MORE INEQUALITIES FOR THE OVERPARTITION FUNCTION

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Abstract. Finding truncated identities is an efficient way to derive inequalities for partition functions. This method was initiated by Andrews and Merca in 2012 and has flourished since then. In this paper, in view of a transformation formula due to Bailey, and combining two new Bailey pairs and an identity given by Wang and Yee, we find some new truncated identities which imply more inequalities for the overpartition function.

1. Introduction

Here and throughout the paper, we assume that $|q| < 1$. The q-shifted factorials are defined by [9]

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \qquad \text{and} \qquad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in \mathbb{Z}.$$

For convenience, the multiple q-shifted factorials can be written as

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or infinity. Moreover, the q-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

The basic hypergeometric series ${}_r\phi_s$ is defined as [9]

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, b_2, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k.$$

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n and the generating function of $p(n)$ is stated as

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

In 2004, Corteel and Lovejoy [8] introduced the definition of overpartitions. An overpartition of n is a partition of n in which the first occurrence of each number may be overlined.

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For example, there are 8 overpartitions of 3: 3, $\overline{3}$, $2+1$, $\overline{2}+1$, $2+\overline{1}$, $\overline{2}+\overline{1}$, $1+1+1$ and $\overline{1}+1+1$. Let $\overline{p}(n)$ be the number of overpartitions of n and $\overline{p}(0)=1$. We always set $\overline{p}(n)=0$ if n is a negative integer. Then the generating function of $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{1.1}$$

In the literature, the study on inequalities for different kinds of partition functions attracted lots of attention. Recall Euler's pentagonal number theorem [3] which is stated as

$$\sum_{n=0}^{\infty}(-1)^n q^{n(3n+1)/2}(1-q^{2n+1})=(q;q)_\infty. \tag{1.2}$$

In 2012, Andrews and Merca [4] established the following truncated version of (1.2):

$$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1})=1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}. \tag{1.3}$$

As a corollary, they deduced the infinite family of inequalities for $p(n)$, namely,

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\left(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1)\right)\geq 0.$$

Since then, seeking truncated identities has become a very useful way to study inequalities for partition functions. Motivated by Andrews and Merca's work, Guo and Zeng [10] proved the truncated forms of the following two identities due to Gauss:

$$1+2\sum_{j=1}^{\infty}(-1)^j q^{j^2}=\frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{1.4}$$

$$\sum_{j=0}^{\infty}(-1)^j q^{j(2j+1)}(1-q^{2j+1})=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}. \tag{1.5}$$

For instance, they obtained that

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right)=1+(-1)^k\sum_{n=k+1}^{\infty}\frac{(-q;q)_k(-1;q)_{n-k}q^{(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k\end{bmatrix}, \tag{1.6}$$

which implies the infinite family of inequalities for $\overline{p}(n)$:

$$(-1)^k\left(\overline{p}(n)+2\sum_{j=1}^{k}(-1)^j\overline{p}(n-j^2)\right)\geq 0.$$

In addition, Andrews and Merca [4] and Guo and Zeng [10] independently posed the following conjecture: for $m, n, k \geq 1$ and $1 \leq r < m/2$,

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\left(J_{m,r}(n-j(mj+m-2r)/2)-J_{m,r}(n-(j+1)(mj+2r)/2)\right)\geq 0, \tag{1.7}$$

where

$$\sum_{n=0}^{\infty}J_{m,r}(n)q^n=\frac{1}{(q^r,q^{m-r},q^m;q^m)_\infty}.$$

This was confirmed by Mao [14] and Yee [21]. Later, Wang and Yee [15] reproved this conjecture by providing an explicit formula which is analogous to (1.3) and (1.6). In particular, taking $m = 2$ and $r = 1$ in (1.7) yields that

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\left(\overline{p}(n-j^2)-\overline{p}(n-(j+1)^2)\right)\geq 0. \tag{1.8}$$

Recently, Xia et al. [19] established some new truncated identities related to (1.2), (1.4) and (1.5) by using a q-series expansion formula due to Liu [12]. For instance, they established that

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j(1-q^{j+1})^2q^{j^2+j}=1+(-1)^{k-1}\frac{(-q;q)_k}{(q;q)_{k-1}}\sum_{j=0}^{\infty}\frac{(-q^{k+j+2};q)_\infty q^{k(k+j+1)}}{(q^{k+j+2};q)_\infty},$$

and then deduced that

$$\overline{p}(n-k(k+1))+(-1)^{k-1}\sum_{j=-k}^{k}(-1)^j\overline{p}(n-j^2)\geq 0.$$

Very recently, in view of a Bailey pair due to Lovejoy [13], Yao [20] derived several generalized truncated identities which include some known truncated forms of (1.2), (1.4) and (1.5) [4, 10, 19] as special cases. For example, it is shown that

$$\begin{aligned}&\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{k-1}(-1)^n(1-q^{2n+2r+1})(q^{n+1};q)_{2r-2}q^{n^2+(r-1)n}\\&=(-q;q)_{r-1}+\frac{(-q;q)_{k+r-1}(-1)^{k-1}}{(q;q)_{k-1}}\sum_{n=0}^{\infty}\frac{(-q^{k+n+r};q)_\infty q^{k(k+r+n-1)}}{(q^{k+n+2r-1};q)_\infty},\end{aligned}$$

which implies the following inequalities for $\overline{p}(n)$: for $k \geq 1$ and $n > r(r-1)/2$,

$$(-1)^{k-1}\sum_{i=0}^{k-1}(-1)^i\sum_{j\geq 0}t_r(i,j)\overline{p}(n-i^2-(r-1)i-j)\geq 0,$$

where

$$\sum_{j\geq 0} t_r(i,j)q^j = (1-q^{2i+2r+1})(q^{i+1};q)_{2r-2}.$$

In addition, Wang and Yee [16] investigated truncated forms of Hecke-Rogers type series and derived three infinite families of inequalities for some interesting partition functions. For more on truncated identities and inequalities for partition functions, one can see [5, 11, 17, 18, 22].

Inspired by the aforementioned work, the aim of this paper is to establish more truncated identities and then derive some new inequalities for the overpartition function $\overline{p}(n)$. The main results are stated as follows.

Theorem 1.1. *For $m \geq 0$, if $t \geq 0$ and $r \geq 4t+1$, then*

$$\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{m}(-1)^n(1-q^{2n+r})(q^{n+1};q)_{r-1}(-q^{n+2t+1};q)_{r-4t-1}q^{n^2+2tn} \\
&= (-q;q)_{r-2t-1} + (-1)^m q^{m(m+1)}(-q;q)_{r-2t-1}\sum_{i,j,i_1,i_2,\ldots,i_t=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i(-q^{r-2t};q^2)_j}{(q^2;q^2)_i(q^2;q^2)_j} \\
&\quad\times q^{(2m+2r+2)i+(2t+1)j+(2t+2)i_1+(2t+4)i_2+\cdots+4ti_t}\begin{bmatrix} i+j+i_1+i_2+\cdots+i_t-1 \\ m\end{bmatrix}_{q^2};
\end{aligned} \tag{1.9}$$

if $t \geq 1$ and $r \geq 4t-1$, then

$$\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{m}(-1)^n(1-q^{2n+r})(q^{n+1};q)_{r-1}(-q^{n+2t};q)_{r-4t+1}q^{n^2+2tn-n} \\
&= (-q;q)_{r-2t} + (-1)^m q^{m(m+1)}(-q;q)_{r-2t}\sum_{i,j,i_1,i_2,\ldots,i_t=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i(-q^{r-2t};q^2)_j}{(q^2;q^2)_i(q^2;q^2)_j} \\
&\quad\times q^{(2m+2r+2)i+(2t+1)j+2ti_1+(2t+2)i_2+\cdots+(4t-2)i_t}\begin{bmatrix} i+j+i_1+i_2+\cdots+i_t-1 \\ m\end{bmatrix}_{q^2}.
\end{aligned} \tag{1.10}$$

Corollary 1.2. *For $m \geq 0$, $k \geq 0$, $r \geq 2k+1$ and $n > \binom{r-k}{2}$, then*

$$(-1)^m\sum_{i=0}^{m}(-1)^i\sum_{j\geq 0} b_{k,r,i}(j)\overline{p}(n-i^2-ki-j) \geq 0, \tag{1.11}$$

where

$$\sum_{j\geq 0} b_{k,r,i}(j)q^j = (1-q^{2i+r})(q^{i+1};q)_{r-1}(-q^{i+k+1};q)_{r-2k-1}.$$

Notice that letting $k=0$ and $r=1$ in Corollary 1.2 yields (1.8). Furthermore, by setting different values of m, k and r, we may establish more inequalities. For example, taking $(m,k,r) = (0,2,5)$ and $(0,1,5)$ in Corollary 1.2, we derive that

$$\overline{p}(n)-\overline{p}(n-1)-\overline{p}(n-2)+\overline{p}(n-5)+\overline{p}(n-6)+\overline{p}(n-7)-\overline{p}(n-8)-\overline{p}(n-9)$$

$$
\begin{aligned}
&- \overline{p}(n-10) + \overline{p}(n-13) + \overline{p}(n-14) - \overline{p}(n-15) \geq 0,\\
&\overline{p}(n) - \overline{p}(n-1) - 2\overline{p}(n-4) + \overline{p}(n-5) + \overline{p}(n-7) + \overline{p}(n-8) + \overline{p}(n-9) - \overline{p}(n-11)\\
&- \overline{p}(n-12) - \overline{p}(n-13) - \overline{p}(n-15) + 2\overline{p}(n-16) + \overline{p}(n-19) - \overline{p}(n-20) \geq 0.
\end{aligned}
$$

Theorem 1.3. *For $m \geq 0$ and $r \geq 1$,*

$$
\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{m} (-1)^n (1-q^{4n+2r})(q^{2n+2};q^2)_{r-1} q^{n^2-n}\\
&= (-1)^m q^{m(m+1)/2} (-q;q)_r \sum_{i=m}^{\infty} \frac{(-q^{-m-r};q)_i}{(q;q)_i} q^{(m+r+1)i} \begin{bmatrix} i \\ m \end{bmatrix}. \qquad (1.12)
\end{aligned}
$$

Corollary 1.4. *For $m \geq 0$ and $r \geq 1$,*

$$
(-1)^m \sum_{i=0}^{m} (-1)^i \sum_{j \geq 0} c_{r,i}(j) \overline{p}(n - i^2 + i - j) \geq 0,
$$

where

$$
\sum_{j \geq 0} c_{r,i}(j) q^j = (1 - q^{4i+2r})(q^{2i+2};q^2)_{r-1}.
$$

For example, setting $r = m = 2$ in Corollary 1.4, we have

$$
\overline{p}(n-6) - \overline{p}(n-12) - \overline{p}(n-14) + \overline{p}(n-20) \geq 0.
$$

Theorem 1.5. *For $m \geq 0$,*

$$
\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{m} \frac{(-1)^n (1+q^{n+1}) q^{n(n+1)/2}}{1+q} \sum_{r=0}^{n} (1-q^{4r+2}) q^{r(r-3)/2}\\
&= (-1)^m q^{m(m+1)/2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-q^{-m-2};q)_j}{(q;q)_j} q^{(m+3)j+k} \begin{bmatrix} k+j \\ m \end{bmatrix}. \qquad (1.13)
\end{aligned}
$$

Corollary 1.6. *For $m \geq 0$,*

$$
(-1)^m \sum_{i=0}^{m} (-1)^i \sum_{j \geq 0} d_i(j) \overline{p}(n - i(i+1)/2 - j) \geq 0,
$$

where

$$
\sum_{j \geq 0} d_i(j) q^j = \frac{(1+q^{i+1})}{1+q} \sum_{r=0}^{i} (1-q^{4r+2}) q^{r(r-3)/2}.
$$

For instance, setting $m = 2$ in the above Corollary yields that

$$
2\overline{p}(n-5) - \overline{p}(n-8) + \overline{p}(n-9) - \overline{p}(n-10) - \overline{p}(n-12) + \overline{p}(n-13) - \overline{p}(n-14) \geq 0.
$$

This paper is organized as follows. In Section 2, some preliminaries are provided. In Section 3, based on a transformation formula due to Bailey, we prove Theorem 1.1 and

Corollary 1.2. Furthermore, combining two new Bailey pairs constructed in Section 2 and an identity given by Wang and Yee, we prove Theorems 1.3 and 1.5. Then Corollaries 1.4 and 1.6 follow immediately.

2. Preliminaries

In this section, we give some preliminaries which will be used to prove the main results.

Lemma 2.1. *[9, Appendix (II.2)] (The q-binomial theorem)*

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty},\quad |z|<1. \tag{2.1}$$

Setting $a=q^{-n}$ in (2.1) yields that

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}z^k=(zq^{-n};q)_n. \tag{2.2}$$

In addition, the following identity [9] is frequently used.

$$(zq^{-n};q)_n=(q/z;q)_n(-z/q)^nq^{-n(n-1)/2}.$$

The next transformation formula is due to Bailey.

Lemma 2.2. *[6, Eq. (6.1)] We have*

$$\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q^2)_n(-aq/b;q)_{2n}}{(q^2,a^2q^2/b^2;q^2)_n(-aq;q)_{2n}}\left(\frac{a^2q^2}{\rho_1\rho_2}\right)^n=\frac{(a^2q^2/\rho_1,a^2q^2/\rho_2;q^2)_\infty}{(a^2q^2,a^2q^2/\rho_1\rho_2;q^2)_\infty}$$
$$\times\left(1+\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(aq;q)_{n-1}(b;q)_n(\rho_1,\rho_2;q^2)_n}{(q,aq/b;q)_n(a^2q^2/\rho_1,a^2q^2/\rho_2;q^2)_n}\left(\frac{a^3q^2}{\rho_1\rho_2 b}\right)^nq^{n^2}\right). \tag{2.3}$$

Lemma 2.3. *We have*

$$\sum_{n=0}^{m}\frac{(1-aq^{2n})(a,b;q)_n}{(1-a)(q,aq/b;q)_n}\left(\frac{a}{b}\right)^nq^{n^2}$$
$$=\frac{(-1)^mq^{m(m+1)}(a^2q^2;q^2)_m}{(q^2;q^2)_m}\sum_{n=0}^{m}\frac{(q^{-2m},a^2q^{2m+2};q^2)_n(-aq/b;q)_{2n}}{(q^2,a^2q^2/b^2;q^2)_n(-aq;q)_{2n}}. \tag{2.4}$$

Proof. First, setting $\rho_1=q^{-2m}$ in (2.3), we obtain that

$$\sum_{n=0}^{m}\frac{(q^{-2m},\rho_2;q^2)_n(-aq/b;q)_{2n}}{(q^2,a^2q^2/b^2;q^2)_n(-aq;q)_{2n}}\left(\frac{a^2q^{2m+2}}{\rho_2}\right)^n=\frac{(a^2q^2/\rho_2;q^2)_m}{(a^2q^2;q^2)_m}$$
$$\times\left(1+\sum_{n=1}^{m}\frac{(1-aq^{2n})(aq;q)_{n-1}(b;q)_n(q^{-2m},\rho_2;q^2)_n}{(q,aq/b;q)_n(a^2q^{2m+2},a^2q^2/\rho_2;q^2)_n}\left(\frac{a^3q^{2m+2}}{\rho_2 b}\right)^nq^{n^2}\right).$$

Then letting $\rho_2 = a^2q^{2m+2}$ in the above identity yields that

$$\begin{aligned}
&\sum_{n=0}^{m} \frac{(q^{-2m}, a^2q^{2m+2}; q^2)_n(-aq/b; q)_{2n}}{(q^2, a^2q^2/b^2; q^2)_n(-aq; q)_{2n}} \\
&= \frac{(q^{-2m}; q^2)_m}{(a^2q^2; q^2)_m} \left(1 + \sum_{n=1}^{m} \frac{(1-aq^{2n})(aq; q)_{n-1}(b; q)_n}{(q, aq/b; q)_n} \left(\frac{a}{b}\right)^n q^{n^2}\right) \\
&= \frac{(-1)^m q^{-m(m+1)}(q^2; q^2)_m}{(a^2q^2; q^2)_m} \left(1 + \sum_{n=1}^{m} \frac{(1-aq^{2n})(aq; q)_{n-1}(b; q)_n}{(q, aq/b; q)_n} \left(\frac{a}{b}\right)^n q^{n^2}\right) \\
&= \frac{(-1)^m q^{-m(m+1)}(q^2; q^2)_m}{(a^2q^2; q^2)_m} \sum_{n=0}^{m} \frac{(1-aq^{2n})(a, b; q)_n}{(1-a)(q, aq/b; q)_n} \left(\frac{a}{b}\right)^n q^{n^2}.
\end{aligned}$$

This completes the proof. □

Definition 2.4. [2] A pair of sequences (α_n, β_n) is called a Bailey pair relative to (a, q) if α_n and β_n satisfy the following identity:

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q; q)_{n-r}(aq; q)_{n+r}}.$$

Moreover, for a Bailey pair relative to (a, q), Andrews [1] gave the following identity:

$$\alpha_n = \frac{(-1)^n(1-aq^{2n})(a; q)_n q^{n(n-1)/2}}{(1-a)(q; q)_n} \sum_{j=0}^{n} (q^{-n}, aq^n; q)_j q^j \beta_j. \tag{2.5}$$

Based on a q-series expansion formula due to Liu [12, Theorem 9.2], Wang and Yee [15] derived the following lemma.

Lemma 2.5. *[15, Lemma 2.1] For a Bailey pair (α_n, β_n) relative to (a, q) and $m \geq 0$, there holds*

$$\frac{(-1)^m q^{m(m+1)/2}(aq; q)_m}{(q; q)_m} \sum_{n=0}^{m} (q^{-m}, aq^{m+1}; q)_n \beta_n = \sum_{n=0}^{m} \alpha_n. \tag{2.6}$$

Lemma 2.6. *[7, Theorem 2.1] If (α_n, β_n) is a Bailey pair relative to (a^2, q^2), then (α'_n, β'_n) is a Bailey pair relative to (a, q), where*

$$\begin{aligned}
\alpha'_n &= \frac{(-B; q)_n}{(-aq/B; q)_n} B^{-n} q^{-n(n-1)/2} \alpha_n, \\
\beta'_n &= \sum_{k=0}^{n} \frac{(-aq; q)_{2k}(B^2; q^2)_k(q^{-k}/B, Bq^{k+1}; q)_{n-k}}{(-aq/B, B; q)_n(q^2; q^2)_{n-k}} B^{-k} q^{-k(k-1)/2} \beta_k.
\end{aligned}$$

Based on the above lemma, we derive the following Bailey pair.

Lemma 2.7. *The pair of the sequences (α_n, β_n) forms a Bailey pair relative to (a, q), where*

$$\alpha_n = \frac{(-1)^n(1-a^2q^{4n})(a^2;q^2)_n q^{n^2-n}}{(1-a^2)(q^2;q^2)_n},$$
$$\beta_n = \frac{q^n}{(q^2;q^2)_n}.$$

Proof. Recall the unit Bailey pair (α_n, β_n) relative to (a, q) [2]

$$\alpha_n = \frac{(-1)^n(1-aq^{2n})(a;q)_n q^{n(n-1)/2}}{(1-a)(q;q)_n},$$
$$\beta_n = \delta_{n,0},$$

where $\delta_{n,0} = \begin{cases} 1, & n=0 \\ 0, & n>0 \end{cases}$. Replacing a, q by a^2 and q^2, respectively, we obtain the Bailey pair $(\overline{\alpha}_n, \overline{\beta}_n)$ relative to (a^2, q^2), where

$$\overline{\alpha}_n = \frac{(-1)^n(1-a^2q^{4n})(a^2;q^2)_n q^{n^2-n}}{(1-a^2)(q^2;q^2)_n},$$
$$\overline{\beta}_n = \delta_{n,0}.$$

Then substituting the above Bailey pair into Lemma 2.6 and letting $B \to \infty$, we derive the desired Bailey pair. □

Lemma 2.8. *[7] For any nonnegative integer n,*

$$\sum_{r=0}^{n} \frac{(1-a^2q^{4r})(q^{-n}, -B, \rho_1, \rho_2; q)_r}{(1-a^2)(aq^{n+1}, -aq/B, aq/\rho_1, aq/\rho_2; q)_r} \left(\frac{aq^{n+1}}{B\rho_1\rho_2}\right)^r$$
$$= \frac{(aq, aq/\rho_1\rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} {}_5\phi_4\left(\begin{matrix} q^{-n}, & Bq, & \rho_1, & \rho_2, & 1/B \\ & -aq/B, & B, & \rho_1\rho_2 q^{-n}/a, & -q \end{matrix}; q, q\right). \tag{2.7}$$

From (2.7), we construct the following Bailey pair.

Lemma 2.9. *The pair of the sequences (α_n, β_n) forms a Bailey pair relative to (a, q), where*

$$\alpha_n = \frac{(-1)^n(1-aq^{2n})(a/\rho_1;q)_n \rho_1^n q^{n(n-1)/2}}{(1-a)(\rho_1 q;q)_n} \sum_{r=0}^{n} \frac{(1-a^2q^{4r-2})(\rho_1;q)_r q^{r(r-1)/2}}{(1-a^2q^{-2})(a/\rho_1;q)_r \rho_1^r},$$
$$\beta_n = \frac{(\rho_1;q)_n q^n}{(q, \rho_1 q, -q; q)_n}. \tag{2.8}$$

Proof. Combining (2.5) and (2.8), we have

$$\alpha_n = \frac{(-1)^n(1-aq^{2n})(a;q)_n q^{n(n-1)/2}}{(1-a)(q;q)_n} \sum_{j=0}^{n} \frac{(q^{-n}, aq^n, \rho_1; q)_j q^{2j}}{(q, \rho_1 q, -q; q)_j}. \tag{2.9}$$

Then letting $\rho_2 = aq^{n+1}$ and $B \to \infty$ in (2.7), we obtain that

$$\sum_{r=0}^{n} \frac{(1-a^2q^{4r})(\rho_1;q)_r q^{r(r-1)/2}}{(1-a^2)(aq/\rho_1;q)_r \rho_1^r} = \frac{(aq,\rho_1 q;q)_n}{(aq/\rho_1,q;q)_n \rho_1^n} \sum_{j=0}^{n} \frac{(q^{-n},aq^{n+1},\rho_1;q)_j q^{2j}}{(q,\rho_1 q,-q;q)_j}.$$

Next, setting $a \to aq^{-1}$ in the above identity yields that

$$\sum_{j=0}^{n} \frac{(q^{-n},aq^{n},\rho_1;q)_j q^{2j}}{(q,\rho_1 q,-q;q)_j} = \frac{(a/\rho_1,q;q)_n}{(a,\rho_1 q;q)_n} \rho_1^n \sum_{r=0}^{n} \frac{(1-a^2q^{4r-2})(\rho_1;q)_r q^{r(r-1)/2}}{(1-a^2q^{-2})(a/\rho_1;q)_r \rho_1^r}. \tag{2.10}$$

Finally, substituting (2.10) into (2.9), we complete the proof. □

3. Proofs of the main results

In this section, we prove the main results.

Proof of Theorem 1.1. For $m,t \geq 0$ and $r \geq 4t+1$, letting $(a,b) = (q^r, -q^{r-2t})$ in (2.4), we have

$$\begin{aligned}
&\sum_{n=0}^{m} \frac{(1-q^{2n+r})(q^r,-q^{r-2t};q)_n}{(1-q^r)(q,-q^{2t+1};q)_n}(-1)^n q^{n^2+2tn} \\
&= \frac{(-1)^m q^{m(m+1)}(q^{2r+2};q^2)_m}{(q^2;q^2)_m} \sum_{n=0}^{m} \frac{(q^{-2m},q^{2m+2r+2},q^{2t+1},q^{2t+2};q^2)_n}{(q^2,q^{4t+2};q^2)_n(-q^{r+1};q)_{2n}}.
\end{aligned} \tag{3.1}$$

Notice that when $0 \leq t \leq n-1$,

$$\frac{(q^{2t+2};q^2)_n}{(q^{4t+2};q^2)_n} = \frac{(q^{2t+2};q^2)_t(q^{4t+2};q^2)_{n-t}}{(q^{4t+2};q^2)_{n-t}(q^{2n+2t+2};q^2)_t} = \frac{(q^{2t+2};q^2)_t}{(q^{2n+2t+2};q^2)_t};$$

when $t \geq n$,

$$\frac{(q^{2t+2};q^2)_n}{(q^{4t+2};q^2)_n} = \frac{(q^{2t+2};q^2)_t}{(q^{2n+2t+2};q^2)_{t-n}} \frac{(q^{2n+2t+2};q^2)_{t-n}}{(q^{2n+2t+2};q^2)_t} = \frac{(q^{2t+2};q^2)_t}{(q^{2n+2t+2};q^2)_t}.$$

So, for $t \geq 0$, there always holds

$$\frac{(q^{2t+2};q^2)_n}{(q^{4t+2};q^2)_n} = \frac{(q^{2t+2};q^2)_t}{(q^{2n+2t+2};q^2)_t}. \tag{3.2}$$

Therefore, substituting (3.2) into (3.1), we obtain that

$$\begin{aligned}
&\sum_{n=0}^{m} \frac{(1-q^{2n+r})(q^r,-q^{r-2t};q)_n}{(1-q^r)(q,-q^{2t+1};q)_n}(-1)^n q^{n^2+2tn} \\
&= \frac{(-1)^m q^{m(m+1)}}{(q^2;q^2)_m} \sum_{n=0}^{m} \frac{(q^{-2m},q^{2t+1};q^2)_n(q^{2r+2};q^2)_{m+n}(q^{2t+2};q^2)_t}{(q^2;q^2)_n(-q^{r+1};q)_{2n}(q^{2n+2t+2};q^2)_t} \\
&= \frac{(-1)^m q^{m(m+1)}(-q^{t+1},q^{t+1};q)_t}{(q^2;q^2)_m} \sum_{n=0}^{m} \frac{(q^{-2m},q^{2t+1};q^2)_n(q^{2r+2};q^2)_{m+n}}{(q^2;q^2)_n(-q^{r+1};q)_{2n}(q^{2n+2t+2};q^2)_t}.
\end{aligned}$$

Then multiplying by $(q;q)_r(-q^{2t+1};q)_{r-4t-1}$ on both sides of the above identity, we arrive at

$$\begin{aligned}
&\sum_{n=0}^{m}(-1)^n(1-q^{2n+r})(q^{n+1};q)_{r-1}(-q^{n+2t+1};q)_{r-4t-1}q^{n^2+2tn}\\
&=\frac{(-1)^m q^{m(m+1)}(q;q)_r(-q^{t+1};q)_{r-3t-1}(q^{t+1};q)_t}{(q^2;q^2)_m}\sum_{n=0}^{m}\frac{(q^{-2m},q^{2t+1};q^2)_n(q^{2r+2};q^2)_{m+n}}{(q^2;q^2)_n(-q^{r+1};q)_{2n}(q^{2n+2t+2};q^2)_t}.
\end{aligned}$$

Next, multiplying by $(-q;q)_\infty/(q;q)_\infty$ on both sides of the above identity yields that

$$\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{m}(-1)^n(1-q^{2n+r})(q^{n+1};q)_{r-1}(-q^{n+2t+1};q)_{r-4t-1}q^{n^2+2tn}\\
&=\frac{(-q^{r+1};q)_\infty}{(q,q^{2r+2};q^2)_\infty}\frac{(-1)^m q^{m(m+1)}(-q^{t+1};q)_{r-3t-1}(q^{t+1};q)_t}{(q^2;q^2)_m}\\
&\quad\times\sum_{n=0}^{m}\frac{(q^{-2m},q^{2t+1};q^2)_n(q^{2r+2};q^2)_{m+n}}{(q^2;q^2)_n(-q^{r+1};q)_{2n}(q^{2n+2t+2};q^2)_t}\\
&=\frac{(-1)^m q^{m(m+1)}(-q^{t+1};q)_{r-3t-1}(q^{t+1};q)_t}{(q^2;q^2)_m(q;q^2)_t}\\
&\quad\times\sum_{n=0}^{m}\frac{(q^{-2m};q^2)_n(-q^{2n+r+1};q)_\infty}{(q^2;q^2)_n(q^{2m+2n+2r+2},q^{2n+2t+1};q^2)_\infty(q^{2n+2t+2};q^2)_t}\\
&=\frac{(-1)^m q^{m(m+1)}(-q^{t+1};q)_{r-3t-1}(q;q)_{2t}}{(q^2;q^2)_m(q;q^2)_t(q;q)_t}\\
&\quad\times\sum_{n=0}^{m}\frac{(q^{-2m};q^2)_n(-q^{2n+r+1},-q^{2n+r+2};q^2)_\infty}{(q^2;q^2)_n(q^{2m+2n+2r+2},q^{2n+2t+1};q^2)_\infty(q^{2n+2t+2};q^2)_t}.
\end{aligned}\tag{3.3}$$

Notice that

$$\frac{(-q^{t+1};q)_{r-3t-1}(q;q)_{2t}}{(q;q^2)_t(q;q)_t}=\frac{(-q^{t+1};q)_{r-3t-1}(q,q^2;q^2)_t}{(q;q^2)_t(q;q)_t}=(-q;q)_{r-2t-1}.\tag{3.4}$$

Furthermore, based on the q-binomial theorem (2.1), we have

$$\frac{(-q^{2n+r+2};q^2)_\infty}{(q^{2m+2n+2r+2};q^2)_\infty}=\sum_{i=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i}{(q^2;q^2)_i}q^{(2m+2n+2r+2)i},\tag{3.5}$$

$$\frac{(-q^{2n+r+1};q^2)_\infty}{(q^{2n+2t+1};q^2)_\infty}=\sum_{j=0}^{\infty}\frac{(-q^{r-2t};q^2)_j}{(q^2;q^2)_j}q^{(2n+2t+1)j}.\tag{3.6}$$

Meanwhile,

$$\frac{1}{(q^{2n+2t+2};q^2)_t}=\prod_{s=1}^{t}\frac{1}{1-q^{2n+2t+2s}}=\prod_{s=1}^{t}\sum_{i_s=0}^{\infty}q^{(2n+2t+2s)i_s}.\tag{3.7}$$

Hence, substituting (3.4)-(3.7) into (3.3) yields that

$$\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{m}(-1)^n(1-q^{2n+r})(q^{n+1};q)_{r-1}(-q^{n+2t+1};q)_{r-4t-1}q^{n^2+2tn}\\
&=\frac{(-1)^m q^{m(m+1)}(-q;q)_{r-2t-1}}{(q^2;q^2)_m}\sum_{n=0}^{m}\frac{(q^{-2m};q^2)_n}{(q^2;q^2)_n}\sum_{i=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i}{(q^2;q^2)_i}q^{(2m+2n+2r+2)i}\\
&\quad\times\sum_{j=0}^{\infty}\frac{(-q^{r-2t};q^2)_j}{(q^2;q^2)_j}q^{(2n+2t+1)j}\prod_{s=1}^{t}\sum_{i_s=0}^{\infty}q^{(2n+2t+2s)i_s}\\
&=\frac{(-1)^m q^{m(m+1)}(-q;q)_{r-2t-1}}{(q^2;q^2)_m}\sum_{i=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i}{(q^2;q^2)_i}q^{(2m+2r+2)i}\\
&\quad\times\sum_{j=0}^{\infty}\frac{(-q^{r-2t};q^2)_j}{(q^2;q^2)_j}q^{(2t+1)j}\prod_{s=1}^{t}\sum_{i_s=0}^{\infty}q^{(2t+2s)i_s}\sum_{n=0}^{m}\frac{(q^{-2m};q^2)_n}{(q^2;q^2)_n}q^{2n(i+j+i_1+i_2+\cdots+i_t)}\\
&=\frac{(-1)^m q^{m(m+1)}(-q;q)_{r-2t-1}}{(q^2;q^2)_m}\sum_{i=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i}{(q^2;q^2)_i}q^{(2m+2r+2)i}\sum_{j=0}^{\infty}\frac{(-q^{r-2t};q^2)_j}{(q^2;q^2)_j}q^{(2t+1)j}\\
&\quad\times\prod_{s=1}^{t}\sum_{i_s=0}^{\infty}q^{(2t+2s)i_s}\left(q^{2(i+j+i_1+i_2+\cdots+i_t-m)};q^2\right)_m,
\end{aligned}\tag{3.8}$$

where the last equality follows from (2.2). Let

$$h=i+j+i_1+i_2+\cdots+i_t.$$

Then according to

$$\frac{\left(q^{2h-2m};q^2\right)_m}{(q^2;q^2)_m}=\begin{cases}(-1)^m q^{-m(m+1)}, & h=0,\\ 0, & 1\le h\le m,\\ {h-1\brack m}_{q^2}, & h\ge m+1,\end{cases}\tag{3.9}$$

the identity (3.8) implies (1.9).

For $m\ge 0$, $t\ge 1$ and $r\ge 4t-1$, letting $(a,b)=(q^r,-q^{r-2t+1})$ in (2.4), we have

$$\begin{aligned}
&\sum_{n=0}^{m}\frac{(1-q^{2n+r})(q^r,-q^{r-2t+1};q)_n}{(1-q^r)(q,-q^{2t};q)_n}(-1)^n q^{n^2+2tn-n}\\
&=\frac{(-1)^m q^{m(m+1)}(q^{2r+2};q^2)_m}{(q^2;q^2)_m}\sum_{n=0}^{m}\frac{(q^{-2m},q^{2m+2r+2},q^{2t},q^{2t+1};q^2)_n}{(q^2,q^{4t};q^2)_n(-q^{r+1};q)_{2n}}.
\end{aligned}\tag{3.10}$$

Notice that

$$\frac{(q^{2t};q^2)_n}{(q^{4t};q^2)_n}=\frac{(1-q^{2t})(q^{2t+2};q^2)_{n-1}}{(1-q^{4t})(q^{4t+2};q^2)_{n-1}}=\frac{(1-q^{2t})}{(1-q^{4t})}\frac{(q^{2t+2};q^2)_t}{(q^{2n+2t};q^2)_t}=\frac{(q^{2t};q^2)_t}{(q^{2n+2t};q^2)_t},\tag{3.11}$$

where the second equality follows from (3.2). Thus, substituting (3.11) into (3.10) yields that

$$\begin{aligned}
&\sum_{n=0}^{m}\frac{(1-q^{2n+r})(q^r,-q^{r-2t+1};q)_n}{(1-q^r)(q,-q^{2t};q)_n}(-1)^nq^{n^2+2tn-n}\\
&=\frac{(-1)^mq^{m(m+1)}}{(q^2;q^2)_m}\sum_{n=0}^{m}\frac{(q^{-2m},q^{2t+1};q^2)_n(q^{2r+2};q^2)_{m+n}(q^{2t};q^2)_t}{(q^2;q^2)_n(-q^{r+1};q)_{2n}(q^{2n+2t};q^2)_t}\\
&=\frac{(-1)^mq^{m(m+1)}(-q^t,q^t;q)_t}{(q^2;q^2)_m}\sum_{n=0}^{m}\frac{(q^{-2m},q^{2t+1};q^2)_n(q^{2r+2};q^2)_{m+n}}{(q^2;q^2)_n(-q^{r+1};q)_{2n}(q^{2n+2t};q^2)_t}.
\end{aligned}$$

Then multiplying by $(q;q)_r(-q^{2t};q)_{r-4t+1}$ on both sides of the above identity, we arrive at

$$\begin{aligned}
&\sum_{n=0}^{m}(-1)^n(1-q^{2n+r})(q^{n+1};q)_{r-1}(-q^{n+2t};q)_{r-4t+1}q^{n^2+2tn-n}\\
&=\frac{(-1)^mq^{m(m+1)}(q;q)_r(-q^t;q)_{r-3t+1}(q^t;q)_t}{(q^2;q^2)_m}\sum_{n=0}^{m}\frac{(q^{-2m},q^{2t+1};q^2)_n(q^{2r+2};q^2)_{m+n}}{(q^2;q^2)_n(-q^{r+1};q)_{2n}(q^{2n+2t};q^2)_t}.
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{m}(-1)^n(1-q^{2n+r})(q^{n+1};q)_{r-1}(-q^{n+2t};q)_{r-4t+1}q^{n^2+2tn-n}\\
&=\frac{(-q^{r+1};q)_\infty}{(q,q^{2r+2};q^2)_\infty}\frac{(-1)^mq^{m(m+1)}(-q^t;q)_{r-3t+1}(q^t;q)_t}{(q^2;q^2)_m}\\
&\quad\times\sum_{n=0}^{m}\frac{(q^{-2m},q^{2t+1};q^2)_n(q^{2r+2};q^2)_{m+n}}{(q^2;q^2)_n(-q^{r+1};q)_{2n}(q^{2n+2t};q^2)_t}\\
&=\frac{(-1)^mq^{m(m+1)}(-q^t;q)_{r-3t+1}(q^t;q)_t}{(q^2;q^2)_m(q;q^2)_t}\\
&\quad\times\sum_{n=0}^{m}\frac{(q^{-2m};q^2)_n(-q^{2n+r+1};q)_\infty}{(q^2;q^2)_n(q^{2m+2n+2r+2},q^{2n+2t+1};q^2)_\infty(q^{2n+2t};q^2)_t}\\
&=\frac{(-1)^mq^{m(m+1)}(-q^t;q)_{r-3t+1}(q;q)_{2t-1}}{(q^2;q^2)_m(q;q^2)_t(q;q)_{t-1}}\\
&\quad\times\sum_{n=0}^{m}\frac{(q^{-2m};q^2)_n(-q^{2n+r+1},-q^{2n+r+2};q^2)_\infty}{(q^2;q^2)_n(q^{2m+2n+2r+2},q^{2n+2t+1};q^2)_\infty(q^{2n+2t};q^2)_t}.
\end{aligned}\tag{3.12}$$

Observe that

$$\frac{(-q^t;q)_{r-3t+1}(q;q)_{2t-1}}{(q;q^2)_t(q;q)_{t-1}}=\frac{(-q^t;q)_{r-3t+1}(q;q^2)_t(q^2;q^2)_{t-1}}{(q;q^2)_t(q;q)_{t-1}}=(-q;q)_{r-2t}.\tag{3.13}$$

Hence, combining (3.5)-(3.7), (3.12) and (3.13), we derive that

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{m}(-1)^n(1-q^{2n+r})(q^{n+1};q)_{r-1}(-q^{n+2t};q)_{r-4t+1}q^{n^2+2tn-n}$$
$$=\frac{(-1)^m q^{m(m+1)}(-q;q)_{r-2t}}{(q^2;q^2)_m}\sum_{n=0}^{m}\frac{(q^{-2m};q^2)_n}{(q^2;q^2)_n}\sum_{i=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i}{(q^2;q^2)_i}q^{(2m+2n+2r+2)i}$$
$$\times\sum_{j=0}^{\infty}\frac{(-q^{r-2t};q^2)_j}{(q^2;q^2)_j}q^{(2n+2t+1)j}\prod_{s=1}^{t}\sum_{i_s=0}^{\infty}q^{(2n+2t+2s-2)i_s}$$
$$=\frac{(-1)^m q^{m(m+1)}(-q;q)_{r-2t}}{(q^2;q^2)_m}\sum_{i=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i}{(q^2;q^2)_i}q^{(2m+2r+2)i}$$
$$\times\sum_{j=0}^{\infty}\frac{(-q^{r-2t};q^2)_j}{(q^2;q^2)_j}q^{(2t+1)j}\prod_{s=1}^{t}\sum_{i_s=0}^{\infty}q^{(2t+2s-2)i_s}\sum_{n=0}^{m}\frac{(q^{-2m};q^2)_n}{(q^2;q^2)_n}q^{2n(i+j+i_1+i_2+\cdots+i_t)}$$
$$=\frac{(-1)^m q^{m(m+1)}(-q;q)_{r-2t}}{(q^2;q^2)_m}\sum_{i=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i}{(q^2;q^2)_i}q^{(2m+2r+2)i}\sum_{j=0}^{\infty}\frac{(-q^{r-2t};q^2)_j}{(q^2;q^2)_j}q^{(2t+1)j}$$
$$\times\prod_{s=1}^{t}\sum_{i_s=0}^{\infty}q^{(2t+2s-2)i_s}\left(q^{2(i+j+i_1+i_2+\cdots+i_t-m)};q^2\right)_m,$$

where the last equality follows from (2.2). By the discussion of (3.9), we prove (1.10). □

Proof of Corollary 1.2. In view of (1.1) and (1.9), for $m\geq 0$, $t\geq 0$ and $r\geq 4t+1$, we have

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n\sum_{i=0}^{m}(-1)^i(1-q^{2i+r})(q^{i+1};q)_{r-1}(-q^{i+2t+1};q)_{r-4t-1}q^{i^2+2ti}$$
$$=(-q;q)_{r-2t-1}+(-1)^m q^{m(m+1)}(-q;q)_{r-2t-1}\sum_{i,j,i_1,i_2,\ldots,i_t=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i(-q^{r-2t};q^2)_j}{(q^2;q^2)_i(q^2;q^2)_j}$$
$$\times q^{(2m+2r+2)i+(2t+1)j+(2t+2)i_1+(2t+4)i_2+\cdots+4ti_t}\begin{bmatrix}i+j+i_1+i_2+\cdots+i_t-1\\ m\end{bmatrix}_{q^2}. \tag{3.14}$$

Then multiplying both sides of (3.14) by $(-1)^m$, we find that for $n>\binom{r-2t}{2}$, the coefficients of q^n on the right-hand side are nonnegative. Therefore, we have

$$(-1)^m\sum_{i=0}^{m}(-1)^i\sum_{j\geq 0}b'_{t,r,i}(j)\overline{p}(n-i^2-2ti-j)\geq 0, \tag{3.15}$$

where

$$\sum_{j\geq 0}b'_{t,r,i}(j)q^j=(1-q^{2i+r})(q^{i+1};q)_{r-1}(-q^{i+2t+1};q)_{r-4t-1}.$$

Similarly, combining (1.1) and (1.10) yields that

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n\sum_{i=0}^{m}(-1)^i(1-q^{2i+r})(q^{i+1};q)_{r-1}(-q^{i+2t};q)_{r-4t+1}q^{i^2+(2t-1)i}$$
$$=(-q;q)_{r-2t}+(-1)^m q^{m(m+1)}(-q;q)_{r-2t}\sum_{i,j,i_1,i_2,\ldots,i_t=0}^{\infty}\frac{(-q^{-2m-r};q^2)_i(-q^{r-2t};q^2)_j}{(q^2;q^2)_i(q^2;q^2)_j}$$
$$\times q^{(2m+2r+2)i+(2t+1)j+2ti_1+(2t+2)i_2+\cdots+(4t-2)i_t}\begin{bmatrix}i+j+i_1+i_2+\cdots+i_t-1\\ m\end{bmatrix}_{q^2}, \tag{3.16}$$

where $m\geq 0$, $t\geq 1$, $r\geq 4t-1$. If we multiply by $(-1)^m$ on both sides of (3.16), then for $n>\binom{r-2t+1}{2}$, the coefficients of q^n on the right-hand side are nonnegative. Hence, we obtain that

$$(-1)^m\sum_{i=0}^{m}(-1)^i\sum_{j\geq 0}b''_{t,r,i}(j)\overline{p}(n-i^2-(2t-1)i-j)\geq 0, \tag{3.17}$$

where

$$\sum_{j\geq 0}b''_{t,r,i}(j)q^j=(1-q^{2i+r})(q^{i+1};q)_{r-1}(-q^{i+2t};q)_{r-4t+1}.$$

Observe that (3.15) and (3.17) are the even and odd cases of k in (1.11), respectively. Hence, we complete the proof. □

Proof of Theorem 1.3. Substituting the Bailey pair in Lemma 2.7 into (2.6), we have

$$\frac{(-1)^m q^{m(m+1)/2}(aq;q)_m}{(q;q)_m}\sum_{n=0}^{m}\frac{(q^{-m},aq^{m+1};q)_n q^n}{(q^2;q^2)_n}=\sum_{n=0}^{m}\frac{(-1)^n(1-a^2q^{4n})(a^2;q^2)_n q^{n^2-n}}{(1-a^2)(q^2;q^2)_n}. \tag{3.18}$$

Setting $a=q^r$ in (3.18) and multiplying by $(q^2;q^2)_r$ on both sides, we derive that

$$\frac{(-1)^m q^{m(m+1)/2}(q^2;q^2)_r}{(q;q)_m}\sum_{n=0}^{m}\frac{(q^{-m};q)_n(q^{r+1};q)_{m+n}q^n}{(q^2;q^2)_n}$$
$$=\sum_{n=0}^{m}(-1)^n(1-q^{4n+2r})(q^{2n+2};q^2)_{r-1}q^{n^2-n}.$$

Therefore,

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{m}(-1)^n(1-q^{4n+2r})(q^{2n+2};q^2)_{r-1}q^{n^2-n}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\frac{(-1)^m q^{m(m+1)/2}(q^2;q^2)_r}{(q;q)_m}\sum_{n=0}^{m}\frac{(q^{-m};q)_n(q^{r+1};q)_{m+n}q^n}{(q;q)_n(-q;q)_n}$$

$$
\begin{aligned}
&= \frac{(-1)^m q^{m(m+1)/2}(-q;q)_r}{(q;q)_m} \sum_{n=0}^{m} \frac{(q^{-m};q)_n q^n}{(q;q)_n} \frac{(-q^{n+1};q)_\infty}{(q^{m+n+r+1};q)_\infty} \\
&= \frac{(-1)^m q^{m(m+1)/2}(-q;q)_r}{(q;q)_m} \sum_{n=0}^{m} \frac{(q^{-m};q)_n}{(q;q)_n} q^n \sum_{i=0}^{\infty} \frac{(-q^{-m-r};q)_i}{(q;q)_i} q^{(m+n+r+1)i} \\
&= \frac{(-1)^m q^{m(m+1)/2}(-q;q)_r}{(q;q)_m} \sum_{i=0}^{\infty} \frac{(-q^{-m-r};q)_i}{(q;q)_i} q^{(m+r+1)i} \sum_{n=0}^{m} \frac{(q^{-m};q)_n}{(q;q)_n} q^{(i+1)n} \\
&= \frac{(-1)^m q^{m(m+1)/2}(-q;q)_r}{(q;q)_m} \sum_{i=0}^{\infty} \frac{(-q^{-m-r};q)_i}{(q;q)_i} q^{(m+r+1)i} \left(q^{i+1-m};q\right)_m, \qquad (3.19)
\end{aligned}
$$

where the third equality follows from (2.1) and the last equality follows from (2.2). Observe that

$$
\frac{(q^{i+1-m};q)_m}{(q;q)_m} = \begin{cases} 0, & 0 \le i \le m-1, \\ {i \brack m}, & i \ge m. \end{cases} \qquad (3.20)
$$

Therefore, (3.19) implies (1.12). □

Proof of Corollary 1.4. Similar to the proof of Corollary 1.2, Corollary 1.4 can be deduced from Theorem 1.3 immediately. □

Proof of Theorem 1.5. Inserting the Bailey pair in Lemma 2.9 into (2.6), we have

$$
\begin{aligned}
&\frac{(-1)^m q^{m(m+1)/2}(aq;q)_m}{(q;q)_m} \sum_{n=0}^{m} \frac{(q^{-m}, aq^{m+1}, \rho_1;q)_n q^n}{(q, \rho_1 q, -q;q)_n} \\
&= \sum_{n=0}^{m} \frac{(-1)^n(1-aq^{2n})(a/\rho_1;q)_n \rho_1^n q^{n(n-1)/2}}{(1-a)(\rho_1 q;q)_n} \sum_{r=0}^{n} \frac{(1-a^2q^{4r-2})(\rho_1;q)_r q^{r(r-1)/2}}{(1-a^2q^{-2})(a/\rho_1;q)_r \rho_1^r}. \qquad (3.21)
\end{aligned}
$$

Setting $(a, \rho_1) = (q^2, q)$ in (3.21) and multiplying by $(1-q^2)$ on both sides, we have

$$
\begin{aligned}
&\frac{(-1)^m q^{m(m+1)/2}}{(q;q)_m} \sum_{n=0}^{m} \frac{(q^{-m};q)_n (q;q)_{m+n+2} q^n}{(q;q)_{n+1}(-q;q)_n} \\
&= \sum_{n=0}^{m} (-1)^n (1-q)(1+q^{n+1}) q^{n(n+1)/2} \sum_{r=0}^{n} q^{r(r-3)/2} \frac{1-q^{2(2r+1)}}{1-q^2} \\
&= \sum_{n=0}^{m} \frac{(-1)^n (1+q^{n+1}) q^{n(n+1)/2}}{1+q} \sum_{r=0}^{n} (1-q^{4r+2}) q^{r(r-3)/2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{m} \frac{(-1)^n (1+q^{n+1}) q^{n(n+1)/2}}{1+q} \sum_{r=0}^{n} (1-q^{4r+2}) q^{r(r-3)/2} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \frac{(-1)^m q^{m(m+1)/2}}{(q;q)_m} \sum_{n=0}^{m} \frac{(q^{-m};q)_n (q;q)_{m+n+2} q^n}{(q;q)_{n+1}(-q;q)_n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^m q^{m(m+1)/2}}{(q;q)_m} \sum_{n=0}^{m} \frac{(q^{-m};q)_n q^n}{(q;q)_n(1-q^{n+1})} \frac{(-q^{n+1};q)_\infty}{(q^{m+n+3};q)_\infty} \\
&= \frac{(-1)^m q^{m(m+1)/2}}{(q;q)_m} \sum_{n=0}^{m} \frac{(q^{-m};q)_n}{(q;q)_n} q^n \sum_{k=0}^{\infty} q^{(n+1)k} \sum_{j=0}^{\infty} \frac{(-q^{-m-2};q)_j}{(q;q)_j} q^{(m+n+3)j} \\
&= \frac{(-1)^m q^{m(m+1)/2}}{(q;q)_m} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-q^{-m-2};q)_j}{(q;q)_j} q^{(m+3)j+k} \sum_{n=0}^{m} \frac{(q^{-m};q)_n}{(q;q)_n} q^{(k+j+1)n} \\
&= \frac{(-1)^m q^{m(m+1)/2}}{(q;q)_m} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-q^{-m-2};q)_j}{(q;q)_j} q^{(m+3)j+k} \left(q^{k+j+1-m};q\right)_m,
\end{aligned}
$$

where the third equality follows from (2.1) and we derive the last equality by (2.2). Similar to the discussion of (3.20), we complete the proof of (1.13). □

Proof of Corollary 1.6. Similar to the proof of Corollary 1.2, we derive Corollary 1.6 from Theorem 1.5. □

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