ON PRESCRIBED HAMILTON LACEABILITY OF HYBRID-FAULTY STAR GRAPHS [*]

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Abstract. In this paper, we investigate Hamilton paths passing through a prescribed linear forest in a hybrid-faulty star graph. Let M be a matching with m edges of the star graph S_n, F be an f-subset of $E(S_n - M)$, $\{x, y\}$ be a 2-subset of $V(S_n - M)$, and L be a linear forest of $S_n - M - F$. Suppose that $m + f \leq n - 4$, neither x nor y is an inner vertex of L, and x, y are not located on the same component of L. We prove that, for any $w \in V(S_n - M) \setminus V(L)$, there exists a Hamilton path of $S_n - M - F - w$ between x and y passing through L provided that x, y belong to the partite set not containing w, and $|E(L)| \leq n - 4 - m - f$. As a consequence, if x, y are from opposite partite sets and $|E(L)| \leq n - 4 - m - f$ then $S_n - M - F$ has a Hamilton path between x and y passing through L. We also proved that $S_n - M - F$ has a Hamilton cycle passing through L if $|E(L)| \leq n - 3 - m - f$.

Key words. Star graph, Hamilton path, linear forests, prescribed Hamilton laceability, prescribed hyper-Hamilton laceability

MSC codes.

1. Introduction.

A crucial factor in the design of multiprocessor systems is the interconnection network, which is often modeled by a simple graph. One minimal requirement for an interconnection network is that the associated graph is sufficiently connected so that the system still works in case of some nodes (processors) or links (communication channels) failure. This inspired an intensive study of embedding long cycles or paths into various specific graphs avoiding certain forbidden edges or vertices, for example, the hypercube and its variants [10, 13, 5, 19, 22, 17, 27], Cayley graphs generated by transposition trees [11, 15, 18, 28, 29, 31], and so on. In an interconnection network, some links or nodes may have better performance than the others, and they may coexist with the faulty ones. Thus the prescribed embedding problem for some famous graphs (networks) has been proposed and widely studied, see [4, 7, 8, 20, 30, 32, 33, 34] for example. In this paper, we focus on long paths passing prescribed linear forests in faulty star graphs.

The star graph [1] possesses many desirable topological properties for building interconnection network of parallel and distributed systems, such as recursiveness, vertex and edge symmetry, maximal fault tolerance, sub-logarithmic degree and diameter, and strong resilience [1, 2]. The star graph has been widely studied in different aspects, such as path routing [12, 24], connectivity and diagnosability [3, 21], broadcasting [9, 23], and embedding problems [11, 18, 28, 30, 31], and so on.

Recall that a *Hamilton path* or *Hamilton cycle* in a graph is a path or cycle which traverses all the vertices, respectively. A graph with a Hamilton cycle is called *Hamiltonian*. Let G be a balanced bipartite graph, and denote $V(G)$ and $E(G)$ the vertex set and edge set, respectively. Then G is called *Hamilton laceable* [25] if (i) any two vertices in opposite partite sets are the ends of some Hamilton path (of G), and *hyper-Hamilton laceable* if (ii) any two distinct vertices in a partite set are the ends of

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some Hamilton path of $G-w$, where w runs over the partite set not containing x and y. It is easy to see that (ii) guarantees the Hamilton laceability of G, our definition of hyper-Hamilton laceability coincides with that given in [14] for balanced bipartite graphs.

A subgraph L of G is called a *liner forest* if either L is null or every components of L is a path of length no less 1. Let $\ell \geq 0$ be an integer. Then G is called *ℓ-prescribed Hamiltonian* if, for any linear forest L with $|E(L)| \leq \ell$, there exists a Hamilton cycle in G passing through L. Two distinct vertices $x, y \in V(G)$ are said to be *compatible* with L (in G) if neither x nor y is an inner vertex of L, and x, y are not located on the same component of L. The graph G is called *ℓ-prescribed Hamilton laceable* if, for any linear forest L with $|E(L)| \leq \ell$ and vertices x, y in opposite partite sets, G has a Hamilton path with ends x, y and passing through L unless x, y are not compatible with L. Moreover, G is called *ℓ-prescribed hyper-Hamilton laceable* if, for any linear forest L with $|E(L)| \leq \ell$, distinct vertices x, y in a partite set and vertex w in the partite set not containing x and y, the graph $G-w$ has a Hamilton path with ends x, y and passing through L unless either $w \in V(L)$ or x, y are not compatible with L. Note that Hamilton laceability and hyper-Hamilton laceability are just 0-prescribed Hamilton laceability and 0-prescribed hyper-Hamilton laceability, respectively.

Recently, Yang and Li [32] investigated the prescribed hyper-Hamilton laceability of the hypercube Q_n with faulty matching and edges. They proved that for a matching M and a set F of edges in Q_n, if $|E(M)| + |F| \leq n-3$ then $Q_n - M - F$ is $(n-3-|E(M)|-|F|)$-prescribed hyper-Hamilton laceable. A natural question which arises here is: How is the prescribed hyper-Hamilton laceability of the star graph S_n with faulty matching and edges? In this paper, we answer this question by proving the next results in Section 4.

THEOREM 1.1. *Let M be a matching with m edges of S_n, and let F be an f-subset of $E(S_n - M)$, where $n \geq 4$ and $m + f \leq n-4$. Then $S_n - M - F$ is $(n-4-m-f)$-prescribed hyper-Hamilton laceable.*

As consequences of Theorem 1.1, the following results are proved in Section 3.

COROLLARY 1.2. *Let M be a matching with m edges of S_n, and let F be an f-subset of $E(S_n - M)$, where $n \geq 4$ and $m + f \leq n-4$. Then $S_n - M - F$ is $(n-4-m-f)$-prescribed Hamilton laceable.*

COROLLARY 1.3. *Let M be a matching with m edges of S_n, and let F be an f-subset of $E(S_n - M)$, where $n \geq 4$ and $m + f \leq n-3$. Then $S_n - M - F$ is $(n-3-m-f)$-prescribed Hamiltonian.*

2. Preliminaries. Let $n > 2$ be an integer, and write $[n] := \{1, 2, \ldots, n\}$. A bijection of $[n]$ onto itself is called a *permutation* of $[n]$. Denote by $\mathcal{S}_n$ the set of permutations of $[n]$. Under composition of mappings, $\mathcal{S}_n$ forms a group of order $n!$, called the *symmetric group* on $[n]$. We always write permutations on the left and compose from right to left, for example, $(x \cdot y)(i) = x(y(i))$. For distinct $i, j \in [n]$, we use $t_{i,j}$ to denote the transposition interchanging i and j. Put

$$\mathcal{T} := \{t_{1,i} \mid 1 \neq i \in [n]\}.$$

Then $\mathcal{T}$ is a set of generators of the symmetric group $\mathcal{S}_n$. The *n-star graph S_n* is defined as a Cayley graph on $\mathcal{S}_n$ such that two vertices $x, y \in \mathcal{S}_n$ are adjacent if and only if $x^{-1} \cdot y \in \mathcal{T}$, refer to [2]. It is easy to see that S_n is an $(n-1)$-regular bipartite graph. The following are several known results about the star graph S_n.

LEMMA 2.1 ([18]). *Let F be a subset of $E(S_n)$.*
(1) *If $n \geq 4$ and $|F| \leq n-4$ then $S_n - F$ is hyper-Hamilton laceable.*
(2) *If $n \geq 4$ and $|F| \leq n-3$ then $S_n - F$ is Hamilton laceable.*

LEMMA 2.2 ([29]). *Let x and y be vertices of opposite parity, and let $e \in E(S_n)$. If $n \geq 4$ and $e \neq xy$ then there exists a Hamilton path $P[x, y]$ of S_n, which passes e.*

LEMMA 2.3 ([31]). *Let M be a matching of size m of star graph S_n.*
(1) *If $n \geq 4$ and $m \leq n-4$ then $S_n - M$ is hyper-Hamilton laceable.*
(2) *If $n \geq 4$ and $m \leq n-3$ then $S_n - M$ is Hamiltonian.*

Recall that a permutation is said to be even or odd if it is a product of an even or odd number of transpositions, respectively. Denote by $\mathcal{E}_n$ and $\mathcal{O}_n$ the sets of even permutations and odd permutations of $[n]$, respectively. Then $\mathcal{S}_n = \mathcal{E}_n \cup \mathcal{O}_n$, $|\mathcal{E}_n| = \frac{n!}{2} = |\mathcal{O}_n|$ and S_n has bipartition $(\mathcal{E}_n, \mathcal{O}_n)$. For convenience, the vertices in $\mathcal{E}_n$ and $\mathcal{O}_n$ are call *even* and *odd* vertices, respectively. For $i, k \in [n]$ with $k \neq 1$, define

$$\mathcal{S}^i_{n,k} := \{x \in \mathcal{S}_n \mid x(k) = i\},$$
$$\mathcal{E}^i_{n,k} := \{x \in \mathcal{E}_n \mid x(k) = i\},$$
$$\mathcal{O}^i_{n,k} := \{x \in \mathcal{O}_n \mid x(k) = i\}.$$

Then we have a partition $\{\mathcal{S}^i_{n,k} \mid i \in [n]\}$ of the symmetric group $\mathcal{S}_n$. Note that $\mathcal{S}^k_{n,k}$ is in fact the subgroup of $\mathcal{S}_n$ fixing the symbol k, and so $\mathcal{S}^k_{n,k}$ is isomorphic to $\mathcal{S}_{n-1}$. In addition, for $i \neq k$, we have

$$\mathcal{S}^i_{n,k} = t_{k,i} \cdot \mathcal{S}^k_{n,k}, \;\; \mathcal{E}^i_{n,k} = t_{k,i} \cdot \mathcal{O}^k_{n,k}, \;\; \mathcal{O}^i_{n,k} = t_{k,i} \cdot \mathcal{E}^k_{n,k}.$$

Denote by $S^i_{n,k}$ the subgraph of S_n induced by $\mathcal{S}^i_{n,k}$. Then $S^i_{n,k}$ is a bipartite graph with bipartition $(\mathcal{E}^i_{n,k}, \mathcal{O}^i_{n,k})$. Moreover, the subgraph $S^k_{n,k}$ is a Cayley graph of the symmetric group on $[n] \setminus \{k\}$ generated by $\mathcal{T} \setminus \{t_{1,k}\}$, and $x \mapsto t_{k,i} \cdot x$ gives an isomorphism from $S^k_{n,k}$ to $S^i_{n,k}$. In particular, every $S^i_{n,k}$ is isomorphic to S_{n-1}.

An edge xy of S_n is called a *k-edge* if $x^{-1} \cdot y = t_{1,k}$, while x (or y) is called a *k-neighbor* of y (or x). The next lemma follows directly from the definition of S_n.

LEMMA 2.4. *Let $i, k \in [n]$ with $k \neq 1$, and $x \in \mathcal{S}^i_{n,k}$. Suppose that y, z are distinct neighbors of x in $S^i_{n,k}$. Then the k-neighbors of x, y and z are all distinct. In particular, the k-neighbor of x and the k-neighbors of the other neighbors of x are scattered into the distinct $n-1$ subgraphs $S^j_{n,k}$, where j runs over $[n] \setminus \{i\}$.*

For $i, j \in [n]$ with $j \neq i$, denote by $E^{i,j}_k$ the set of edges between $S^i_{n,k}$ and $S^j_{n,k}$, and put

$$E_k := \bigcup_{i,j \in [n], i \neq j} E^{i,j}_k = E(S_n) \setminus \bigcup_{i \in [n]} E(S^i_{n,k}).$$

Let $V(E^{i,j}_k)$ be the set of ends of edges in $E^{i,j}_k$. It is easily to see that E_k consists of all k-edges of S_n, $\{V(E^{i,j}_k) \cap \mathcal{O}^i_{n,k} \mid i \neq j \in [n]\}$ is a partition of $\mathcal{O}^i_{n,k}$, and $\{V(E^{i,j}_k) \cap \mathcal{E}^i_{n,k} \mid i \neq j \in [n]\}$ is a partition of $\mathcal{E}^i_{n,k}$. Moreover, the following lemma holds, refer to [6, Theorem 2.1].

LEMMA 2.5. *Let $k, i, j \in [n]$ with $n \geq 3$, $k \neq 1$ and $i \neq j$. Then*
(1) *$S^i_{n,k}$ is isomorphic to the $(n-1)$-star graph S_{n-1}; and*
(2) *E_k indices a perfect matching of S_n, and $E_{k_1} \cap E_{k_2} = \emptyset$ for distinct $k_1, k_2 \in [n] \setminus \{1\}$; and*

(3) $|V(S^i_{n,k}) \cap \mathcal{E}_n| = \frac{(n-1)!}{2} = |V(S^i_{n,k}) \cap \mathcal{O}_n|$, *and* $|V(E^{i,j}_k) \cap \mathcal{E}^i_{n,k}| = \frac{(n-2)!}{2} = |V(E^{i,j}_k) \cap \mathcal{O}^i_{n,k}|$.

LEMMA 2.6. *Let* $H \subset E(S_n)$ *with* $|H| \le n-2$. *Then there exists an automorphism* ϕ *of* S_n *such that* $\phi(\mathcal{E}_n) = \mathcal{E}_n$, $\phi(\mathcal{O}_n) = \mathcal{O}_n$ *and* $\phi(H)$ *contains no* n*-edges.*

Proof. First, we observe from (2) of Lemma 2.5 that $E_k \cap H = \emptyset$ for some $k \ge 2$. If $k = n$ then the lemma is true by letting ϕ be the identity transformation of $\mathcal{S}_n$. Now let $k \ne n$. Considering the conjugation of $t_{i,j}$ on $\mathcal{S}_n$, we have an automorphism of S_n, say $\phi : x \mapsto t_{i,j} \cdot x \cdot t_{i,j}$. Then $\phi(\mathcal{E}_n) = \mathcal{E}_n$, $\phi(\mathcal{O}_n) = \mathcal{O}_n$ and $\phi(E_k) = E_n$, and so the lemma follows. ◻

To simplify the notation, we write $S^i_{n,n}$ as S^i_n, $\mathcal{S}^i_{n,n}$ as $\mathcal{S}^i_n$, $\mathcal{E}^i_{n,n}$ as $\mathcal{E}^i_n$, and $\mathcal{O}^i_{n,n}$ as $\mathcal{O}^i_n$, respectively. Then S^i_n has bipartition $(\mathcal{E}^i_n, \mathcal{O}^i_n)$.

LEMMA 2.7. *Let* $i, j \in [n]$ *with* $i \ne j$, *and let* $H \subset E(S_n) \setminus E_n$ *with* $|H| \le n-4$, *for* $n \ge 4$. *Then* $E^{i,j}_n$ *contains at least* $\frac{(n-2)!}{2} - |H|$ *edges ends in* $\mathcal{E}^i_n \cup \mathcal{O}^j_n \setminus V(H)$, *and at least* $\frac{(n-2)!}{2} - |H|$ *edges with ends in* $\mathcal{O}^i_n \cup \mathcal{E}^j_n \setminus V(H)$.

Proof. By Lemma 2.5, $E^{i,j}_n$ contains $\frac{(n-2)!}{2}$ many n-edges that have ends in $\mathcal{E}^i_n \cup \mathcal{O}^j_n$ (resp., $\mathcal{O}^i_n \cup \mathcal{E}^j_n$). Let $H_k = H \cap E(S^k_n)$, and $h_k = |H_k|$, where $k \in \{i, j\}$. Then $V(H_i)$ contributes at most h_i even (resp., odd) ends and $V(H_j)$ contributes at most h_j odd (resp., even) ends for the edges in $E^{i,j}_n$. Thus $E^{i,j}_n$ contains at least $\frac{(n-2)!}{2} - (h_i + h_j)$ edges in $E^{i,j}_n$ with ends in $\mathcal{E}^i_n \cup \mathcal{O}^j_n \setminus V(H)$ (resp., $\mathcal{O}^i_n \cup \mathcal{E}^j_n \setminus V(H)$). Noting that $h_i + h_j \le |H|$, the lemma follows. ◻

3. Consequences of Theorem 1.1. In the next section, we shall proceed by induction on n to prove Theorem 1.1. For this purpose, let's take a look at some conclusions which are deduced from Theorem 1.1 and in turn play an important role in the induction process. Thus, in this section, suppose that $S_n - M_0 - F_0$ is hyper-Hamilton laceable, where M_0 is an arbitrary matching of S_n and F_0 is an arbitrary subset of $E(S_n - M_0)$ such that $|E(M_0)| + |F_0| \le n - 4$.

In this and next sections, for a path P and its two vertices u and v, the symbol $P[u, v]$ means the path on P between u and v.

LEMMA 3.1. *Let* M *be a matching of* S_n *and* F *be a subset of* $E(S_n - M)$. *Suppose that* $n \ge 4$ *and* $|E(M)| + |F| \le n - 3$. *Then* $S_n - M - F$ *is Hamiltonian.*

Proof. If $|F| = 0$ then, by Lemma 2.3(2), $S_n - M$ is Hamiltonian, and the lemma is true. Now let $|F| \ne 0$, and pick $xy \in F$. Since $|E(M)| + |F| \le n - 3$, there exist two neighbors z and w of y in $S_n - M$ such that neither yz nor yw lies in F. Choosing $F_0 = F \setminus \{xy\}$. Noth that $|E(M)| + |F_0| \le n - 4$ and since $S_n - M - F_0$ is hyper-Hamilton laceable, there exists a Hamilton path $P[z, w]$ of $S_n - M - F_0 - y$. Then $S_n - M - F$ has a desired Hamilton cycle constructed by $P[z, w]$, yz and yw. Thus the lemma follows. ◻

Proof of Corollary 1.2. Let M be a matching of S_n, F be a subset of $E(S_n - M)$, and L be a linear forest in $S_n - M - F$. Suppose that $n \ge 4$ and $|E(M)| + |F| + |E(L)| \le n - 4$. Pick two vertices x, y with opposite parity and compatible with L. Since $|E(L)| + |E(M) + |F| \le n - 4$, there exists a neighbor z of y in $S_n - M$ such that $yz \notin F$, and either $yz \in E(L)$ or $z \notin V(L)$. Let L_0 be the subgraph of L induced by $E(L) \setminus \{yz\}$. Then $\{x, z\}$ and L_0 are compatible in $S_n - M - F - y$. By the assumption, $S_n - M - F - y$ has a Hamilton path $P[x, z]$ passing through L_0. Thus $S_n - M - F$ has a desired Hamilton path constructed by $P[x, z]$ and zy, which passes through L. This completes the proof.

Proof of Corollary 1.3. Note that prescribed hyper-Hamilton laceability leads to hyper-Hamilton laceability. Let M be a matching of S_n, F be a subset of $E(S_n - M)$, and L be a linear forest in $S_n - M - F$. Suppose that $n \geq 4$ and $|E(M)| + |F| + |E(L)| \leq n-3$. If $E(L) = \emptyset$ then, by Lemma 3.1, $S_n - M - F$ has a Hamilton cycle passes through L. Assume that $E(L) \neq \emptyset$, and pick $xy \in E(L)$ with x not an inner vertex of L. Let L_0 be the subgraph of L induced by $E(L) \setminus \{xy\}$. Then $\{x, y\}$ and L_0 are compatible in $S_n - M - F$. Noting that $|E(M)| + |F| \leq (n-3) - |E(L)| \leq n-4$ and $|L_0| \leq n - 4 - |E(M)| - |F|$, by Corollary 1.2, there exists a Hamilton path $P[x, y]$ of $S_n - M - F$ that passes through L_0. Then $S_n - M - F$ has a desired Hamilton cycle constructed by $P[x, y]$ and xy, which passes through L. This completes the proof.

4. The proof of Theorem 1.1. We proceed by induction on n. By Lemma 2.1, the result is trivial when $n = 4$, so suppose that $n \geq 5$, and Theorem 1.1 holds for the $(n-1)$-star graph S_{n-1}.

Let M be a matching of S_n with m edges, and F an f-subset F of $E(S_n - M)$ such that

$$m + f \leq n - 4.$$

Pick a linear forest L of $S_n - M - F$ with $|E(L)| = \ell$ such that

$$\ell \leq n - 4 - m - f.$$

Put

$$H := E(M) \cup F \cup E(L). \tag{4.1}$$

Then $|H| \leq n-4$. In view of Lemma 2.6, replacing $S_n - M - F$ and L by their images under some automorphism of S_n, we may suppose that H contains no n-edges, i.e.,

$$H \subseteq E(S_n) - E_n = \bigcup_{i \in [n]} E(S_n^i) = E(S_n) - \bigcup_{i \neq j} E_n^{i,j}. \tag{4.2}$$

4.1. Symbols and lemmas. For each $i \in [n]$, put $F_i := F \cap E(S_n^i)$, denote by M_i and L_i the subgraphs of M and L contained in S_n^i, respectively. Define

$$f_i := |F_i|,\ m_i := |E(M_i)|,\ \ell_i := |E(L_i)|. \tag{4.3}$$

Since $m + f + \ell \leq n - 4$, we have the following observation.

Lemma 4.1. *Either*

(1) $\ell_i \leq n - 5 - m_i - f_i$ *for all* $i \in [n]$*; or*

(2) *there exists a unique* $j \in [n]$ *such that* $M_j = M$*,* $F_j = F$*,* $L_j = L$ *and* $\ell = n - 4 - m - f$*.*

For a vertex $u \in \mathcal{S}_n$, we write

$$i_u := u(n), \tag{4.4}$$

and denote by $\bar{u}$ the n-neighbor of u in S_n, i.e.,

$$\bar{u} = u \cdot t_{1,n}. \tag{4.5}$$

Lemma 4.2. *Let* $i \in [n]$*, and let* E *be the edge set of some linear forest of* $S_n^i - M_i - F_i$ *with* $|E| > 3n-12$*. Let* n' *be the number of edges* $uv \in E$ *such that* $uv \notin E(L)$*,* $\bar{u}, \bar{v} \notin V(M)$ *and* $\{\bar{u}, \bar{v}\}$ *is compatible with* L*. Suppose that* E *contains no* n*-edges. Then* $n' \geq |E| + 2 - 2m - 2\ell - \ell_i \geq |E| - 3n + 14$*.*

Proof. Let $V_0 = \{u \in V(E) \mid \bar{u} \in V(M)\}$, and let $E_0 = \{uv \in E \mid \bar{u}, \bar{v} \notin V(M)\} \setminus E(L_i)$. For $u_1, u_2 \in V_0$, since $u_1\bar{u}_1$ and $u_2\bar{u}_2$ are n-edges, if $u_1 \neq u_2$ then $\bar{u}_1 \neq \bar{u}_2$ by (2) of Lemma 2.5. Moreover, for $uv \in E$, by Lemma 2.4, $i_{\bar{u}} \neq i_{\bar{v}}$. It follows that $|E_0| \geq |E| - 2m - \ell_i$. Further, each edge uv in E_0 if exists is a desired edge in this lemma, or else at least one of $\bar{u}$ and $\bar{v}$ is an inner vertex of L. Noting that L has at most $\ell - 1$ inner vertices, we deduce that $n' \geq |E_0| - 2(\ell - 1)$. Since $m + f + \ell \leq n - 4$, we have $n' \geq |E| - 2m - \ell_i - 2(\ell - 1) \geq |E| - 3n + 14$. ◻

For $I \subseteq [n]$, denote by S_n^I the subgraph of S_n induced by $\mathcal{S}_n^I := \cup_{i \in I}(\mathcal{S}_n^i)$, and put

$$M_I := S_n^I \cap M, \; F_I := E(S_n^I) \cap F, \; L_I := S_n^I \cap L.$$

LEMMA 4.3. *Let $I \subseteq [n]$ with $|I| \geq 2$, and $i, j \in I$ with $i \neq j$ for $n \geq 5$. Let $u, v \in S_n^I - M_I$ with $u \in \mathcal{O}_n^i$ and $v \in \mathcal{E}_n^j$ such that $\{u, v\}$ is compatible with L. Suppose that $\ell_i \leq n - 5 - m_i - f_i$ for all $i \in I$. Then $S_n^I - M_I - F_I$ has a Hamilton path $P_I[u, v]$ passing through L_I.*

Proof. Write $I = \{i_1, i_2, \ldots, i_s\}$, where $s = |I|$, $i_1 = i$ and $i_s = j$. Let $u_1 = u$ and $v_s = v$. For each integer k with $1 \leq k \leq s - 1$, by Lemma 2.7, we may choose $v_k u_{k+1} \in E^{i_k, i_{k+1}}$ with $v_k \in \mathcal{E}_n^{i_k} \setminus V(H)$ and $u_{k+1} \in \mathcal{O}_n^{i_{k+1}} \setminus V(H)$. By the induction hypothesis and Corollary 1.2, $S_n^{i_k} - M_{i_k} - F_{i_k}$ has a Hamilton path $P_k[u_k, v_k]$ passing through L_{i_k}, where $1 \leq k \leq s$. Then $S_n^I - M_I - F_I$ has a desired Hamilton path constructed by $P_1[u_1, v_1]$, $v_1 u_2$, $P_2[u_2, v_2]$, …, $v_{s-1} u_s$ and $P_s[u_s, v_s]$. ◻

LEMMA 4.4. *Let $I \subseteq [n]$ with $|I| \geq 3$ and $i, j \in I$ for $n \geq 5$. Let $u, v \in S_n^I$ with $u \in \mathcal{O}_n^i$ and $v \in \mathcal{E}_n^j$. Then S_n^I has a Hamilton path $P_I[u, v]$.*

Proof. If $i \neq j$, then, by Lemma 4.3, a desired Hamilton path is guaranteed. Now let $i = j$. Choose $u_0 v_0 \in S_n^i$ with $i_{\bar{u}_0}, i_{\bar{v}_0} \in I \setminus \{i\}$. Clearly, by Lemma 2.4, we have $i_{\bar{u}_0} \neq i_{\bar{v}_0}$. By Lemma 2.2, S_n^i has a Hamilton path $P_1[u, v]$ that passes $u_0 v_0$ with u_0 lying between u and v_0. Set $I' = I \setminus \{i\}$. Lemma 4.3 guarantees a Hamilton path $P_{I'}[\bar{u}_0, \bar{v}_0]$ of $S_n^{I'}$. Then S_n^I has a desired Hamilton path constructed by $P_1[u, u_0]$, $u_0 \bar{u}_0$, $P_{I'}[\bar{u}_0, \bar{v}_0]$, $\bar{v}_0 v_0$, and $P_1[v_0, v]$. ◻

LEMMA 4.5. *Let $I \subseteq [n]$, and $l, i, j \in I$ with $i \neq j, l$. Let $u, v, z \in S_n^I - M_I$ with $u \in \mathcal{O}_n^i$, $v \in \mathcal{O}_n^j$ and $z \in \mathcal{E}_n^l \setminus V(L)$ such that $\{u, v\}$ is compatible with L. Suppose that $\ell_i \leq n - 5 - m_i - f_i$ for all $i \in I$. Then $S_n^I - M_I - F_I - z$ has a Hamilton path $P_I[u, v]$ passing through L_I.*

Proof. Suppose first that $l = j$ and $I = \{i, j\}$. Choose $u_0 \bar{u}_0 \in E^{i,j}$ with $u_0, \bar{u}_0 \notin V(H)$, $u_0 \in \mathcal{E}_n^i$ and $\bar{u}_0 \in \mathcal{O}_n^j$. By the induction hypothesis, $S_n^j - M_j - F_j - z$ has a Hamilton path $P_1[\bar{u}_0, v]$ passing through L_j. And by Corollary 1.2, $S_n^i - M_i - F_i$ has a Hamilton path $P_2[u, u_0]$ passing through L_i. Then $S_n^I - M_I - F_I - z$ has a desired Hamilton path constructed by $P_2[u, u_0]$, $u_0 \bar{u}_0$ and $P_1[\bar{u}_0, v]$.

Suppose that $l = j$ and $|I| > 2$. Fix a $k \in I \setminus \{i, j\}$. Choose $v_0 \bar{v}_0 \in E^{k,j}$ with $v_0, \bar{v}_0 \notin V(H)$, $v_0 \in \mathcal{E}_n^k$ and $\bar{v}_0 \in \mathcal{O}_n^j$. Let $P_3[\bar{v}_0, v]$ be a Hamilton path of $S_n^j - M_j - F_j - z$ passing through L_j. Set $I' = I \setminus \{j\}$. By Lemma 4.3, $S_n^{I'} - M_{I'} - F_{I'}$ has a Hamilton path $P_{I'}[u, v_0]$ passing through $L_{I'}$. We obtain a desired Hamilton path of $S_n^I - M_I - F_I - z$, which is constructed by $P_{I'}[u, v_0]$, $v_0 \bar{v}_0$ and $P_3[\bar{v}_0, v]$.

Finally, let $l \neq j$. Choose $v_1 \bar{v}_1 \in E^{l,j}$ with $v_1, \bar{v}_1 \notin V(H)$, $v_1 \in \mathcal{O}_n^l$ and $\bar{v}_1 \in \mathcal{E}_n^j$, and choose a Hamilton path $P_4[\bar{v}_1, v]$ of $S_n^j - M_j - F_j$ passing through L_j. Set $I'' = I \setminus \{j\}$. A similar argument as above implies that $S_n^{I''} - M_{I''} - F_{I''} - z$ has a Hamilton path $P_{I''}[u, v_1]$ passing through $L_{I''}$. Then $S_n^I - M_I - F_I - z$ has a desired Hamilton path constructed by $P_{I''}[u, v_1]$, $v_1 \bar{v}_1$ and $P_4[\bar{v}_1, v]$. This completes the proof. ∎

Let $w \in \mathcal{E}_n \setminus V(L)$, and $x, y \in \mathcal{O}_n \setminus V(M)$ such that $\{x, y\}$ is compatible with L in $S_n - M - F$. For convenience, define

$$I_0 := \{i_x, i_y, i_w\},\ J_0 := \{j \in [n] \mid \ell_j = n - 4 - m - f\}\}. \tag{4.6}$$

Then $|J_0| \leq 1$ by Lemma 4.1.

LEMMA 4.6. *Suppose that $i_x = i_y \neq i_w$ and $I_0 \cap J_0 = \emptyset$. Then there exist $u\bar{u} \in E_n^{i_x,i_w}$ and a neighbor u' of $\bar{u}$ such that*

(1) *$u, \bar{u}, u' \notin V(H) \cup \{x, y\}$, $u \in \mathcal{O}_n^{i_w}$, $\bar{u} \in \mathcal{E}_n^{i_x}$ and $u' \in \mathcal{O}_n^{i_x}$; and*
(2) *$S_n^{i_x} - M_{i_x} - F_{i_x} - \bar{u}$ has a Hamilton path $P_u[x, y]$ passing through L_{i_x}; and*
(3) *$P_u[x, y]$ has an edge $u'z$ with the n-neighbor $\bar{z}$ of z neither an inner vertex of L nor a vertex of $V(M)$, and $i_{\bar{z}} \notin J_0 \cup \{i_w\}$.*

Proof. Let $E = \{u\bar{u} \in E_n^{i_x,i_w} \mid u, \bar{u} \notin V(H), u \in \mathcal{O}_n^{i_w}, \bar{u} \in \mathcal{E}_n^{i_x}\}$. Then $|E| \geq \frac{(n-2)!}{2} - (n-4) \geq 2$. For each $u\bar{u} \in E$, by the induction hypothesis, $S_n^{i_x} - M_{i_x} - F_{i_x} - \bar{u}$ has a Hamilton path $P_u[x, y]$ passing through L_{i_x}. Note that $\bar{u}$ has $n - 2$ (odd) neighbors in $S_n^{i_x}$, and we hope that these neighbors are not contained in $V(H) \cup \{x, y\}$. Since $m_{i_x} + f_{i_x} + \ell_{i_x} \leq n-5$, there are at least $n' = (\frac{(n-2)!}{2} - (n-4))(n-2) - (n-5) - 2$ distinct vertices u' satisfying the above hopes, and further produce at least n' vertex-disjoint 2-path $u\bar{u}u'$.

Let $v_k u'_k z_k$ be 2-path on $P_{u_k}[x, y]$ with mid vertex u'_k, where $1 \leq k \leq n'$. By Lemma 2.4, $i_{\bar{v}_k} \neq i_{\bar{z}_k}$ and $i_x \notin \{i_{\bar{v}_k}, i_{\bar{z}_k}\}$. Recalling that $|J_0| \leq 1$, without loss of generality, we let $i_{\bar{z}_k} \notin J_0$. Noting that $\bar{u}_k u'_k z_k$ is 2-path of $S_n^{i_x}$, by Lemma 2.4, $i_{\bar{z}_k} \neq i_{u_k} = i_w$, where $1 \leq k \leq n'$. In particular, $\bar{z}_k \neq u_k$. Then we have n' distinct 4-paths $u_k\bar{u}_k u'_k z_k \bar{z}_k$, and these paths produce at least $n'' := \lceil \frac{n'}{n-2} \rceil$ distinct edges $z_k\bar{z}_k$, say $z_1\bar{z}_1, \ldots, z_{n''}\bar{z}_{n''}$ without loss of generality. Note that every $\bar{z}_k$ is an odd vertex.

On the other hand, recalling that $m + \ell \leq n - 4$, we deduce that at most $n - 4$ either inner vertices of L or vertices of M is odd. Since $n \geq 5$, calculation shows that $n'' > n - 4$. In particular, for $n = 5$, we have $n^{''} = 2 > 5 - 4 = 1$. Thus, the above construction process produces $u\bar{u}$, $P_u[x, y]$, $u'z$ and $\bar{z}$, which are desired as in the lemma. ∎

LEMMA 4.7. *Suppose that $I_0 \cap J_0 = \emptyset$. Let $I = [n] \setminus J_0$. Then $S_n^I - M_I - F_I - w$ has a Hamilton path $P_I[x, y]$ passing through L_I.*

Proof. Noting that $m_i + f_i + \ell_i \leq n - 5$ for all $i \in I$, if $i_x \neq i_y$ then the result is true by Lemma 4.5. Thus the left cases are $i_x = i_y = i_w$, and $i_x = i_y \neq i_w$.

Case 1. Assume that $i_x = i_y = i_w$. By the induction hypothesis, $S_n^{i_w} - M_{i_w} - F_{i_w} - w$ has a Hamilton path $P_1[x, y]$ which passes through L_{i_w}. Note that $P_1[x, y]$ has length at least $(n-1)! - 1 - 2(n-5) - 1$, which is larger than $3n - 12$. By Lemma 4.2, we may choose an edge uv on $P_1[x, y]$ with $uv \notin E(L)$, $\bar{u}, \bar{v} \notin V(M)$ and $\{\bar{u}, \bar{v}\}$ is compatible with L. Without loss of generality, we assume that u lies between x and v, where $x = u$ is allowed. By Lemma 2.4, $i_{\bar{u}}$, $i_{\bar{v}}$ and i_w are pairwise distinct integers. Let $J = I \setminus \{i_w\}$. Since u and v has opposite parity, so do $\bar{u}$ and $\bar{v}$. By Lemma 4.3, $S_n^J - M_J - F_J$ has a Hamilton path $P_J[\bar{u}, \bar{v}]$ passing through L_J. Then we obtain a desired Hamilton path of $S_n^I - M^I - F^I - w$, which is constructed by $P_1[x, u]$, $u\bar{u}$, $P_J[\bar{u}, \bar{v}]$, $\bar{v}v$ and $P_1[v, y]$.

Case 2. Assume that $i_x = i_y \neq i_w$. Choose $u\bar{u} \in E_n^{i_x,i_w}$, $P_u[x, y]$, $u'z$ and $\bar{z}$ which are described as in Lemma 4.6. Note that $i_{\bar{z}} \notin J_0 \cup \{i_x, i_w\}$, and $u, \bar{z} \in \mathcal{O}_n$. Let

$J = I \setminus \{i_x\}$. By Lemma 4.5, $S_n^J - M_J - F_J - w$ has a Hamilton path $P_J[u, \bar{z}]$ passing through L_J. If z is located on $P_u[x, y]$ between x and u' then $S_n^I - M^I - F^I - w$ has a desired Hamilton path constructed by $P_u[x, z]$, $z\bar{z}$, $P_J[\bar{z}, u]$, $u\bar{u}u'$ and $P_u[u', y]$. If z is located on $P_u[x, y]$ between y and u' then $S_n^I - M^I - F^I - w$ has a desired Hamilton path constructed by $P_u[x, u']$, $u'\bar{u}u$, $P_J[u, \bar{z}]$, $\bar{z}z$ and $P_u[z, y]$. This completes the proof. ∎

It is not difficult to prove the mirror versions of Lemmas 4.5, 4.6 and 4.7 for even vertices compatible with L in $S_n - M - F$. In particular, we have the following lemma.

Lemma 4.8. *Let $u, v, z \in S_n^I - M_I$ with $u, v \in \mathcal{E}_n$ and $z \in \mathcal{O}_n \setminus V(L)$ such that $\{u, v\}$ is compatible with L in S_n. Suppose that $\{i_u, i_v, i_z\} \cap J_0 = \emptyset$, and let $I \subseteq [n] \setminus J_0$. Then $S_n^I - M_I - F_I - z$ has a Hamilton path $P_I[u, v]$ passing through L_I if either*

(1) $\{i_u, i_v, i_z\} \subseteq I$ and $i_u \neq i_v, i_z$; or

(2) $I = [n] \setminus J_0$.

4.2. The proof. Let $x, y, w \in \mathcal{S}_n - V(M)$ such that $\{x, y\}$ is chosen from one partite set of $S_n - M$ and compatible with L in $S_n - M - F$, while w lies in the other partite set and $w \notin V(L)$. Our task is to prove that

(†) $S_n - M - F - w$ has a Hamilton path with ends x, y and passing through L.

If either $m = \ell = 0$ or $f = \ell = 0$ then (†) follows from Lemma 2.1 or 2.3, respectively. Thus, in the following, we suppose that

$$m + \ell > 0,\ f + \ell > 0. \tag{4.7}$$

Now choose $z_1 = \imath$, the identity of $\mathcal{S}_n$, or $z_1 = t_{w(n),n}$ depending on whether $w(n) = n$ or not, and choose $z_2 = \imath$ or $t_{j,k}$ depending on whether $z_1 {\cdot} w \in \mathcal{E}_n$ or not, where $j, k \in [n] \setminus \{i, n\}$. Let $z = z_2 {\cdot} z_1$. We have $z {\cdot} w \in \mathcal{E}_n^n$. Let $\tilde{z}$ be the automorphism of S_n induced by z by left multiplication. Then $\tilde{z}$ permutes the subgraphs $S_n^1, S_n^2, \ldots, S_n^n$, and so $\tilde{z}(H) \subseteq E(S_n) - E_n$. Clearly, $\tilde{z}(w) \in \mathcal{E}_n^n \setminus V(\tilde{z}(L))$, $\tilde{z}(x), \tilde{z}(y) \in \mathcal{O}_n$, $\tilde{z}(M)$ is a matching of size m and $\{\tilde{z}(x), \tilde{z}(y)\}$ is compatible with $\tilde{z}(L)$ in $S_n - \tilde{z}(M) - \tilde{z}(F)$. In view of these observations, we may let

$$x, y \in \mathcal{O}_n,\ w \in \mathcal{E}_n^n. \tag{4.8}$$

Thus, it suffices to prove that (†) holds under the assumptions (4.2), (4.7) and (4.8).

Recall that $I_0 = \{i_x, i_y, n\}$, $J_0 = \{j \in [n] \mid \ell_j = n - 4 - m - f\}$ and $|J_0| \leq 1$. If $J_0 = \emptyset$ then, by Lemma 4.7, we are done. Thus we suppose further that $J_0 \neq \emptyset$, and set

$$J_0 = \{j_0\}.$$

Then

$$m_{j_0} = m,\ f_{j_0} = f,\ \ell_{j_0} = \ell,\ m + f + \ell = n - 4.$$

By the induction hypothesis, applying Lemmas 3.1 (if $\ell = 0$) and Corollary 1.3 (if $\ell \neq 0$), $S_n^{j_0} - M - F$ has a Hamiltonian cycle, which passes through L. Next we discuss in five cases: $j_0 \notin I_0$; $n \neq j_0 \in \{i_x, i_y\}$; $n = j_0 \notin \{i_x, i_y\}$; $n = j_0 \in \{i_x, i_y\}$ and $i_x \neq i_y$; and $n = j_0 = i_x = i_y$.

Lemma 4.9. *If $j_0 \notin I_0$ then (†) holds.*

Proof. Suppose that $j_0 \notin I_0$. By Lemma 2.7, there exists $u_0\bar{u}_0 \in E_n^{j_0,n}$ with $\bar{u}_0 \notin V(H)$, $u_0 \in \mathcal{O}_n^n \setminus \{x, y\}$ and $\bar{u}_0 \in \mathcal{E}_n^{j_0}$. Let C be a Hamiltonian cycle of $S_n^{j_0} - M - F$ passing through L, and let $u\bar{u}_0v$ be the 2-path on C with mid vertex

$\bar{u}_0$. Clearly, $\bar{u}_0 v, \bar{u}_0 u \notin E(L)$. By Lemma 2.4, $i_{\bar{v}} \neq i_{\bar{u}}$ and $i_{\bar{v}}, i_{\bar{u}} \notin \{j_0, n\}$. Thus, we let $i_{\bar{v}} \neq i_y$ in the following.

Case 1. Suppose that $i_x \neq i_y$. Without loss of generality, we may assume that (i) $i_x = n$, or (ii) $i_x = i_{\bar{v}}$, or (iii) $i_x \notin \{n, i_{\bar{v}}\}$.

1.1. Suppose that $i_x = n$. By Lemma 2.1, $S_n^n - w$ has a Hamilton path $P_1[x, u_0]$. Let $I_1 = [n] \setminus \{n, j_0\}$. Note that $i_{\bar{v}} \neq i_y$. Lemma 4.3 yields a Hamilton path $P_{I_1}[\bar{v}, y]$ of $S_n^{I_1}$. Then $S_n - M - F - w$ has desired Hamilton path constructed by $P_1[x, u_0]$, $u_0\bar{u}_0$, $C - \bar{u}_0 v$, $v\bar{v}$ and $P_{I_1}[\bar{v}, y]$.

1.2. Suppose that $i_x = i_{\bar{v}}$. Lemma 2.1 guarantees a Hamilton path $P_2[x, \bar{v}]$ of $S_n^{i_x}$. Let $I_2 = [n] \setminus \{j_0, i_x\}$. By Lemma 4.7, $S_n^{I_2} - w$ has a Hamilton path $P_{I_2}[\bar{u}_0, y]$. Then $S_n - M - F - w$ has desired Hamilton path constructed by $P_2[x, \bar{v}]$, $\bar{v}v$, $C - v\bar{u}_0$, $\bar{u}_0 u_0$ and $P_{I_2}[\bar{u}_0, y]$.

1.3. Suppose that $i_x \notin \{n, i_{\bar{v}}\}$. Let $I_3 = \{n, i_y\}$ and $I_4 = [n] \setminus \{j_0, n, i_y\}$. By Lemma 4.7, $S_n^{I_3} - w$ has a Hamilton path $P_{I_3}[u_0, y]$. Note that $i_x \neq i_{\bar{v}}$. Lemma 4.3 guarantees a Hamilton path $P_{I_4}[x, \bar{v}]$ of $S_n^{I_4}$. Then $S_n - M - F - w$ has desired Hamilton path constructed by $P_{I_4}[x, \bar{v}]$, $\bar{v}v$, $C - v\bar{u}_0$, $\bar{u}_0 u_0$ and $P_{I_3}[u_0, y]$.

Case 2. Suppose that $i_x = i_y < n$. We need consider either $i_{\bar{u}} = i_y$ or $i_{\bar{u}} \neq i_y$.

2.1. Suppose that $i_{\bar{u}} = i_y$. Choose $u\bar{u} \in E_n^{j_0, i_x}$, $P_u[x, y]$, $u'z$ and $\bar{z}$ which are described as in Lemma 4.6. Without loss of generality, we assume that z lies between u' and y on $P_u[x, y]$. In particular, $i_{\bar{z}} \notin \{j_0, n\}$. Note that $\bar{z}, u_0 \in \mathcal{O}_n$ and $i_{\bar{z}} \neq i_{u_0}$. Let $I_1 = [n] \setminus \{j_0, i_x\}$. By Lemma 4.5, $S_n^{I_1} - M_{I_1} - F_{I_1} - w$ has a Hamilton path $P_{I_1}[u_0, \bar{z}]$. Then we obtain a desired Hamilton path of $S_n - M - F - w$, which is constructed by $P_u[x, u']$, $u'\bar{u}u$, $C - u\bar{u}_0$, $\bar{u}_0 u_0$, $P_{I_1}[u_0, \bar{z}]$, $\bar{z}z$ and $P_u[z, y]$.

2.2. Suppose that $i_{\bar{u}} \neq i_y$. Choose $u_1\bar{u}_1 \in E_n^{i_x, n}$, $P_{u_1}[x, y]$, $u_1'z$ and $\bar{z}$ which are described as in Lemma 4.6. Without loss of generality, we assume that z lies between u_1' and y on $P_{u_1}[x, y]$. In particular, $i_{\bar{z}} \notin \{j_0, n\}$. Choose a Hamilton path $P_3[u_0, u_1]$ of $S_n^n - w$, and let $I_2 = [n] \setminus \{i_x, j_0, n\}$. Since $i_{\bar{v}} \neq i_{\bar{u}}$, swapping u and v if necessary, we suppose $i_{\bar{z}} \neq i_{\bar{v}}$. Noting that $\bar{v} \in \mathcal{E}_n$ and $\bar{z} \in \mathcal{O}_n$, by Lemma 4.3, $S_n^{I_2}$ has a Hamilton path $P_{I_2}[\bar{v}, \bar{z}]$. Then we have a desired Hamilton path of $S_n - M - F - w$ constructed by $P_{u_1}[x, u_1{}']$, $u_1{}'\bar{u}_1 u_1$, $P_3[u_1, u_0]$, $u_0\bar{u}_0$, $C - \bar{u}_0 v$, $v\bar{v}$, $P_{I_2}[\bar{v}, \bar{z}]$, $\bar{z}z$ and $P_{u_1}[z, y]$.

Case 3. Suppose that $i_x = i_y = n$. By Lemma 2.1, we choose a Hamilton path $P_4[x, y]$ of $S_n^n - w$. Let $u_0 v_0$ be the edge on $P_4[x, y]$ with v_0 lying between u_0 and y. By Lemma 2.4, $i_{\bar{v}_0} \notin \{n, j_0\}$. Since $i_{\bar{v}} \neq i_{\bar{u}}$, swapping v and u if necessary, we suppose $i_{\bar{v}_0} \neq i_{\bar{v}}$. Let $I = [n] \setminus \{j_0, n\}$. Noting that $\bar{v} \in \mathcal{E}_n$ and $\bar{v}_0 \in \mathcal{O}_n$, by Lemma 4.3, S_n^I has a Hamilton path $P_I[\bar{v}, \bar{v}_0]$. Then a desired Hamilton path of $S_n - M - F - w$ is constructed by $P_4[x, u_0]$, $u_0\bar{u}_0$, $C - \bar{u}_0 v$, $v\bar{v}$, $P_I[\bar{v}, \bar{v}_0]$, $\bar{v}_0 v_0$ and $P_4[v_0, y]$. ∎

LEMMA 4.10. *If $n \neq j_0 \in \{i_x, i_y\}$ then (†) holds.*

Proof. Suppose that $n \neq j_0 \in \{i_x, i_y\}$. Without of generality, we let $j_0 = i_x$. Let C be a Hamiltonian cycle of $S_n^{j_0} - M - F$ passing through L. Let $y_0 x x_0$ be the 2-path on C with mid vertex x. By Lemma 2.4, $i_{\bar{x}_0}, i_{\bar{y}_0} \neq i_x = j_0$ and $i_{\bar{x}_0} \neq i_{\bar{y}_0}$. Since x is not an inner vertex of L, we assume that $xx_0 \notin E(L)$. Put $P_0[x, x_0] := C - xx_0$.

Case 1. Assume that $j_0 = i_x \neq i_y$. Let $I = [n] \setminus \{j_0\}$. By Lemma 4.7, either $y = \bar{x}_0$, or $S_n^I - w$ has a Hamiltonian path $P_I[\bar{x}_0, y]$. For the latter case, $S_n - M - F - w$ has a desired Hamiltonian path constructed by $P_0[x, x_0]$, $x_0\bar{x}_0$ and $P_I[\bar{x}_0, y]$.

Suppose now that $y = \bar{x}_0$. Then $i_{\bar{y}_0} \neq i_y$. If $xy_0 \notin E(L)$ then our result holds by a similar argument as above. Now assume that $xy_0 \in E(L)$. Let $L_0 = L - x$. Then $m + f = n - 4 - \ell \leq n - 5$, and $|L_0| + m + f = n - 5$. Let x_0' be a neighbor of x in $S_n^{i_x} - M - F$ with $x_0' \notin V(L_0) \cup \{x_0, y_0\}$. By the hypothesis induction, $S_n^{i_x} - M - F - x$ has a Hamilton path $P_1[y_0, x_0']$ passing through L_0. By Lemma 2.4, we have $i_{\bar{x}_0'} \neq i_{\bar{x}_0} = i_y$. Considering the Hamilton cycle constructed by $P_1[y_0, x_0']$ and y_0xx_0', our result follows from a similar argument as in the first paragraph of the case.

Case 2. Assume that $j_0 = i_x = i_y < n$. Let x_1yy_1 be the 2-path on $P_0[x, x_0]$ with y_1 lying between y_0 and y. Then one of x_1y and yy_1 is not contained in $E(L)$.

Suppose first that $yy_1 \notin E(L)$. Noting that $x_0\bar{x}_0$ and $y_1\bar{y}_1$ are distinct n-edges of S_n, we deduce from Lemma 2.5 that $\bar{x}_0 \neq \bar{y}_1$. Let $I = [n] \setminus \{j_0\}$. By Lemma 4.7, $S_n^I - w$ has a Hamiltonian path $P_I[\bar{y}_1, \bar{x}_0]$. Then $S_n - M - F - w$ has a desired Hamiltonian path constructed by $P_0[x, y_1]$, $y_1\bar{y}_1$, $P_I[\bar{y}_1, \bar{x}_0]$, $\bar{x}_0x_0$ and $P_0[x_0, y]$.

Suppose now that $yy_1 \in E(L)$. Then $x_1y \notin E(L)$. If $xy_0 \notin E(L)$ then our result is true by a similar argument as above. Thus we suppose further that $xy_0 \in E(L)$; in particular, $\ell \geq 2$ and $m + f \leq n - 6$. Let $F_0 = F \cup \{yy_1\}$, and let L_0 be subgraph of L induced by $E(L) \setminus \{xy_0, yy_1\}$. Then $m + |F_0| \leq n - 5$ and $|L_0| = n - 5 - m - |F_0|$. Clearly, $x \notin V(L_0)$, and $\{y_0, y_1\}$ is compatible with L_0 in $S_n^{j_0} - M - F_0$. By the induction hypothesis, $S_n^{j_0} - M - F_0 - x$ has a Hamilton path $P[y_0, y_1]$. Let z_0yz_1 be the 2-path on $P[y_0, y_1]$ with mid vertex y. Since yy_1 is not an edge on $P[y_0, y_1]$, we have $y_1 \notin \{z_0, z_1\}$. Without loss of generality, assume that z_0 lies between y_0 and y on $P[y_0, y_1]$. Let $I = [n] \setminus \{j_0\}$. Noting that $\bar{z}_0$ and $\bar{z}_1$ are distinct odd vertices, by Lemma 4.7, $S_n^I - w$ has a Hamiltonian path $P_I[\bar{z}_0, \bar{z}_1]$. Then $S_n - M - F - w$ has a desired Hamiltonian path constructed by xy_0, $P[y_0, z_0]$, $z_0\bar{z}_0$, $P_I[\bar{z}_0, \bar{z}_1]$, $\bar{z}_1z_1$, $P[z_1, y_1]$ and y_1y. ∎

LEMMA 4.11. *If $n = j_0 \notin \{i_x, i_y\}$ then (†) holds.*

Proof. Suppose that $n = j_0 \notin \{i_x, i_y\}$. Let C be a Hamiltonian cycle of $S_n^n - M - F$ passing through L, and let uwv be the 2-path on C. Since $w \notin V(L)$, we have $uw, wv \notin E(L)$. By Lemma 2.4, $i_{\bar{u}} \neq i_{\bar{v}}$ and $n \notin \{i_{\bar{u}}, i_{\bar{v}}\}$.

Case 1. Assume that $i_x \neq i_y$. Swapping u and v if necessary, we assume that (i) $\{i_{\bar{u}}, i_{\bar{v}}\} \cap \{i_x, i_y\} = \emptyset$, or (ii) $i_{\bar{u}} = i_x$ and $i_{\bar{v}} \neq i_y$, or (iii) $i_{\bar{u}} = i_x$ and $i_{\bar{v}} = i_y$.

1.1. Suppose that $\{i_{\bar{u}}, i_{\bar{v}}\} \cap \{i_x, i_y\} = \emptyset$. Let $I_1 = \{i_x, i_{\bar{u}}\}$ and $I_2 = [n] \setminus \{i_x, i_{\bar{u}}, n\}$. By Lemma 4.3, $S_n^{I_1}$ has a Hamilton path $P_{I_1}[x, \bar{u}]$, and $S_n^{I_2}$ has a Hamilton path $P_{I_2}[\bar{v}, y]$. Then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_{I_1}[x, \bar{u}]$, $\bar{u}u$, $C - w$, $v\bar{v}$ and $P_{I_2}[\bar{v}, y]$.

1.2. Suppose that $i_{\bar{u}} = i_x$ and $i_{\bar{v}} \neq i_y$. Let $I_3 = [n] \setminus \{i_x, n\}$. Then $S_n^{I_3}$ has a Hamilton path $P_{I_3}[\bar{v}, y]$ by Lemma 4.3, and $S_n^{i_x}$ has a Hamilton path $P_1[x, \bar{u}]$ by Lemma 2.1. Thus we obtain a desired Hamilton path of $S_n - M - F - w$, which is constructed by $P_1[x, \bar{u}]$, $\bar{u}u$, $C - w$, $v\bar{v}$ and $P_{I_3}[\bar{v}, y]$.

1.3. Suppose that $i_{\bar{u}} = i_x$ and $i_{\bar{v}} = i_y$. By Lemma 2.1, $S_n^{i_x}$ has a Hamilton path $P_2[x, \bar{u}]$. Let $I_4 = [n] \setminus \{i_x, n\}$. By Lemma 4.4, S_n^I has a Hamilton path $P_{I_4}[\bar{v}, y]$. Then $S_n - M - F - w$ has desired Hamilton path constructed by $P_2[x, \bar{u}]$, $\bar{u}u$, $C - w$, $v\bar{v}$ and $P_{I_4}[\bar{v}, y]$.

Case 2. Now let $i_x = i_y$. Without loss of generality, we may suppose that either $i_x = i_{\bar{u}}$ or $i_x \notin \{i_{\bar{u}}, i_{\bar{v}}\}$.

2.1. Suppose that $i_x = i_{\bar{u}}$. By Lemma 2.1, $S_n^{i_x} - \bar{u}$ has a Hamilton path $P_1[x, y]$. Let $u' \in \mathcal{O}_n^{i_x}$ be a neighbor of $\bar{u}$ in $S_n^{i_x}$, and $x_1u'y_1$ be the 2-path on $P_1[x, y]$ with mid vertex u' and y_1 lying between u' and y. Then $i_{\bar{x}_1} \neq i_{\bar{y}_1}$ by Lemma 2.4. In particular, $i_{\bar{x}_1} \neq i_{\bar{v}}$ or $i_{\bar{y}_1} \neq i_{\bar{v}}$. Without loss of generality, we let $i_{\bar{y}_1} \neq i_{\bar{v}}$. Considering the 2-path $\bar{u}u'y_1$ of $S_n^{i_x}$, by Lemma 2.4, we have $i_{\bar{y}_1} \neq n$. Let $I = [n] \setminus I_0$. By Lemma 4.4, S_n^I has a Hamilton path $P_I[\bar{v}, \bar{y}_1]$. Then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_1[x, u']$, $u'\bar{u}u$, $C - w$, $v\bar{v}$, $P_I[\bar{v}, \bar{y}_1]$, $\bar{y}_1y_1$ and $P_1[y_1, y]$.

2.2. Suppose that $i_x \notin \{i_{\bar{u}}, i_{\bar{v}}\}$. Fix $\bar{u}_0u_0 \in E^{i_x, i_{\bar{u}}}$ with $u_0 \in \mathcal{O}_n^{i_{\bar{u}}}$ such that neither x nor y is a neighbor of $\bar{u}_0$ in $S_n^{i_x}$. Choose a Hamilton path $P_2[u_0, \bar{u}]$ of $S_n^{i_{\bar{u}}}$, and a Hamilton path $P_3[x,y]$ of $S_n^{i_x} - \bar{u}_0$ by Lemma 2.1. Considering the n-neighbors of the $n-2$ neighbors of $\bar{u}_0$ in $S_n^{i_x}$, by Lemma 2.4, we may choose one neighbor z of $\bar{u}_0$ in $S_n^{i_x}$ with $i_{\bar{z}} = n$. Noting that $z \notin \{x,y\}$, let x_0zy_0 be the 2-path on $S_n^{i_x} - \bar{u}_0$ with mid vertex z. Then, by Lemma 2.4, $i_{\bar{x}_0} \neq i_{\bar{y}_0}$ and $i_x, i_{\bar{u}}, n \notin \{i_{\bar{x}_0}, i_{\bar{y}_0}\}$. Without loss of generality, we assume that $i_{\bar{x}_0} \neq i_{\bar{v}}$. Let $I = [n] \setminus \{i_x, i_{\bar{u}}, n\}$. Then, by Lemma 4.3, S_n^I has a Hamilton path $P_I[\bar{x}_0, \bar{v}]$. If x_0 lies between x and z on $P_3[x,y]$ then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_3[x,x_0]$, $x_0\bar{x}_0$, $P_I[\bar{x}_0,\bar{v}]$, $\bar{v}v$, $C-w$, $u\bar{u}$, $P_2[\bar{u},u_0]$, $u_0\bar{u}_0z$, and $P_3[z,y]$. If x_0 lies between z and y on $P_3[x,y]$ then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_3[x,z]$, $z\bar{u}_0u_0$, $P_2[u_0,\bar{u}]$, $\bar{u}u$, $C-w$, $v\bar{v}$, $P_I[\bar{v},\bar{x}_0]$, $\bar{x}_0x_0$ and $P_3[x_0,y]$. ∎

LEMMA 4.12. *If $n = j_0 \in \{i_x, i_y\}$ and $i_x \neq i_y$ then (†) holds.*

Proof. Suppose that $n = j_0 \in \{i_x, i_y\}$ and $i_x \neq i_y$. Without loss of generality, we let $i_y = n$. Let C be a Hamiltonian cycle of $S_n^n - M - F$ passing through L, and let uwv be the 2-path on C. Then $uw, wv \notin E(L)$, $i_{\bar{u}} \neq i_{\bar{v}}$ and $n \notin \{i_{\bar{u}}, i_{\bar{v}}\}$.

Suppose first that $y \in \{u,v\}$. Let $u = y$ without loss of generality. Pick $J = [n] \setminus \{n\}$. Note that $x \in \mathcal{O}_n$ and $\bar{v} \in \mathcal{E}_n$, by Lemma 4.4, S_n^J has a Hamilton path $P_J[x,\bar{v}]$, and then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_J[x,\bar{v}]$, $\bar{v}v$ and $(C-w)[v,y]$.

Next let $y \notin \{u,v\}$, and let u_0yv_0 be the 2-path on C with mid vertex y and u_0 lying between u and y. Then $i_{\bar{u}_0} \neq i_{\bar{v}_0}$ and $n \notin \{i_{\bar{u}_0}, i_{\bar{v}_0}\}$. Moreover, since y is not an inner vertex of L, at least one of u_0y and yv_0 is not contained in $E(L)$. We next discuss in two cases, say $i_x \in \{i_{\bar{u}}, i_{\bar{v}}\}$, and $i_x \notin \{i_{\bar{u}}, i_{\bar{v}}\}$.

Case 1. Assume that $i_x \in \{i_{\bar{u}}, i_{\bar{v}}\}$, and let $i_x = i_{\bar{u}}$ without loss of generality.

1.1. Suppose that $u_0y \notin E(L)$. Choose a Hamilton path $P_1[x,\bar{u}]$ of $S_n^{i_x}$, and let $I = [n] \setminus \{j_0, i_x\}$. For $i_{\bar{u}_0} \neq i_x$, since $\bar{u}_0 \in \mathcal{O}_n^{i_{\bar{u}_0}}$ and $\bar{v} \in \mathcal{E}_n^{i_{\bar{v}}}$, by Lemma 4.4, S_n^I has a Hamilton path $P_I[\bar{u}_0,\bar{v}]$, and then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_1[x,\bar{u}]$, $\bar{u}u$, $(C-w)[u,u_0]$, $u_0\bar{u}_0$, $P_I[\bar{u}_0,\bar{v}]$, $\bar{v}v$ and $(C-w)[v,y]$.

Now let $i_{\bar{u}_0} = i_x$. For $\bar{u}_0 = x$, letting $I_1 = [n] \setminus \{n\}$, by Lemma 4.8, $S_n^{I_1} - x$ has a Hamilton path $P_{I_1}[\bar{u},\bar{v}]$, and thus $S_n - M - F - w$ has a desired Hamilton path constructed by xu_0, $(C-w)[u_0,u]$, $u\bar{u}$, $P_{I_1}[\bar{u},\bar{v}]$, $\bar{v}v$ and $(C-w)[v,y]$. Thus suppose that $\bar{u}_0 \neq x$. By Lemma 2.4, we have $\bar{u}$ and $\bar{u}_0$ are not adjacent in $S_n^{i_x}$. Let $z\bar{u}_0$ be the edge on $P_1[x,\bar{u}]$ with z lying between $\bar{u}$ and $\bar{u}_0$. Let $I_2 = [n] \setminus \{j_0, i_x\}$. By Lemma 4.4, $S_n^{I_2}$ has a Hamilton path $P_{I_2}[\bar{v},\bar{z}]$, and then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_1[x,\bar{u}_0]$, $\bar{u}_0u_0$, $(C-w)[u_0,u]$, $u\bar{u}$, $P_1[\bar{u},z]$, $z\bar{z}$, $P_{I_2}[\bar{z},\bar{v}]$, $\bar{v}v$ and $(C-w)[v,y]$.

1.2. Suppose that $yv_0 \notin E(L)$ and $i_{\bar{v}_0} \neq i_x$. By Lemma 2.7, there exists $u_1\bar{u}_1 \in E_n^{i_x, i_{\bar{v}_0}}$ with $u_1 \in \mathcal{O}_n^{i_x} \setminus \{x\}$ and $\bar{u}_1 \in \mathcal{E}_n^{i_{\bar{v}_0}} \setminus \{\bar{v}\}$. Choose a neighbor $u_2 \in \mathcal{E}_n^{i_x}$ of u_1 with $i_{\bar{u}_2} \notin \{j_0, i_x, i_{\bar{v}_0}, i_{\bar{v}}\}$. By Lemma 2.2, $S_n^{i_x}$ has a Hamilton path $P_2[x,\bar{u}]$ that passes u_1u_2 with u_2 lying between x and u_1.

For $i_{\bar{v}} = i_{\bar{v}_0}$. Note that $\bar{u}_1 \neq \bar{v}$. Choose a neighbor $z \in \mathcal{E}_n^{i_{\bar{v}_0}}$ of $\bar{u}_1$ with $i_{\bar{z}} \notin \{j_0, i_x, i_{\bar{v}_0}, i_{\bar{u}_2}\}$. By Lemma 2.2, $S_n^{i_{\bar{v}}}$ has a Hamilton path $P_3[\bar{v},\bar{v}_0]$ that passes $\bar{u}_1z$ with z lying between $\bar{v}$ and $\bar{u}_1$. Let $I_3 = I \setminus \{i_x, j_0, i_{\bar{v}}\}$. Since $i_{\bar{z}} \neq i_{\bar{u}_2}$, by Lemma 4.3, $S_n^{I_3}$ has a Hamilton path $P_{I_3}[\bar{u}_2,\bar{z}]$, and then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_2[x,u_2]$, $u_2\bar{u}_2$, $P_{I_3}[\bar{u}_2,\bar{z}]$, $\bar{z}z$, $P_3[z,\bar{v}]$, $\bar{v}v$, $(C-w)[v,v_0]$, $v_0\bar{v}_0$, $P_3[\bar{v}_0,\bar{u}_1]$, $\bar{u}_1u_1$, $P_2[u_1,\bar{u}]$, $\bar{u}u$ and $(C-w)[u,y]$.

Now let $i_{\bar{v}} \neq i_{\bar{v}_0}$. By Lemma 2.1, $S_n^{i_{\bar{v}_0}}$ has a Hamilton path $P_4[\bar{v}_0,\bar{u}_1]$. Note that $i_{\bar{u}_2} \neq i_{\bar{v}}$. Let $I_4 = [n] \setminus \{i_x, j_0, i_{\bar{v}_0}\}$. By Lemma 4.3, $S_n^{I_4}$ has a Hamilton path $P_{I_4}[\bar{u}_2,\bar{v}]$, and then $S_n - M - F - w$ has a desired Hamilton path constructed by

$P_2[x,u_2]$, $u_2\bar{u}_2$, $P_{I_4}[\bar{u}_2,\bar{v}]$, $\bar{v}v$, $(C-w)[v,v_0]$, $v_0\bar{v}_0$, $P_4[\bar{v}_0,\bar{u}_1]$, $\bar{u}_1u_1$, $P_2[u_1,\bar{u}]$, $\bar{u}u$ and $(C-w)[u,y]$.

1.3. Suppose that $yv_0 \notin E(L)$ and $i_{\bar{v}_0} = i_x$. Let $I_5 = [n] \setminus \{j_0\}$. If $\bar{v}_0 = x$, by Lemma 4.7, $S_n^{I_5} - x$ has a Hamilton path $P_{I_5}[\bar{v},\bar{u}]$. In this time, a desired Hamilton path is constructed by xv_0, $(C-w)[v_0,v]$, $v\bar{v}$, $P_{I_5}[\bar{v},\bar{u}]$, $\bar{u}u$ and $(C-w)[u,y]$. Otherwise $\bar{v}_0 \neq x$. Choose a neighbor $x_1 \in \mathcal{E}_n^{i_x}$ of $\bar{v}_0$ with $i_{\bar{x}_1} \notin \{j_0, i_x, i_{\bar{v}}\}$. By Lemma 2.2, $S_n^{i_x}$ has a Hamilton path $P_5[x,\bar{u}]$ that passes $\bar{v}_0x_1$ with x_1 lying between $\bar{v}_0$ and $\bar{u}$. Note that $\bar{x}_1 \in \mathcal{O}_n$ and $\bar{v} \in \mathcal{E}_n$. Let $I_6 = [n] \setminus \{j_0, i_x\}$. By Lemma 4.3, $S_n^{I_6}$ has a Hamilton path $P_{I_6}[\bar{v},\bar{x}_1]$, and then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_5[x,\bar{v}_0]$, $\bar{v}_0v_0$, $(C-w)[v_0,v]$, $v\bar{v}$, $P_{I_6}[\bar{v},\bar{x}_1]$, $\bar{x}_1x_1$, $P_5[x_1,\bar{u}]$, $\bar{u}u$ and $(C-w)[u,y]$.

Case 2. Assume that $i_x \notin \{i_{\bar{u}}, i_{\bar{v}}\}$. Without loss of generality, we let $u_0y \notin E(L)$ and further assume that (i) $i_{\bar{u}_0} = i_{\bar{u}}$, or (ii) $i_{\bar{u}_0} = i_x$, or (iii) $i_{\bar{u}_0} \notin \{i_x, i_{\bar{u}}\}$.

1.1. Suppose that $i_{\bar{u}_0} = i_{\bar{u}}$. By Lemma 2.7, there exists $u_1\bar{u}_1 \in E_n^{i_{\bar{u}},i_x}$ with $u_1 \in \mathcal{O}_n^{i_{\bar{u}}} \setminus \{\bar{u}_0\}$ and $\bar{u}_1 \in \mathcal{E}_n^{i_x}$. Choose a neighbor $u_2 \in \mathcal{E}_n^{i_{\bar{u}}}$ of u_1 with $i_{\bar{u}_2} \notin \{j_0, i_x, i_{\bar{u}}, i_{\bar{v}}\}$. By Lemma 2.2, $S_n^{i_{\bar{u}}}$ has a Hamilton path $P_1[\bar{u},\bar{u}_0]$ that passes u_1u_2 with u_2 lying between $\bar{u}_0$ and u_1. And Lemma 2.1 guarantees a Hamilton path $P_2[x,\bar{u}_1]$ of $S_n^{i_x}$. Note that $i_{\bar{v}} \neq i_{\bar{u}_2}$. Let $I_1 = [n] \setminus \{j_0, i_x, i_{\bar{u}}\}$. By Lemma 4.3, $S_n^{I_1}$ has a Hamilton path $P_{I_1}[\bar{u}_2,\bar{v}]$, and then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_2[x,\bar{u}_1]$, $\bar{u}_1u_1$, $P_1[u_1,\bar{u}]$, $\bar{u}u$, $(C-w)[u,u_0]$, $u_0\bar{u}_0$, $P_1[\bar{u}_0,u_2]$, $u_2\bar{u}_2$, $P_{I_1}[\bar{u}_2,\bar{v}]$ and $(C-w)[v,y]$.

1.2. Suppose that $i_{\bar{u}_0} = i_x$. By Lemma 2.7, there exists $u_1\bar{u}_1 \in E_n^{i_{\bar{v}},i_x}$ with $u_1 \in \mathcal{O}_n^{i_{\bar{v}}}$ and $\bar{u}_1 \in \mathcal{E}_n^{i_x}$. Choose a neighbor $u_2 \in \mathcal{E}_n^{i_x}$ of $\bar{u}_0$ with $i_{\bar{u}_2} \notin \{j_0, i_x, i_{\bar{u}}, i_{\bar{v}}\}$. By Lemma 2.2, $S_n^{i_x}$ has a Hamilton path $P_3[x,\bar{u}_1]$ that passes $\bar{u}_0u_2$ with u_2 lying between $\bar{u}_0$ and $\bar{u}_1$. And Lemma 2.1 guarantees a Hamilton path $P_4[u_1,\bar{v}]$ of $S_n^{i_{\bar{v}}}$. Note that $i_{\bar{u}} \neq i_{\bar{u}_2}$. Let $I_2 = [n] \setminus \{j_0, i_x, i_{\bar{v}}\}$. By Lemma 4.3, $S_n^{I_2}$ has a Hamilton path $P_{I_2}[\bar{u},\bar{u}_2]$, and then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_3[x,\bar{u}_0]$, $\bar{u}_0u_0$, $(C-w)[u_0,u]$, $u\bar{u}$, $P_{I_2}[\bar{u},\bar{u}_2]$, $\bar{u}_2u_2$, $P_3[u_2,\bar{u}_1]$, $\bar{u}_1u_1$, $P_4[u_1,\bar{v}]$ and $(C-w)[v,y]$.

1.3. Suppose that $i_{\bar{u}_0} \notin \{i_x, i_{\bar{u}}\}$. Let $I_3 = \{i_{\bar{v}}, i_{\bar{u}_0}\}$ and $I_4 = [n] \setminus \{j_0, i_{\bar{v}}, i_{\bar{u}_0}\}$. By Lemma 2.1 or Lemma 4.3 depending on whether $i_{\bar{v}} = i_{\bar{u}_0}$ or not, $S_n^{I_3}$ has a Hamilton path $P_{I_3}[\bar{u}_0,\bar{v}]$. And Lemma 4.3 also guarantee a Hamilton path $P_{I_4}[\bar{u},x]$ of $S_n^{I_4}$. Then $S_n - M - F - w$ has a desired Hamilton path constructed by $P_{I_4}[x,\bar{u}]$, $\bar{u}u$, $(C-w)[u,u_0]$, $u_0\bar{u}_0$, $P_{I_3}[\bar{u}_0,\bar{v}]$, $\bar{v}v$ and $(C-w)[v,y]$. ∎

By Lemmas 4.9-4.12, to complete the proof Theorem 1.1 it remains to show that (†) holds when $n = j_0 = i_x = i_y$.

LEMMA 4.13. *If $n = j_0 = i_x = i_y$ then (†) holds.*

Proof. Suppose that $n = j_0 = i_x = i_y$. We discuss in two cases: $\ell = 0$, and $\ell > 0$.

Case 1. Assume that $\ell = 0$. Recalling that $\ell + f > 0$, we have $f > 0$. Pick $uv \in F$, and let $F_0 = F \setminus \{uv\}$. By the induction hypothesis, $S_n^n - M - F_0 - w$ has a Hamilton path $P_1[x,y]$. Fix an edge x_0y_0 on $P_1[x,y]$, and let $x_0y_0 = uv$ when uv lies on $P_1[x,y]$. Without loss of generality, we assume that x_0 is an even vertex and lies between x and y_0 on $P_1[x,y]$. Let $I = [n] \setminus J_0$. By Lemma 4.3, S_n^I has a Hamilton path $P_I[\bar{x}_0,\bar{y}_0]$, and thus $S_n - M - F - w$ has a desired Hamilton path constructed by $P_1[x,x_0]$, $x_0\bar{x}_0$, $P_I[\bar{x}_0,\bar{y}_0]$, $\bar{y}_0y_0$ and $P_1[y_0,y]$.

Case 2. Assume that $\ell \neq 0$. Then $m + f \leq n - 5$. Pick $x_1y_1 \in E(L)$ such that x_1 not an inner vertex of L, and let $L_0 = L - x_1$. Then, by the induction hypothesis, $S_n^n - M - F - w$ has a Hamilton path $P_2[x,y]$ passing through L_0. Since $n \geq 5$, we have $(n-1)! - 2(n-4) - 2 > 1$. Then we may choose an edge u_1v_1 on $P_2[x,y]$ with $u_1v_1 \notin E(L)$. If x_1y_1 lies on $P_2[x,y]$ then the lemma holds by a similar argument as in Case 1. Thus we suppose next that x_1y_1 is not an edge on $P_2[x, y]$.

By the choices of x, y and x_1y_1, we have $\{x_1, y_1\} \neq \{x, y\}$. Without loss of generality, we suppose further that x_1 is an even vertex on $P_2[x, y]$ lying between x and y_1. Let $x_3x_1x_2$ be the 2-path on $P_2[x, y]$ with mid vertex x_1 and x_3 lying between x and x_1, and let y_2y_1 be the edge on $P_2[x, y]$ with y_2 lying between x_2 and y_1. Then $y_2 \in \mathcal{E}_n^n$, and $x_2, x_3 \in \mathcal{O}_n^n$. Recalling x_1 is not an inner vertex of L, we have $x_2x_1, x_3x_1 \notin E(L)$.

Suppose that $y_2y_1 \notin E(L)$. Let $I_1 = [n] \setminus J_0$. By Lemma 4.4, S_n^I has a Hamilton path $P_{I_1}[\bar{x}_3, \bar{y}_2]$, and thus $S_n - M - F - w$ has a desired Hamilton path constructed by $P_2[x, x_3]$, $x_3\bar{x}_3$, $P_{I_1}[\bar{x}_3, \bar{y}_2]$, $\bar{y}_2y_2$, $P_2[y_2, x_1]$, x_1y_1 and $P_1[y_1, y]$.

Suppose that $y_2y_1 \in E(L)$. Then $y_1 \neq y$ as y is not an inner vertex of L. Let $y_2y_1y_3$ be the 2-path on $P_2[x, y]$ with mid vertex y_1. Since L is a linear forest, we have $y_3y_1 \notin E(L)$. Then a similar argument as above implies that $S_n - M - F - w$ has a desired Hamilton path. This completes the proof. □

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