Spectral analysis of normalized Hermitian Laplacian matrices in random mixed graphs *

Dan Hu[a,†], Hajo Broersma[b], Xueliang Li[c], Shenggui Zhang[d,e]

[a]Department of Mathematics, Shantou University,
Shantou, Guangdong 515063, P.R. China

[b]Faculty of EEMCS, University of Twente,
P.O. Box 217, 7500 AE Enschede, The Netherlands

[c]Center for Combinatorics, Nankai University,
Tianjin 300071, P.R. China

[d]School of Mathematics and Statistics, Northwestern Polytechnical University,
Xi'an, Shaanxi 710129, P.R. China

[e]Xian-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University,
Xian, Shaanxi 710129, P.R. China

Abstract

A mixed graph is a graph that can be obtained from a simple undirected graph by replacing some of the edges by arcs in precisely one of the two possible directions. Let $p = p(n)$ be a function of n such that $0 < p < 1$. Let $\widehat{G}_n(p)$ be a random mixed graph with n vertices in which all arcs are chosen independently with probability p (and an edge is regarded as two oppositely oriented arcs joining the same pair of vertices). In this paper we study the spectral properties of the normalized Hermitian Laplacian matrices of the random mixed graphs $\widehat{G}_n(p)$ for large n. We characterize the limiting spectral distribution of the normalized Hermitian Laplacian matrices of random mixed graphs. In fact, under the case that $p \in (0, 1)$ and $np/\ln^4 n \to \infty$, we prove that the empirical distribution of the eigenvalues of the normalized Hermitian Laplacian matrix converges to the semicircle law.

Keywords: Random mixed graphs; normalized Hermitian Laplacian matrix; Empirical spectral distribution; Limiting spectral distribution; semicircle law

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†Corresponding author. E-mail addresses: hudan@stu.edu.cn

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1 Introduction

Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of $n \times n$ random Hermitian matrices. Suppose that $\lambda_1(M_n)$, $\lambda_2(M_n), \ldots, \lambda_n(M_n)$ are the eigenvalues of M_n. The *empirical spectral distribution* (ESD) of M_n is defined by

$$F^{M_n}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{\lambda_i(M_n) \leq x\}, x \in \mathbb{R},$$

where $\mathbb{I}\{\lambda_i(M_n) \leq x\}$ is the indicator function, which is 1 if $\lambda_i(M_n) \leq x$ and 0 otherwise. The distribution to which the ESD of M_n converges as $n \to \infty$ is called the *limiting spectral distribution* (LSD) of $\{M_n\}_{n=1}^{\infty}$.

The ESD of a random Hermitian matrix has a very complicated form when the order of the matrix is large. In particular, it seems more difficult to characterize the LSD of an arbitrary given sequence of random Hermitian matrices. A pioneer work on the spectral distribution of random Hermitian matrices [5, 23] was owed to Wigner, which is called the *Wigner's semicircle law* [29, 30]. Wigner's semicircle law characterizes the LSD of a sort of random Hermitian matrices. This sort of random Hermitian matrices is usually called the *Wigner matrices*. Wigner's semicircle law has been greatly generalized to more general random matrices by lots of researchers, including Arnold [1, 2], Grenander [20], Bai and Yin [3, 4, 5, 6, 7, 24], Geman [16], Girko [17, 18, 19], Loève [22] and so on.

The spectral properties of the graph-relevant matrices such as adjacency matrices, Laplacian matrices and normalized Laplacian matrices have many applications in graph theory. For example, the spectrum of the normalized Laplacian matrix is related to the graph discrepancy and diffusion on graphs. The second smallest eigenvalue of the normalized Laplacian matrix relates to the Cheeger constant and the rate of convergence of random walks on the graph, see, e.g., [10, 11].

In recent years, also for random graphs, spectral distributions of their adjacency matrices, Laplacian matrices and normalized Laplacian matrices are well-studied. We next present a brief account of some of the results that were obtained for random graphs. Ding et al. [13] considered the spectral distributions of adjacency and Laplacian matrices of random graphs; Du et al. [14, 15] considered the spectral distributions of adjacency and Laplacian matrices of Erdős-Rényi model and the spectral distribution of adjacency matrices of random multipartite graphs; and Chen et al. [9] considered the spectral distribution of skew adjacency matrices of random oriented graphs and the spectral distribution of adjacency matrices of random regular oriented graphs. Jiang [28] studied the spectral properties of the Laplacian matrices and the normalized Laplacian matrices of the Erdős-Rényi random graph $G(n, p_n)$ for large n. Under the dilute case, that is, $p_n \in (0, 1)$ and $np_n \to \infty$. Jiang proved that the empirical distribution of the eigenvalues of the Laplacian matrix converges to a deterministic distribution, which is the free convolution of the semicircle law and $N(0, 1)$. However, for its normalized version, Jiang proved that the empirical distribution converges to the semicircle law. Hu et al. [27] proved that the empirical distribution of the eigenvalues of the Hermitian adjacency matrix converges to the Wigner's semicircle law.

The purpose of our paper is to study the spectral distribution of the normalized Hermitian Laplacian matrices of random mixed graphs. A graph is called a *mixed graph* if it contains both directed and undirected edges. We usually use $G = (V(G), E(G), A(G))$ to denote a mixed graph with a set $V(G)$ of vertices, a set $E(G)$ of undirected edges, and a set $A(G)$ of directed edges (or arcs). If we regard each undirected edge $uv \in E(G)$ in $G = (V(G), E(G), A(G))$ as two directed edges (u, v) and (v, u), then G is indeed a directed graph. Throughout this paper, we regard mixed graphs as directed graphs by keeping this thought in mind. Define the *underlying graph* of a mixed graph G, denoted $\Gamma(G)$, to be the graph with vertex set $V(\Gamma(G))$, which is same as $V(G)$, and edge set

$$E(\Gamma(G)) = \{uv | uv \in E(G) \text{ or } (u, v) \in A(G) \text{ or } (v, u) \in A(G)\}.$$

In [21], the *Hermitian adjacency matrix* of a mixed graph G of order n was defined to be the $n \times n$ matrix $H(G) = (h_{uv})_{n\times n}$, where

$$h_{uv} = \begin{cases} 1, & \text{if } uv \in E(G); \\ \mathrm{i}, & \text{if } (u, v) \in A(G) \text{ and } (v, u) \notin A(G); \\ -\mathrm{i}, & \text{if } (u, v) \notin A(G) \text{ and } (v, u) \in A(G); \\ 0, & \text{otherwise}, \end{cases}$$

and $\mathrm{i} = \sqrt{-1}$. This matrix was also introduced independently by Guo and Mohar in [25].

Moreover, we give the definition of a random mixed graph $\widehat{G}_n(p)$. Let K_n be a complete graph on n vertices. A *complete directed graph* DK_n is the graph obtained from K_n by replacing each edge of K_n with two opposite directed edges. Let $p = p(n)$ be a function of n such that $0 < p < 1$. The random mixed graph model $\widehat{\mathcal{G}}_n(p)$ consists of all random mixed graphs $\widehat{G}_n(p)$ in which the directed edges are chosen randomly and independently, with probability p from the set of the directed edges of DK_n. Then the *Hermitian adjacency matrix* of $\widehat{G}_n(p)$, denoted by $H(\widehat{G}_n(p)) = (h_{ij})$ (or H_n, for brevity), satisfies that:

• H_n is a random Hermitian matrix, particularly, $h_{ii} = 0$ for $1 \leq i \leq n$;

• the upper-triangular entries h_{ij}, $1 \leq i < j \leq n$ are independently identically distributed (i.i.d.) copies of a random variable ξ which takes value 1 with probability p^2, i with probability $p(1-p)$, $-$i with probability $p(1-p)$, and 0 with probability $(1-p)^2$.

Let $D_n = \text{diag}(d_1, d_2, \ldots, d_n)$ be a diagonal matrix where d_i is the degree of vertex v_i in the underlying graph $\Gamma(\widehat{G}_n(p))$. The matrix $\mathcal{L}_n = I_n - D_n^{-\frac{1}{2}} H_n D_n^{-\frac{1}{2}}$ is said the *normalized Hermitian Laplacian matrix* of $\widehat{G}_n(p)$, where I_n is the identity matrix and $D_n^{-\frac{1}{2}}$ is the $n \times n$ diagonal matrix whose (i,i)-th entry is $d_i^{-1/2}$ if $d_i \neq 0$, or 0 if $d_i = 0$.

In this paper, we characterize the LSD of the normalized Hermitian Laplacian matrices of random mixed graphs. Our main result is stated as follows.

Theorem 1. *Let $\{\mathcal{L}_n\}_{n=1}^{\infty}$ be a sequence of normalized Hermitian Laplacian matrices of random mixed graphs $\{\widehat{G}_n(p)\}_{n=1}^{\infty}$ with $p = p(n)$, $0 < p < 1$ and $\sup\{p(n); n \geq 2\} < 1$. Let $\sigma = \sqrt{2p - p^2 - p^4}$, and $\delta = (n-1)(2p - p^2)$. If $np/\ln^4 n \to \infty$ as $n \to \infty$. Then almost surely, the ESD of $\frac{\delta}{\sigma\sqrt{n}}(I_n - \mathcal{L}_n)$ converges weakly to the standard semicircle distribution whose density is given by*

$$\rho(x) := \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{for } |x| \leq 2, \\ 0, & \text{for } |x| > 2. \end{cases}$$

If p_n in Theorem 1 is equal to a constant, we get the following result.

Corollary 1. *Suppose $p(n) \equiv p \in (0,1)$ for all $n \geq 2$. Let $\sigma = \sqrt{2p - p^2 - p^4}$ and $\delta = (n-1)(2p - p^2)$. Then almost surely, the ESD of $\frac{\delta}{\sigma\sqrt{n}}(I_n - \mathcal{L}_n)$ converges weakly to the standard semicircle distribution whose density is given by*

$$\rho(x) := \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{for } |x| \leq 2, \\ 0, & \text{for } |x| > 2. \end{cases}$$

If $p = p(n) \to 0$, then $\frac{\delta}{\sigma\sqrt{n}} = \frac{(n-1)(2p-p^2)}{\sqrt{n(2p-p^2-p^4)}} = \sqrt{\frac{(n-1)^2 p(2-p)^2}{n(2-p-p^3)}} \sim \sqrt{2np}$ as $n \to \infty$. From Theorem 1 we immediately obtain the following corollary.

Corollary 2. *Suppose $p = p(n) \to 0$ and $np/\ln^4 n \to \infty$ as $n \to \infty$. Then almost surely, the ESD of $\sqrt{2np}(I_n - \mathcal{L}_n)$ converges weakly to the standard semicircle distribution whose density is given by*

$$\rho(x) := \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{for } |x| \leq 2, \\ 0, & \text{for } |x| > 2. \end{cases}$$

We postpone the proof of Theorem 1 to Section 3. In the next section, we state some existing results that we need in our proof of Theorem 1.

2 Key tools

Before giving the proof of Theorem 1, we collect some notations and results that will be used in the sequel of the paper.

For each matrix A, The *spectral norm* $\|A\|$ is the largest singular value of A, i.e.,

$$\|A\| = \sqrt{\lambda_{\max}(A^*A)}.$$

Here A^* is the conjugate transpose of A and $\lambda_{\max}(A^*A)$ is the largest eigenvalue of A^*A. When A is an $n \times n$ Hermitian matrix, we denote by $\lambda_i(A)(1 \le i \le n)$ the i-th largest eigenvalue of A (multiplicities counted), and we have $\|A\| = \max\{|\lambda_i(A)| : 1 \le i \le n\}$. Moreover, $\mathrm{Tr}(A)$ (the trace of A) is the sum of the eigenvalues of A.

A random matrix A is a matrix in which each entry is a random variable. We write $\mathbb{E}(A)$ to denote the coordinate-wise expectation of A, so $\mathbb{E}(A)_{ij} = \mathbb{E}(A_{ij})$. In the following $\|\mathbf{x}\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$ denotes the 2-norm in the unitary space $\mathbb{C}^n$, where $\mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{C}^n$ is a complex vector. Let $f(n), g(n)$ be two functions of n. Then $f(n) = o(g(n))$ means that $f(n)/g(n) \to 0$, as $n \to \infty$; $f(n) = \omega(g(n))$ means that $f(n)/g(n) \to \infty$, as $n \to \infty$, or equivalently, $g(n) = o(f(n))$. For two functions $f(n)$ and $g(n)$ taking nonnegative values, we say $f(n) \gg g(n)$ (or $g(n) = o(f(n))$) if $g(n)/f(n) \to 0$, as $n \to \infty$. The proof of Theorem 1 is based on the following theorem and lemmas.

An useful concentration inequality is Bernstein's inequality, originally proven by Bernstein in 1924 [8]. We state here a version somewhat simpler than Bernstein's original result. It can, e.g., be found as Lemma A in [12].

Lemma 1. *(Bernstein's Inequality) Let $X_1, X_2, \ldots, X_m$ be independent random variables satisfying $|X_i| \le c$ for all i. Let $X = \sum_{i=1}^m X_i$. Then for any $a > 0$,*

$$\Pr(|X - \mathbb{E}(X)| \ge a) \le \exp\Big(-\frac{a^2}{2\sum_{i=1}^m \mathrm{Var}(X_i) + 2ac/3}\Big).$$

Lemma 2 (Borel-Cantelli Lemma)**.** *Let $\{E_n\}_{n=1}^\infty$ be a sequence of events in a probability space. If $\sum_{n=1}^\infty \Pr(E_n) < \infty$, then $\Pr(\limsup_{n\to\infty} E_n) = 0$.*

Lemma 3 (Rank Inequality (See [4]))**.** *Let A and B be two $n \times n$ Hermitian matrices. Then*

$$\|F^A - F^B\| \le \frac{1}{n}\mathrm{rank}(A - B),$$

where $\|f(x)\| := \sup_x |f(x)|$ for a function $f(x)$, and F^A means the ESD of A.

$L(F,G)$ denotes the Levy distance between distribution functions F and G defined by $L(F,G) = \inf\{\epsilon \ge 0 : \forall x \in \mathbb{R}, F(x-\epsilon) \le G(x) \le F(x+\epsilon)\}$, which characterizes the weak convergence of probability distributions.

Lemma 4 (Norm Inequality (See [5])). *Let A and B be two $n \times n$ Hermitian matrices. Then*

$$L(F^A, F^B) \leq \|A - B\|,$$

Lemma 5 (Courant-Fischer [26]). *Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. For any integer k and $1 \leq k \leq n$, then*

$$\lambda_k = \min_{w_1, w_2, \ldots, w_{n-k} \in \mathbb{C}^n} \max_{\substack{x \neq 0, x \in \mathbb{C}^n \\ x \perp w_1, w_2, \ldots, w_{n-k}}} \frac{x^* A x}{x^* x},$$

and

$$\lambda_k = \max_{w_1, w_2, \ldots, w_{k-1} \in \mathbb{C}^n} \min_{\substack{x \neq 0, x \in \mathbb{C}^n \\ x \perp w_1, w_2, \ldots, w_{k-1}}} \frac{x^* A x}{x^* x}.$$

Theorem 2. ([27]) *Let $\sigma = \sqrt{2p - p^2 - p^4}$, and $M_n = \frac{1}{\sigma}[H_n - p^2(J_n - I_n)]$. Then the ESD of $n^{-1/2} M_n$ converges to the standard semicircle distribution whose density is given by*

$$\rho(x) := \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{for } |x| \leq 2, \\ 0, & \text{for } |x| > 2. \end{cases}$$

Let F be an absolute continuous distribution function with (semicircle) density $\rho(x)$.

3 Proof of Theorem 1

Let $\widehat{G}_n(p)$ be a random mixed graph as described in the statement of Section 1. The normalized Hermitian Laplacian matrix of a mixed graph $\widehat{G}_n(p)$ is defined to be

$$\mathcal{L}_n = I_n - D_n^{-\frac{1}{2}} H_n D_n^{-\frac{1}{2}},$$

where I_n is the identity matrix, H_n is the Hermitian adjacency matrix of $\widehat{G}_n(p)$, and D_n denotes the diagonal degree matrix of the underlying graph $\Gamma(\widehat{G}_n(p))$. We can write $\mathcal{L}_n$ as

$$\mathcal{L}_n = I_n - (D_n^{-\frac{1}{2}} H_n D_n^{-\frac{1}{2}} - D_n^{-\frac{1}{2}} \mathbb{E} H_n D_n^{-\frac{1}{2}}) - D_n^{-\frac{1}{2}} \mathbb{E} H_n D_n^{-\frac{1}{2}}.$$

Set

$$C_n = D_n^{-\frac{1}{2}} H_n D_n^{-\frac{1}{2}} - D_n^{-\frac{1}{2}} \mathbb{E} H_n D_n^{-\frac{1}{2}}.$$

Instead of directly dealing with C_n, we consider the simpler matrix

$$R_n = (\mathbb{E} D_n)^{-\frac{1}{2}} H_n (\mathbb{E} D_n)^{-\frac{1}{2}} - (\mathbb{E} D_n)^{-\frac{1}{2}} \mathbb{E} H_n (\mathbb{E} D_n)^{-\frac{1}{2}}.$$

R_n can be seen as the expectation of C_n, and we shall consider the spectrum of R_n carefully.

3.1 The spectral norm of R_n

In this section, we will prove a bound on the spectral norm of R_n.

Theorem 3. *Let* $\sigma = \sqrt{2p - p^2 - p^4}$ *and* $R_n = (\mathbb{E}D_n)^{-\frac{1}{2}} H_n (\mathbb{E}D_n)^{-\frac{1}{2}} - (\mathbb{E}D_n)^{-\frac{1}{2}} \mathbb{E}H_n (\mathbb{E}D_n)^{-\frac{1}{2}}$. *Assume that* $\delta = (n-1)(2p - p^2)$. *If* $np/\ln^4 n \to \infty$ *as* $n \to \infty$ *and* $\sup\{p(n); n \geq 2\} < 1$, *then we have*

$$\|R_n\| \leq (1 + o(1)) \frac{2\sigma}{\delta} \sqrt{n}.$$

Proof of Theorem 3. We rely on Wigner's high moment method. Recall that $H_n = (h_{ij})_{n\times n}$ is the Hermitian adjacency matrix of $\widehat{G}_n(p)$. Then h_{ij} $(1 \leq i < j \leq n)$ are independent random variables with the following properties:

- $\mathbb{E}(h_{ij}) = p^2$;
- $\mathrm{Var}(h_{ij}) = 2p - p^2 - p^4 = \sigma^2 < 2$;
- $h_{ij}, h_{i'j'}$ are independent, unless $(i,j) = (j',i')$. If $i > j$, we have $\overline{h_{ji}} = h_{ij}$, i.e., h_{ji} is the complex conjugate of h_{ij};
- $\mathbb{E}D_n = \mathrm{diag}(t_1, t_2, \ldots, t_n)$, where $t_i = \mathbb{E}(d_i) = \sum_{j=1}^n \mathbb{E}|h_{ij}| = (n-1)(2p - p^2) = \delta$ for $1 \leq i \leq n$;
- $|h_{ij}| \leq 1$. Then

$$\begin{aligned} R_n &= (\mathbb{E}D_n)^{-\frac{1}{2}} H_n (\mathbb{E}D_n)^{-\frac{1}{2}} - (\mathbb{E}D_n)^{-\frac{1}{2}} \mathbb{E}H_n (\mathbb{E}D_n)^{-\frac{1}{2}} \\ &= \frac{1}{\delta}[H_n - p^2(J_n - I_n)], \end{aligned}$$

where J_n be the all 1's matrix. Let r_{ij} denote the (i,j)th entry of R_n. Then r_{ij} $(1 \leq i < j \leq n)$ are independent random variables with the following properties:

- $\mathbb{E}(r_{ij}) = 0$;
- $\mathrm{Var}(r_{ij}) = \frac{1}{t_i t_j}\mathrm{Var}(h_{ij}) = \frac{1}{t_i t_j}(2p - p^2 - p^4) = \frac{1}{\delta^2}\sigma^2 < \frac{2}{\delta^2} \leq 1$;
- $r_{ij}, r_{i'j'}$ are independent, unless $(i,j) = (j',i')$. If $i > j$, we have $\overline{r_{ji}} = r_{ij}$;
- $|r_{ij}| \leq \frac{1}{\sqrt{t_i t_j}}\sqrt{1 + p^4} \leq \frac{1}{\delta}\sqrt{2} \leq 1$.

Now let $k \geq 2$ be an even integer. We estimate

$$\begin{aligned} \mathrm{Tr}(R_n^k) &= \sum_{i=1}^n \lambda_i(R_n)^k \\ &\geq \max\{\lambda_1(R_n)^k, \lambda_n(R_n)^k\} \\ &= \|R_n\|^k. \end{aligned}$$

A standard fact in linear algebra tells us that for any positive integer k,

$$\mathrm{Tr}(R_n^k) = \sum_{i_1,\ldots,i_k \in [n]} r_{i_1 i_2} r_{i_2 i_3} \cdots r_{i_k i_1}, \tag{3.1}$$

where $[n] = \{1, 2, \ldots, n\}$.

Let us now take a closer look at $\mathrm{Tr}(R_n^k)$. This is a sum where a typical term is $r_{i_1i_2}r_{i_2i_3}\ldots r_{i_{k-1}i_k}r_{i_ki_1}$, where $W := i_1i_2\ldots i_{k-1}i_ki_1$ corresponds to a closed directed walk of length k in the complete directed graph DK_n of order n. In other words, each term corresponds to a closed walk of length k (containing k, not necessarily different, directed edges) of the complete directed graph DK_n on $[n]$. For each directed edge $(i, j) \in W$, let q_{ij} be the number of occurrence of the directed edge (i, j) in the walk W. Note that all directed edges of a mixed graph are mutually independent. Then we rewrite (3.1) as

$$\mathrm{Tr}(R_n^k) = \sum_W \prod_{i<j} r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}. \tag{3.2}$$

Then

$$\mathbb{E}(\mathrm{Tr}(R_n^k)) = \mathbb{E}\Bigg(\sum_W \prod_{i<j} r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\Bigg) = \sum_W \prod_{i<j} \mathbb{E}\Big(r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\Big),$$

here the summation is taken over all directed closed walks of length k.

We decompose $\mathbb{E}(\mathrm{Tr}(R_n^k))$ into parts $\mathbb{E}_{n,k,t}, t = 2, \ldots, k$, containing the t-fold sums,

$$\mathbb{E}(\mathrm{Tr}(R_n^k)) = \sum_{t=2}^{k} \mathbb{E}_{n,k,t}, \tag{3.3}$$

where

$$\mathbb{E}_{n,k,t} = \sum_{\{W:|V(W)|=t\}} \prod_{i<j} \mathbb{E}\left(r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\right), \tag{3.4}$$

and $|V(W)| = t$ means the cardinality of the vertex set of W is t. (Note that as $r_{ii} = 0$ by construction of R_n we have that $\mathbb{E}_{n,k,1} = 0$.) Here the summation in (3.4) is taken over all closed directed walks W of length k using exactly t different vertices.

Recall that the entries r_{ij} of R_n are independent random variables with mean zero, i.e., $\mathbb{E}(r_{ij}) = 0$, for all $1 \le i < j \le n$, and recall also that q_{ij} denotes the number of occurrence of the directed edge (i, j) in the closed walk W. So, if $q_{ij} + q_{ji} = 1$, that is, $q_{ij} = 1, q_{ji} = 0$ or $q_{ij} = 0, q_{ji} = 1$, then $\prod_{i<j} \mathbb{E}\left(r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\right) = 0$. Thus, the expectation of a term is nonzero if and only if the total number of occurrence of each directed edge and its inverse edge of DK_n in the directed walk W is at least 2. i.e., we need only to consider the case of $q_{ij} + q_{ji} \ge 2$. We call such a closed directed walk a *good* directed walk. The set of all good closed directed walks of length k in DK_n is denoted by $\mathcal{G}(n, k)$. Consider a closed good directed walk W, the underlying graph $\Gamma(W)$ of W uses l different edges $e_1, \ldots, e_l$, i.e., $|E(\Gamma(W))| = l$, with corresponding multiplicities $s_1, \ldots, s_l$ (the s_hs are positive integers at least 2 summing up to k). Without loss of generality, we set $e_h = v_iv_j$

and then $s_h = q_{ij} + q_{ji}$. The (expected) contribution of the term defined by this directed walk in $\mathbb{E}(\mathrm{Tr}(R_n^k))$ is

$$\prod_{\substack{i<j \\ |E(\Gamma(W))|=l}} \mathbb{E}\left(r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\right). \tag{3.5}$$

Next, we will compute $\mathbb{E}\left(r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\right)$. Note that $q_{ij} + q_{ji} \geq 2$ implies that $q_{ij} \geq 1, q_{ji} \geq 1$ or $q_{ij} \geq 2, q_{ji} = 0$ or $q_{ij} = 0, q_{ji} \geq 2$. Since $|r_{ij}| \leq \frac{\sqrt{2}}{\delta} \leq 1$ and $E(r_{ij}) = 0$. Then, we consider these three cases separately.

If $q_{ij} \geq 1, q_{ji} \geq 1$, then we have

$$\begin{aligned}
\left|\mathbb{E}\left(r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\right)\right| &\leq \mathbb{E}|r_{ij}^{q_{ij}-1} \cdot r_{ji}^{q_{ji}-1} \cdot r_{ij} \cdot r_{ij}| \\
&\leq \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}+q_{ji}-2} \mathbb{E}|r_{ij} r_{ji}| \\
&= \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}+q_{ji}-2} \mathbb{E}|r_{ij}|^2 \\
&= \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}+q_{ji}-2} \mathbb{E}(r_{ij}\overline{r_{ij}}) \\
&= \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}+q_{ji}-2} \mathbb{E}\{[r_{ij} - \mathbb{E}(r_{ij})][\overline{r_{ij} - \mathbb{E}(r_{ij})}]\} \\
&= \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}+q_{ji}-2} \mathrm{Var}(r_{ij}) \\
&= \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}+q_{ji}-2} \frac{\sigma^2}{\delta^2}.
\end{aligned} \tag{3.6}$$

If $q_{ij} \geq 2, q_{ji} = 0$, then we have

$$\begin{aligned}
\left|\mathbb{E}\left(r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\right)\right| &= \mathbb{E}|r_{ij}^{q_{ij}}| \\
&\leq \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}-2} \mathbb{E}|r_{ij}^2| \\
&= \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}-2} \mathbb{E}(|r_{ij}|^2) \\
&= \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ij}-2} \frac{\sigma^2}{\delta^2}.
\end{aligned} \tag{3.7}$$

If $q_{ij} = 0, q_{ji} \geq 2$, by a similar discussion as above, we have

$$\left|\mathbb{E}\left(r_{ij}^{q_{ij}} r_{ji}^{q_{ji}}\right)\right| \leq \left(\frac{\sqrt{2}}{\delta}\right)^{q_{ji}-2} \frac{\sigma^2}{\delta^2}. \tag{3.8}$$

Since $\frac{\sigma^2}{\delta^2} < 1$. Let $\mathcal{G}(n,k,t)$ denote the set of closed good directed walks on DK_n of length k using exactly t different vertices. Notice that for each directed walk W in $\mathcal{G}(n,k,l+1)$, the underlying graph $\Gamma(W)$ of W must have at least l different edges. By (3.5)-(3.8), the contribution of a term correspond to such a good directed walk toward $\mathbb{E}(\mathrm{Tr}(R_n^k))$ is at most

$$\left(\frac{\sqrt{2}}{\delta}\right)^{k-2l} \frac{\sigma^{2l}}{\delta^{2l}} = \frac{\sqrt{2}^{k-2l}\sigma^{2l}}{\delta^k}.$$

By the pigeon hole principle, if $l+1 > \frac{k}{2}+1$, then there must be a directed edge (i,j) that the total number of occurrence of this directed edge and its inverse edge of DK_n in the directed walk W is 1, i.e., $q_{ij}+q_{ji}=1$. Therefore, $\mathbb{E}_{n,k,l+1}=0$ for $l > \frac{k}{2}$.

So, in the following, we only consider the case of $l \leq \frac{k}{2}$ and $q_{ij}+q_{ji} \geq 2$. It follows that

$$\begin{aligned}\mathbb{E}(\mathrm{Tr}(R_n^k)) &\leq \sum_{l=1}^{\frac{k}{2}} |\mathcal{G}(n,k,l+1)| \frac{\sqrt{2}^{k-2l}\sigma^{2l}}{\delta^k} \\ &= \sum_{m=2}^{\frac{k}{2}+1} |\mathcal{G}(n,k,m)| \frac{\sqrt{2}^{k-2(m-1)}\sigma^{2(m-1)}}{\delta^k}.\end{aligned} \tag{3.9}$$

The key of the trace method is a good estimate on $|\mathcal{G}(n,k,m)|$. We need the following result of Hu [27].

Lemma 6 ([27])**.** *Let $\mathcal{G}(n,k,m)$ be the set of good closed directed walks in DK_n using k edges and m vertices. Then*

$$|\mathcal{G}(n,k,m)| \leq n(n-1)\cdots(n-m+1)\binom{k}{2m-2} 2^{2k-2m+3} m^{k-2m+2}(k-2m+4)^{k-2m+2}. \tag{3.10}$$

Substituting (3.10) into (3.9) yields

$$\begin{aligned}\mathbb{E}(\mathrm{Tr}(R_n^k)) &\leq \sum_{m=2}^{\frac{k}{2}+1} \frac{\sqrt{2}^{k-2(m-1)}\sigma^{2(m-1)}}{\delta^k} n^m \binom{k}{2m-2} 2^{2k-2m+3} m^{k-2m+2}(k-2m+4)^{k-2m+2} \\ &= \sum_{m=2}^{\frac{k}{2}+1} S(n,k,m).\end{aligned}$$

where the final equality defines $S(n,k,m)$. Now fix $k = g(n)\ln n$, where $g(n)$ tends to infinity (with n) arbitrarily slowly. Let us consider the ratio $S(n,k,m-1)/S(n,k,m)$ for some $m \leq \frac{k}{2}+1$:

$$\begin{aligned}\frac{S(n,k,m-1)}{S(n,k,m)} &= \frac{\frac{\sqrt{2}^{k-2(m-2)}\sigma^{2(m-2)}}{\delta^k} n^{m-1}\binom{k}{2m-4} 2^{2k-2m+5}(m-1)^{k-2m+4}(k-2m+6)^{k-2m+4}}{\frac{\sqrt{2}^{k-2(m-1)}\sigma^{2(m-1)}}{\delta^k} n^{m}\binom{k}{2m-2} 2^{2k-2m+3}m^{k-2m+2}(k-2m+4)^{k-2m+2}} \\ &= \frac{2(2m-2)(2m-3)2^2(m-1)^{k-2m+4}(k-2m+6)^{k-2m+4}}{\sigma^2 n(k-2m+4)(k-2m+3)m^{k-2m+2}(k-2m+4)^{k-2m+2}} \\ &\leq \frac{8(m-1)^2 2^2(m-1)^{k-2m+4}(k-2m+6)^{k-2m+4}}{\sigma^2 n(k-2m+3)m^{k-2m+2}(k-2m+4)^{k-2m+3}} \\ &= \frac{32(m-1)^{k-2m+6}(k-2m+6)^{k-2m+4}}{\sigma^2 n(k-2m+3)m^{k-2m+2}(k-2m+4)^{k-2m+3}} \\ &\leq \frac{32m^{k-2m+6}(k-2m+6)^{k-2m+4}}{\sigma^2 n(k-2m+3)m^{k-2m+2}(k-2m+4)^{k-2m+3}} \\ &= \frac{32m^4(k-2m+6)^{k-2m+4}}{\sigma^2 n(k-2m+3)(k-2m+4)^{k-2m+3}}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{32m^4(k-2m+6)^{k-2m+4}}{\sigma^2 n(k-2m+3)^{k-2m+4}} \\
&\to \frac{32C_0 m^4}{\sigma^2 n} \\
&= \frac{2C_0(k+2)^4}{\sigma^2 n} \\
&\leq \frac{32C_0 k^4}{\sigma^2 n}
\end{aligned}
$$

for some constant C_0 independent of σ. This implies that

$$S(n,k,m-1) \leq \frac{32C_0k^4}{\sigma^2 n} S(n,k,m).$$

By the assumption $\sup\{p(n);\ n \geq 2\} < 1$, we have $0 < p < 1$. Then, $2 - p - p^3 > 0$. Furthermore, there exists a constant $c > 0$ such that for all $n \geq 2$, $2 - p - p^3 \geq c$. Thus,

$$\sigma^2 = p(2-p-p^3) \geq cp$$

and

$$\frac{32C_0k^4}{\sigma^2 n} = \frac{32C_0 \cdot (g(n)\ln n)^4}{\sigma^2 n} \leq \frac{32C_0}{c} \cdot (g(n))^4 \cdot \frac{\ln^4 n}{np}.$$

By the assumption $\frac{np}{\ln^4 n} \to \infty$ as $n \to \infty$, this is equivalent to $\frac{\ln^4 n}{np} \to 0$. Since $g(n)$ is a function that tends to infinity arbitrarily slowly, we can choose the growth rate of $g(n)$ such that the growth of $(g(n))^4$ is much slower than that of $\frac{np}{\ln^4 n}$. Therefore, for sufficiently large n, it can be guaranteed that

$$\frac{32C_0}{c} \cdot (g(n))^4 \cdot \frac{\ln^4 n}{np} \leq \frac{1}{2}.$$

Thus, $\frac{32C_0k^4}{\sigma^2 n} \leq \frac{1}{2}$. It follows that

$$S(n,k,m-1) \leq \frac{1}{2} S(n,k,m).$$

Then

$$
\begin{aligned}
\mathbb{E}(\mathrm{Tr}(R_n^k)) &\leq \sum_{m=2}^{\frac{k}{2}+1} S(n,k,m) \\
&\leq S(n,k,\frac{k}{2}+1) \sum_{m=2}^{\frac{k}{2}+1} \left(\frac{1}{2}\right)^{\frac{k}{2}+1-m} \\
&\leq 2S(n,k,\frac{k}{2}+1) \\
&= 2\frac{\sigma^k}{\delta^k} n^{\frac{k}{2}+1} 2^{k+1} \\
&= 4n\Big(\frac{2\sigma}{\delta}\sqrt{n}\Big)^k.
\end{aligned}
$$

Then

$$\mathbb{E}(\|R_n^k\|) \le \mathbb{E}(\mathrm{Tr}(R_n^k)) \le 4n\Big(\frac{2\sigma}{\delta}\sqrt{n}\Big)^k.$$

Using Markov's inequality, we get

$$\begin{aligned}
\Pr\left(\|R_n\| \ge (1+\epsilon)\frac{2\sigma}{\delta}\sqrt{n}\right) &= \Pr\left(\|R_n\|^k \ge \left((1+\epsilon)\frac{2\sigma}{\delta}\sqrt{n}\right)^k\right)\\
&\le \frac{\mathbb{E}(\|R_n^k\|)}{((1+\epsilon)\frac{2\sigma}{\delta}\sqrt{n})^k}\\
&\le \frac{4n(\frac{2\sigma}{\delta}\sqrt{n})^k}{((1+\epsilon)\frac{2\sigma}{\delta}\sqrt{n})^k}\\
&= \frac{4n}{(1+\epsilon)^k}.
\end{aligned}$$

Since $k = \omega(\ln n)$, we can find an $\epsilon = \epsilon(n)$ tending to 0 with n so that $\frac{n}{(1+\epsilon)^k} = o(1)$. Then asymptotically almost surely we have

$$\|R_n\| \le (1+o(1))\frac{2\sigma}{\delta}\sqrt{n}.$$

This completes the proof. □

3.2 The semicircle law

In this section, we provide proof of Theorem 1.

Proof of Theorem 1. Let $\widehat{G}_n(p)$ and $H_n = (h_{ij})$ be defined as before. Set

$$M_n = \frac{1}{\sigma}[H_n - p^2(J_n - I_n)].$$

Recall that

$$\begin{aligned}
R_n &= (\mathbb{E}D_n)^{-\frac{1}{2}}H_n(\mathbb{E}D_n)^{-\frac{1}{2}} - (\mathbb{E}D_n)^{-\frac{1}{2}}\mathbb{E}H_n(\mathbb{E}D_n)^{-\frac{1}{2}}\\
&= \frac{1}{\delta}[H_n - p^2(J_n - I_n)].
\end{aligned}$$

It is clear that

$$\frac{\delta}{\sigma}\lambda_i(R_n) = \lambda_i(M_n), i = 1, 2, \ldots, n.$$

Thus, by Theorem 2, almost surely,

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\Big\{\frac{\delta}{\sigma\sqrt{n}}\lambda_i(R_n) \le x\Big\}, x \in \mathbb{R},$$

i.e., $F^{\frac{\delta}{\sigma\sqrt{n}}R_n}(x)$ converges weakly to to the standard semicircle distribution $F(x)$ with density $\rho(x)$ as $n \to \infty$.

Recall that

$$C_n = D_n^{-\frac{1}{2}} H_n D_n^{-\frac{1}{2}} - D_n^{-\frac{1}{2}} \mathbb{E} H_n D_n^{-\frac{1}{2}}.$$

We rewrite C_n as follows:

$$C_n = R_n + B_n,$$

where $B_n = D_n^{-\frac{1}{2}}(H_n - \mathbb{E}H_n)D_n^{-\frac{1}{2}} - (\mathbb{E}D_n)^{-\frac{1}{2}}(H_n - \mathbb{E}H_n)(\mathbb{E}D_n)^{-\frac{1}{2}}$. Letting b_{ij} denote the (i,j)th entry of B_n. To bound $\|B_n\|$, by Lemma 5, we have almost surely

$$\begin{aligned}
\|B_n\| &= \sup_{\|x\|=1} |x^* B_n x| \\
&= \sup_{\|x\|=1} \Big| \sum_{i,j} x_i^* b_{ij} x_j \Big| \\
&= \sup_{\|x\|=1} \Big| \sum_{i,j} x_i^* r_{ij} x_j \frac{\sqrt{t_i t_j} - \sqrt{d_i d_j}}{\sqrt{d_i d_j}} \Big| \\
&\le \sup_{\|x\|=1} \Big(\Big| \sum_{i,j} x_i^* r_{ij} x_j \frac{\sqrt{t_j} - \sqrt{d_j}}{\sqrt{d_j}} \Big| + \Big| \sum_{i,j} x_i^* \frac{\sqrt{t_i} - \sqrt{d_i}}{\sqrt{d_i}} r_{ij} x_j \frac{\sqrt{t_j}}{\sqrt{d_j}} \Big| \Big) \\
&=: \sup_{\|x\|=1} (|x^* R_n y| + |y^* R_n z|),
\end{aligned}$$

where $y = \left(x_1 \frac{\sqrt{t_1} - \sqrt{d_1}}{\sqrt{d_1}}, \ldots, x_n \frac{\sqrt{t_n} - \sqrt{d_n}}{\sqrt{d_n}} \right)^T, z = \left(x_1 \frac{\sqrt{t_1}}{\sqrt{d_1}}, \ldots, x_n \frac{\sqrt{t_n}}{\sqrt{d_n}} \right)^T$. Then we have

$$\begin{aligned}
\|B_n\| &\le \sup_{\|x\|=1} (\|R_n\| \|y\| + \|R_n\| \|y\| \|z\|) \\
&= \|R_n\| \sup_{\|x\|=1} (\|y\| + \|y\| \|z\|),
\end{aligned}$$

where $\|y\|^2 = \sum_{i=1}^n |x_i|^2 \left(\frac{\sqrt{t_i} - \sqrt{d_i}}{\sqrt{d_i}} \right)^2, \|z\|^2 = \sum_{i=1}^n |x_i|^2 \frac{t_i}{d_i}$.

Note that $|h_{i1}|, |h_{i2}|, \ldots, |h_{in}|$ are independent random variables, and for all $j \in \{1, 2, \ldots, n\}$, with

$$\Pr(|h_{ij}| = 1) = 2p - p^2, \Pr(|h_{ij}| = 0) = (1-p)^2.$$

Recall that $d_i = \sum_{j=1}^n |h_{ij}|$, and $t_i = \mathbb{E}(d_i) = \sum_{j=1}^n \mathbb{E}|h_{ij}| = (n-1)(2p - p^2) = \delta$. Since $|h_{ij}| \le 1$ and

$$\begin{aligned}
\sum_{j=1}^n \mathrm{Var}(|h_{ij}|) &= \sum_{j=1}^n [\mathbb{E}(|h_{ij}|^2) - (\mathbb{E}(|h_{ij}|))^2] \\
&\le \sum_{j=1}^n \mathbb{E}(|h_{ij}|^2) \\
&= (n-1)(2p - p^2) \\
&= \delta.
\end{aligned}$$

By the assumption that $\delta = t_i = (n-1)(2p-p^2) \gg \ln n$. By Lemma 1 with $b = 3\sqrt{t_i \ln n}$, we have for all i,

$$\Pr(|d_i - t_i| \geq b) \leq e^{-\frac{b^2}{2(t_i + b/3)}} = \frac{1}{n^{9/4}}.$$

Thus asymptotically almost surely, for all i we have $|d_i - t_i| \leq 3\sqrt{t_i \ln n}$.

Note that

$$\Pr\Big(\max_{1\leq i\leq n} \frac{t_i}{d_i} > (1+\epsilon)\Big) \leq n \cdot \max_{1\leq i\leq n} \Pr\Big(\frac{t_i}{d_i} > (1+\epsilon)\Big),$$

since $\Pr(\bigcup_i A_i) \leq \sum_i \Pr(A_i)$. Choose $0 < a = 3\sqrt{\frac{\ln n}{t_i}} < 1$ such that $\frac{1}{1-a} < 1+\epsilon$. Recall that $\delta \gg \ln n$. Applying Lemma 1, we have

$$\begin{aligned}
\Pr\Big(\frac{t_i}{d_i} > (1+\epsilon)\Big) &\leq \Pr\Big(\frac{t_i}{d_i} > \frac{1}{1-a}\Big) \\
&= \Pr\Big(d_i - t_i < -at_i\Big) \\
&\leq \Pr\Big(|d_i - t_i| \geq at_i\Big) \\
&\leq e^{-\frac{(at_i)^2}{2(t_i + at_i/3)}} \\
&\leq e^{-\frac{(at_i)^2}{2(t_i + t_i)}} \\
&= e^{-\frac{a^2 t_i}{4}} \\
&= e^{-\frac{9 \ln n}{4}} \\
&= \frac{1}{n^{9/4}}.
\end{aligned}$$

So

$$\Pr\Big(\max_{1\leq i\leq n} \frac{t_i}{d_i} > (1+\epsilon)\Big) \leq \frac{1}{n^{5/4}}.$$

Then we have

$$\sum_{n=1}^{\infty} \Pr\Big(\max_{1\leq i\leq n} \frac{t_i}{d_i} > (1+\epsilon)\Big) < \infty.$$

By Borel-Cantelli Lemma, we have

$$\Pr\Big(\limsup_{n\to\infty} \max_{1\leq i\leq n} \frac{t_i}{d_i} > (1+\epsilon)\Big) = 0.$$

i.e.,

$$\limsup_{n\to\infty} \frac{t_i}{d_i} \leq 1.$$

Then we have

$$\|z\| = \Big(\sum_{i=1}^{n} |x_i|^2 \frac{t_i}{d_i}\Big)^{\frac{1}{2}} \leq \max_{1\leq i\leq n} (\frac{t_i}{d_i})^{\frac{1}{2}} \cdot \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{\frac{1}{2}} = \max_{1\leq i\leq n} (\frac{t_i}{d_i})^{\frac{1}{2}} \leq 1.$$

And

$$\begin{aligned}
\|y\| &= \Big(\sum_{i=1}^{n}|x_i|^2\Big(\frac{\sqrt{t_i}-\sqrt{d_i}}{\sqrt{d_i}}\Big)^2\Big)^{\frac{1}{2}} \\
&\le \max_{1\le i\le n}\Big(\frac{(\sqrt{t_i}-\sqrt{d_i})^2}{d_i}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{i=1}^{n}|x_i|^2\Big)^{\frac{1}{2}} \\
&= \max_{1\le i\le n}\Big(\frac{(t_i-d_i)^2}{d_i(\sqrt{t_i}+\sqrt{d_i})^2}\Big)^{\frac{1}{2}} \\
&\le \max_{1\le i\le n}\Big(\frac{(t_i-d_i)^2}{t_i(\sqrt{t_i}+\sqrt{t_i})^2}\Big)^{\frac{1}{2}} \\
&= \max_{1\le i\le n}\frac{|t_i-d_i|}{2t_i} \\
&\le \max_{1\le i\le n}\frac{3\sqrt{t_i\ln n}}{2t_i} \\
&\le \max_{1\le i\le n}\frac{3}{2}\sqrt{\frac{\ln n}{t_i}} \\
&= o(1).
\end{aligned}$$

Since $\frac{t_i}{\ln n}=(n-1)(2p-p^2)/\ln n\to\infty$.

Then by Theorem 3, we have

$$\begin{aligned}
\|B_n\| &\le \|R_n\|\sup_{\|x\|=1}(\|y\|+\|y\|\|z\|) \\
&\le o(\|R_n\|) \\
&\le o\Big((1+o(1))\frac{2\sigma}{\delta}\sqrt{n}\Big).
\end{aligned}$$

Recall that almost surely,

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\Big\{\frac{\delta}{\sigma\sqrt{n}}\lambda_i(R_n)\le x\Big\}, x\in\mathbb{R},$$

i.e., $F^{\frac{\delta}{\sigma\sqrt{n}}R_n}(x)$ converges weakly to to the standard semicircle distribution $F(x)$ with density $\rho(x)$ as $n\to\infty$. Note that

$$C_n=R_n+B_n.$$

Then by Lemma 4, we have

$$L(F^{\frac{\delta}{\sigma\sqrt{n}}C_n},F^{\frac{\delta}{\sigma\sqrt{n}}R_n})\le\frac{\delta}{\sigma\sqrt{n}}\|B_n\|\le\frac{\delta}{\sigma\sqrt{n}}o((1+o(1))\frac{2\sigma}{\delta}\sqrt{n})\to 0.$$

This implies that the LSDs of $\frac{\delta}{\sigma\sqrt{n}}C_n$, $\frac{\delta}{\sigma\sqrt{n}}R_n$ are the same. Thus, by Theorem 2, almost surely,

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\Big\{\frac{\delta}{\sigma\sqrt{n}}\lambda_i(C_n)\le x\Big\}, x\in\mathbb{R},$$

converges i.e., $F^{\frac{\delta}{\sigma\sqrt{n}}C_n}(x)$ weakly to to the standard semicircle distribution $F(x)$ with density $\rho(x)$ as $n \to \infty$.

$$\begin{aligned}\mathcal{L}_n &= I_n - C_n - D_n^{-\frac{1}{2}}\mathbb{E}HD_n^{-\frac{1}{2}} \\ &= I_n - C_n - D_n^{-\frac{1}{2}}p^2(J_n - I_n)D_n^{-\frac{1}{2}}\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}&\|F^{\frac{\delta}{\sigma\sqrt{n}}(I_n-\mathcal{L}_n+D_n^{-\frac{1}{2}}p^2I_nD_n^{-\frac{1}{2}})} - F^{\frac{\delta}{\sigma\sqrt{n}}C_n}\| \\ =&\|F^{\frac{\delta}{\sigma\sqrt{n}}(I_n-\mathcal{L}_n+p^2D_n^{-1})} - F^{\frac{\delta}{\sigma\sqrt{n}}C_n}\| \\ \le&\frac{1}{n}\mathrm{rank}(\frac{\delta}{\sigma\sqrt{n}}D_n^{-\frac{1}{2}}p^2J_nD_n^{-\frac{1}{2}}) \\ \le&\frac{1}{n}\mathrm{rank}(J_n) \\ =&\frac{1}{n} \\ \to&0.\end{aligned}$$

This implies that the LSDs of $\frac{\delta}{\sigma\sqrt{n}}(I_n - \mathcal{L}_n + p^2D_n^{-1})$, $\frac{\delta}{\sigma\sqrt{n}}C_n$ are the same.

By Lemma 4, we have

$$\begin{aligned}&L(F^{\frac{\delta}{\sigma\sqrt{n}}(I_n-\mathcal{L}_n+D_n^{-\frac{1}{2}}p^2I_nD_n^{-\frac{1}{2}})}, F^{\frac{\delta}{\sigma\sqrt{n}}(I_n-\mathcal{L}_n+\mathbb{E}D_n^{-\frac{1}{2}}p^2I_n\mathbb{E}D_n^{-\frac{1}{2}})}) \\ \le&\frac{\delta}{\sigma\sqrt{n}}p^2\|D_n^{-1} - \mathbb{E}D_n^{-1}\| \\ =&\frac{\delta}{\sigma\sqrt{n}}p^2\max_{1\le i\le n}\left|\frac{1}{d_i} - \frac{1}{t_i}\right| \\ =&\frac{\delta}{\sigma\sqrt{n}}p^2\max_{1\le i\le n}\frac{|t_i - d_i|}{t_id_i} \\ \le&\frac{1}{\sigma\sqrt{n}}p^2\max_{1\le i\le n}\frac{|t_i - d_i|}{t_i} \\ \le&\frac{1}{\sigma\sqrt{n}}p^2\max_{1\le i\le n}\frac{3\sqrt{t_i\ln n}}{t_i} \\ \le&\frac{1}{\sigma\sqrt{n}}p^2\max_{1\le i\le n}3\sqrt{\frac{\ln n}{t_i}} \\ \to&0.\end{aligned}$$

This implies that the LSDs of $\frac{\delta}{\sigma\sqrt{n}}(I_n - \mathcal{L}_n + p^2D_n^{-1})$, $\frac{\delta}{\sigma\sqrt{n}}(I_n - \mathcal{L}_n + p^2\mathbb{E}D_n^{-1})$, $\frac{\delta}{\sigma\sqrt{n}}C_n$ are the same.

By Lemma 4, we have

$$\begin{aligned}&L\Big(F^{\frac{\delta}{\sigma\sqrt{n}}(I_n-\mathcal{L}_n+p^2\mathbb{E}D_n^{-1})}, F^{\frac{\delta}{\sigma\sqrt{n}}(I_n-\mathcal{L}_n)}\Big) \\ \le&\frac{\delta}{\sigma\sqrt{n}}p^2\|\mathbb{E}D_n^{-1}\|\end{aligned}$$

$$
\begin{aligned}
&=\frac{\delta}{\sigma\sqrt{n}}p^2\frac{1}{\delta}\\
&=\frac{p^2}{\sigma\sqrt{n}}\\
&\to 0.
\end{aligned}
$$

This implies that the LSDs of $\frac{\delta}{\sigma\sqrt{n}}(I_n - \mathcal{L}_n + p^2\mathbb{E}D_n^{-1})$, $\frac{\delta}{\sigma\sqrt{n}}(I_n - \mathcal{L}_n)$ are the same. So the LSDs of $\frac{\delta}{\sigma\sqrt{n}}(I_n - \mathcal{L}_n)$, $\frac{\delta}{\sigma\sqrt{n}}C_n$ are the same. Equivalently, almost surely,

$$
\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\Big\{\frac{\delta}{\sigma\sqrt{n}}(1-\lambda_i(\mathcal{L}_n)) \le x\Big\}, x \in \mathbb{R},
$$

i.e., $F^{\frac{\delta}{\sigma\sqrt{n}}(I_n-\mathcal{L}_n)}(x)$ converges weakly to the standard semicircle distribution $F(x)$ with density $\rho(x)$ as $n \to \infty$. This completes the proof. □

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