On extremal problems on multigraphs

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Abstract

An (n, s, q)-graph is an n-vertex multigraph in which every s-set of vertices spans at most q edges. Erdős initiated the study of maximum number of edges of (n, s, q)-graphs, and the extremal problem on multigraphs has been considered since the 1990s. The problem of determining the maximum product of the edge multiplicities in (n, s, q)-graphs was posed by Mubayi and Terry in 2019. Recently, Day, Falgas-Ravry and Treglown settled a conjecture of Mubayi and Terry on the case $(s, q) = (4, 6a + 3)$ of the problem (for $a \geq 2$), and they gave a general lower bound construction for the extremal problem for many pairs (s, q), which they conjectured is asymptotically best possible. Their conjecture was confirmed exactly or asymptotically for some specific cases. In this paper, we consider the case that $(s, q) = (5, \binom{5}{2}a + 4)$ and $d = 2$ of their conjecture, partially solve an open problem raised by Day, Falgas-Ravry and Treglown. We also show that the conjecture fails for $n = 6$, which indicates for the case that $(s, q) = (5, \binom{5}{2}a + 4)$ and $d = 2$, n needs to be sufficiently large for the conjecture to hold.

Keywords: Multigraphs; Turán problems; Extremal graphs

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1 Introduction

In 1963, Erdős [3, 4] posed the extremal question on $ex(n, s, q)$, the maximum number of edges in an n-vertex graph in which every s-set of vertices spans at most q edges, where q is an integer satisfying that $0 \le q \le \binom{n}{2}$. In the 1990s, Bondy and Tuza [1] and Kuchenbrod [7] raised an analogous extremal problem on multigraphs. A multigraph is a pair (V, w), where V is a vertex set and w is a function $w : \binom{V}{2} \to \mathbb{Z}_{\ge 0}$.

Definition 1 *Given integers $s \ge 2$ and $q \ge 0$, we say a multigraph $G = (V, w)$ is an (s, q)-graph if every s-set of vertices in V spans at most q edges; i.e. $\sum_{xy \in \binom{X}{2}} w(xy) \le q$ for every $X \in \binom{V}{s}$. An (n, s, q)-graph is an n-vertex (s, q)-graph. We write $\mathcal{F}(n, s, q)$ for the set of all (n, s, q)-graphs with vertex set $[n] := \{1, ..., n\}$.*

Bondy and Tuza [1], Kuchenbrod [7] and Füredi and Kündgen [6] studied the problem of the maximum of the sum of the edge multiplicities in an (n, s, q)-graph. Especially, Füredi and Kündgen [6] obtained an asymptotically tight upper bound $m\binom{n}{2} + O(n)$ for the maximum of edges in (n, s, q)-graphs, where $m = m(s, q)$ is an explicit constant. Recently, Mubayi and Terry [8, 9] introduced a version of multiplicity product of the problem as follows.

Definition 2 *Given a multigraph $G = (V, w)$, we define*

$$P(G) := \prod_{xy \in \binom{V}{2}} w(xy),$$

$$ex_\Pi(n, s, q) := \max\{P(G) : G \in \mathcal{F}(n, s, q)\},$$
$$ex_\Pi(s, q) := \lim_{n \to +\infty} (ex_\Pi(n, s, q))^{\binom{n}{2}^{-1}}.$$

Mubayi and Terry showed in [8, Theorem 2.2], that for $q \ge \binom{s}{2}$,

$$\left|\mathcal{F}(n, s, q - \binom{s}{2})\right| = ex_\Pi(s, q)^{\binom{n}{2} + o(n^2)}.$$

As Mubayi and Terry pointed out estimating the size of the multigraph family $\mathcal{F}\left(n, s, q - \binom{s}{2}\right)$ is equivalent to the Turán-type extremal problem of determining $ex_\Pi(n, s, q)$. Therefore, Mubayi and Terry raised the general problem of determining $ex_\Pi(n, s, q)$ as below.

Problem 1 *[8, 9] Given positive integers $s \ge 2$ and q, determine $ex_\Pi(n, s, q)$.*

Mubayi and Terry in [8] proved that

$$ex_{\Pi}(n,4,15) = 2^{\gamma n^2 + O(n)}$$

where γ is defined by

$$\gamma := \frac{\beta^2}{2} + \beta(1-\beta)\frac{\log 3}{\log 2} \quad \text{where} \quad \beta := \frac{\log 3}{2\log 3 - \log 2}.$$

This is the first example of a 'fairly natural extremal graph problem' whose asymptotic answer is given by an explicitly defined transcendental number. In response to a question of Alon which asked whether this transcendental behaviour is an isolated case, Mubayi and Terry [8] made a conjecture on the value of $ex_{\Pi}(n,4,\binom{4}{2}a+3)$. In [2], Day, Falgas-Ravry and Treglown resolved their conjecture fully, consequently providing infinitely many examples on Alon's question.

Mubayi and Terry [9] exactly or asymptotically determined $ex_{\Pi}(n,s,q)$ for pairs (s,q) where $a\binom{s}{2} - \frac{s}{2} \leq q \leq a\binom{s}{2} + s - 2$ for some $a \in \mathbb{N}$. Day, Falgas-Ravry and Treglown [2] investigated $ex_{\Pi}(n,s,q)$ for a further range of values of (s,q). In particular, they gave a potential extremal structure related to $ex_{\Pi}(n,s,q)$. Their constructions may be seen as a class of multigraphs analogues of the well known Turán graphs.

Construction 1 *[2] Let $a, r \in \mathbb{N}$ and $d \in [0, a-1]$. Given $n \in \mathbb{N}$, let $\mathcal{T}_{r,d}(a,n)$ denote the collection of multigraphs G on $[n]$ for which $V(G)$ can be partitioned into r parts $V_0, \ldots, V_{r-1}$ such that:*
(i) all edges in $G[V_0]$ have multiplicity $a-d$;
(ii) for all $i \in [r-1]$, all edges in $G[V_i]$ have multiplicity a;
(iii) all other edges of G have multiplicity $a+1$.

Given $G \in \mathcal{T}_{r,d}(a,n)$, we refer to $\cup_{i=0}^{r-1} V_i$ as the canonical partition of G.

Fig. 1 is an example of what these graphs look like when $r = 4$. We write

$$\Sigma_{r,d}(a,n) := \max\{e(G) : G \in \mathcal{T}_{r,d}(a,n)\},$$

and

$$\Pi_{r,d}(a,n) := \max\{P(G) : G \in \mathcal{T}_{r,d}(a,n)\}.$$

In [2], Day, Falgas-Ravry and Treglown raised a new conjecture as below.

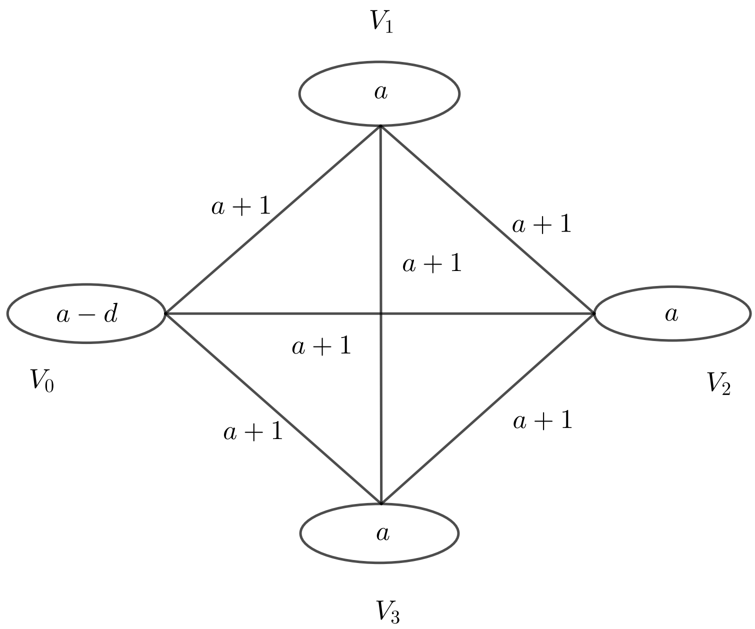

Figure 1: An example of the structure of graphs in $\mathcal{T}_{4,d}(a,n)$.

Conjecture 1 *For all integers a, r, s, d with $a, r \geq 1$, $d \in [0, a-1]$,$s \geq (r-1)(d+1)+2$ and all n sufficiently large,*

$$ex_\Pi(n, s, \Sigma_{r,d}(a, s)) = \Pi_{r,d}(a, n). \quad (1)$$

As the definition of $\Sigma_{r,d}(a, n)$, we know $ex_\Pi(n, s, \Sigma_{r,d}(a, s)) \geq \Pi_{r,d}(a, n)$. Conjecture 1 roughly states that for any $d \in [0, a-1]$ and other conditions of n, s, a, r, when we take $q = \Sigma_{r,d}(a, s)$, it is a graph G from $\mathcal{T}_{r,d}(a, n)$ maximises the edge-product $P(G)$ amongst all(n, s, q)-graphs. Mubayi and Terry [9] proved that for all r such that $\frac{s}{2} \leq r \leq s-1$ and $n \geq s$, $ex_\Pi(n, s, \Sigma_{r,0}(a, n)) = \Pi_{r,0}(a, n)$, which partly confirms Conjecture 1 for $d = 0$. In [2], Day, Falgas-Ravry and Treglown completed the proof of the $d = 0$ case of Conjecture 1. Combine the results in [5] and [2], Conjecture 1 asymptotically holds for $d = 1$ and sufficiently large a.

Note that the known results on Conjecture 1 are considering the cases $d = 0$ or $d = 1$. In this paper, we consider (1) for the case of $d = 2$ and the special pair $(s, q) = (5, \binom{5}{2}a + 4)$ which is mentioned as an open problem by Day, Falgas-Ravry and Treglown in [2]. We obtain the results as follows.

Theorem 1 *For $(s, q) = (5, \binom{5}{2}a + 4)$, we have*

(i) For $n \geq 8$ and $a \to \infty$, $ex_\Pi(n, 5, \binom{5}{2}a + 4) \to \Pi_{2,2}(a, n)$.

(ii) For $n = 7$ and $a \geq 3$, we have

$$(a-2)(a+1)^{10}a^{10} \leq ex_{\Pi}(7, 5, \binom{5}{2}a+4) < (a-1)a^{11}(a+1)^{9}.$$

Particularly, $ex_{\Pi}(7, 5, \binom{5}{2}a+4) \to \Pi_{2,2}(a, 7)$ as $a \to \infty$.

(iii) For $n = 6$ and $a \geq 3$, $ex_{\Pi}(6, 5, \binom{5}{2}a+4) = a^9(a+1)^6 > \Pi_{2,2}(a, 6)$. And

$$\Pi_{2,2}(a, 6) = \begin{cases} (a+1)^5a^{10}, a = 3, 4; \\ (a-2)(a+1)^8a^6, a \geq 5. \end{cases}$$

(iv) For $n = 5$ and $a \geq 3$, $ex_{\Pi}(5, 5, \binom{5}{2}a+4) = \Pi_{2,2}(a, 5) = (a+1)^4a^6$.

It is not difficult to verify that $\Sigma_{2,2}(a, 5) = \binom{5}{2}a + 4$, therefore, our results above imply that (1) does not hold for any $a \geq 3$ when $n = 6$. This shows that for $(s, q) = (5, \binom{5}{2}a + 4)$ and $d = 2$, n needs to be sufficiently large for (1) to hold.

The rest of the paper is organized as follows. In Section 2, we introduce some more notation and preliminaries. We give the proof of Theorem 1 in Section 3.

2 Preliminaries

The following integral version of the AM-GM inequality was presented in [2], we will use it frequently.

Lemma 1 *[2] Let $a, n \in [0, n]$, and let $\omega_1, ..., \omega_n$ be non-negative integers with $\sum_{i=1}^n \omega_i = an + t$. Then the following hold:*
(i) $\Pi_{i=1}^n \omega_i \leq a^{n-t}(a+1)^t$;
(ii) if $t \leq n-2$ and $\omega_1 = a-1$ then $\Pi_{i=1}^t \omega_i \leq (a-1)a^{n-t-2}(a+1)^{t+1}$.

Considering the maximum $P(G)$ among graphs in $\mathcal{T}_{2,2}(a, n)$, we obtain the following lemma.

Lemma 2 *Let $a \in [3, +\infty)$, $n \in [5, +\infty)$, for the graph $G \in \mathcal{T}_{2,2}(a, n)$, let $\cup_{i=0}^1 V_i$ be the canonical partition of G. Set $|V_0| = x$, then $P(G)$ is maximum among all the graphs in $\mathcal{T}_{2,2}(a, n)$ if and only if*

$$n \in \left((x-1)\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x, x\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x + 1 \right].$$

Moreover,

$$\Pi_{2,2}(a, n) = (a-2)^{\binom{x}{2}}a^{\binom{n-x}{2}}(a+1)^{x(n-x)}$$

for $n \in \left((x-1)\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x, x\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x + 1\right]$.

Proof. Let G be an n-vertex graph from $\mathcal{T}_{2,2}(a,n)$. Suppose that V_0 of G has x vertices, and V_1 has $n-x$ vertices. By simple calculation, when V_0 has x vertices, the product of the edge multiplicities is $(a-2)^{\binom{x}{2}}a^{\binom{n-x}{2}}(a+1)^{x(n-x)}$. Now we define the new graph G' to be a graph obtained from G by moving a vertex from V_1 to V_0. Considering the changing of the product of the edge multiplicities by moving a vertex from V_1 to V_0, we have that $\frac{P(G)}{P(G')} = \frac{a^{n-x-1}(a+1)^x}{(a-2)^x(a+1)^{n-x-1}}$, which means that the product of the edge multiplicities is increased when the quantity is less than one and otherwise decreased.

If $n \in \left((x-1)\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})}+x, x\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})}+x+1\right]$, then we find that n satisfies the following inequalities:

$$\begin{cases} \frac{a^{n-x}(a+1)^{x-1}}{(a-2)^{x-1}(a+1)^{n-x}} < 1, \\ \frac{a^{n-x-1}(a+1)^{x}}{(a-2)^{x}(a+1)^{n-x-1}} \geq 1, \end{cases}$$

which means that the product of the edge multiplicities in G is decreased whether moving a vertex from V_1 to V_0 or moving a vertex from V_0 to V_1.

As $(a+1)^2 \geq a(a-2)$ holds for any $a \geq 3$, the inequality $\frac{a^{n-k}(a+1)^{k-1}}{(a-2)^{k-1}(a+1)^{n-k}} \leq \frac{a^{n-k-1}(a+1)^{k}}{(a-2)^{k}(a+1)^{n-k-1}}$ established when k is a positive integer. Therefore, n satisfies:

$$\begin{cases} \frac{a^{n-2}(a+1)}{(a-2)(a+1)^{n-2}} < 1, \\ \frac{a^{n-3}(a+1)^2}{(a-2)^2(a+1)^{n-3}} < 1, \\ \dots \\ \frac{a^{n-x}(a+1)^{x-1}}{(a-2)^{x-1}(a+1)^{n-x}} < 1, \\ \frac{a^{n-x-1}(a+1)^{x}}{(a-2)^{x}(a+1)^{n-x-1}} \geq 1, \\ \dots \\ \frac{a^{1}(a+1)^{n-2}}{(a-2)^{n-2}(a+1)^{1}} \geq 1, \\ \frac{a^{0}(a+1)^{n-1}}{(a-2)^{n-1}(a+1)^{0}} \geq 1. \end{cases} \tag{2}$$

This system of inequalities means that the quantity of the product of the edge multiplicities is maximum when $|V_0| = x$. Thus we have proved the sufficiency of Lemma 2. Now we prove the necessity of Lemma 2.

For the necessity of Lemma 2, if $P(G)$ is maximum when $|V_0| = x$, then both adding a vertex to V_0 and taking out a vertex from V_0 will decrease the quantity of the product of the edge multiplicities, which means that:

$$\begin{cases} \frac{a^{n-x}(a+1)^{x-1}}{(a-2)^{x-1}(a+1)^{n-x}} < 1, \\ \frac{a^{n-x-1}(a+1)^{x}}{(a-2)^{x}(a+1)^{n-x-1}} \geq 1, \end{cases}$$

Solving the above set of inequalities, we have that:

$$n \in \left((x-1)\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x, x\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x + 1 \right].$$

As a result, when $n \in \left((x-1)\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x, x\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x + 1 \right]$, we have $\Pi_{2,2}(a,n) = (a-2)^{\binom{x}{2}} a^{\binom{n-x}{2}} (a+1)^{x(n-x)}$.

□

Lemma 3 *Let G be an n-vertex graph from $\mathcal{T}_{2,2}(a,n)$, and let $\cup_{i=0}^{1} V_i$ be the canonical partition of G. Then among all the graphs in $\mathcal{T}_{2,2}(a,n)$, the product of the edge multiplicities will be maximum when $|V_0|$=2 if and only if*

$$n \in \left(\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + 2, 2\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + 3 \right].$$

Proof. By substituting $x = 2$ into the system of the inequalities (2), we get

$$n \in \left(\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + 2, 2\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + 3 \right],$$

which proves the lemma. □

If we let $F(a) = \frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})}$, $a \geq 3$, we find the function is monotone decreasing for a. Therefore, the maximum of $F(a)$ is $F(3) = 4.82$. Also, it is not difficult to obtain that $\lim_{a\to+\infty} F(a) = 3$.

3 Proof of Theorem 1

3.1 The case when $n \geq 7$

For $n \geq 7$, let G be a graph from $\mathcal{T}_{2,2}(a,n)$ and n be a positive integer. Let $\cup_{i=0}^{1} V_i$ be the canonical partition of G. Suppose $P(G)$ is maximum when $|V_0| = x$, among all the graphs in $\mathcal{T}_{2,2}(a,n)$. By Lemma 2, we know:

$$n \in \left((x-1)\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x, x\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + x + 1 \right]. \tag{3}$$

Moreover, $\Pi_{2,2}(a,n) = (a-2)^{\binom{x}{2}} a^{\binom{n-x}{2}} (a+1)^{x(n-x)}$.

Let G be a graph from $\mathcal{F}(n,5,\binom{5}{2}a+4)$. By averaging over all 5-sets, we obtain that the number of edges of G satisfying that

$$e(G) \le \left\lfloor \frac{\binom{n}{5}}{\binom{n-2}{3}} \left(\binom{5}{2}a+4 \right) \right\rfloor = \frac{n(n-1)}{2}a + \left\lfloor \frac{n(n-1)}{5} \right\rfloor.$$

By Lemma 1 (i), we have

$$P(G) \le a^{\frac{n(n-1)}{2}-\lfloor \frac{n(n-1)}{5} \rfloor}(a+1)^{\lfloor \frac{n(n-1)}{5} \rfloor}.$$

As a result, we have

$$\begin{aligned} \Pi_{2,2}(a,n) &= (a-2)^{\binom{x}{2}}a^{\binom{n-x}{2}}(a+1)^{x(n-x)} \\ &\le ex_{\Pi}(n,5,\binom{5}{2}a+4) \\ &\le a^{\frac{n(n-1)}{2}-\lfloor \frac{n(n-1)}{5} \rfloor}(a+1)^{\lfloor \frac{n(n-1)}{5} \rfloor}. \end{aligned}$$

We will prove

$$\lim_{a\to+\infty} \frac{a^{\frac{n(n-1)}{2}-\lfloor \frac{n(n-1)}{5} \rfloor}(a+1)^{\lfloor \frac{n(n-1)}{5} \rfloor}}{(a-2)^{\binom{x}{2}}a^{\binom{n-x}{2}}(a+1)^{x(n-x)}} = 1. \tag{4}$$

Note that the case (i) in Theorem 1 follows if the equality (4) holds. Let

$$\begin{cases} h_1(a) = a^{\frac{n(n-1)}{2}-\lfloor \frac{n(n-1)}{5} \rfloor}(a+1)^{\lfloor \frac{n(n-1)}{5} \rfloor}, \\ h_2(a) = (a-2)^{\binom{x}{2}}a^{\binom{n-x}{2}}(a+1)^{x(n-x)}. \end{cases}$$

Expanding $h_1(a)$ and $h_2(a)$, we have

$$h_1(a) = a^{\frac{n(n-1)}{2}} + \dots + a^{\frac{n(n-1)}{2}-\lfloor \frac{n(n-1)}{5} \rfloor},$$

which is a polynomial on variable a with the highest degree $\frac{n(n-1)}{2}$, and the coefficient of $a^{\frac{n(n-1)}{2}}$ is 1. Also,

$$h_2(a) = a^{\frac{n(n-1)}{2}} + \dots + (-2)^{\frac{x(x-1)}{2}}a^{\frac{(n-x)(n-x-1)}{2}},$$

which is a polynomial on variable a with the highest degree $\frac{n(n-1)}{2}$, and the coefficient of $a^{\frac{n(n-1)}{2}}$ is 1. It follows that $\lim_{a\to+\infty}\frac{h_1(a)}{h_2(a)} = 1$.

3.2 The case when $n = 7$

For $n = 7$, note that

$$7 \in \left(\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + 2, 2\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})} + 3 \right].$$

Applying Lemma 3 for graph $H \in \mathcal{T}_{2,2}(a,7)$, if H maximizes the product of the edge multiplicities, then V_0 of H consists of two vertices. Therefore, $\Pi_{2,2}(a,7) = (a-2)a^{10}(a+1)^{10}$. And

$$ex_{\Pi}(7,5,\binom{5}{2}a+4) \geq \Pi_{2,2}(a,7) = (a-2)a^{10}(a+1)^{10}.$$

Let G be a product-extremal graph in $\mathcal{F}(7,5,\binom{5}{2}a+4)$. By averaging over all 5-sets, we see that

$$e(G) \leq \left\lfloor \frac{\binom{7}{5}}{\binom{5}{3}} \left(\binom{5}{2}a+4\right) \right\rfloor = 21a + \left\lfloor \frac{42}{5} \right\rfloor = 21a+8.$$

If $e(G) \leq 21a+7$, then by Lemma 1 (i),

$$P(G) \leq a^{14}(a+1)^{7}.$$

Suppose now that

$$e(G) = 21a+8. \tag{5}$$

If G contains at least one edge of multiplicity at most $a-1$, then by Lemma 1 (ii), $P(G) \leq (a-1)a^{11}(a+1)^{9}$. The equation holds if and only if there are one edge with multiplicity $a-1$ and 11 edges with multiplicity a and 9 edges with multiplicity $a+1$.

On the other hand, suppose all the edges of G have multiplicity at least a. Since $G \in \mathcal{F}(7,5,\binom{5}{2}a+4)$, the edges in G have multiplicity between a and $a+4$. We need to consider the various values of the edge multiplicity of G, beginning with the easiest cases.

Case I: Every edge of G has multiplicity either a or $a+1$.

From (5), we obtain that G has exactly 8 edges of multiplicity $a+1$. We denote by G^{a+1} the graph spanned by edges of multiplicity $a+1$ of G. Then we claim that there is a cycle on 4 vertices in G^{a+1}.

Claim 1 *There must have a C_4 in G^{a+1}.*

Proof. Above all, there must be a path of length 3 in G^{a+1}. Indeed, since G^{a+1} has 7 vertices and 8 edges, it must have a path of length 2. If there is not a path of length 3, then G^{a+1} must have as many as possible paths of length 2 to cover the 8 edges in G^{a+1}. As there are 7 vertices, there must be a vertex adjacent to the midpoint of only one path of length 2. Then there are at most 5 edges in G^{a+1}, a contradiction. So G^{a+1} must have a path of length 3.

Suppose there is not a C_4 in G^{a+1}, we want to get a contradiction. We denote by P a path of length 3 in G^{a+1}, and its vertices set is $\{1, 2, 3, 4\}$, such that i and $i+1$ are adjacent in P for $1 \leq i \leq 3$. Then the endpoints of P are 1 and 4 and the remained vertices are $5, 6, 7$.

Since G^{a+1} does not have a C_4, the vertices 1 and 4 are not adjacent in G^{a+1}. If $\{1, 3\}$ and $\{2, 4\}$ are edges in G^{a+1}, we find that the 5-vertex set $\{1, 2, 3, 4, 5\}$ spans edges with multiplicities at least $10a + 5$, a contradiction with $G \in \mathcal{F}(7, 5, \binom{2}{2}a + 4)$. If one of $\{1, 3\}$ and $\{2, 4\}$ is an edge in G^{a+1}, then there exists an edge between two sets $\{1, 2, 3, 4\}$ as $\{5, 6, 7\}$ can span at most 3 edges in G^{a+1}. Hence we can obtain a 5-set containing vertices $1, 2, 3, 4$, which spans edges with multiplicities summation at least $10a + 5$.

The statement above means that the vertices of P spanned no edges in G^{a+1} except the edges of the path. And we claim that for any vertex $u \in \{5, 6, 7\}$, it can send at most one edge into P in G^{a+1}. Otherwise $\{u, 1, 2, 3, 4\}$ is a 5-vertex set which spans edges with multiplicities summation at least $10a + 5$, a contradiction with $G \in \mathcal{F}(7, 5, \binom{5}{2}a + 4)$.

Note that there are 8 edges in G^{a+1} and any vertex in $\{5, 6, 7\}$ can send at most one edge into P in G^{a+1} and $\{5, 6, 7\}$ can span at most 3 edges in G^{a+1}, then we find that either $\{5, 6, 7\}$ span 3 edges or $\{5, 6, 7\}$ span 2 edges.

Case a: If $\{5, 6, 7\}$ span 2 edges, then all vertices in $\{5, 6, 7\}$ must send one edge into P in G^{a+1}. Consequently, there must either exist a C_4 or a 5-set which spans edges with multiplicities summation at least $10a + 5$, a contradiction.

Case b: If $\{5, 6, 7\}$ span 3 edges, without loss of generality we suppose that the vertex 5 sends no edges into P. There are two cases need to be consider. If there are two vertices in $\{5, 6, 7\}$ are adjacent to the same vertex (without loss of generality we let it be 1) in $\{1, 2, 3, 4\}$, then we find the two vertices with $\{1, 2, 3\}$ form a 5-set which spans edges with multiplicities summation at least $10a+5$. Otherwise, without loss of generality we suppose the vertex 6 is adjacent to 1 and the vertex 7 is adjacent to 4 in G^{a+1}. Now we find a 5-set $\{1, 4, 5, 6, 7\}$ which spans edges with multiplicities summation at least $10a+5$. Both case make a contradiction with $G \in \mathcal{F}(7, 5, \binom{2}{2}a+4)$. Thus, there is a C_4 in G^{a+1}. $\square$

Let C be a cycle of length 4 in G^{a+1}, and its vertices set is $\{1, 2, 3, 4\}$, such that i and $i + 1$ are adjacent in C for $1 \leq i \leq 3$ and 1 is adjacent to 4. As $G \in \mathcal{F}(7, 5, \binom{5}{2}a + 4)$, the spanning graph of $\{1, 2, 3, 4\}$ in G^{a+1} is only the C and $\{5, 6, 7\}$ send no edges into C in G^{a+1}. However, $\{5, 6, 7\}$ can span at most 3 edges with multiplicity $a+1$, which means there are at most 7 edges with multiplicity $a+1$, a contradiction. As a result, edges in G can not only have multiplicity either a or $a+1$.

Case II : G contains an edge e_0 with multiplicity $a+2$.

Since any 5 vertices can span edges with multiplicities at most $10a+4$, then the multiplicity of other edges are one of $a, a+1, a+2$. We consider the following three subcases.

Subcase 1: Suppose no edge with multiplicity $a+1$ or $a+2$ is incident with e_0. Then

$$
\begin{aligned}
P(G) &\leq \omega(e_0)a^{10}ex_{\Pi}(5,5,\binom{5}{2}a+4) \\
&= (a+2)a^{10}(a+1)^4a^6 \\
&= a^{16}(a+1)^4(a+2).
\end{aligned}
$$

Subcase 2: Suppose there is some vertex v sending an edge of multiplicity $a+2$ to one of endpoints of e_0 . Then the vertex is unique, it sends exactly one such edge into e_0, and every other edges must have multiplicity a as any 5 points can span edges with multiplicities at most $10a+4$. Thus $P(G) \leq (a+2)^2a^{19}$.

Subcase 3: Suppose there is some vertex v sending an edge e_1 of multiplicity $a+1$ to one of the endpoints of e_0. Let the endpoints of e_0 be $1, 2$ and v be 3. Without losing generality we suppose 2 is adjacent to 3. Denote by P the path with vertices 1, 2, 3.

If 1 and 3 are adjacent by an edge e_2 with multiplicity $a+1$, then all other edges must have multiplicity a. Otherwise we suppose the vertex 4 sends an edge with multiplicity more than a, we find $\{1,2,3,4\}$ with any vertex in $\{5,6,7\}$ form a 5-vertex set which spans edges with multiplicities summation at least $10a+5$. A contradiction with $G \in \mathcal{F}(7,5,\binom{5}{2}a+4)$. Then $P(G) \leq (a+2)(a+1)^2a^{18}$.

If 1 and 3 are adjacent by an edge e_3 with multiplicity a. Then there is at most one vertex in $\{4,5,6,7\}$ which can send an edge to P with multiplicity at most $a+1$.

- If there is a vertex 4 sending an edge e_4 with multiplicity $a+1$ to P, then the rest edges except $\{e_0, e_1, e_3, e_4\}$ in $G[\{1,2,3,4\}]$ must have multiplicity a and the rest vertices $5, 6, 7$ only can send edge with multiplicity a into $G[\{1,2,3,4\}]$. Now we consider the edges in $G[\{5,6,7\}]$. If there is one edge e_5, without losing generality we assume that the endpoints of e_5 are $6, 7$, and e_5 has multiplicity $a+2$. Then $\{1,2,3,6,7\}$ is a 5-vertex set which spans edges with multiplicities summation exceed $10a+4$, a contradiction. If there is no edge having multiplicity $a+2$, then $\{5,6,7\}$ can span at most 2 edges with multiplicity $a+1$. So

$P(G) \leq a^{16}(a+1)^4(a+2)$.

- If there is no vertex in $\{4,5,6,7\}$ sends an edge with multiplicity $a+1$ to P. We claim that there is no edge with multiplicity $a+2$ in $G[\{5,6,7\}]$. Otherwise we suppose that $G[\{5,6,7\}]$ has an edge $e^{'}$ incident with $5,6$ with multiplicity $a+2$, then $\{1,2,3,5,6\}$ is a 5-vertex set which spans edges with multiplicities summation at least $10a+5$, a contradiction. Meanwhile we find there are at most 3 edges with multiplicity $a+1$, so $P(G) \leq a^{17}(a+1)^3(a+2)$.

In conclusion, if G contains a edge e_0 with multiplicity $a+2$, then we have the following inequalities

$$\begin{cases} P(G) \leq a^{16}(a+1)^4(a+2), \\ P(G) \leq a^{19}(a+2)^2, \\ P(G) \leq a^{18}(a+1)^2(a+2), \\ P(G) \leq a^{16}(a+1)^4(a+2), \\ P(G) \leq a^{17}(a+1)^3(a+2). \end{cases}$$

Through comparing the right-hand side of these inequalities, we find $a^{16}(a+1)^4(a+2)$ is the maximum value. So if G contains an edge e_0 with multiplicity $a+2$, then $P(G) \leq a^{16}(a+1)^4(a+2)$.

Case III : G contains an edge e_0 of multiplicity $a+3$.

Since any 5 vertices can span edges with multiplicities at most $10a+4$, every edge except for e_0 has multiplicity either a or $a+1$. Suppose there is some vertex v sending an edge of multiplicity $a+1$ to one of the endpoints of e_0. As any 5 vertices can span edges with multiplicities at most $10a+4$, every other edge has multiplicity exactly a. Then

$$P(G) \leq P(G[e_0 \cup v])a^{19} = a^{19}(a+1)(a+3).$$

On the other hand suppose there is no edge with multiplicity $a+1$ connect to e_0. We claim that there is no path of length 2 in G^{a+1}, otherwise there will be a 5-set which span edges with multiplicities at most $10a+4$. Hence there is at most two edge of multiplicity $a+1$. Then

$$P(G) \leq a^{18}(a+1)^2(a+3).$$

As

$$\frac{a^{18}(a+1)^2(a+3)}{a^{19}(a+1)(a+3)} > 1 \quad \text{when} \quad a \geq 3,$$

then if G contains an edge e_0 of multiplicity $a+3$, we have

$$P(G) \leq a^{18}(a+1)^2(a+3).$$

Case IV : G contains an edge of multiplicity $a+4$.

Since any 5 vertices can span edges with multiplicities at most $10a+4$, every other edge has multiplicity exactly a, and $P(G) = (a+4)a^{20}$.

Consider all the upper bounds for $P(G)$ we obtained above, we have

$$\begin{cases} P(G) \leq a^{14}(a+1)^7, \\ P(G) \leq (a-1)a^{11}(a+1)^9, \\ P(G) \leq a^{16}(a+1)^4(a+2), \\ P(G) \leq a^{18}(a+1)^2(a+3), \\ P(G) \leq a^{20}(a+4). \end{cases}$$

Through comparison we finally get that $(a-1)a^{11}(a+1)^9$ is the maximum upper bound for $P(G)$. However, this bound is not achievable. In fact, we obtained this bound $(a-1)a^{11}(a+1)^9$ when G contains at least one edge of multiplicity at most $a-1$. The equality holds if and only if there is one edge, denoted by e with multiplicity $a-1$ and 11 edges with multiplicity a and 9 edges with multiplicity $a+1$. By claim 1, we know that G^{a+1} must have a C_4. Let the vertex set of C_4 be $\{1,2,3,4\}$.

Suppose $\{1,2,3,4\}$ contains both of the endpoints of e, let the endpoints of e be $1,3$. As $\{1,2,3,4\}$ can span at most 5 edges with multiplicity $a+1$ and $\{5,6,7\}$ can span at most 3 edges with multiplicity $a+1$, there is at least one vertex $u \in \{5,6,7\}$ sending edge with multiplicity $a+1$ into $\{1,2,3,4\}$. Actually, $\{1,2,3,4,u\}$ is a 5-vertex set which spans edges with multiplicities summation at least $10a+5$, a contradiction.

Suppose $\{1,2,3,4\}$ contains one of the endpoints of e, then let the endpoints of e be $1,5$. As $\{1,2,3,4\}$ can span only 4 edges with multiplicity $a+1$ and $\{5,6,7\}$ can span at most 3 edges with multiplicity $a+1$. So there is at least one vertex in $\{5,6,7\}$ sending edge with multiplicity $a+1$ into $\{1,2,3,4\}$. If 6 or 7 sends edges with multiplicity $a+1$ into $\{1,2,3,4\}$, then $\{1,2,3,4\}$ with 6 or 7 form a 5-vertex set which spans edges with multiplicities summation at least $10a+5$, a contradiction. Otherwise 5 must send exactly 2 edges with multiplicity $a+1$ into $\{1,2,3,4\}$. Then $\{5,6,7\}$ with the endpoints of the two edges form a 5-vertex set which spans edges with multiplicities summation at least $10a+5$, a contradiction.

Suppose $\{1,2,3,4\}$ contains no endpoints of e, let the endpoints of e_0 be $5,6$. As $\{1,2,3,4\}$ can span only 4 edges with multiplicity $a+1$ and $\{5,6,7\}$ can span at most 2 edges with multiplicity $a+1$. So there is at least one vertex $u \in \{5,6,7\}$ sending edge with multiplicity $a+1$ into $\{1,2,3,4\}$. Actually, $\{1,2,3,4,u\}$ is a 5-vertex set which spans edges with multiplicities summation at least $10a+5$, a contradiction. Therefore, we have that $P(G)$ is strictly less than $(a-1)a^{11}(a+1)^9$.

As $\Pi_{2,2}(a,7) \in \mathcal{F}(7,5,\binom{5}{2}a+4)$, we have

$$\begin{aligned}\Pi_{2,2}(a,7) &= (a-2)(a+1)^{10}a^{10} \\ &\leq ex_{\Pi}(7,5,\binom{5}{2}a+4) \\ &< (a-1)a^{11}(a+1)^9,\end{aligned}$$

and

$$\lim_{a\to+\infty}\frac{(a-1)a^{11}(a+1)^9}{(a-2)(a+1)^{10}a^{10}} = 1$$

which proves the case (ii) in Theorem 1.

3.3 The case when $n = 6$

For $n=6$, let H be a product extremal graph in $\mathcal{F}(6,5,\binom{5}{2}a+4)$. By averaging over all 5-sets, we see that

$$e(H) \leq \frac{\binom{6}{5}}{\binom{4}{3}}\left(\binom{5}{2}a+4\right) = 15a+6.$$

By Lemma 1 (i), $P(H) \leq a^9(a+1)^6$. Therefore, $ex_{\Pi}(6,5,\binom{5}{2}a+4) \leq a^9(a+1)^6$. On the other hand, consider the 6-vertex multigraph H' whose edges of multiplicity $a+1$ form a 6-cycle C_6, and all other edges are of multiplicity a. We have $P(H') = a^9(a+1)^6$ and $H \in \mathcal{F}(6,5,\binom{5}{2}a+4)$. Hence, $ex_{\Pi}(6,5,\binom{5}{2}a+4) \geq a^9(a+1)^6$. In conclusion, we have $ex_{\Pi}(6,5,\binom{5}{2}a+4) = a^9(a+1)^6$.

Let $G \in \mathcal{T}_{2,2}(a,6)$. When $a=3,4$, we have

$$6 \in \left(1, \frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})}+2\right],$$

which means the quantity $P(G)$ is maximised when $V_0=\{1\}$ and $V_1=\{2,3,4,5,6\}$ by lemma 2. Then $\mathcal{T}_{2,2}(a,6) = (a+1)^5a^{10}$ when $a=3,4$.

When $a \geq 5$, we have

$$6 \in \left(\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})}+2, 2\frac{\ln(1-\frac{3}{a+1})}{\ln(1-\frac{1}{a+1})}+3\right],$$

which means the quantity $P(G)$ is maximised when $V_0 = \{1, 2\}$ and $V_1 = \{3, 4, 5, 6\}$ by lemma 2. Then $\mathcal{T}_{2,2}(a, 6) = (a-2)(a+1)^8 a^6$ when $a \geq 5$. So we have that

$$\Pi_{2,2}(a, 6) = \begin{cases} (a+1)^5 a^{10}, a = 3, 4; \\ (a-2)(a+1)^8 a^6, a \geq 5. \end{cases}$$

Comparing $\Pi_{2,2}(a, 6)$ and $a^9(a+1)^6$, we find that $(a+1)^5 a^{10}$ is strictly less than $a^9(a+1)^6$ when $a = 3, 4$, and $(a-2)(a+1)^8 a^6$ is strictly less than $a^9(a+1)^6$ when $a \geq 5$. Hence

$$ex_{\Pi}(6, 5, \binom{5}{2}a + 4) = a^9(a+1)^6 > \Pi_{2,2}(a, 6), \quad \text{when} \quad a \geq 3.$$

3.4 The case when $n = 5$

For $n = 5$, let G be a product-extremal graph from $\mathcal{F}(5, 5, \binom{5}{2}a+4)$. Then $e(G) \leq \binom{5}{2}a + 4 = 10a + 4$. By Lemma 1 (i), we have $P(G) \leq a^{(10-4)}(a+1)^4 = a^6(a+1)^4$. Hence, $ex_{\Pi}(5, 5, \binom{5}{2}a + 4) \leq a^6(a+1)^4$.

On the other hand, partitioning [5] into $V_0 = \{1\}$ and $V_1 = \{2, 3, 4, 5\}$, we have that

$$(a+1)^4 a^6 \leq \Pi_{2,2}(a, 5) \leq ex_{\Pi}(5, 5, \binom{5}{2}a + 4) \leq a^6(a+1)^4.$$

Therefore, we have proved the case (iv) in Theorem 1.

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4 Competing interest and date availability statement

All authors disclosed no relevant relationships.

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