BUMPLESS PIPE DREAMS MEET PUZZLES

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ABSTRACT. Knutson and Zinn-Justin recently found a puzzle rule for the expansion of the product $\mathfrak{G}_u(x,t)\cdot\mathfrak{G}_v(x,t)$ of two double Grothendieck polynomials indexed by permutations with separated descents. We establish its triple Schubert calculus version in the sense of Knutson and Tao, namely, a formula for expanding $\mathfrak{G}_u(x,y)\cdot\mathfrak{G}_v(x,t)$ in different secondary variables. Our rule is formulated in terms of pipe puzzles, incorporating the structures of both bumpless pipe dreams and classical puzzles. As direct applications, we recover the separated-descent puzzle formula by Knutson and Zinn-Justin (by setting $y=t$) and the bumpless pipe dream model of double Grothendieck polynomials by Weigandt (by setting $v=\mathrm{id}$ and $x=t$). Moreover, we utilize the formula to partially confirm a positivity conjecture of Kirillov about applying a skew operator to a Schubert polynomial.

Keywords: Schubert calculus, Grothendieck polynomial, bumpless pipe dream, puzzle, lattice model, Yang–Baxter equation

MSC: 14M15, 14N15, 05E14, 05E05

CONTENTS

1. INTRODUCTION

The core of this paper is to provide a combinatorial rule for the *triple* Schubert calculus in the torus-equivariant K-theory of flag manifolds, with respect to the basis of structure sheaves indexed by permutations with separated descents. The geometry of triple Schubert calculus (particularly for the case of cohomology of Grassmannians) was revealed by Knutson and Tao [9]. Combinatorially, we shall give a formula for expanding the product of two double Grothendieck polynomials in different secondary variables

$$\mathfrak{G}_u(x,y)\cdot\mathfrak{G}_v(x,t)=\sum_{w} c_{u,v}^{w}(t,y)\cdot\mathfrak{G}_w(x,t), \tag{1.1}$$

for two permutations u and v of $\{1, 2, \dots, n\}$ with separated descents at position k, that is,

$$\mathrm{maxdes}(u) \le k \le \mathrm{mindes}(v), \tag{1.2}$$

where $\mathrm{maxdes}(u) = \max\{i\colon u(i) > u(i+1)\}$ and $\mathrm{mindes}(v) = \min\{i\colon v(i) > v(i+1)\}$. Here, for the identity permutation $\mathrm{id} = 12\cdots n$, we use the convention that $\mathrm{maxdes}(\mathrm{id}) = 0$ and $\mathrm{mindes}(\mathrm{id}) = +\infty$.

Our formula for $c^w_{u,v}(t, y)$, see Theorem 2.6, is described in terms of "*pipe puzzles*", see Section 2 for the precise definition. Pipe puzzles enjoy the features of both bumpless pipe dreams and puzzles, as will be explained in Section 6. The formula includes the following specializations and applications.

(i) The case $y = t$. Theorem 2.6 recovers the puzzle rule for permutations with separated descents by Knutson and Zinn-Justin [12, Theorem 1], which is manifestly positive in the sense of Anderson, Griffeth and Miller [1] (an equivariant K-theory extension of Graham's positivity theorem [6]).

(ii) The case $\beta = 0$. Theorem 2.6 becomes a combinatorial rule for the expansion of the product $\mathfrak{S}_u(x, y) \cdot \mathfrak{S}_v(x, t)$ of two Schubert polynomials in different secondary variables, see Theorem 2.2. We point out that in the case $y = t = 0$, Huang [7] derived a tableau formula for the product $\mathfrak{S}_u(x)\cdot\mathfrak{S}_v(x)$ of two single Schubert polynomials for u, v with separated descents.

(iii) The case that both u and v are k-Grassmannian permutations. Theorem 2.6 extends the puzzle formula for the product $\mathfrak{G}_\lambda(x, t) \cdot \mathfrak{G}_\mu(x, t)$ by Wheeler and Zinn-Justin [20, Theorem 2] (The latter formula on the one hand is an equivariant extension of Vakil's puzzle formula [21] for the product $\mathfrak{G}_\lambda(x) \cdot \mathfrak{G}_\mu(x)$ of two single Grothendieck polynomials, and on the other hand is a K-theory extension of the Knutson–Tao puzzle formula [9] for the product $s_\lambda(x, t) \cdot s_\mu(x, t)$ of two double Schur polynomials).

We remark that (1) an alternative puzzle formula (different from the one in [20]) for $\mathfrak{G}_\lambda(x, t) \cdot \mathfrak{G}_\mu(x, t)$ was conjectured by Knutson and Vakil, and proved by Pechenik and Yong [18] (after a modification) based on their earlier tableau formula established in [17], (2) Wheeler and Zinn-Justin [20, Theorems 2″ and 3″] gave puzzle formulas for the product of two *dual* Grothendieck polynomials in different secondary variables, and (3) puzzle formulations of the Molev–Sagan tableau formula [16] for the product $s_\lambda(x, y)\cdot s_\mu(x, t)$ of two double Schur polynomials in different secondary variables were given by Knutson and Tao [9, Section 6] and Zinn-Justin [23].

(iv) The case that $k = n$ (this means u may be any permutation of $\{1, 2, \dots, n\}$), $v = \mathrm{id}$, and $x = t$. Theorem 2.6 reduces to the bumpless pipe dream model of double Grothendieck polynomials by Weigandt [22], which, by setting $\beta = 0$, leads to the bumpless pipe dream model of double Schubert polynomials by Lam, Lee and Shimozono [13]. An alternative proof of Weigandt's model was given by Buciumas and Scrimshaw [4] based on colored lattice models.

(v) Kirillov [8, Conjecture 1] conjectured that after applying the skew divided difference operator $\partial_{w/v}$ to a single Schubert polynomial $\mathfrak{S}_u(x)$, the resulting polynomial will have nonnegative integer coefficients. Setting $y = 0$ in Theorem

2.2 leads to a positive expansion of the product $\mathfrak{S}_u(x) \cdot \mathfrak{S}_v(x,t)$, which, as will be explained in Section 6.3, allows us to confirm Kirillov's prediction for u and v with separated descents and arbitrary w.

An innovation in our approach is finding that $c_{u,v}^w(t,y)$ satisfies two kinds of recurrence relations, as given in Section 3. Such recurrence relations work well when u and v are restricted to permutations with separated descents. This could essentially simplify the proof of Theorem 2.6. Specifically, to prove Theorem 2.6, it suffices to show that our pipe puzzle formula obeys the same recurrence relations (together with an initial condition). This could be done (without too much efforts) by realizing pipe puzzles as an integrable lattice model.

We remark that the above mentioned recurrence relations are no longer available in the case $y = t$. The proof of the $y = t$ case in [12, Theorem 1] is achieved by studying the geometric representation of quantized loop algebras and quiver varieties [10,11]. It turns out that while the problem of computing triple Schubert structure constants is broader, its proof could be simpler. From this point of view, our approach may provide new insights into the study of Schubert calculus for flag manifolds.

This paper is arranged as follows. In Section 2, we state the pipe puzzle formula for $c_{u,v}^w(t,y)$ in the case that u, v are permutations with separated descents, see Theorem 2.6. In Section 3, we provide two recurrence relations for $c_{u,v}^w(t,y)$, and explain that such recurrence relations still work when restricted to permutations with separated descents. In Section 4, we realize pipe puzzles as a lattice model, and show that it satisfies two types of Yang–Baxter equations. In Section 5, based on the lattice model, we show that our pipe puzzle formula satisfies the same recurrence relations as $c_{u,v}^w(t,y)$, thus completing the proof of Theorem 2.6. Section 6 is devoted to applications of Theorem 2.6, mainly including those aforementioned.

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2. Main result

The main result is given in Theorem 2.6, a separated-descent pipe puzzle formula for the coefficients $c_{u,v}^w(t,y)$. Let us begin by giving the definition of Grothendieck polynomials. As usual, we use S_n to denote the symmetric group of permutations of $\{1, 2, \ldots, n\}$. Let β be a formal variable. Denote

$$x \ominus y = \frac{x - y}{1 + \beta y}.$$

Let π_i be the *Demazure operator*:

$$\pi_i f = \frac{(1+\beta x_{i+1})f - (1+\beta x_i)f|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}.$$

The *double Grothendieck polynomial* $\mathfrak{G}_w(x,t)$ for $w \in S_\infty = \bigcup_{n\geq 0} S_n$ is determined by the following two properties:

$$\mathfrak{G}_{n\cdots 21}(x,t) = \prod_{i+j\leq n} (x_i \ominus t_j);$$

$$\pi_i \mathfrak{G}_w(x,t) = \mathfrak{G}_{ws_i}(x,t), \qquad \text{if } w(i) > w(i+1).$$

Here, $s_i = (i, i+1)$ is the simple transposition, and ws_i is obtained from w by swapping $w(i)$ and $w(i+1)$. Since $\pi_i^2 = -\beta\pi_i$, it follows that

$$\pi_i \mathfrak{G}_w(x,t) = \begin{cases} \mathfrak{G}_{ws_i}(x,t), & \text{if } w(i) > w(i+1), \\ -\beta \mathfrak{G}_w(x,t), & \text{if } w(i) < w(i+1). \end{cases} \tag{2.1}$$

Letting $t_i = 0$ defines the *single Grothendieck polynomial*

$$\mathfrak{G}_w(x) = \mathfrak{G}_w(x,0).$$

Setting $\beta = 0$, we get the *double (resp., single) Schubert polynomial*

$$\mathfrak{S}_w(x,t) = \mathfrak{G}_w(x,t)|_{\beta=0}, \qquad (\text{resp., } \mathfrak{S}_w(x) = \mathfrak{G}_w(x)|_{\beta=0}).$$

Remark 2.1. *There appear different definitions for Grothendieck polynomials in the literature, which will be equivalent after appropriate changes of variables. For example,* [12] *adopts the following operator and initial condition:*

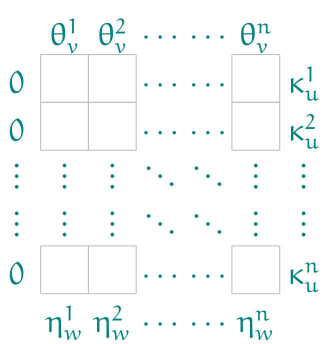
$$\bar{\partial}_i f = \frac{X_{i+1} f - X_i f|_{X_i \leftrightarrow X_{i+1}}}{X_{i+1} - X_i}, \qquad \mathcal{G}_{n\cdots 21}(X,T) = \prod_{i+j\leq n} (1 - X_i/T_j).$$

It can be checked that $\mathcal{G}_w(X,T)$ can be obtained from $\mathfrak{G}_w(x,t)$ by the following replacements:

$$\beta = -1, \qquad X_i = 1 - x_i, \qquad T_i = 1 - t_i.$$

Our definition is consistent with that used in [14, Section 5.1].

In the remaining of this section, we assume that u and v are permutations of S_n with separated descents at position k. We are going to describe our pipe puzzle formula for $c_{u,v}^w(t,y)$. To begin, consider an n by n grid with labeled boundary:



$$\kappa_u^i = \begin{cases} u^{-1}(i), & u^{-1}(i) \leq k, \\ 0, & u^{-1}(i) > k. \end{cases}$$

$$\theta_v^i = \begin{cases} 0, & v^{-1}(i) \leq k, \\ v^{-1}(i), & v^{-1}(i) > k. \end{cases} \tag{2.2}$$

$$\eta_w^i = w^{-1}(i).$$

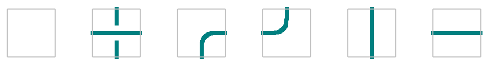
We see that the nonzero labels on the right side are $1, \ldots, k$, and the nonzero labels on the top side are $k+1, \ldots, n$. There is no obstruction to rebuilding u, v and w from the boundary labeling, because of the separated-descent assumption. For the sake of brevity, the label 0 on the boundary will often be omitted. See Example 2.5 for the boundary labeling for $u = 42135$, $v = 14532$, $w = 53412$, and $k = 2$.

Our formula is a weighted counting of tilings of the n by n grid by unit tiles (with pipes), subject to certain conditions. To warm up, we first give the formula for double Schubert polynomials.

2.1. **Statement for double Schubert polynomials.** Assume that

$$\mathfrak{S}_u(x, y) \cdot \mathfrak{S}_v(x, t) = \sum_{w} \overline{c}_{u,v}^{w}(t, y) \cdot \mathfrak{S}_w(x, t). \tag{2.3}$$

The admissible tiles are



(2.4)

The curves drawn on the tiles are referred to as *pipes*. A tiling of (2.2) built upon the tiles in (2.4) is a network of pipes such that

(1) there are a total of n pipes, among which k pipes enter horizontally from rows on the right side labeled $1, \ldots, k$, and $n-k$ pipes enter vertically from columns on the top side labeled $k+1, \ldots, n$. The pipes inherit the labels of the corresponding rows and columns.
(2) the n pipes end vertically on the bottom side, such that the label of each pipe matches the label of the column where it ends.

A *Schubert pipe puzzle* for u, v, w is a tiling of (2.2) with the tiles in (2.4), subject to the following restriction on the tiles ┼:

> The horizontal pipe in ┼ must receive a smaller label. For example,
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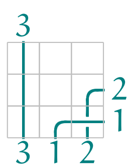
> 
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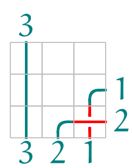
> 
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> is not allowed. (2.5)

Denote by $\mathrm{PP}_0(u, v, w)$ the set of Schubert pipe puzzles for u, v, w. For each $\pi \in \mathrm{PP}_0(u, v, w)$, define its *Schubert weight* by

$$\mathrm{wt}_0(\pi; t, y) = \prod_{(i,j)} (t_j - y_i),$$

where the sum is over empty tiles □ at the (i, j)-positions (in the matrix coordinate).

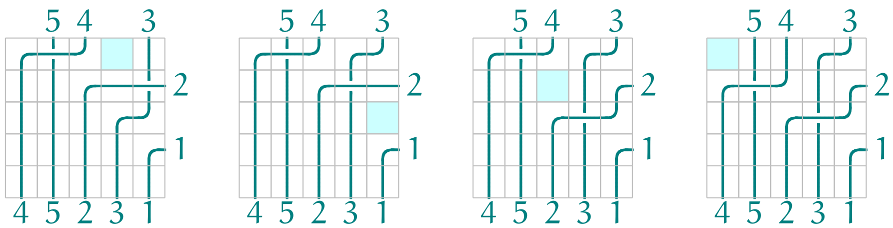
Theorem 2.2. *Let $u, v \in S_n$ be permutations with separated descents at position k. For $w \in S_n$, we have*

$$\overline{c}_{u,v}^{w}(t, y) = \sum_{\pi \in \mathrm{PP}_0(u,v,w)} \mathrm{wt}_0(\pi; t, y). \tag{2.6}$$

Remark 2.3. *It may happen that $\mathfrak{S}_w(x, t)$, $w \in S_{n'}$ with $n < n'$, appears in the expansion of $\mathfrak{S}_u(x, y) \cdot \mathfrak{S}_v(x, t)$. In such a case, to compute $\overline{c}_{u,v}^{w}(t, y)$, one needs only to embed naturally S_n into $S_{n'}$, and then apply Theorem 2.2 (u and v are now viewed as permutations in $S_{n'}$).*

Remark 2.4. *An alternative expresssion for the coefficients in Theorem 2.2 was recently given by Samuel* [19, Theorem 3.1]*, which is described in terms of certain paths in the Bruhat order. Its proof relies on a Pieri formula using three sets of variables as in* (2.3)*. The formula in* [19, Theorem 3.1] *also implies Kirillov's conjecture in the separated-descent case, as discussed in Section 6.3.*

Example 2.5. *Let* $u = 42135$*,* $v = 14532$*, and set* $k = 2$*. For* $w = 53412$*, there are four Schubert pipe puzzles in* $\mathrm{PP}_0(u, v, w)$*:*



Here, the empty tiles are colored. So it follows from (2.6) *that*

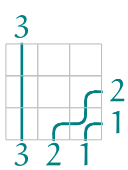
$$\bar{c}_{42135,14532}^{53412} = (t_4 - y_1) + (t_5 - y_3) + (t_3 - y_2) + (t_1 - y_1).$$

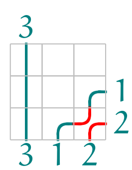
2.2. **Statement for double Grothendieck polynomials.** We allow one more admissible tile than (2.4):

 (2.7)

The extra tile in (2.7) is a "bumping" tile . We shall have the following restrictions on the usage of :

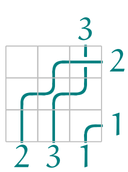
If the two pipes in  are from the same side, then the northwest pipe must receive a larger label. For example, (2.8)

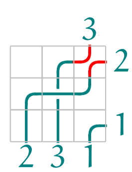


is allowed, while


is not allowed.

If the two pipes in  are from different sides, then the northwest pipe must enter from the right side (equivalently, it receives a smaller label). For example, (2.9)



is allowed, while


is not allowed.

A *(Grothendieck) pipe puzzle* for u, v, w is a tiling of (2.2) with the tiles in (2.7) obeying the restriction (2.5) on , as well as the restrictions (2.8) and (2.9) on .

Let $\mathrm{PP}(u, v, w)$ be the set of pipe puzzles for u, v, w. For $\pi \in \mathrm{PP}(u, v, w)$, its *weight* $\mathrm{wt}(\pi; t, y)$ is the product of factors contributed by all tiles of π: at the (i, j)-position,

(1) an empty tile  contributes $t_j \ominus y_i$;

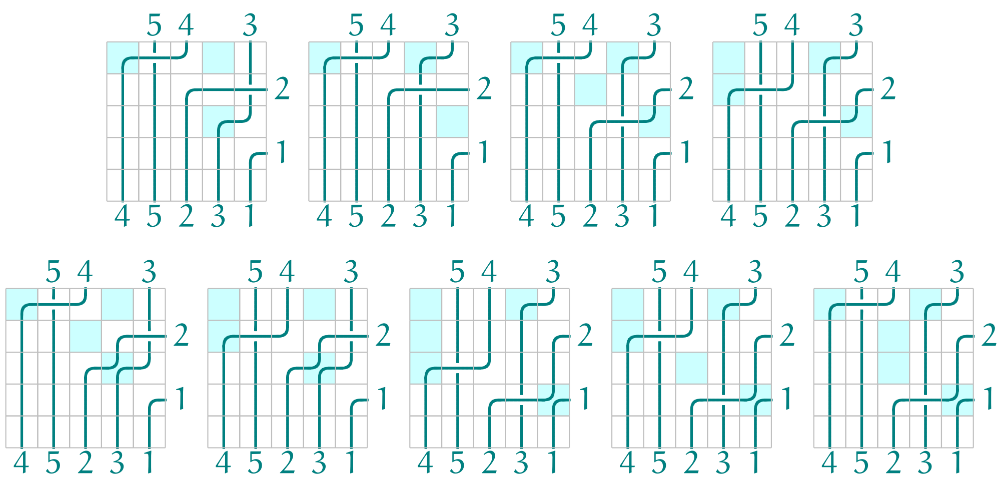
(2) an elbow tile, in which the pipe is from the right side, contributes $1+\beta(t_j \ominus y_i)$;
(3) an elbow tile, in which the pipe is from the top side, contributes $1+\beta(t_j \ominus y_i)$;
(4) a bumping tile, in which the two pipes are from the same side, contributes β;
(5) a bumping tile, in which the two pipes are from different sides, contributes $\beta(1+\beta(t_j \ominus y_i))$;
(6) any other tile except for the above cases contributes 1.

Theorem 2.6. *Let $u, v \in S_n$ be permutations with separated descents at position k. For $w \in S_n$, we have*

$$c_{u,v}^{w}(t,y) = \sum_{\pi \in \mathrm{PP}(u,v,w)} \mathrm{wt}(\pi; t, y). \tag{2.10}$$

Note that Remark 2.3 is still valid for Theorem 2.6. We also remark that Theorem 2.6 specializes to Theorem 2.2 in the case $\beta = 0$ by noticing that $\mathrm{wt}(\pi; t, y)|_{\beta=0} = 0$ whenever $\pi \notin \mathrm{PP}_0(u, v, w)$, and $\mathrm{wt}(\pi; t, y)|_{\beta=0} = \mathrm{wt}_0(\pi; t, y)$ for $\pi \in \mathrm{PP}_0(u, v, w)$.

Example 2.7. *Take the same setting as in Example 2.5. There are nine pipe puzzles in $\mathrm{PP}(u, v, w)$, among which the pipe puzzles in the top row are those appearing in Example 2.5. Here, the tiles with weights not equal to 1 are colored.*



As a result,

$$\begin{aligned}
c_{42135,14532}^{53412} = {} & (t_4 \ominus y_1)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_3)) \\
& + (t_5 \ominus y_3)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_1)) \\
& + (t_3 \ominus y_2)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_1))(1+\beta(t_5 \ominus y_3)) \\
& + (t_1 \ominus y_1)(1+\beta(t_4 \ominus y_1))(1+\beta(t_1 \ominus y_2))(1+\beta(t_5 \ominus y_3)) \\
& + \beta(t_4 \ominus y_1)(t_3 \ominus y_2)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_3)) \\
& + \beta(t_1 \ominus y_1)(t_4 \ominus y_1)(1+\beta(t_1 \ominus y_2))(1+\beta(t_4 \ominus y_3)) \\
& + \beta(t_1 \ominus y_1)(t_1 \ominus y_2)(1+\beta(t_4 \ominus y_1))(1+\beta(t_1 \ominus y_3)) \\
& + \beta(t_1 \ominus y_1)(t_3 \ominus y_3)(1+\beta(t_4 \ominus y_1))(1+\beta(t_1 \ominus y_2))
\end{aligned}$$

$$+\beta(t_3\ominus y_2)(t_3\ominus y_3)(1+\beta(t_1\ominus y_1))(1+\beta(t_4\ominus y_1)).$$

3. Recurrence relations

In this section, we present two recurrence relations, as well as an initial condition, for $c^w_{u,v}(t,y)$, and explain that they can be used to determine the computation of $c^w_{u,v}(t,y)$ for $u,v\in S_n$ with separated descents.

Let us first review the definition of Bruhat order on S_n. Let t_{ij} $(1\le i<j\le n)$ denote the transpositions of S_n. Then S_n is generated by the set of simple transpositions $s_i=t_{i\,i+1}$ for $1\le i<n$. The length $\ell(w)$ of $w\in S_n$ is the minimum number of simple transpositions appearing in any decomposition $w=s_{i_1}\cdots s_{i_m}$. It is well known that $\ell(w)$ equals the number of inversions of w:

$$\ell(w)=\#\{(i,j)\colon 1\le i<j\le n,\ w(i)>w(j)\}.$$

Notice that wt_{ij} (resp., $t_{ij}w$) is obtained from w by swapping $w(i)$ and $w(j)$ (resp., the values i and j). Write $w<wt_{ij}$ if $\ell(w)<\ell(wt_{ij})$ (namely, $w(i)<w(j)$). The transitive closure of all relations $w<wt_{ij}$ forms the Bruhat order $\le$ on S_n. It should be noted that the Bruhat order can be defined equivalently as the transitive closure of relations $w<t_{ij}w$ (which means $\ell(w)<\ell(t_{ij}w)$).

In the rest of this section, we shall often encounter the situation $s_iw<w$ or $s_iw>w$. By definition, $s_iw<w$ means i appears after $i+1$ in w, while $s_iw>w$ means i appears before $i+1$ in w.

To describe the two recurrence relations for $c^w_{u,v}(t,y)$, we define two operators acting on a rational function $f:=f(x,t,y)$ in three sets of variables. The *(left) Demazure operator* ϖ_i acts on the t variables:

$$\varpi_i f=-\frac{(1+\beta t_i)f-(1+\beta t_{i+1})f|_{t_i\leftrightarrow t_{i+1}}}{t_i-t_{i+1}}.$$

Similarly, we define φ_i which acts on the y variables:

$$\varphi_i f=-\frac{(1+\beta y_i)f-(1+\beta y_{i+1})f|_{y_i\leftrightarrow y_{i+1}}}{y_i-y_{i+1}}.$$

It should be noted that if the polynomial is in two sets of variables, it means that the third set is set to 0.

The two recurrence relations for $c^w_{u,v}(t,y)$ can be stated as follows. If there is no confusion occurring, we sometimes simply write $c^w_{u,v}$ for $c^w_{u,v}(t,y)$.

Proposition 3.1. *If $s_iu<u$, then*

$$c^w_{s_iu,v}=\varphi_i\,c^w_{u,v}=-\frac{1+\beta y_i}{y_i-y_{i+1}}c^w_{u,v}+\frac{1+\beta y_{i+1}}{y_i-y_{i+1}}c^w_{u,v}|_{y_i\leftrightarrow y_{i+1}}. \tag{3.1}$$

Proposition 3.2. *If $s_iw>w$, then*

$$c^{s_iw}_{u,v}=\begin{cases}-\varpi_i\,c^w_{u,v}+c^w_{u,s_iv}|_{t_i\leftrightarrow t_{i+1}}, & s_iv<v,\\[2ex] -\varpi_i\,c^w_{u,v}-\beta c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}}, & s_iv>v,\end{cases}$$

that is,

$$c_{u,v}^{s_i w} = \begin{cases} \dfrac{1+\beta t_i}{t_i - t_{i+1}} c_{u,v}^{w} - \dfrac{1+\beta t_{i+1}}{t_i - t_{i+1}} c_{u,v}^{w}|_{t_i \leftrightarrow t_{i+1}} + c_{u,s_i v}^{w}|_{t_i \leftrightarrow t_{i+1}}, & s_i v < v, \\ \dfrac{1+\beta t_i}{t_i - t_{i+1}} c_{u,v}^{w} - \dfrac{1+\beta t_i}{t_i - t_{i+1}} c_{u,v}^{w}|_{t_i \leftrightarrow t_{i+1}}, & s_i v > v. \end{cases} \tag{3.2}$$

In order to more quickly go into the proof of Theorem 2.6, we put the proofs of Propositions 3.1 and 3.2 in Appendix A.

To give the initial condition, we need the following localization

$$\mathfrak{G}_w(t,t) = \begin{cases} 1, & w = \mathrm{id}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

This is the very special case of the general localization formula for Grothendieck polynomials, see for example Buch and Rimányi [3] and the references therein. Taking $x = t$ in (1.1) and then applying (3.3), we obtain the following relationship.

Lemma 3.3. *We have*

$$c_{u,v}^{\mathrm{id}}(t,y) = \begin{cases} \mathfrak{G}_u(t,y), & \textit{if } v = \mathrm{id}, \\ 0, & \textit{otherwise}. \end{cases}$$

Denote by $u_0^{k,n} = n(n-1)\cdots(n-k+1)\,12\cdots(n-k) \in S_n$ the unique longest permutation among those $u \in S_n$ with $\mathrm{maxdes}(u) \le k$:

$$u_0^{k,n}(i) = \begin{cases} n+1-i, & i \le k, \\ i-k, & k < i \le n. \end{cases} \tag{3.4}$$

By direct computation, we have

$$\mathfrak{G}_{u_0^{k,n}}(x,t) = \prod_{i=1}^{k} \prod_{j=1}^{n-i} (x_i \ominus t_j).$$

Indeed, this is clearly true for $k = n$. If the statement is true for k, then applying operators $\pi_1 \cdots \pi_k$, we can compute the case of $k-1$. This, along with Lemma 3.3, leads to the initial condition.

Proposition 3.4. *For $v \in S_n$,*

$$c_{u_0^{k,n},v}^{\mathrm{id}} = \begin{cases} \displaystyle\prod_{i=1}^{k} \prod_{j=1}^{n-i} (t_i \ominus y_j), & \textit{if } v = \mathrm{id}, \\ 0, & \textit{otherwise}. \end{cases}$$

Propositions 3.1 and 3.2 are valid for any $u, v, w \in S_n$. We explain that such recurrences are closed when restricting $u, v \in S_n$ to permutations with separated descents at k. In other words, we could use Propositions 3.1 and 3.2 (only applied to permutations with separated descents at k), along with the initial condition in Proposition 3.4, to compute $c_{u,v}^{w}(t,y)$ for any $u, v \in S_n$ with separated descents at k.

- First, compute $c_{u,v}^{\mathrm{id}}(t,y)$ for $w=\mathrm{id}$. The initial case is for the longest permutation $u=u_0^{k,n}$, as done in Proposition 3.4. We next consider $c_{u,v}^{\mathrm{id}}(t,y)$ with $\ell(u)<\ell(u_0^{k,n})$. Since $u\neq u_0^{k,n}$, one can always choose an integer i among the first k values $u(1),\ldots,u(k)$, such that i appears before $i+1$ in u. For example, given $u=7423156$ and $k=4$, we may choose $i=4$ or $i=2$.

  Now we have $u<s_iu\in S_n$. It is easily checked that $\mathrm{maxdes}(s_iu)\leq k$. Set $u'=s_iu$. By backwards induction on the length of u, the value of $c_{u',v}^{\mathrm{id}}(t,y)$ is known, which allows us to compute $c_{u,v}^{\mathrm{id}}(t,y)=c_{s_iu',v}^{\mathrm{id}}(t,y)$ from $c_{u',v}^{\mathrm{id}}(t,y)$ by means of Proposition 3.1.
- Second, compute $c_{u,v}^{w}(t,y)$ for $\ell(w)>0$. In this case, choose any s_i such that $s_iw<w$. It is also easily checked that if $s_iv<v$, then we still have $\mathrm{mindes}(s_iv)\geq k$. Set $w'=s_iw$. By induction on the length of w, the values of $c_{u,v}^{w'}(t,y)$ and $c_{u,s_iv}^{w'}(t,y)$ are known. Applying Proposition 3.2, we may deduce $c_{u,v}^{w}(t,y)=c_{u,v}^{s_iw'}(t,y)$ from $c_{u,v}^{w'}(t,y)$ and $c_{u,s_iv}^{w'}(t,y)$.

4. Integrable lattice models

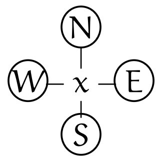
Throughout this section, we assume that $u,v\in S_n$ are permutations with separated descents at k, and w is any permutation in S_n. We shall realize the pipe puzzles in $\mathrm{PP}(u,v,w)$ as a (colored) lattice model, denoted $\mathrm{L}(u,v,w)$, so that the right-hand side of (2.6) is equal to the partition function of $\mathrm{L}(u,v,w)$. For more background about lattice models, we refer the reader to, for example, [2, 4, 5, 24]. We verify that $\mathrm{L}(u,v,w)$ is integrable, in the sense that it satisfies Yang–Baxter equations with respect to particular choices of R-matrices.

4.1. **Lattice model.** Consider a square grid with n horizontal lines and n vertical lines. The intersection point of two lines will be a vertex (so there are a total of n^2 vertices). The lines between two vertices are called edges. We shall also attach additional half edges to the vertices on the boundary, so that there are four half edges around each vertex.

A *state* is a labeling of all the (half) edges with labels from $\{0,1,2\ldots,n\}$, with a fixed boundary condition which is consistent with that in (2.2): the left half edges are all labeled 0, the right half edges are labeled $\kappa_u^1,\ldots,\kappa_u^n$ from top to bottom, the top (resp., bottom) half edges are labeled $\theta_v^1,\ldots,\theta_v^n$ (resp., $\eta_w^1,\ldots,\eta_w^n$) from left to right. The label of each (half) edge will be marked with a circle, and a vertex will be formally assigned a parameter x. A state is *admissible* if the local configurations around each vertex (namely, the labeled half edges adjacent to each vertex) satisfy exactly one of the conditions as listed in the middle column of Table 1. Moreover, each allowable local configuration is assigned a weight as given in the first column of Table 1.

Each configuration around a vertex naturally corresponds to a tile that is used to define a pipe puzzle, as illustrated in the last column of Table 1, with pipes inheriting the labels of edges. We display the information in Table 1 more intuitively in Table 2. Therefore, each admissible state generates a pipe puzzle, and vice versa. See Figure 1 for an admissible state and its corresponding pipe puzzle.

The lattice model $\mathrm{L}(u,v,w)$ we are considering is defined as the set of all admissible states ($\mathrm{L}(u,v,w)$ can be regarded as a colored lattice model if the labels $1,2,\ldots,n$ are

 :

weights	conditions	tiles
x	$N = E = W = S = 0$	
1	$E = W = 0 < N = S$	
1	$N = S = 0 < E = W$	
1	$0 < E = W < N = S$	
$1 + \beta x$	$E = S = 0 < N = W \leq k$	
1	$E = S = 0$ and $k < N = W$	
1	$N = W = 0 < E = S \leq k$	
$1 + \beta x$	$N = W = 0$ and $k < E = S$	
β	$0 < E = S < N = W \leq k$	
β	$k < E = S < N = W$	
$\beta(1 + \beta x)$	$0 < N = W \leq k < E = S$	

TABLE 1. Weights, local configurations, and tiles.

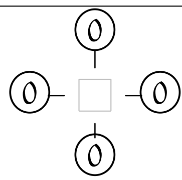
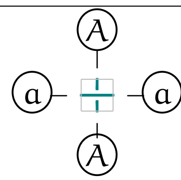
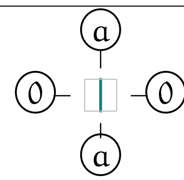
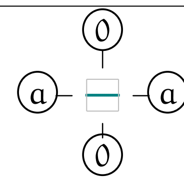
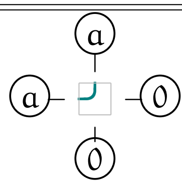
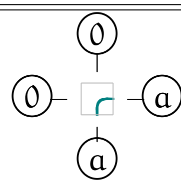
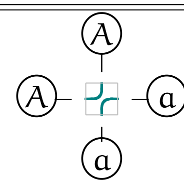
		 
x	$1 \quad (0 < a < A)$	$1 \quad (0 < a)$
		
$1 + \beta x \quad (0 < a \leq k)$ $1 \quad (k < a)$	$1 \quad (0 < a \leq k)$ $1 + \beta x \quad (k < a)$	$\beta \quad (0 < a < A \leq k)$ $\beta \quad (k < a < A)$ $\beta(1 + \beta x) \quad (0 < A \leq k < a)$

TABLE 2. Diagram illustration of Table 1.

viewed as n colors). The weight $\mathrm{wt}(S; t, y)$ of a state S in $L(u, v, w)$ is the product of all the weights of vertices with $x = t_j \ominus y_i$ in row i and column j. The *partition function* of $L(u, v, w)$ is defined by

$$Z^w_{u,v}(t, y) = \sum_{S \in L(u,v,w)} \mathrm{wt}(S; t, y).$$

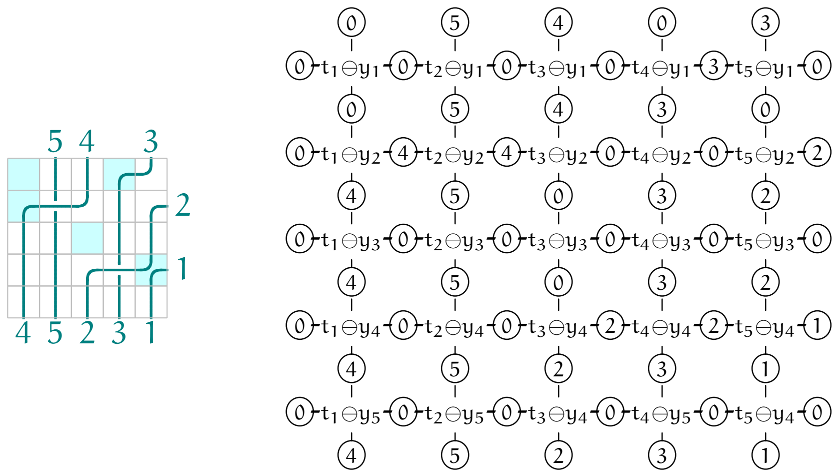
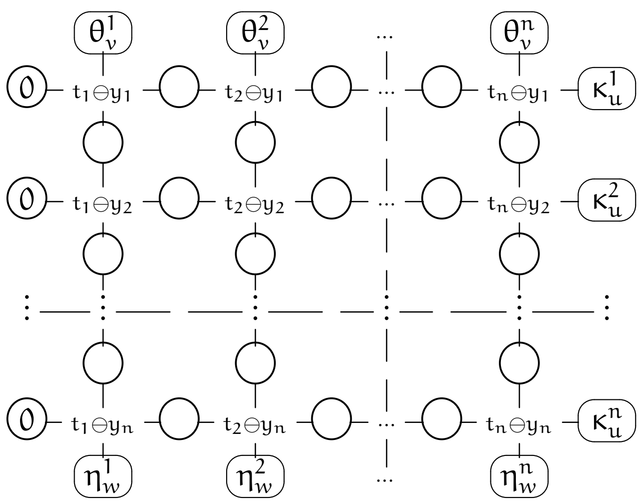


FIGURE 1. Correspondence between a pipe puzzle and an admissible state.

That is,

$Z^w_{u,v}(t,y) =$



Remark 4.1. *It can be checked directly from Table 2 that the weights of vertices (with* $x = t_j \ominus y_i$ *in row* i *and column* j*) are consistent with the weights of the corresponding tiles as defined above Theorem 2.6.*

Collecting the above observations, we summarize the following facts.

Proposition 4.2. *Let* $u, v \in S_n$ *be permutations with separated descents at position* k*. Then, for* $w \in S_n$*,*

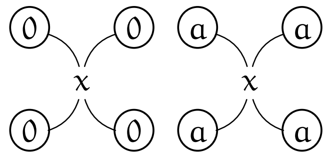
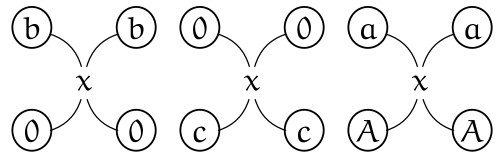
(1) *The set* $L(u, v, w)$ *of admissible states are in bijection with the set* $PP(u, v, w)$ *of pipe puzzles.*
(2) *We have*

$$Z^w_{u,v}(t,y) = \sum_{\pi \in PP(u,v,w)} \mathrm{wt}(\pi; t, y).$$

That is, the partition function $Z^w_{u,v}(t,y)$ *coincides with the right-hand side of* (2.10).

We next introduce two types of R-matrices: R_{row} and R_{col}, and check that the lattice model satisfies the Yang–Baxter equation when attaching an R_{row} (resp., an R_{col}) to rows (resp., columns).

4.2. **The R-matrix R_{row}.** The R-matrix R_{row} is given in Table 3.

A	B_1
	
$1 \quad (0 < a)$	$1 \quad (0 < b \leq k < c$ and $0 < a < A)$

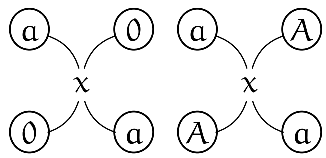
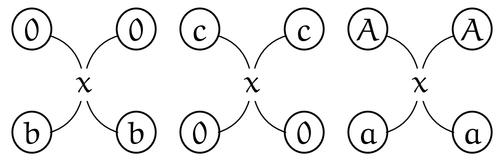
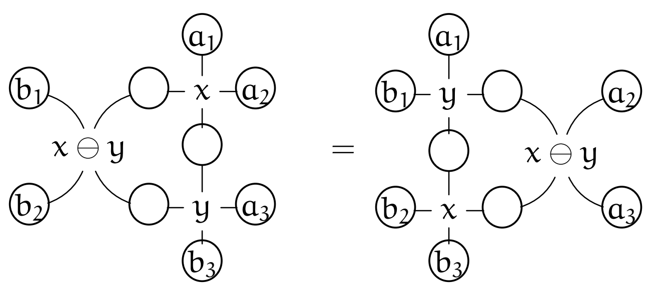
C	B_2
	
$x \quad (0 < a < A)$	$1 + \beta x \quad (0 < b \leq k < c$ and $0 < a < A)$

TABLE 3. The R-matrix R_{row}.

Theorem 4.3 (Yang–Baxter Equation for R_{row})**.** *For the R-matrix R_{row}, the partition functions of the following two models are equal for any given boundary condition with $a_1, a_2, a_3, b_1, b_2, b_3 \in \{0, 1, 2, \ldots, n\}$.*

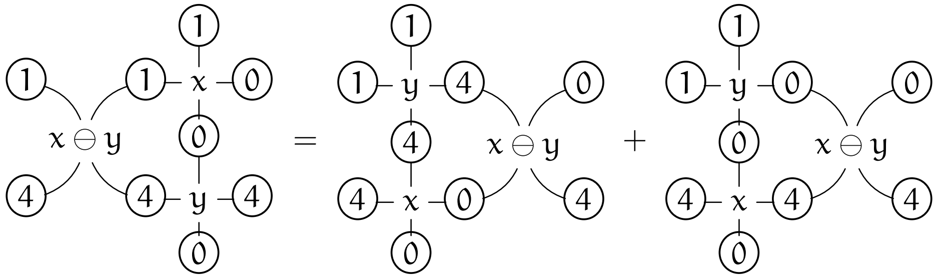


(4.1)

Here, the partition function of the left (resp., right) model is the sum of all weights of admissible configurations of the left (resp., right) diagram with the given boundary condition.

Proof. Note that (4.1) only depends on the relative values of $k, a_1, a_2, a_3, b_1, b_2, b_3$. By Tables 2 and 3, it suffices to assume $k = 3$ and $a_1, a_2, a_3, b_1, b_2, b_3 \in \{0, 1, 2, \ldots, 6\}$ with $\#\{a_1, a_2, a_3, b_1, b_2, b_3\} \leq 3$. So there are only finitely many cases to consider, which can be directly dealt with via computer verification. The SageMath code that we used to verify this theorem as well as Theorem 4.5 is available at `https://cubicbear.github.io/PipePuzzle.html`. □

Example 4.4. *Take* $k = 3$. *Let* $(a_1, a_2, a_3, b_1, b_2, b_3) = (1, 0, 4, 1, 4, 0)$. *By Tables 2 and 3, it can be checked that the admissible configurations of both sides are illustrated below.*



Again, in view of Tables 2 and 3, the partition function of the left model is $1 + \beta x$, *while the partition function of the right model is* $\beta(1 + \beta y)(x \ominus y) + (1 + \beta y)$. *These two partition functions are indeed the same. This agrees to the assertion in* (4.1).

4.3. **The R-matrix** R_{col}. The R-matrix R_{col} is given in Table 4.

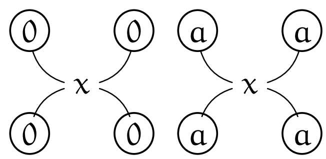
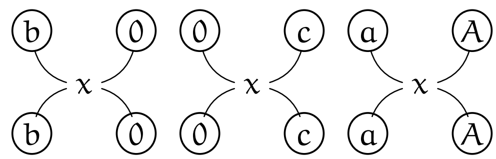
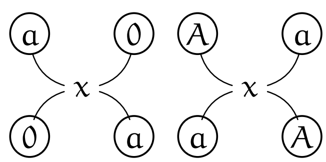
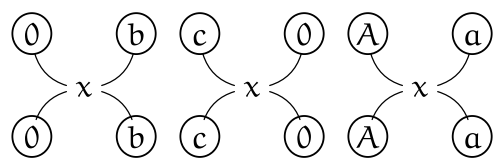
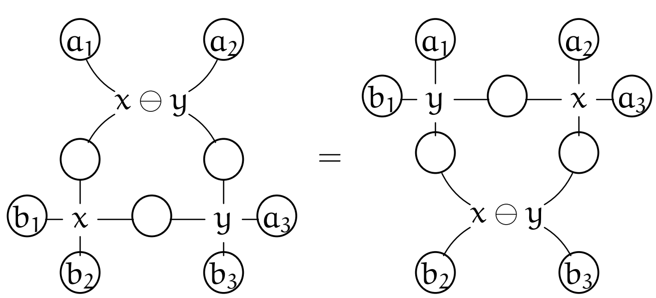
A	B_1
	
$1 \quad (0 < a)$	$1 \quad (0 < b \le k < c$ and $0 < a < A)$
C	**B_2**
	
$x \quad (0 < a < A)$	$1 + \beta x \quad (0 < b \le k < c$ and $0 < a < A)$

TABLE 4. The R-matrix R_{col}.

Theorem 4.5 (Yang–Baxter Equation for R_{col}). *For the R-matrix* R_{col}, *the partition functions of the following two models are equal for any given boundary condition with* $a_1, a_2, a_3, b_1, b_2, b_3 \in \{0, 1, 2, \ldots, n\}$.

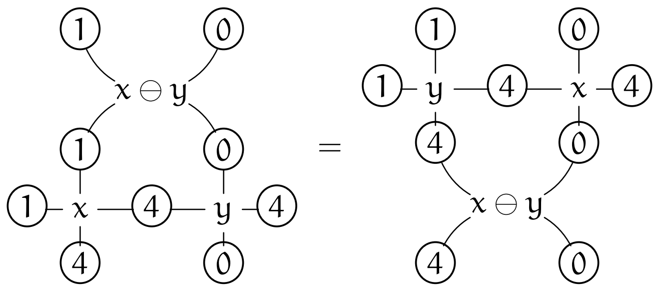


(4.2)

Example 4.6. *Still take* $k = 3$, *and* $(a_1, a_2, a_3, b_1, b_2, b_3) = (1, 0, 4, 1, 4, 0)$. *Equality* (4.2) *tells*

$$\beta(1 + \beta x) = \beta(1 + \beta y)(1 + \beta(x \ominus y)),$$

as implied by the following admissible configurations



5. Proof of the main result

We always set $u, v, w \in S_n$ where u and v have separated descents at position k. We finish the proof of Theorem 2.6 by showing that $Z^w_{u,v}(t, y)$ satisfies the same recurrence relations (Propositions 3.1 and 3.2) and initial condition (Proposition 3.4) as $c^w_{u,v}(t, y)$.

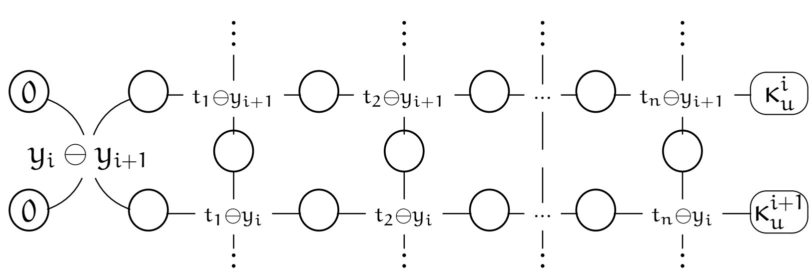
5.1. **Induction on** u. Suppose that $s_i u < u$. It is easily checked that $\mathrm{maxdes}(s_i u) \le k$. Recalling the definition in (2.2), we see that $0 < \kappa^{i+1}_u < \kappa^i_u \le k$ or $\kappa^i_u = 0 < \kappa^{i+1}_u \le k$, depending on the positions where i and $i + 1$ lie. Clearly, $\kappa_{s_i u}$ is obtained from κ_u by interchanging κ^i_u and κ^{i+1}_u. For example, for $n = 7$ and $k = 3$, we list a descending chain as follows:

$$\begin{aligned} u &= 5431267\cdots > 5341267\cdots > 5241367\cdots > 4251367\cdots \\ \kappa_u &= 0032100\cdots \to 0023100\cdots \to 0203100\cdots \to 0201300\cdots \end{aligned}$$

Theorem 5.1. *If* $s_i u < u$, *then*

$$Z^w_{s_i u,v} = -\frac{1 + \beta y_i}{y_i - y_{i+1}} Z^w_{u,v} + \frac{1 + \beta y_{i+1}}{y_i - y_{i+1}} Z^w_{u,v}|_{y_i \leftrightarrow y_{i+1}}. \tag{5.1}$$

Proof. Consider the lattice model $L(u, v, w)$. We attach an R_{row} to the left boundary of row i and row $i + 1$ (Meanwhile, we make the variable exchange $y_i \leftrightarrow y_{i+1}$ in the states of $L(u, v, w)$), as illustrated in (5.2).

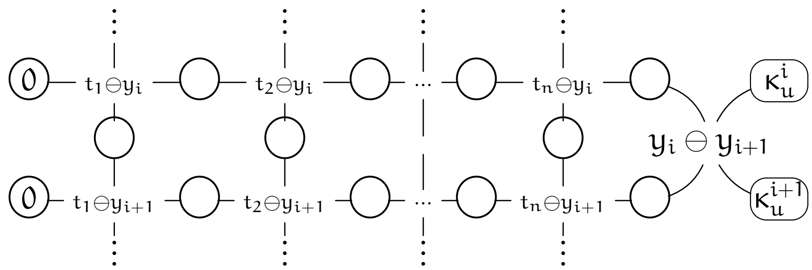


(5.2)

By Table 3, there is exactly one admissible configuration for the R-matrix R_{row} (from A in Table 3). So the partition function of (5.2) reads as

$$Z^w_{u,v}|_{y_i \leftrightarrow y_{i+1}}. \tag{5.3}$$

Noticing that $(t_j \ominus y_{i+1}) \ominus (t_j \ominus y_i) = y_i \ominus y_{i+1}$, we may apply repeatedly the Yang–Baxter equation in Theorem 4.3 to (5.2), resulting in a model depicted in (5.4), with an R-matrix R_{row} attached on the right boundary.



(5.4)

Consider the partition function of (5.4). Keep in mind that $0 < \kappa_u^{i+1} < \kappa_u^i \le k$ or $\kappa_u^i = 0 < \kappa_u^{i+1} \le k$. For each situation, there are two admissible configurations for the R-matrix R_{row} respectively from B_2 and C in Table 3, corresponding respectively to the models $L(u, v, w)$ and $L(s_i u, v, w)$. Thus, the partition function of (5.4) is

$$(1 + \beta(y_i \ominus y_{i+1}))Z_{u,v}^w + (y_i \ominus y_{i+1})Z_{s_i u,v}^w. \tag{5.5}$$

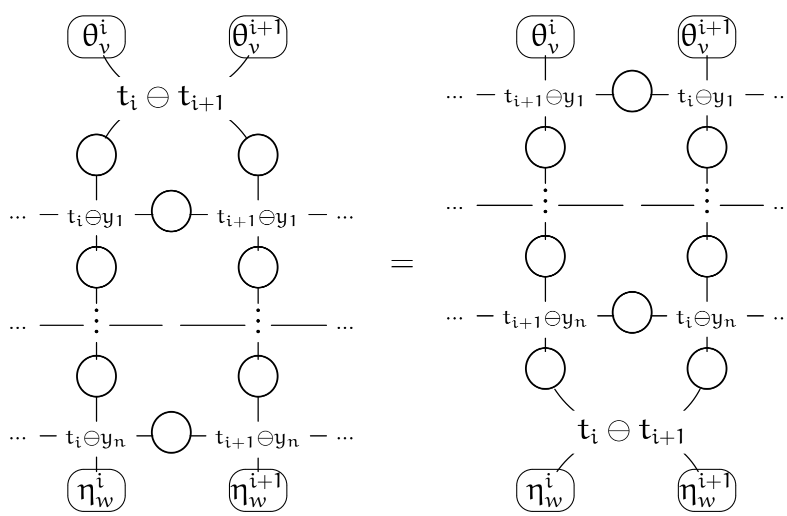
Equating (5.3) and (5.5), we get the desired formula in (5.1). □

5.2. **Induction on** w. We now establish the recurrence relation for $Z_{u,v}^w$, which is parallel to Proposition 3.2.

Theorem 5.2. *If* $s_i w > w$, *then*

$$Z_{u,v}^{s_i w} = \begin{cases} \dfrac{1+\beta t_i}{t_i - t_{i+1}} Z_{u,v}^w - \dfrac{1+\beta t_{i+1}}{t_i - t_{i+1}} Z_{u,v}^w|_{t_i \leftrightarrow t_{i+1}} + Z_{u,s_i v}^w|_{t_i \leftrightarrow t_{i+1}}, & s_i v < v, \\[2ex] \dfrac{1+\beta t_i}{t_i - t_{i+1}} Z_{u,v}^w - \dfrac{1+\beta t_i}{t_i - t_{i+1}} Z_{u,v}^w|_{t_i \leftrightarrow t_{i+1}}, & s_i v > v. \end{cases} \tag{5.6}$$

Proof. This time we attach an R_{col} to the top boundary of $L(u, v, w)$. Applying the Yang–Baxter equation in Theorem 4.5, we obtain equivalent models given in (5.7).



(5.7)

We first consider the partition function of the right model in (5.7). The assumption $s_i w > w$ implies $0 < \eta_w^i < \eta_w^{i+1}$. Notice also that $\eta_{s_i w}$ is obtained from η_w by interchanging η_w^i and η_w^{i+1}. In view of Table 4, there are two admissible configurations for the R-matrix R_{col} (one is from B_1 in Table 4, and the other is from C in Table 4), corresponding respectively to the models $L(u, v, w)$ and $L(u, v, s_i w)$. So the partition function of the right model in (5.7) is

$$Z_{u,v}^{w}|_{t_i \leftrightarrow t_{i+1}} + (t_i \ominus t_{i+1}) Z_{u,v}^{s_i w}|_{t_i \leftrightarrow t_{i+1}}. \tag{5.8}$$

We next consider the partition function of the left model in (5.7). There are two cases.

Case 1. $s_i v < v$. In this case, notice that $k < \theta_u^{i+1} < \theta_u^i$ or $0 = \theta_u^{i+1} < \theta_u^i$, and that $\theta_{s_i v}$ is obtained from θ_v by interchanging θ_v^i and θ_v^{i+1}. By Table 4, for either $k < \theta_u^{i+1} < \theta_u^i$ or $0 = \theta_u^{i+1} < \theta_u^i$, there are two choices for the configurations of R_{col} (one is from B_2, and the other is from C), corresponding respectively to the models $L(u, v, w)$ and $L(u, s_i v, w)$. So, the partition function of the left model in (5.7) is

$$(1 + \beta(t_i \ominus t_{i+1})) Z_{u,v}^{w} + (t_i \ominus t_{i+1}) Z_{u,s_i v}^{w}. \tag{5.9}$$

Equating (5.8) and (5.9), we deduce that

$$\begin{aligned} Z_{u,v}^{s_i w}|_{t_i \leftrightarrow t_{i+1}} &= \frac{1 + \beta(t_i \ominus t_{i+1})}{t_i \ominus t_{i+1}} Z_{u,v}^{w} + Z_{u,s_i v}^{w} - \frac{1}{t_i \ominus t_{i+1}} Z_{u,v}^{w}|_{t_i \leftrightarrow t_{i+1}} \\ &= \frac{1 + \beta t_i}{t_i - t_{i+1}} Z_{u,v}^{w} + Z_{u,s_i v}^{w} - \frac{1 + \beta t_{i+1}}{t_i - t_{i+1}} Z_{u,v}^{w}|_{t_i \leftrightarrow t_{i+1}}, \end{aligned}$$

which, after the variable exchange $t_i \leftrightarrow t_{i+1}$, becomes the first equality in (5.6).

Case 2. $s_i v > v$. In this case, i appears before $i + 1$ in v. So we have $0 = \theta_v^i = \theta_v^{i+1}$, or $0 = \theta_v^i$ and $k < \theta_v^{i+1}$, or $k < \theta_v^i < \theta_v^{i+1}$. By Table 4, for each of these situations, there is exactly one admissible configuration (from A or B_1) of R_{col}, and we see that the partition function of the left model in (5.7) is precisely equal to $Z_{u,v}^{w}$. By equating with (5.8), we obtain that

$$\begin{aligned} Z_{u,v}^{s_i w}|_{t_i \leftrightarrow t_{i+1}} &= \frac{1}{t_i \ominus t_{i+1}} Z_{u,v}^{w} - \frac{1}{t_i \ominus t_{i+1}} Z_{u,v}^{w}|_{t_i \leftrightarrow t_{i+1}} \\ &= \frac{1 + \beta t_{i+1}}{t_i - t_{i+1}} Z_{u,v}^{w} - \frac{1 + \beta t_{i+1}}{t_i - t_{i+1}} Z_{u,v}^{w}|_{t_i \leftrightarrow t_{i+1}}. \end{aligned}$$

After the variable exchange $t_i \leftrightarrow t_{i+1}$ on both sides, we reach the second equality in (5.6). □

5.3. **Initial condition.** We finally verify the initial case for u_0 (as defined in (3.4)) and $w = \mathrm{id}$.

Theorem 5.3. *For $v \in S_n$, we have*

$$Z_{u_0,v}^{\mathrm{id}} = \begin{cases} \prod_{i=1}^{k} \prod_{j=1}^{n-i} (t_i \ominus y_j), & \text{if } v = \mathrm{id}, \\ 0, & \text{otherwise}. \end{cases} \tag{5.10}$$

Proof. Here, we go back to the pipe puzzle model $\mathrm{PP}(u_0, v, \mathrm{id})$ for the computation of $Z^{\mathrm{id}}_{u_0,v}$. The boundary condition is illustrated in the left diagram in Figure 2. Evidently,

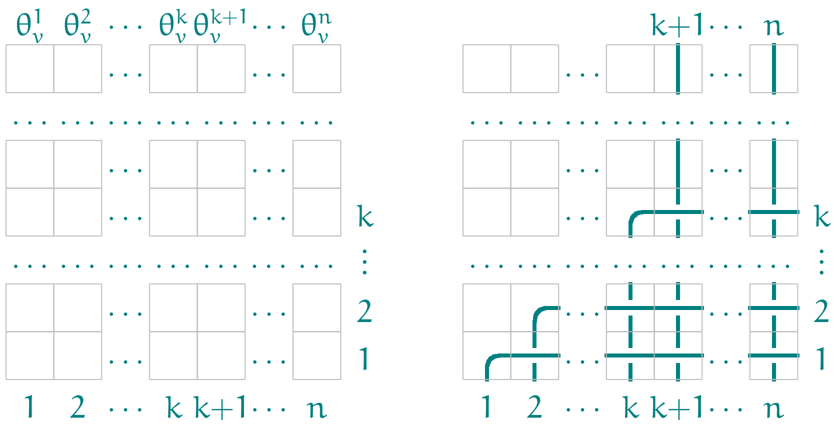


FIGURE 2. Boundary condition and the unique pipe puzzle in $\mathrm{PP}(u_0, \mathrm{id}, \mathrm{id})$.

the pipes labeled $k+1, \ldots, n$ must go vertically from the top side down to the bottom side. So we have $Z^{\mathrm{id}}_{u_0,v} = 0$ whenever $v \neq \mathrm{id}$. It remains to check the case $v = \mathrm{id}$. It is easily checked that there is exactly one pipe puzzle in $\mathrm{PP}(u_0, \mathrm{id}, \mathrm{id})$, see the right diagram of Figure 2. This pipe puzzle contributes a weight as displayed in (5.10). □

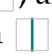
6. APPLICATIONS

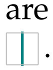
We list three main applications of Theorem 2.6. The first application is to recover the puzzle formula discovered by Knutson and Zinn-Justin [12, Theorem 1].

6.1. **Separated-descent puzzles.** Consider (1.1) by setting $y = t$:

$$\mathfrak{G}_u(x, t) \cdot \mathfrak{G}_v(x, t) = \sum_w c^w_{u,v}(t, t) \cdot \mathfrak{G}_w(x, t).$$

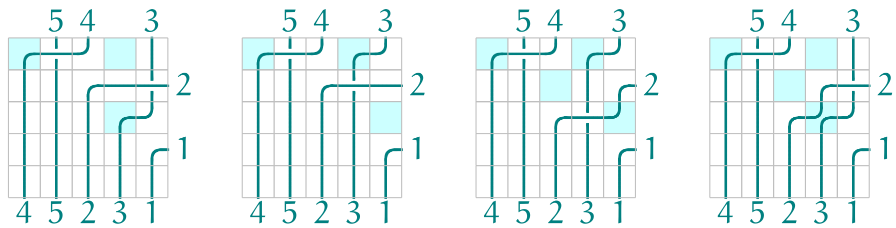
Assume that $u, v \in S_n$ have separated descents at k. For $w \in S_n$, a pipe puzzle $\pi \in \mathrm{PP}(u, v, w)$ has weight zero if and only if π has (at least) one empty tile  on the diagonal. This implies that $c^w_{u,v}(t, t)$ is a weighted counting of pipe puzzles $\pi \in \mathrm{PP}(u, v, w)$ such that π has no empty tile on the diagonal. For such pipe puzzles, we have the following observation:

- Each position on the diagonal is tiled with either  or , and each position lying strictly to the southwest of the diagonal is tiled with .

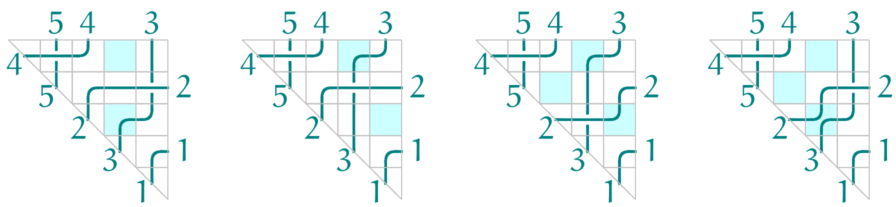
This can be checked as follows. First, the tile at the position $(1, 1)$ must be tiled with either  or  since (1) the tile cannot be empty, and (2) the labels on the left boundary are all 0. Therefore, all positions below $(1, 1)$ in the first column must be tiled with . The same analysis applies to the remaining positions $(2, 2), \ldots, (n, n)$.

Let $\pi \in \mathrm{PP}(u, v, w)$ be a pipe puzzle without empty tile on the diagonal. Cut π along its diagonal into two triangles, and denote by $P(\pi)$ the upper-right triangle. By the above observation, π can be recovered from $P(\pi)$. To get the puzzle visualization of Knutson and Zinn-Justin [12, Theorem 1], we rotate $P(\pi)$ counterclockwise by 45 degrees, and then warp it into an equilateral triangle. If further assuming that u and v are both k-Grassmannian, there is a bijection to the classical Grassmannian puzzles, see [12, §5.1] for more details.

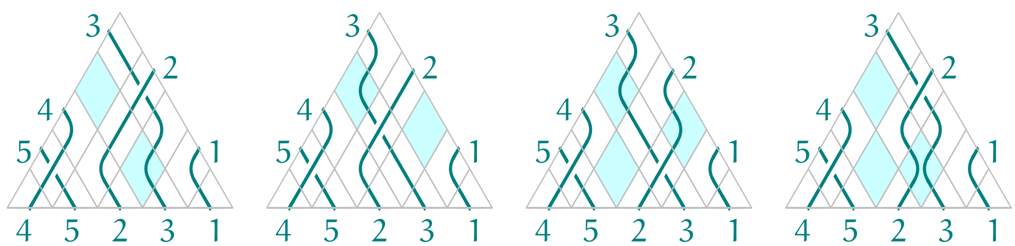
Example 6.1. *Consider the pipe puzzles in Example 2.7. The following four puzzles survive after setting* $y = t$.



Their upper-right triangular regions are



After rotation and warping, the corresponding puzzles are



In the second application, we explain that Theorem 2.6 could be used to recover the bumpless pipe dream model of double Grothendieck polynomials by Weigandt [22].

6.2. **Bumpless pipe dreams.** Let $k = n$ and $v = \mathrm{id}$. In this case, arbitrary $u \in S_n$ satisfies the separated-descent condition in (1.2). By Lemma 3.3,

$$c^{\mathrm{id}}_{u,\mathrm{id}}(t, y) = \mathfrak{G}_u(t, y).$$

Let $\pi \in \mathrm{PP}(u, \mathrm{id}, \mathrm{id})$. Then all pipes enter into π from the right side. Apply the following operations to π:

- reflecting π across the diagonal;
- replacing $\kappa^i_u = u^{-1}(i)$ by i, and $\eta^i_w = i$ by $u(i)$.

The resulting diagram is denoted as $B(\pi)$. Write

$$\mathrm{BP}(u) = \{B(\pi) \colon \pi \in \mathrm{PP}(u, \mathrm{id}, \mathrm{id})\}.$$

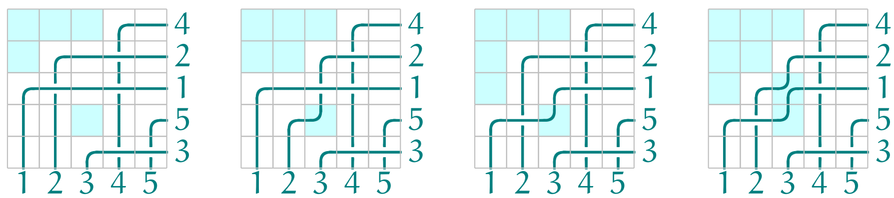
By the restriction (2.5) on ┼ along with the restriction (2.8) on ╯╭, it can be checked that for a diagram in $\mathrm{BP}(u)$: (1) two pipes cross at most once, and (2) if two pipes have a "bumping" ╯╭ at position (i, j), then they must cross at a position to the northeast of (i, j). This implies that the set $\mathrm{BP}(u)$ is precisely the set of bumpless pipe dreams of u, as defined in [22].

Remark 6.2. *As bumpless pipe dreams in* $\mathrm{BP}(u)$ *are obtained from pipe puzzles in* $\mathrm{PP}(u, \mathrm{id}, \mathrm{id})$ *after a reflection, a tile at position* (i, j) *is assigned a weight in the following way:*

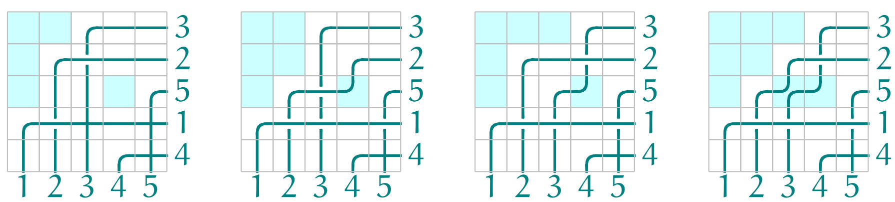
(1) an empty tile ☐ *contributes* $t_i \ominus y_j$*;*
(2) an elbow tile ╯ *contributes* $1 + \beta(t_i \ominus y_j)$*;*
(3) a bumping tile ╯╭ *contributes* β*;*
(4) any other tile contributes 1.

The weights described above are slightly different from the weights adopted in [22]*. It seems that when setting* $\beta = 0$*, the weights we used imply more explicitly the bumpless pipe dream model of double Schubert polynomials due to Lam, Lee and Shimozono* [13]*.*

Example 6.3. *Let* $u = 32514$*. Below are pipe puzzles in* $\mathrm{PP}(u, \mathrm{id}, \mathrm{id})$*.*



After reflection and relabeling, the resulting bumpless pipe dreams of u *are*



We finally apply Theorem 2.2 to investigate a conjecture posed by Kirillov [8].

6.3. **Kirillov's conjecture.** Let us restrict to Schubert polynomials. Setting $\beta = 0$, the operator π_i is usually denoted as ∂_i:

$$\partial_i f = \frac{f - f|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}.$$

The operator ∂_i is also called the *divided difference operator*. For $w \in S_\infty$, define $\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$ for any reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$(this is well defined since the ∂_i's satisfy the braid relations). It can be deduced that [8, Proposition 2]

$$\partial_w \mathfrak{S}_u(x, t) = \begin{cases} \mathfrak{S}_{uw^{-1}}(x, t), & \text{if } \ell(uw^{-1}) = \ell(u) - \ell(w), \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

The *skew operator* $\partial_{w/v}$ is characterized by

$$\partial_w(fg) = \sum_v (\partial_{w/v} f)(\partial_v g). \tag{6.2}$$

See [8, Definition 4] for a more concrete description of $\partial_{w/v}$. Kirillov [8, Conjecture 1] conjectured that for any u, v, w, the polynomial $\partial_{w/v}\mathfrak{S}_u(x)$ has nonnegative integer coefficients:

$$\partial_{w/v}\mathfrak{S}_u(x) \in \mathbb{Z}_{\geq 0}[x_1, x_2, \ldots].$$

Setting $y = 0$ in (2.3) yields that

$$\mathfrak{S}_u(x) \cdot \mathfrak{S}_v(x,t) = \sum_w \bar{c}_{u,v}^w(t,0) \cdot \mathfrak{S}_w(x,t). \tag{6.3}$$

Proposition 6.4. *We have*

$$\partial_{w/v}\mathfrak{S}_u(x) = \bar{c}_{u,v}^w(x,0).$$

Proof. Apply ∂_w to both sides of (6.3), and then take the specialization $x = t$. In view of (6.1), (6.2) and the localization formula in (3.3) (which is still valid for double Schubert polynomials), the left-hand side becomes $\partial_{w/v}\mathfrak{S}_u(x)$, and the right-hand side is left with $\bar{c}_{u,v}^w(x,0)$. □

Setting $y = 0$ in Theorem 2.2, we arrive at the following conclusion.

Corollary 6.5. *Let $u, v \in S_n$ be permutations with separated descents. Then*

$$\mathfrak{S}_u(x) \cdot \mathfrak{S}_v(x,t) \in \sum_w \mathbb{Z}_{\geq 0}[t] \cdot \mathfrak{S}_w(x,t).$$

Combining Proposition 6.4 with Corollary 6.5 enables us to confirm Kirillov's conjecture for permutations with separated descents.

Corollary 6.6. *Kirillov's conjecture is true for u and v with separated descents and arbitrary w.*

APPENDIX A. LEFT DEMAZURE OPERATORS

Recall that the (left) Demazure operator ϖ_i and φ_i are as defined in Section 3.

Proposition A.1. *We have*

$$\varpi_i\mathfrak{G}_w(x,t) = \begin{cases} \mathfrak{G}_{s_iw}(x,t), & \text{if } s_iw < w, \\ -\beta\mathfrak{G}_w(x,t), & \text{if } s_iw > w. \end{cases} \tag{A.1}$$

A geometric proof of Proposition A.1 can be found in [15]. Here, we provide an algebraic proof. To this end, we need the *Hecke product* on permutations:

$$s_i * w = \begin{cases} s_iw, & \text{if } s_iw > w, \\ w, & \text{if } s_iw < w, \end{cases} \quad \text{and} \quad w * s_i = \begin{cases} ws_i, & \text{if } ws_i > w, \\ w, & \text{if } ws_i < w. \end{cases}$$

This defines a monoid structure over S_∞ called the *0-Hecke monoid*.

Proof of Proposition A.1. Let $w_0 = n \cdots 21$ be the longest element in S_n. For $w \in S_n$, denote

$$\mathfrak{G}^w(x,t) = \mathfrak{G}_{w_0w}(x,t).$$

Then (2.1) can be rewritten as

$$\pi_i\mathfrak{G}^w = (-\beta)^{\ell(w)+1-\ell(w*s_i)}\mathfrak{G}^{w*s_i}. \tag{A.2}$$

Note that the identity in (A.1) can be restated as

$$\varpi_i\mathfrak{G}^w = (-\beta)^{\ell(w)+1-\ell(s_{n-i}*w)}\mathfrak{G}^{s_{n-i}*w}. \tag{A.3}$$

We prove (A.3) by induction on the length of w. When $\ell(w) = 0$ (namely, $w = \mathrm{id}$), it follows from direct computation that

$$\varpi_i \mathfrak{G}^{\mathrm{id}}(x,t) = \varpi_i \mathfrak{G}_{w_0}(x,t) = \prod_{\substack{a+b\leq n \\ (a,b)\neq(n-i,i)}} (x_a \ominus t_b),$$

which coincides with $\mathfrak{G}^{s_{n-i}}(x,t) = \mathfrak{G}_{w_0 s_{n-i}}(x,t) = \pi_{n-i}\mathfrak{G}_{w_0}(x,t)$.

For $\ell(w) > 0$, one can find an index j such that $ws_j < w$, and so by induction,

$$\begin{aligned}
\varpi_i \mathfrak{G}^{w} &= \varpi_i \pi_j \mathfrak{G}^{ws_j} = \pi_j \varpi_i \mathfrak{G}^{ws_j} = (-\beta)^{\ell(ws_j)+1-\ell(s_{n-i}*ws_j)} \pi_j \mathfrak{G}^{s_{n-i}*ws_j} \\
&= (-\beta)^{\ell(ws_j)+1-\ell(s_{n-i}*ws_j)} (-\beta)^{\ell(s_{n-i}*ws_j)+1-\ell(s_{n-i}*ws_j*s_j)} \mathfrak{G}^{s_{n-i}*ws_j*s_j} \\
&= (-\beta)^{\ell(ws_j)+2-\ell(s_{n-i}*w)} \mathfrak{G}^{s_{n-i}*w} \\
&= (-\beta)^{\ell(w)+1-\ell(s_{n-i}*w)} \mathfrak{G}^{s_{n-i}*w}.
\end{aligned}$$

Here, we used (A.2) in the first and fourth equality, and used the fact that the operators π_j and ϖ_j commute in the second equality. □

Now, we can present proofs of Propositions 3.1 and 3.2.

Proof of Proposition 3.1. Assume that $s_i u < u$. Applying φ_i to (1.1), by Proposition A.1, the left-hand side is

$$\mathfrak{G}_{s_i u}(x,y) \cdot \mathfrak{G}_v(x,t) = \sum_w c^w_{s_i u, v}(t,y) \cdot \mathfrak{G}_w(x,t).$$

While the right-hand side is

$$\sum_w \varphi_i c^w_{u,v}(t,y) \cdot \mathfrak{G}_w(x,t).$$

Comparing the coefficients of $\mathfrak{G}_w(x,t)$, we are given $c^w_{s_i u, v} = \varphi_i c^w_{u,v}$, as desired. □

Proof of Proposition 3.2. Apply ϖ_i to (1.1). By Proposition A.1, the left-hand side is

$$\begin{cases} \mathfrak{G}_u(x,y) \cdot \mathfrak{G}_{s_i v}(x,t) = \sum_w c^w_{u, s_i v}(t,y) \cdot \mathfrak{G}_w(x,t), & s_i v < v, \\ -\beta \mathfrak{G}_u(x,y) \cdot \mathfrak{G}_v(x,t) = \sum_w -\beta c^w_{u,v}(t,y) \cdot \mathfrak{G}_w(x,t), & s_i v > v. \end{cases}$$

To compute the right-hand side, we use the following property of ϖ_i:

$$\varpi_i(fg) = (f|_{t_i \leftrightarrow t_{i+1}})(\varpi_i g) - \frac{1+\beta t_i}{t_i - t_{i+1}}(f - f|_{t_i \leftrightarrow t_{i+1}})g. \tag{A.4}$$

By (A.4) and Proposition A.1, the right-hand side is

$$\begin{aligned}
&\sum_w \varpi_i \big(c^w_{u,v} \cdot \mathfrak{G}_w(x,t)\big) \\
&= \sum_w \left((c^w_{u,v}|_{t_i \leftrightarrow t_{i+1}}) \varpi_i \mathfrak{G}_w - \frac{1+\beta t_i}{t_i - t_{i+1}} (c^w_{u,v} - c^w_{u,v}|_{t_i \leftrightarrow t_{i+1}}) \mathfrak{G}_w \right) \\
&= \sum_{s_i w < w} \left((c^w_{u,v}|_{t_i \leftrightarrow t_{i+1}}) \mathfrak{G}_{s_i w} - \frac{1+\beta t_i}{t_i - t_{i+1}} (c^w_{u,v} - c^w_{u,v}|_{t_i \leftrightarrow t_{i+1}}) \mathfrak{G}_w \right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s_iw>w}\left(-\beta(c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w-\frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w\right)\\
=&-\sum_{s_iw<w}\frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w\\
&+\sum_{s_iw>w}\left(c^{s_iw}_{u,v}|_{t_i\leftrightarrow t_{i+1}}-\beta c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}}-\frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\right)\mathfrak{G}_w\\
=&-\sum_{s_iw<w}\frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w\\
&+\sum_{s_iw>w}\left(c^{s_iw}_{u,v}|_{t_i\leftrightarrow t_{i+1}}-\frac{1+\beta t_i}{t_i-t_{i+1}}c^w_{u,v}+\frac{1+\beta t_{i+1}}{t_i-t_{i+1}}c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}}\right)\mathfrak{G}_w.
\end{aligned}
$$

Extracting the coefficients of $\mathfrak{G}(w)$ with $s_iw > w$ on both sides, we deduce that

$$
c^{s_iw}_{u,v}|_{t_i\leftrightarrow t_{i+1}}=\frac{1+\beta t_i}{t_i-t_{i+1}}c^w_{u,v}-\frac{1+\beta t_{i+1}}{t_i-t_{i+1}}c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}}+\begin{cases}c^w_{u,s_iv}, & s_iv<v,\\ -\beta c^w_{u,v}, & s_iv>v,\end{cases}
$$

which coincides with (3.2) after the variable exchange $t_i \leftrightarrow t_{i+1}$. □

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