\gcd(k, l)^2}{k}, \qquad C_4(l) = \sqrt{\frac{\gcd(k, l)}{k}}. \tag{64}$$

We also need to check the conditions of theorem 4.1. The condition $C_1 \leq 0$ naturally holds for any $k \geq 2$, while inequality (60) is satisfied only for $2 \leq k \leq 24$. Consequently, it is feasible to apply Chern's theorem to asymptotically estimate $p_k(n)$ for $2 \leq k \leq 24$. We have the following result.

Proposition 4.2. *For any* $k \geq 2$*, let* $c_2(k) = k-1, c_3(k,l) = 1 - \frac{\gcd(k,l)^2}{k}, c_4(k,l) = \sqrt{\frac{\gcd(k,l)}{k}}$ *and*

$$\mu_k(n) = \frac{\pi}{6}\sqrt{c_3(k,1)(24n + c_2(k))}.$$

Then, for $2 \leq k \leq 24$ *and any positive integers* n, N*, we have*

$$p_k(n) = \frac{(k-1)\pi^2}{3k\sqrt{k}\mu_k(n)} I_1(\mu_k(n)) + R_k(n), \tag{65}$$

where $R_k(n) = E_k(n) + B_k(n)$ *and*

$$|E_k(n)| \leq \frac{24N^2}{\pi(24n + c_2(k))} \exp\left(\frac{\pi\,(24n + c_2(k))}{12N^2}\right) \sum_{l \in \mathcal{L}_{>0}} \exp\left(\frac{c_3(k,l)\pi}{3}\right) + 2\exp\left(\frac{\pi\,(24n + c_2(k))}{12N^2}\right)$$

$$\times \left(\sum_{1 \leq l \leq k} c_4(k,l) \exp\left(\frac{\pi c_3(k,l)}{24} + \frac{\exp(-\pi)}{(1-\exp(-\pi))^2} + \frac{\exp\left(-\pi \gcd^2(k,l)/k\right)}{\left(1 - \exp\left(-\pi\gcd^2(k,l)/k\right)\right)^2}\right)\right.$$

$$\left. - \sum_{l \in \mathcal{L}_{>0}} c_4(k,l) \exp\left(\frac{\pi c_3(k,l)}{24}\right)\right). \tag{66}$$

and

$$|B_k(n)| \leq \frac{\pi^2 c_4(k,l')c_3(k,1)}{3y_k} \sum_{l \in \mathcal{L}_{>0}} \sum_{\substack{2 \leq t \leq N \\ t \equiv_k l}} I_1\left(\frac{y_k}{t}\right), \tag{67}$$

with $c_4(k,l') = \max\{c_4(k,l) \mid l \in \mathcal{L}_{>0}\}$.

Proof. By substituting the values of (64) into Theorem 4.1, we get

$$p_k(n) = E_k(n) + \sum_{l \in \mathcal{L}_{>0}} 2\pi \frac{c_4(l)\sqrt{c_3(l)}}{\sqrt{24n + c_2}} \times \sum_{\substack{1 \leq t \leq N \\ t \equiv_k l}} I_1\left(\frac{\pi}{6t}\sqrt{c_3(l)(24n + c_2)}\right) \frac{\hat{A}_t(n)}{t}, \tag{68}$$

where $E_k(n)$ plays the role of $E(n)$ of (61) and satisfies (66). We find that for any $2 \leq k \leq 24$, $1 \in \mathcal{L}_{>0}$. Moreover, $1 \equiv_k 1$ for any k, and hence $t = 1$ will appear in the second sumnation of (68). Based on the fact that $\hat{A}_1(n) = 1$, (68) can be written as

$$p_k(n) = \frac{\pi^2 c_4(1)c_3(1)}{3y_k} I_1(y_k) + E_k(n) + B_k(n),$$

where

$$B_k(n) = \frac{\pi^2 c_4(1)c_3(1)}{3y_k} \sum_{\substack{2 \leq t \leq N \\ t \equiv_k 1}} I_1\left(\frac{y_k}{t}\right) \frac{\hat{A}_t(n)}{t}$$

$$+\sum_{l\in\mathcal{L}_{>0}\setminus\{1\}} \frac{\pi^2 c_4(l)\sqrt{c_3(l)}\sqrt{c_3(1)}}{3y_k} \times \sum_{\substack{2\le t\le N\\ t\equiv_k l}} I_1\left(\frac{\sqrt{c_3(l)}y_k}{\sqrt{c_3(1)}t}\right)\frac{\hat{A}_t(n)}{t}.$$

Noting that $c_3(l) \le c_3(1)$ for $l \ge 1$, $I_1(s)$ is increasing for $s > 0$ and $|\hat{A}_t(n)| \le t$ for $t \ge 1$, we get

$$|B_k(n)| \le \frac{\pi^2 c_4(1)c_3(1)}{3y_k} \sum_{\substack{2\le t\le N\\ t\equiv_k 1}} I_1\left(\frac{y_k}{t}\right) + \sum_{l\in\mathcal{L}_{>0}\setminus\{1\}} \frac{\pi^2 c_4(l)c_3(1)}{3y_k} \times \sum_{\substack{2\le t\le N\\ t\equiv_k l}} I_1\left(\frac{y_k}{t}\right).$$

We immediately obtain the desired result. □

One can see that the above bound of $\hat{R}_k(n)$ in (65) is complicated, and it is not sufficient for our purpose. By using the following upper bound on the first modified Bessel function of the first kind $I_1(s)$:

$$I_1(s) \le \sqrt{\frac{2}{\pi s}} e^s, \tag{69}$$

due to Bringmann, Kane, Rolen and Trippin [6], we are able to give a simpler bound of $\hat{R}_k(n)$. We have the following result.

Theorem 4.3. *For $2 \le k \le 9$, let $\mu_k(n)$ and $R_k(n)$ be given as in Proposition (4.2), and let n_k and $\hat{R}_k(n)$ be given as in Table 2. If $\mu_k(n) \ge n_k$, then $|R_k(n)| \le \hat{R}_k(n)$.*

k	n_k	$\hat{R}_k(n)$	k	n_k	$\hat{R}_k(n)$
2	15	$\frac{\pi^{\frac{3}{2}}}{3\sqrt{2}\sqrt{y_2}} \exp\left(\frac{y_2}{2}\right)$	6	16	$\frac{20\pi^{\frac{3}{2}}}{27\sqrt{3y_6}} \exp\left(\frac{y_6}{2}\right)$
3	14	$\frac{16\pi^{\frac{3}{2}}}{27\sqrt{3y_3}} \exp\left(\frac{y_3}{2}\right)$	7	21	$\frac{48\pi^{\frac{3}{2}}}{49\sqrt{7y_7}} \exp\left(\frac{y_7}{2}\right)$
4	15	$\frac{\pi^{\frac{3}{2}}}{4\sqrt{y_4}} \exp\left(\frac{y_4}{2}\right)$	8	23	$\frac{7\pi^{\frac{3}{2}}}{16\sqrt{y_8}} \exp\left(\frac{y_8}{2}\right)$
5	16	$\frac{64\pi^{\frac{3}{2}}}{75\sqrt{5y_5}} \exp\left(\frac{y_5}{2}\right)$	9	28	$\frac{64\pi^{\frac{3}{2}}}{243\sqrt{y_9}} \exp\left(\frac{y_9}{2}\right)$

Table 2: Values of n_k and $\hat{R}_k(n)$ for $2 \le k \le 9$.

Proof. We prove the theorem for $k = 6$ and omit the details for other values of k. By Proposition 4.2, it is enough to show that for $\mu_6(n) \ge 16$,

$$|E_6(n)| + |B_6(n)| \le \frac{20\pi^{\frac{3}{2}}}{27\sqrt{3\mu_6(n)}},$$

where $E_6(n)$ satisifies (66) and $B_6(n)$ satisfies (67).

To further estimate $E_6(n)$ and $B_6(n)$, we first determine the values of $c_3(6,l)$ and $c_4(6,l)$, which are listed in Table 3. For notational convenience, set $a(x)=\frac{e^x}{(1-e^x)^2}$. By

l	1	2	3	4	5	6
$c_3(l)$	$\frac{5}{6}$	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{6}$	-5
$c_4(l)$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$	1

Table 3: The values of $c_3(6,l)$ and $c_4(6,l)$ for $1\le l\le 6$.

(66) we obtain

$$|E_6(n)|\le \frac{24N^2}{(24n+5)\pi}\exp\left(\frac{(24n+5)\,\pi}{12N^2}\right)\left(2e^{\frac{5\pi}{18}}+2e^{\frac{\pi}{9}}\right)+2\exp\left(\frac{\pi\,(24n+5)}{12N^2}\right)\times\Upsilon$$

where

$$\begin{aligned}\Upsilon=&\frac{2}{\sqrt{6}}\exp\left(a(-\pi)+a(-\pi/6)+\frac{5\pi}{144}\right)-\frac{2}{\sqrt{6}}\exp\left(\frac{5\pi}{144}\right)\\&+\frac{2}{\sqrt{3}}\exp\left(a(-\pi)+a(-2\pi/3)+\frac{\pi}{72}\right)-\frac{2}{\sqrt{3}}\exp\left(\frac{\pi}{72}\right)\\&+\frac{1}{\sqrt{2}}\exp\left(a(-\pi)+a(-3\pi/2)-\frac{\pi}{48}\right)+\exp\left(a(-\pi)+a(-6\pi)-\frac{5\pi}{24}\right).\end{aligned}$$

Now let $y=\mu_6(n)$, implying that $24n+5=\frac{6^3y^2}{5\pi^5}$. By taking $N=\lfloor y\rfloor$, one can verify that

$$|E_6(n)|\le\frac{5\pi\lfloor y\rfloor^2}{9y^2}\exp\left(\frac{18y^2}{5\pi\lfloor y\rfloor^2}\right)\left(2e^{\frac{5\pi}{18}}+2e^{\frac{\pi}{9}}\right)+2\exp\left(\frac{18y^2}{5\pi\lfloor y\rfloor^2}\right)\times\Upsilon.$$

Then, using the following two inequalities:

$$\frac{\lfloor y\rfloor^2}{y^2}\le 1\quad\text{and}\quad\frac{y^2}{\lfloor y\rfloor^2}<\frac{y^2}{(y-1)^2}<2\quad\text{for } y\ge 4,$$

we deduce that

$$|E_6(n)|\le\frac{10\pi}{9}e^{\frac{36}{5\pi}}\left(e^{\frac{5\pi}{18}}+e^{\frac{\pi}{9}}\right)+2e^{\frac{36}{5\pi}}\times\Upsilon\le 812$$

holds when n satisfies $y=\mu_6(n)\ge 4$.

Next we focus on estimating the value of $|B_6(n)|$. From Proposition 4.2 it follows that

$$|B_6(n)|\le\frac{5\pi^2}{18\sqrt{3}y}\sum_{l\in\{1,2,4,5\}}\sum_{\substack{2\le t\le\lfloor y\rfloor\\ t\equiv_6 l}}I_1\left(\frac{y}{t}\right)=\frac{5\pi^2}{18\sqrt{3}y}\sum_{\substack{2\le t\le\lfloor y\rfloor\\ 3\nmid t}}I_1\left(\frac{y}{t}\right).$$

Thus,

$$|B_6(n)| \leq \frac{5\pi^2}{18\sqrt{3}y}\frac{2\lfloor y \rfloor}{3} I_1\left(\frac{y}{2}\right) \leq \frac{5\pi^2}{27\sqrt{3}} I_1\left(\frac{y}{2}\right) \leq \frac{10\pi^{\frac{3}{2}} e^{\frac{y}{2}}}{27\sqrt{3}y^{\frac{1}{2}}},$$

where the last inequality is obtaind by using (69). Combining the bounds of $E_6(n)$ and $B_6(n)$, we get

$$|R_6(n)| = |E_6(n)| + |B_6(n)| \leq 812 + \frac{10\pi^{\frac{3}{2}} e^{\frac{y}{2}}}{27\sqrt{3}y^{\frac{1}{2}}}$$

whenever n satisfies $y = \mu_6(n) \geq 4$. Thus, it remains to show for $y \geq 16$,

$$\gamma(y) := \frac{10\pi^{\frac{3}{2}} e^{\frac{y}{2}}}{27\sqrt{3}y^{\frac{1}{2}}} \geq 812. \tag{70}$$

By studying the derivative of $\gamma(y)$, one can show that it is increasing on the interval $[1, +\infty)$. Thus if $y \geq 16$ then $\gamma(y) \geq \gamma(16) > 812$, as desired. This completes the proof. □

Finally, we are able to prove Theorem 3.2.

Proof of Theorem 3.2. As before, we only proof the case of $k = 6$. According to Theorem 4.3, if n satisfies the condition that $\mu_6(n) \geq 17$, then

$$p_6(n) = M_6(n) + R_6(n),$$

where $|R_6(n)| \leq \hat{R}_6(n) = \frac{20\pi^{\frac{3}{2}} e^{\frac{y}{2}}}{27\sqrt{3}y^{\frac{1}{2}}}$ and $y = \mu_6(n)$.

If we let

$$G_6(n) := \frac{\hat{R}_6(n)}{M_6(n)} = \frac{8\sqrt{2}}{3\sqrt{\pi}} \cdot \frac{\sqrt{y}e^{\frac{y}{2}}}{I_1(y)},$$

then

$$M_6(n)(1 - G_6(n)) \leq p_6(n) \leq M_6(n)(1 + G_6(n)).$$

Note that (29) allows us to deduce that

$$G_6(n) \leq \frac{32y^2 e^{-\frac{y}{2}}}{3(2y-1)}$$

whenever n satisfies that $y \geq 26$. It remains to show

$$\frac{32y^2 e^{-\frac{y}{2}}}{3(2y-1)} \leq \frac{1}{y^6},$$

or equivalently,

$$32y^8 - 3(2y-1)e^{\frac{y}{2}} \leq 0,$$

holds for $y \geq 61$. But this can be verified by showing the derivative of $L(y) = 32y^8 - 3(2y-1)e^{\frac{y}{2}}$ is negative and $L(y) \leq L(61)$ on the interval $[61, +\infty)$ with the help of mathematical software. This completes the proof. □

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