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Rainbow and Gallai–Rado numbers involving binary function equations[☆]

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ABSTRACT

The rainbow number $\mathrm{rb}([n], \mathcal{E})$ is defined as the minimum number of colors such that for every exact $(\mathrm{rb}([n], \mathcal{E}))$-coloring of $[n]$, there exists a rainbow solution of the equation $\mathcal{E}$. The Gallai–Rado number $\mathrm{GR}_k(\mathcal{E}_1 : \mathcal{E}_2)$ is defined as the minimum positive integer N, if it exists, such that for all $n \geq N$, every k-colored $[n]$ contains either a rainbow solution of the equation $\mathcal{E}_1$ or a monochromatic solution of the equation $\mathcal{E}_2$. Our main results are providing some exact values of rainbow and Gallai–Rado numbers involving binary function equations. We also provide an algorithm to calculate the rainbow numbers of nonlinear binary function equations.

1. Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive natural numbers and $n \in \mathbb{N}$. For convenience, we denote the set $\{1, 2, \ldots, n\}$ as $[n]$. A *k-coloring* of a set $[n]$ is a function $\chi : [n] \to [k]$, where $[k]$ is the set of colors. We usually use different numbers to represent different colors, and when the number of colors is relatively small, we can use specific color names, such as red, blue, green, and so on. If χ is surjective, then we call the coloring χ *exact*. In other words, an exact coloring requires that each color be used at least once. Obviously, if χ is exact, then the number of colors k does not exceed the number of elements in the colored set $[n]$, that is, $k \leq n$. All the colorings we consider in this paper are exact. In this paper, we use $y = f(x)$ to represent a binary function equation. If x_0, y_0 is a solution of $y = f(x)$, then we usually write the solution as (x_0, y_0) or $(x_0, f(x_0))$.

1.1. Schur and Rado numbers

Ramsey theory is an important theory developed in the 20th century. In particular, Ramsey theory is widely used in graph theory and number theory, and is currently one of the hot and difficult research areas. In brief, Ramsey theory states that under sufficiently large structures, there must exist a substructure with specific properties. The main contribution is attributed to Ramsey, who published a pioneering paper [18] in 1930. In fact, the study of Ramsey theory predates 1930. In 1916, Schur gave a theorem, later known as Schur's theorem, which is one of the most important theorems in the early era of Ramsey theory. For a historical introduction to Ramsey theory, we refer to the first chapter of the monograph [25].

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Theorem 1.1 (*[24]*). *For each $k \in \mathbb{N}$, there is a positive integer $S(k)$ such that every k-colored $[S(k)]$ contains a monochromatic solution of the equation $x + y = z$.*

Generally speaking, if the colors of all numbers in a solution of an equation are the same, then we call the solution a *monochromatic solution*; if the colors of all numbers in a solution of an equation are different, then we call the solution a *rainbow solution*.

The $S(k)$ in Theorem 1.1 is called the *Schur number*. To date, the exact values of the Schur numbers only for $1 \le k \le 5$ have been determined. That is, $S(1) = 2$, $S(2) = 5$, $S(3) = 14$, $S(4) = 45$, and $S(5) = 161$. For details, we refer to [1,10].

Rado was one of the best Ph.D. students of Schur. In 1933, Rado generalized Schur's theorem to general linear equations in his Ph.D thesis [17]. Rado called a linear equation $\sum_{i=1}^{n} a_i x_i = b$ *k-regular* if there exists a monochromatic solution of the equation in any k-colored $\mathbb{N}$, where k is a positive integer. If a linear equation $\sum_{i=1}^{n} a_i x_i = b$ is k-regular for all $k \in \mathbb{N}$, then the equation is *regular*. According to the definition of regular equation and Theorem 1.1, the equation $x + y = z$ is regular. Rado gave a necessary and sufficient condition for judging whether a homogeneous linear equation is regular.

Theorem 1.2 (*[17]*). *A homogeneous linear equation $\sum_{i=1}^{n} a_i x_i = 0$ is regular if and only if there exists $I \subseteq [n]$ such that $\sum_{i\in I} a_i = 0$.*

Inspired by Rado, the definition of regularity of equations can be extended beyond linear equations, as shown in the following definition.

Definition 1.3. Let $k \in \mathbb{N}$. An equation $\mathcal{E}$ is k-regular if there exists a monochromatic solution of the equation in any k-colored $\mathbb{N}$. If an equation $\mathcal{E}$ is k-regular for all $k \in \mathbb{N}$, then the equation is regular.

Rado number is a generalization based on Schur number. The following is the definition of Rado number. Noticing that, unlike Schur number, given a positive integer k and an equation $\mathcal{E}$, the Rado number of $\mathcal{E}$ may not always exist.

Definition 1.4. Let $\mathcal{E}$ be an equation and $k \ge 2$ be an integer. The Rado number $\mathrm{R}_k(\mathcal{E})$ is defined as the minimum positive integer, if it exists, such that every k-colored $[\mathrm{R}_k(\mathcal{E})]$ contains a monochromatic solution of $\mathcal{E}$.

The following observation gives a basic method for proving the upper and lower bounds of the Rado number.

Observation 1. *Let $\mathcal{E}$ be an equation and $k \ge 2$ be an integer.*

- *If there exists an exact k-coloring of $[n-1]$ such that there is no monochromatic solution of $\mathcal{E}$, then $\mathrm{R}_k(\mathcal{E}) \ge n$.*
- *If every exact k-coloring of $[n]$ ensures that there is a monochromatic solution of $\mathcal{E}$, then $\mathrm{R}_k(\mathcal{E}) \le n$.*
- *Moreover, if there exists an exact k-coloring of $\mathbb{N}$ such that there is no monochromatic solution of $\mathcal{E}$, then $\mathrm{R}_k(\mathcal{E})$ does not exist.*

Solving the exact value or upper and lower bounds of the Rado number of an equation has always been a hot topic.

Many scholars have studied the Rado number of homogeneous linear equations. In 1997, Harborth and Maasberg [9] studied the Rado numbers of three variables equations $a(x + y) = 2z$ and $a(x + y) = (a + 1)z$ with two colors. In 2005, Hopkins and Schaal [11] studied the Rado number of multivariate equation $\sum_{i=1}^{m-1} a_i x_i = x_m$ with two colors. In 2008, Robertson and Myers [19] studied the Rado number of four variables equation $x + y + kz = \ell w$ with two colors. In 2008, Saracino and Wynne [21] studied the Rado number of four variables equation $x + y + kz = 3w$ with two colors. In 2016, Saracino [20] studied the Rado number of multivariate equation $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_k$ with two colors.

Many scholars have also conducted research on the Rado number of non-homogeneous linear equations. In 2001, Kosek and Schaal [14] studied the two colors Rado number of multivariate equation $\sum_{i=1}^{m-1} x_i + c = x_m$ for negative values of c. In 2004, Jones and Schaal [13] studied the Rado number of three variables equation $x + y + c = kz$ with two colors. In 2012, Schaal and Zinter [22] studied the Rado number of three variables equation $x_1 + 3x_2 + c = x_3$ with two colors. In 2020, Dwivedi and Tripathi [3] studied the Rado number of three variables equation $x_1 + ax_2 - x_3 = c$ with two colors. Recently, Yang, Mao, He, and Wang [26] studied the two colors Rado numbers for two different equations (i.e., the exact values of $\mathrm{R}_2(x + y = z, \ell x = y)$ and $\mathrm{R}_2(x + y = z, x + a = y)$), as well as other related results.

The problem of regularity for the nonlinear equation $x^2 + y^2 = z^2$ was proposed by Erdős. Graham [7] used to be very concerned about this problem. There is a reward of \$250 for the person who can solve this problem. In fact, there are many other related articles on the regularity of equations and Rado numbers that we not provide in this article.

1.2. Rainbow and Gallai–Rado numbers

In graph theory, a *monochromatic subgraph* refers to a subgraph where the colors of all edges are the same, while a *rainbow subgraph* refers to a subgraph where the colors of all edges are different. The definition of rainbow number in graph theory is relatively early, and we refer to the definition given by Schiermeyer [23] in 2004. However, the study of rainbow numbers in equations is later than that in graph theory, and we refer to the definition given by Fallon, Giles, Rehm, Wagner, and Warnberg [4] in 2020.

Definition 1.5 ([23]). Let G be a graph and $n \in \mathbb{N}$. The rainbow number $\mathrm{rb}(n, G)$ is defined as the minimum number of colors such that for every exact $(\mathrm{rb}(n, G))$-coloring of complete graph K_n, there exists a rainbow subgraph G.

Definition 1.6 ([4]). Let $\mathcal{E}$ be an equation and $n \in \mathbb{N}$. The rainbow number $\mathrm{rb}([n], \mathcal{E})$ is defined as the minimum number of colors such that for every exact $(\mathrm{rb}([n], \mathcal{E}))$-coloring of $[n]$, there exists a rainbow solution of $\mathcal{E}$.

Fallon, Giles, Rehm, Wagner, and Warnberg also gave the following observation, which is the basic method for proving the upper and lower bounds of the rainbow number.

Observation 2 ([4]). *Let $\mathcal{E}$ be an equation and $n \in \mathbb{N}$.*

- *If there exists an exact $(k-1)$-coloring of $[n]$ such that the equation $\mathcal{E}$ has no rainbow solution in $[n]$, then $\mathrm{rb}([n], \mathcal{E}) \geq k$.*
- *If every exact k-coloring of $[n]$ ensures that the equation $\mathcal{E}$ has a rainbow solution in $[n]$, then $\mathrm{rb}([n], \mathcal{E}) \leq k$.*

In 1967, Gallai's paper [6] for the first time revealed the structure of colored complete graphs without rainbow triangles, and thus gave rise to a new research direction in graph theory known as the *Gallai–Ramsey number*. Gallai's result was restated in [8] by the terminology of graphs.

Theorem 1.7 ([6,8]). *In any edge-colored complete graph without rainbow triangle, there exists a nontrivial partition of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In 2010, Faudree, Gould, Jacobson, and Magnant in [5] defined the Gallai–Ramsey number $\mathrm{gr}_k(G : H)$. For more information about Gallai–Ramsey numbers, we refer to the monograph [15]. In addition, there are the latest research results on the Gallai–Ramsey numbers, which we can refer to in the article [12].

Definition 1.8 ([5]). Given two non-empty graphs G and H, and $k \in \mathbb{N}$, define the Gallai–Ramsey number $\mathrm{gr}_k(G : H)$ to be the minimum integer N such that for all $n \geq N$, every k-edge-colored complete graph K_n contains either a rainbow subgraph G or a monochromatic subgraph H.

Due to the rapid development of the Gallai–Ramsey number in the past decade, it has become a hot research area in graph theory. Inspired by this problem, the definition of the *Gallai–Schur number* was proposed in Budden's paper [2] in 2020, introducing this type of problem from graph theory to number theory. Since the Gallai–Schur number only studies the equation $x + y = z$, it can be generalized to other equations, and similarly defined as the *Gallai–Rado number*. The Gallai–Rado numbers were first studied by Mao, Robertson, Wang, Yang, and Yang [16] (In this article, the authors refer to it as the Gallai–Schur triples). It can be said that the Gallai–Rado number is a generalized definition of the Gallai–Schur number.

Definition 1.9. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two equations, and $k \in \mathbb{N}$. The Gallai–Rado number $\mathrm{GR}_k(\mathcal{E}_1 : \mathcal{E}_2)$ is defined as the minimum positive integer N, if it exists, such that for all $n \geq N$, every k-colored $[n]$ contains either a rainbow solution of $\mathcal{E}_1$ or a monochromatic solution of $\mathcal{E}_2$.

For simplicity, without causing confusion, we sometimes say that $\mathcal{E}_1$ in $\mathrm{GR}_k(\mathcal{E}_1 : \mathcal{E}_2)$ is the *rainbow equation* and $\mathcal{E}_2$ is the *monochromatic equation*. Noticing that one of the biggest differences between the Gallai–Rado number and the Gallai–Ramsey number is that the Gallai–Rado number does not always exist. Next, we present an observation on the basic method for proving the upper and lower bounds of the Gallai–Rado number.

Observation 3. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two equations, and $k \in \mathbb{N}$.*

- *If there exists an exact k-coloring of $[N-1]$ such that there is neither a rainbow solution of $\mathcal{E}_1$ nor a monochromatic solution of $\mathcal{E}_2$, then $\mathrm{GR}_k(\mathcal{E}_1 : \mathcal{E}_2) \geq N$.*
- *If every exact k-coloring of $[n] (n \geq N)$ ensures that there is either a rainbow solution of $\mathcal{E}_1$ or a monochromatic solution of $\mathcal{E}_2$, then $\mathrm{GR}_k(\mathcal{E}_1 : \mathcal{E}_2) \leq N$.*
- *Moreover, if there exists an exact k-coloring of $\mathbb{N}$ such that there is neither a rainbow solution of $\mathcal{E}_1$ nor a monochromatic solution of $\mathcal{E}_2$, then $\mathrm{GR}_k(\mathcal{E}_1 : \mathcal{E}_2)$ does not exist.*

1.3. Article structure and main results

In Section 2, we introduce some new definitions that are important in Sections 3 and 4. For example, the λ-class defined in Section 2 is used to describe the colored structure of $[n]$ without rainbow solution of the equation $y = ax + b$.

In Section 3, we first prove the exact value of the rainbow number of the binary linear equation $y = ax + b$. Then, for nonlinear binary function equations, the formulas for their rainbow numbers are not easily given directly. Therefore, we define a parameter called the monochromatic parameter. Due to the close relationship between the monochromatic parameter we defined and the rainbow number, we also give an algorithm to solve the monochromatic parameters of nonlinear binary function equations.

In Section 4, we first give the general properties and related connections of the Rado numbers and Gallai–Rado numbers of the binary linear equation $y = ax + b$, and then we prove one of the main results, which gives the exact values of the Gallai–Rado numbers of the rainbow equation $y = x+b$ versus monochromatic general multivariate linear equations with the fixed number of colors b. Next, we keep the rainbow equation as $y = x + b$, consider the monochromatic nonlinear binary function equations, and prove the exact values of the Gallai–Rado numbers.

In Section 5, as a discussion for further research, we present two problems that need to be studied and solved in the future.

Our main results are presented as follows:

Theorem 1.10. *For integers $a \geq 1$, $b \geq 0$, and $n \geq a + b$, we have*

$$\mathrm{rb}([n], y = ax + b) = n - \left\lfloor \frac{n-b}{a} \right\rfloor + 1.$$

Theorem 1.11. *For integers $a_i \geq 1$ for all $i \in [t]$, $t \geq 1$, $b \geq 2$, and $c \geq 0$, we have*

$$\mathrm{GR}_b\left(y = x + b : y = \sum_{i=1}^{t} a_i x_i + c\right) = \sum_{i=1}^{t} (\lambda_{\min} + (i-1)b)\, a_i + c,$$

where $\lambda_{\min} = \min\left\{\lambda : \lambda \in [b] \text{ and } \left(\sum_{i=1}^{t} a_i\lambda + c - \lambda\right) \equiv 0 \pmod b\right\}$. Moreover, if $\lambda_{\min}$ does not exist, then $\mathrm{GR}_b\left(y = x + b : y = \sum_{i=1}^{t} a_i x_i + c\right)$ also does not exist.

Theorem 1.12. *Let the integers $a \geq 1$, $b \geq 0$, and $c \geq 2$. Then*

$$\mathrm{GR}_2\left(y = x + 2 : y = ax^c + b\right) \begin{cases} \text{does not exist}, & \text{if } a \text{ and } b \text{ are odd;} \\ = a + b, & \text{if } a \text{ and } b \text{ have different parity;} \\ = a \cdot 2^c + b, & \text{if } a \text{ and } b \text{ are even.} \end{cases}$$

Theorem 1.13. *For a strictly monotonically increasing binary function equation $y = f(x)$ such that $f(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$, and integer $b \geq 2$, we have*

$$\mathrm{GR}_b\left(y = x + b : y = f(x)\right) = f(x_{\min}),$$

where $x_{\min} = \min\left\{x \in \mathbb{N} : \frac{f(x)-x}{b} \text{ is an integer}\right\}$. Moreover, if $x_{\min}$ does not exist, then $\mathrm{GR}_b\left(y = x + b : y = f(x)\right)$ also does not exist.

2. Preliminaries

In the following definition, the λ-class $\mathcal{C}_{(y=f(x),\lambda)}$ we provide is crucial for some of our results.

Definition 2.1. Let $y = f(x)$ be a strictly monotonically increasing binary function equation such that $f(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$, and let the integer $n \geq f(1)$.

- For each $\lambda \in [n]$, we define the λ-class of $y = f(x)$ as

  $$\mathcal{C}_{(y=f(x),\lambda)} = \{\lambda, f(\lambda), f(f(\lambda)), \ldots\} \subseteq [n].$$

- The λ-class can also be defined as a subset of $\mathbb{N}$, that is,

  $$\mathcal{C}_{(y=f(x),\lambda)} = \{\lambda, f(\lambda), f(f(\lambda)), \ldots\} \subseteq \mathbb{N}$$

  for each $\lambda \in \mathbb{N}$.
- If all the numbers in $\mathcal{C}_{(y=f(x),\lambda)}$ are of the same color, then we call the set $\mathcal{C}_{(y=f(x),\lambda)}$ monochromatic, and the color of the set $\mathcal{C}_{(y=f(x),\lambda)}$ is the same as the color of the numbers in it.

Lemma 2.2. *Let the integers $a \geq 1$ and $b \geq 0$. If the recurrence of the sequence $\{x_i : i \geq 1\}$ is $x_{i+1} = ax_i + b$, then the general term is $x_{i+1} = a^{i+1}\left(\sum_{j=1}^{i} \frac{b}{a^{j+1}} + \frac{x_1}{a}\right)$.*

Proof. The recurrence of $\{x_i : i \geq 1\}$ can be rewritten as

$$\begin{aligned}\frac{x_{i+1}}{a^{i+1}} &= \frac{b}{a^{i+1}} + \frac{x_i}{a^i}\\ &= \frac{b}{a^{i+1}} + \frac{b}{a^i} + \frac{x_{i-1}}{a^{i-1}}\\ &\ \ \vdots\\ &= \sum_{j=1}^{i} \frac{b}{a^{j+1}} + \frac{x_1}{a}.\end{aligned}$$

The result thus follows. □

Based on Lemma 2.2 and Definition 2.1, we give an example of the λ-class of $y = ax + b$.

Example 1. For integers $a \geq 1$, $b \geq 0$, and $n \geq a + b$, and for each $\lambda \in [n]$, we have

$$\mathcal{C}_{(y=ax+b,\lambda)} = \left\{ a^i \left(\sum_{j=1}^{i-1} \frac{b}{a^{j+1}} + \frac{\lambda}{a} \right) : i \in \mathbb{N} \text{ and } \sum_{j=1}^{0} \frac{b}{a^{j+1}} \overset{\textbf{def}}{=} 0 \right\} \subseteq [n].$$

Next, we present the colored structure theorem for $[n]$ without rainbow solution of $y = f(x)$.

Theorem 2.3. *Let $y = f(x)$ be a strictly monotonically increasing binary function equation such that $f(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$, and let the integer $n \geq f(1)$. A colored set $[n]$ contains no rainbow solution of $y = f(x)$ if and only if for each $\lambda \in [n]$, the λ-class $\mathcal{C}_{(y=f(x),\lambda)} \subseteq [n]$ is monochromatic.*

Proof. Firstly, we assume that for each $\lambda \in [n]$, the λ-class $\mathcal{C}_{(y=f(x),\lambda)} \subseteq [n]$ is monochromatic. Let (x_0, y_0) be an arbitrary solution of the binary function equation $y = f(x)$ in $[n]$. Since $x_0, y_0 \in \mathcal{C}_{(y=f(x),x_0)} = \{x_0, f(x_0), f(f(x_0)), \ldots\} \subseteq [n]$, it follows that $[n]$ contains no rainbow solution of $y = f(x)$.

Next, we assume that the colored set $[n]$ contains no rainbow solution of $y = f(x)$. Thus, for each $\lambda \in [n]$, λ and $f(\lambda)$ are of the same color. Similarly, $f(\lambda)$ and $f(f(\lambda))$ must also be of the same color. According to the recursion, we get all the numbers in $\{\lambda, f(\lambda), f(f(\lambda)), \ldots\} \subseteq [n]$ are of the same color, which implies that the λ-class $\mathcal{C}_{(y=f(x),\lambda)} \subseteq [n]$ is monochromatic. The result thus follows. □

In order to maximize the number of colors used without the rainbow solution of $y = f(x)$ in the colored set $[n]$, based on Theorem 2.3, we construct a coloring of $[n]$ as follows.

Definition 2.4. Let χ be a coloring of $[n]$ that satisfies the following two conditions:

- For each $\lambda \in [n]$, the λ-class $\mathcal{C}_{(y=f(x),\lambda)} \subseteq [n]$ is monochromatic;
- For each pair of different λ_i and λ_j satisfies $\mathcal{C}_{(y=f(x),\lambda_i)} \cap \mathcal{C}_{(y=f(x),\lambda_j)} = \emptyset$, $\mathcal{C}_{(y=f(x),\lambda_i)}$ and $\mathcal{C}_{(y=f(x),\lambda_j)}$ are of different colors.

Let $y = ax + b$. We provide an example based on Definition 2.4.

Example 2. Let the integers $a \geq 1$, $b \geq 0$, and $n \geq a + b$. According to Definition 2.4, we provide the following coloring of $[n]$:

All integers in

$$\mathcal{C}_{(y=ax+b,1)} = \{1, a+b, a^2+ab+b, a^3+a^2b+ab+b, \ldots\} \subseteq [n]$$

have a color of 1. If $t < a + b$, then all integers in

$$\mathcal{C}_{(y=ax+b,t)} = \{t, ta+b, ta^2+ab+b, ta^3+a^2b+ab+b, \ldots\} \subseteq [n]$$

have a color of t. If $t = a + b$, then all integers in

$$\mathcal{C}_{(y=ax+b,t)} = \mathcal{C}_{(y=ax+b,a+b)} = \{a+b, a^2+ab+b, a^3+a^2b+ab+b, \ldots\} \subseteq [n]$$

have a color of 1.

For $a + b + 1 \leq t \leq n$, if $t \notin \mathcal{C}_{(y=ax+b,\lambda)}$ for all $\lambda \in [t]$, then assign a new color to $\mathcal{C}_{(y=ax+b,t)}$; if $t \in \mathcal{C}_{(y=ax+b,\lambda_0)}$, then the color assigned to $\mathcal{C}_{(y=ax+b,t)}$ is the same as the color of $\mathcal{C}_{(y=ax+b,\lambda_0)}$. In fact, we can use an algorithm to describe this coloring process. The detailed content is shown in Algorithm 1.

Algorithm 1 The coloring χ of $[n]$.

Input: Integers $a \geq 1$, $b \geq 0$, and $n \geq a+b$.
Output: A colored set $[n]$ without rainbow solution of $y = ax + b$.

Let $t = 1$ and $S = \emptyset$.
while $t \leq n$ **do**
  if $t \notin \mathcal{C}_{(y=ax+b,\lambda)}$ for all $\lambda \in S$ **then**
    assign a new color to t
    $S = S \cup \{t\}$, $t = t + 1$
  else
    the color assigned to t is the same as the color of $\mathcal{C}_{(y=ax+b,\lambda_0)}$, where $t \in \mathcal{C}_{(y=ax+b,\lambda_0)}$
    $t = t + 1$
  end if
end while

According to Theorem 2.3, the coloring χ of $[n]$ given in Definition 2.4 is the coloring that maximize number of colors used and ensure that $y = f(x)$ has no rainbow solution.

Corollary 2.5. *Let the integers $b \geq 2$ and $n \geq b+1$. A b-colored set $[n]$ contains no rainbow solution of $y = x + b$ if and only if for each $\lambda \in [b]$, the λ-class $\mathcal{C}_{(y=x+b,\lambda)} \subseteq [n]$ is monochromatic. Furthermore, for different λ_i and λ_j in $[b]$, $\mathcal{C}_{(y=x+b,\lambda_i)}$ and $\mathcal{C}_{(y=x+b,\lambda_j)}$ have different colors.*

In order to clearly demonstrate the relationship between the main results of this article, we provide the logical connections between definitions, theorems, observations, corollary, examples and algorithm in Fig. 1.

The meaning of the directional line segment labels in Fig. 1 is as follows:

① Example 2 is a special case of Definition 2.4, which can be directly derived from Definition 2.4.

② The coloring described in Example 2 can be used for the lower bound of Theorem 1.10.

③ From Definition 1.6, Observation 2 can be directly obtained.

④ Observation 2 provides a fundamental method for proving the upper and lower bounds of Theorem 1.10.

⑤ According to Theorem 2.3, we provide an important coloring in Definition 2.4.

⑥ Theorem 2.3 provides important coloring structure for proving the upper bound of Theorem 1.10.

⑦ Inspired by Theorem 1.10, further research can lead to Algorithm 2.

⑧ Example 4 is a special case of Algorithm 2 and an example of using Python to solve problems.

⑨ By using the Python code given in Example 4, we can use a computer to verify Theorem 1.10.

⑩, ⑪ Corollary 2.5 can be directly obtained from Theorem 2.3 and Definition 2.4.

⑫ From Definition 1.9, Observation 3 can be directly obtained.

⑬ Observation 3 provides a fundamental method for proving the upper and lower bounds of Theorem 1.11.

⑭ Corollary 2.5 provides important coloring structure for proving the lower bound of Theorem 1.11.

⑮ Observation 3 provides a fundamental method for proving the upper and lower bounds of Theorem 1.12.

⑯ Corollary 2.5 provides important coloring structure for proving the upper bound of Theorem 1.12.

⑰ Inspired by Theorem 1.12, further research can lead to Theorem 1.13.

3. Results for rainbow numbers

At first, we give the proof of Theorem 1.10.

Proof of Theorem 1.10. For the lower bound, we only need to construct a $\left(n - \left\lfloor \frac{n-b}{a} \right\rfloor\right)$-coloring of $[n]$, so that the equation $y = ax + b$ has no rainbow solution in $[n]$. In fact, the coloring we constructed is the coloring described in Example 2. Next, we calculate how many colors are used for this coloring. According to recursion, for each $\lambda \in [n]$, each number in $\mathcal{C}_{(y=ax+b,\lambda)} \setminus \{\lambda\}$ can be written as $ia + b\,(i \in \mathbb{N})$, so all numbers in the subset $\{ia + b : i \in \mathbb{N}\} \subseteq [n]$ cannot be assigned new colors. Therefore, we only need to calculate the number of numbers in $[n]$ that cannot be assigned new colors. For the convenience of counting, we list the following table. Numbers with lightgray boxes in the table cannot be assigned new colors, while other numbers are assigned different colors. It is easy to see that there are $\left\lfloor \frac{n-b}{a} \right\rfloor$ numbers that cannot be assigned new colors, that is, the coloring uses $n - \left\lfloor \frac{n-b}{a} \right\rfloor$ colors.

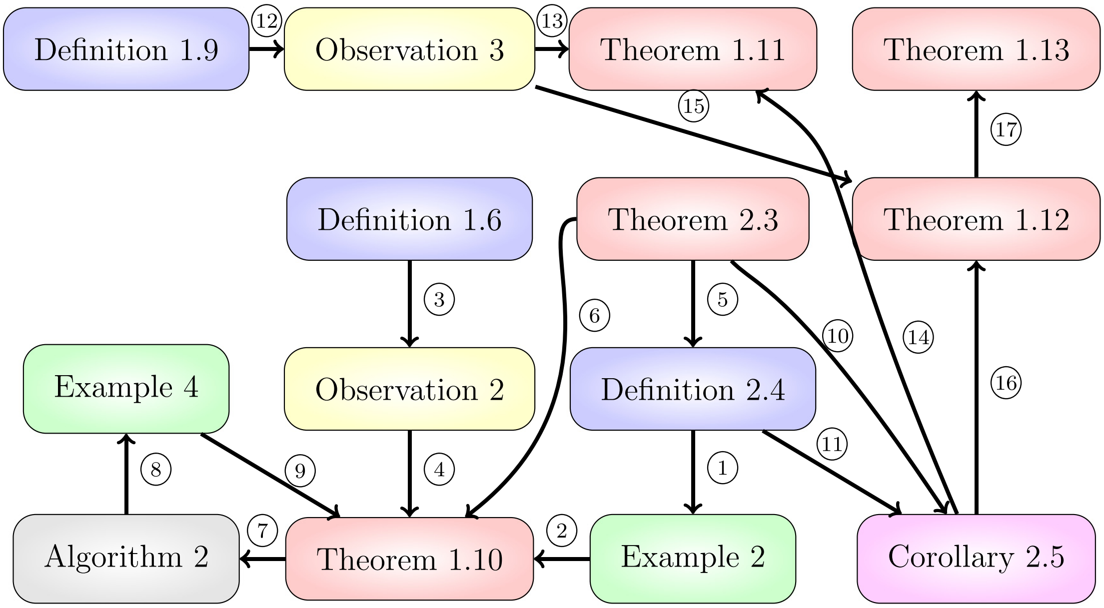

Fig. 1. The logical connections between definitions, theorems, observations, corollary, examples and algorithm.

Count the number of colors used for coloring χ						
1	…	b	$b+1$	…	$a+b-1$	$a+b$
	…		$a+b+1$	…	$2a+b-1$	$2a+b$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
	…		$(i-1)a+b+1$	…	$ia+b-1$	$ia+b$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
	…		…	n	…	…

For the upper bound, we arbitrarily color $[n]$ with $n-\left\lfloor\frac{n-b}{a}\right\rfloor+1$ colors. Recall that if there is no rainbow solution of $y=ax+b$ in $[n]$, then according to Theorem 2.3, each λ-class is monochromatic. Noticing that under the coloring χ constructed above, all the numbers in $\mathcal{C}_{(y=ax+b,\lambda)}$ have the same color for each $\lambda\in[n]$, and if each pair of different λ_i and λ_j satisfies $\mathcal{C}_{(y=ax+b,\lambda_i)}\cap\mathcal{C}_{(y=ax+b,\lambda_j)}=\emptyset$, $\mathcal{C}_{(y=ax+b,\lambda_i)}$ and $\mathcal{C}_{(y=ax+b,\lambda_j)}$ are of different colors. We have counted $n-\left\lfloor\frac{n-b}{a}\right\rfloor$ colors used for the coloring χ, and if an additional color is added, then according to pigeonhole principle, it will inevitably lead to the existence of an integer $\lambda_0\in[n]$, resulting in two numbers with different colors in $\mathcal{C}_{(y=ax+b,\lambda_0)}$, which implies that there is a rainbow solution of $y=ax+b$ in $[n]$. The result thus follows. □

We consider the rainbow number of binary function equations. Assuming that strictly monotonically increasing binary function equations satisfy $f(x)\in\mathbb{N}$ for all $x\in\mathbb{N}$. Thus, there is an inverse function $x=f^{-1}(y)$ for $y=f(x)$, and if y is not a positive integer, then $f^{-1}(y)$ must also be not a positive integer. If $y=f(x)$ has no solution in $[n]$, then naturally there is no rainbow solution. Therefore, it is meaningful to study the rainbow number of $y=f(x)$ only when $y=f(x)$ has at least one solution in $[n]$, that is, $n\geq f(1)$. Now we know that $y=f(x)$ has a solution in $[n]$, and the natural idea is that if all numbers in $[n]$ are assigned different colors, then there must be a rainbow solution of $y=f(x)$, but in this case, the number of colors is the largest. Next, we try to color some of the numbers in $[n]$ the same color in an attempt to reach the extreme value. Definition 3.1 is to define this extreme value characteristic.

Definition 3.1. Let $y=f(x)$ be a strictly monotonically increasing binary function equation such that $f(x)\in\mathbb{N}$ for all $x\in\mathbb{N}$, and integers $n\geq f(1)$ and $\mu\geq 1$. If there exists a $(n-\mu)$-coloring of $[n]$ such that $y=f(x)$ has no rainbow solution in $[n]$, but for all $(n-\mu+1)$-coloring of $[n]$, $y=f(x)$ always has a rainbow solution in $[n]$, then we call μ the monochromatic parameter with respect to $y=f(x)$ and n.

Based on Theorem 1.10 and Definition 3.1, we give an example of the monochromatic parameter μ with respect to $y=ax+b$ and n.

Example 3. Let the integers $a \geq 1$, $b \geq 0$, and $n \geq a+b$. The monochromatic parameter with respect to $y = ax + b$ and n is

$$\mu = \left\lfloor \frac{n-b}{a} \right\rfloor.$$

Combining Definition 3.1 and Observation 2, we directly provide the following observation.

Observation 4. *Let $y = f(x)$ be a strictly monotonically increasing binary function equation such that $f(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$, and integer $n \geq f(1)$. If μ is the monochromatic parameter with respect to $y = f(x)$ and n, then*

$$\mathrm{rb}([n], y = f(x)) = n - \mu + 1.$$

We can obtain the explicit expression for the monochromatic parameter μ with respect to $y = ax + b$ and n due to the linear properties of $y = ax + b$. However, for a nonlinear binary function equation $y = f(x)$ that satisfies strictly monotonically increasing and $f(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$, it is not easy to directly obtain the explicit expression of its monochromatic parameter μ. The Algorithm 2 we provide next can be applied to find the monochromatic parameters of the general nonlinear binary function equations.

Algorithm 2 Calculate the monochromatic parameter μ.

Input: A strictly monotonically increasing binary function equation $y = f(x)$ such that $f(1) \geq 2$ and $f(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$, and integer $n \geq f(1)$.
Output: The monochromatic parameter μ with respect to $y = f(x)$ and n.

1: Let $\mu = 0$, $t = \lfloor f^{-1}(n) \rfloor$, and $S = \emptyset$.
2: **while** $t \geq 1$ **do**
3:   **if** $t \notin S$ **then**
4:     $\mu = \mu + 1$
5:     **if** $f^{-1}(t) \geq 1$ is an integer **then**
6:       $s = t$
7:       **while** $f^{-1}(s) \geq 1$ is an integer **do**
8:         $S = S \cup \left\{s, f^{-1}(s)\right\}$, $\mu = \mu + 1$, $s = f^{-1}(s)$
9:       **end while**
10:       $t = t - 1$
11:     **else**
12:       $S = S \cup \{t\}$, $t = t - 1$
13:     **end if**
14:   **else**
15:     $t = t - 1$
16:   **end if**
17: **end while**
18: **return** μ.

Remark 1. In order for Algorithm 2 to run properly, the While-Do loop from step 7 to step 9 cannot be a dead loop, that is, there cannot be a $x_0 \in \mathbb{N}$ such that $f(x_0) = x_0$. In fact, we do not need to worry about this problem. Since $y = f(x)$ is strictly monotonically increasing, $f(1) \geq 2$ and $f(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$, it follows that $f(2) \geq 3$. Otherwise, this contradicts the strictly monotonically increasing of $y = f(x)$. According to recursion, for all $x \in \mathbb{N}$, $f(x) \geq x + 1$. Therefore, there is no such $x_0 \in \mathbb{N}$ that $f(x_0) = x_0$.

In addition, for binary function equations such that $f(1) = 1$, $f(2) \geq 3$, and strictly monotonically increasing (for example, $y = x^a$, where integer $a \geq 2$), we only need to modify some of the conditions for Algorithm 2, that is, to replace $t \geq 1$, $f^{-1}(t) \geq 1$, and $f^{-1}(s) \geq 1$ in steps 2, 5, and 7 with $t \geq 2$, $f^{-1}(t) \geq 2$, and $f^{-1}(s) \geq 2$, respectively.

Example 4. We implement Algorithm 2 in Python to solve $\mathrm{rb}([n], y = ax + b)$ for integers $a \geq 1$ and $b \geq 0$. The computational results of the computer are completely consistent with the exact values given in Theorem 1.10. Here is the complete Python code.

```
import math
a=int(input("Solving the rainbow number of y=ax+b in [n] \na="))
b=int(input("b="))
n=int(input("n="))
if n<a+b:
  print("Please re-enter n")
```

```
else:
  def f(y):
    return (y-b)/a
  mu=0
  t=math.floor(f(n))
  S={""}
  while t>=1:
    if t not in S:
      mu=mu+1
      if f(t)>=1 and isinstance(f(t),int):
          s=t
          while f(s)>=1 and isinstance(f(s),int):
           S=S.union({s,f(s)})
           mu=mu+1
           s=f(s)
      else:
         S=S.union({t})
         t=t-1
    else:
      t=t-1
rb=n-mu+1
print(f"The rainbow number of y={a}x+{b} in [{n}] is {rb}")
```

4. Results for Gallai–Rado numbers

We first prove that the Rado number for equation $y = ax + b$ does not exist, except for $y = x$.

Lemma 4.1. *Let the integers $k \geq r \geq 2$. If $\mathcal{E}$ is not r-regular, then $\mathrm{R}_k(\mathcal{E})$ does not exist.*

Proof. Since the equation $\mathcal{E}$ is not r-regular, there exists an r-coloring χ of $\mathbb{N}$, so that the equation $\mathcal{E}$ has no monochromatic solution in $\mathbb{N}$. For each integer $k \geq t$, we assign new colors to some numbers of the same color based on the r-coloring χ of $\mathbb{N}$, thereby constructing a k-coloring χ' of $\mathbb{N}$. Noticing that $\mathbb{N}$ is an infinite set, while k is a finite number of colors, so we can always add new colors to construct a k-coloring χ' of $\mathbb{N}$, which implies that the equation $\mathcal{E}$ is not k-regular. It follows from Observation 1 that $\mathrm{R}_k(\mathcal{E})$ does not exist. □

Theorem 4.2. *For integers $k \geq 2$, $a \geq 1$, $b \geq 0$, and $(a, b) \neq (1, 0)$, $\mathrm{R}_k(y = ax + b)$ does not exist.*

Proof. From Lemma 4.1, it is sufficient to show that $y = ax + b$ is not 2-regular. The red/blue-coloring of $\mathbb{N}$ constructed is as follows: For integer $i \geq 1$, we consider the following set and agree that $\sum_{j=1}^{0} \frac{b}{a^{j+1}} = 0$.

$$\mathcal{A}_i = \left\{ a^i \left(\sum_{j=1}^{i-1} \frac{b}{a^{j+1}} + \frac{1}{a} \right), a^i \left(\sum_{j=1}^{i-1} \frac{b}{a^{j+1}} + \frac{1}{a} \right) + 1, \ldots, a^{i+1} \left(\sum_{j=1}^{i} \frac{b}{a^{j+1}} + \frac{1}{a} \right) - 1 \right\}.$$

As can be seen, $\bigcup_{i=1}^{\infty} \mathcal{A}_i = \mathbb{N}$. When the positive integer i is odd, we color all the numbers in set $\mathcal{A}_i$ red; and when the positive integer i is even, we color all the numbers in set $\mathcal{A}_i$ blue. It is easy to verify that under this coloring, the equation $y = ax + b$ does not have a monochromatic solution in $\mathbb{N}$. The result thus follows. □

Obviously, with exact k-coloring, $\mathrm{R}_k(y = x) = k$ and there never be any rainbow solution of $y = x$, so the following theorem is straightforward.

Theorem 4.3. *For integer $k \geq 2$ and an arbitrary equation $\mathcal{E}$, we have*

$$\mathrm{GR}_k\,(y = x : \mathcal{E}) = \mathrm{R}_k(\mathcal{E}).$$

Let the integers $n \geq k \geq b + 1$ and $b \geq 1$. According to Corollary 2.5, when $k \geq b + 1$, any k-coloring of $[n]$, the equation $y = x + b$ must have a rainbow solution in $[n]$. The following theorem is directly given.

Theorem 4.4. *For integers $k \geq b + 1$, $b \geq 1$, and an arbitrary equation $\mathcal{E}$, we have*

$$\mathrm{GR}_k\,(y = x + b : \mathcal{E}) = k.$$

Next, we provide the proof of Gallai–Rado numbers involving monochromatic general linear equations.

Proof of Theorem 1.11. Let

$$N = \sum_{i=1}^{t} (\lambda_{\min} + (i-1)b)\, a_i + c$$

and

$$\mathcal{C}_{(y=x+b,\lambda)} = \{\lambda + (i-1)b : i \in \mathbb{N}\} \subseteq [N-1],$$

where

$$\lambda \in [b] \text{ and } \lambda_{\min} = \min\left\{\lambda : \ \lambda \in [b] \text{ and } \left(\sum_{i=1}^{t} a_i\lambda + c - \lambda\right) \equiv 0 \pmod b\right\}.$$

For the lower bound, we only need to construct a b-colored $[N-1]$ such that there is neither a rainbow solution of $y = x + b$ nor a monochromatic solution of $y = \sum_{i=1}^{t} a_i x_i + c$ in $[N-1]$. Since there is no rainbow solution of $y = x + b$ in $[N-1]$, it follows from Corollary 2.5 that for each $\lambda \in [b]$, the set $\mathcal{C}_{(y=x+b,\lambda)}$ is monochromatic. Therefore, the colored structure of $[N-1]$ is uniquely determined without considering the order of colors. Next, we only need to prove that $[N-1]$ does not contain a monochromatic solution of $y = \sum_{i=1}^{t} a_i x_i + c$.

To the contrary, if the equation $y = \sum_{i=1}^{t} a_i x_i + c$ has a monochromatic solution in $[N-1]$, then there exists $\lambda_0 \in [b]$ so that there are integers $x'_1, x'_2, \ldots, x'_t$ and $y' = \sum_{i=1}^{t} a_i x'_i + c$ are all in $\mathcal{C}_{(y=x+b,\lambda_0)}$. Since $x'_1, x'_2, \ldots, x'_t$ in $\mathcal{C}_{(y=x+b,\lambda_0)}$, it follows that there are positive integers $j_1, j_2, \ldots, j_t$ such that

$$\begin{aligned}
x'_1 &= \lambda_0 + j_1 b,\\
x'_2 &= \lambda_0 + j_2 b,\\
&\ \ \vdots\\
x'_t &= \lambda_0 + j_t b.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y' &= \sum_{i=1}^{t} a_i x'_i + c\\
&= \sum_{i=1}^{t} a_i(\lambda_0 + j_i b) + c\\
&= \sum_{i=1}^{t} a_i\lambda_0 + \left(\sum_{i=1}^{t} a_i j_i\right) \cdot b + c\\
&= \lambda_0 + \left(\sum_{i=1}^{t} a_i j_i\right) \cdot b + \sum_{i=1}^{t} a_i\lambda_0 + c - \lambda_0
\end{aligned}$$

in $\mathcal{C}_{(y=x+b,\lambda_0)}$ if and only if $\left(\sum_{i=1}^{t} a_i\lambda_0 + c - \lambda_0\right) \equiv 0 \pmod b$. We can see that if such λ_0 does not exist, then the equation $y = \sum_{i=1}^{t} a_i x_i + c$ cannot have a monochromatic solution in $\mathbb{N}$, which implies that $\mathrm{GR}_b\left(y = x + b : y = \sum_{i=1}^{t} a_i x_i + c\right)$ does not exist. If such λ_0 exists, then we choose the smallest one, $\lambda_{\min}$, to obtain

$$\begin{aligned}
y' &= \sum_{i=1}^{t} a_i x'_i + c\\
&\geq \lambda_{\min} a_1 + (\lambda_{\min} + b)\, a_2 + \cdots + (\lambda_{\min} + (t-1)b)\, a_t + c\\
&= \sum_{i=1}^{t} (\lambda_{\min} + (i-1)b)\, a_i + c = N,
\end{aligned}$$

which contradicts with $[N-1]$.

For the upper bound, we consider any b-colored $[n]\,(n \geq N)$. If the equation $y = x + b$ has a rainbow solution in $[n]$, then it is done. Otherwise, the equation $y = x + b$ does not have a rainbow solution in $[n]$, and in this case, the colored structure of $[n]$ is uniquely determined. This means that for each $\lambda \in [b]$, the set $\mathcal{C}_{(y=x+b,\lambda)} = \{\lambda + (i-1)b : i \in \mathbb{N}\} \subseteq [n]$ is monochromatic. As discussed above, if the defined $\lambda_{\min}$ does not exist, then $\mathrm{GR}_b\left(y = x + b : y = \sum_{i=1}^{t} a_i x_i + c\right)$ also does not exist. Therefore, we assume that $\lambda_{\min}$ exists. In this case, the equation $y = \sum_{i=1}^{t} a_i x_i + c$ has a monochromatic solution in $\mathcal{C}_{(y=x+b,\lambda_{\min})}$, where one of the monochromatic solution is

$$\begin{aligned} x_1 &= \lambda_{\min}, \\ x_2 &= \lambda_{\min} + b, \\ &\vdots \\ x_t &= \lambda_{\min} + (t-1)b, \\ y &= \sum_{i=1}^{t} a_i x_i + c = \sum_{i=1}^{t} (\lambda_{\min} + (i-1)b)\, a_i + c. \end{aligned}$$

The result thus follows. □

Regarding monochromatic nonlinear binary function equations, we prove the following results.

Proof of Theorem 1.12. We distinguish the following three cases to show this theorem.

Case 1. *a and b are odd.*

According to Observation 3, we only need to construct a red/blue-coloring of $\mathbb{N}$ such that there is neither a rainbow solution of $y = x + 2$ nor a monochromatic solution of $y = ax^c + b$. In fact, we only need to color all odd numbers red and all even numbers blue in $\mathbb{N}$. Under this coloring, there is clearly no rainbow solution of $y = x + 2$. Let (x_0, y_0) be any positive integer solution of $y = ax^c + b$. If x_0 is even, then x_0^c is also even, indicating that ax_0^c is also even. Since b is odd, it follows that $y_0 = ax_0^c + b$ is odd, which implies that the colors of x_0 and y_0 are different. If x_0 is odd, then x_0^c is also odd, indicating that ax_0^c is also odd. Since b is odd, it follows that $y_0 = ax_0^c + b$ is even, which implies that the colors of x_0 and y_0 are different. Therefore, there is no monochromatic solution of $y = ax^c + b$ in such red/blue-colored $\mathbb{N}$. The result thus follows.

Case 2. *a and b have different parity.*

For the lower bound, we color all odd numbers red and all even numbers blue in $[a + b - 1]$. Since $y = ax^c + b$ has no solution in $[a + b - 1]$, it follows that there is no monochromatic solution of $y = ax^c + b$ in $[a + b - 1]$. Under this coloring, there is no rainbow solution of $y = x + 2$. Thus, $\mathrm{GR}_2\,(y = x + 2 : y = ax^c + b) \geq a + b$.

For the upper bound, we consider any red/blue-coloring of $[a + b]$. To the contrary, suppose that there is a red/blue-coloring of $[a+b]$ such that there is neither a rainbow solution of $y = x+2$ nor a monochromatic solution of $y = ax^c + b$. Since there is no rainbow solution of $y = x+2$, it follows from Corollary 2.5 that without considering the order of colors, the coloring is uniquely determined. Without loss of generality, we assume that all odd numbers red and all even numbers blue in $[a+b]$. Noticing that due to the different parity of a and b, $a+b$ is odd. In this case, $(1, a+b)$ is a monochromatic solution of $y = ax^c + b$, which is a contradiction. The result thus follows.

Case 3. *a and b are even.*

For the lower bound, we color all odd numbers red and all even numbers blue in $[a\cdot 2^c + b - 1]$. Noticing that $y = ax^c + b$ has only one solution in $[a \cdot 2^c + b - 1]$, which is $(1, a + b)$. But since both a and b are even, $a + b$ is also even, that is, the solution $(1, a + b)$ is not a monochromatic solution. Also, under this coloring, there is no rainbow solution of $y = x + 2$. Thus, $\mathrm{GR}_2\,(y = x + 2 : y = ax^c + b) \geq a \cdot 2^c + b$.

For the upper bound, we consider any red/blue-coloring of $[a\cdot 2^c + b]$. To the contrary, suppose that there is a red/blue-coloring of $[a \cdot 2^c + b]$ such that there is neither a rainbow solution of $y = x + 2$ nor a monochromatic solution of $y = ax^c + b$. Since there is no rainbow solution of $y = x + 2$, it follows from Corollary 2.5 that without considering the order of colors, the coloring is uniquely determined. Without loss of generality, we assume that all odd numbers red and all even numbers blue in $[a \cdot 2^c + b]$. Since both a and b are even, it follows that $a \cdot 2^c + b$ is also even. In this case, $(2, a \cdot 2^c + b)$ is a monochromatic solution of $y = ax^c + b$, which is a contradiction. The result thus follows. □

It follows from Corollary 2.5 that a b-colored $\mathbb{N}$ contains no rainbow solution of $y = x+b$ if and only if for each $\lambda \in [b]$, the λ-class $\mathcal{C}_{(y=x+b,\lambda)} \subseteq \mathbb{N}$ is monochromatic. For example, $\mathcal{C}_{(y=x+2,1)}$ is the set of all positive odd numbers, and $\mathcal{C}_{(y=x+2,2)}$ is the set of all positive even numbers. Since $\mathcal{C}_{(y=x+2,1)}$ and $\mathcal{C}_{(y=x+2,2)}$ are arithmetic sequences, it follows that $(x_0, ax_0^c + b)$ is a solution of $y = ax^c + b$ in $\mathcal{C}_{(y=x+2,1)}$ or $\mathcal{C}_{(y=x+2,2)}$ if and only if $\frac{ax_0^c+b-x_0}{2}$ is an integer. If a and b are odd, then $\frac{ax_0^c+b-x_0}{2}$ is not an integer for any $x_0 \in \mathbb{N}$, which implies that there is no monochromatic solution of $y = ax^c + b$ in $\mathbb{N}$. If a and b have different parity, then $\frac{a\cdot 1^c+b-1}{2}$ is an integer, which implies that one of the monochromatic solution of $y = ax^c + b$ in $\mathbb{N}$ is $(1, a + b)$. If a and b are even, then $\frac{ax_0^c+b-x_0}{2}$ is not an integer for $x_0 = 1$ but is an integer for $x_0 = 2$, which implies that one of the monochromatic solution of $y = ax^c + b$ in $\mathbb{N}$ is $(2, a \cdot 2^c + b)$. Based on the above ideas, we prove the following result.

Proof of Theorem 1.13. Let the λ-class $\mathcal{C}_{(y=x+b,\lambda)} = \{\lambda, \lambda+b, \lambda+2b, \ldots\} \subseteq \mathbb{N}$ for each $\lambda \in [b]$, and the coloring χ of $\mathbb{N}$ is as described in Corollary 2.5. Specifically, the λ-class $\mathcal{C}_{(y=x+b,\lambda)}$ is monochromatic for each $\lambda \in [b]$, and for different λ_i and λ_j in $[b]$, $\mathcal{C}_{(y=x+b,\lambda_i)}$ and $\mathcal{C}_{(y=x+b,\lambda_j)}$ have different colors. Obviously, under the coloring χ, $y = x+b$ does not have a rainbow solution in $\mathbb{N}$. Noticing that for each $\lambda \in [b]$, $\mathcal{C}_{(y=x+b,\lambda)}$ is an arithmetic sequence. Therefore, $(x_0, f(x_0))$ is a solution of $y = ax^c + b$ in $\mathcal{C}_{(y=x+b,\lambda)}$ for some $\lambda \in [b]$ if and only if $\frac{f(x_0)-x_0}{b}$ is an integer. Let $x_{\min} = \min\left\{x \in \mathbb{N} : \frac{f(x)-x}{b} \text{ is an integer}\right\}$. If $x_{\min}$ does not exist, then $\frac{f(x_0)-x_0}{b}$ is not an integer for any $x_0 \in \mathbb{N}$, which implies that there is no monochromatic solution of $y = ax^c + b$ in $\mathbb{N}$, and thus $\mathrm{GR}_b\,(y = x+b : y = f(x))$ does not exist.

Next, we assume that $x_{\min}$ exists. For the lower bound, we apply the coloring χ to $\mathcal{C}_{(y=x+b,\lambda)} \subseteq [f(x_{\min})-1]$. If there is a monochromatic solution of $y = f(x)$ in $[f(x_{\min})-1]$, then it contradicts the minimality of $x_{\min}$. Therefore, under the coloring χ, there is neither a rainbow solution of $y = x+b$ nor a monochromatic solution of $y = f(x)$ in $[f(x_{\min})-1]$. For the upper bound, we consider any b-coloring of $[f(x_{\min})]$. To the contrary, suppose that there is a b-coloring of $[f(x_{\min})]$ such that there is neither a rainbow solution of $y = x+b$ nor a monochromatic solution of $y = f(x)$. Since there is no rainbow solution of $y = x+b$, it follows from Corollary 2.5 that without considering the order of colors, the coloring χ is uniquely determined. But under the coloring χ, we can find a monochromatic solution of $y = f(x)$ in $[f(x_{\min})]$, which is $(x_{\min}, f(x_{\min}))$, a contradiction. The result thus follows. □

5. Further research

Although we have provided a calculation method for rainbow numbers of strictly monotonically increasing nonlinear binary function equations in Algorithm 2, we have not provided a formula expression for rainbow numbers, so the following problem can be considered.

Problem 1. For a strictly monotonically increasing nonlinear binary function equation $y = f(x)$, determine the formula expression for $\mathrm{rb}([n], y = f(x))$.

Just as the Gallai–Ramsey numbers for rainbow triangle depend on Gallai's coloring structure theorem (Theorem 1.7), our research on the Gallai–Rado number is also based on the coloring structure theorem (Theorem 2.3). Since the number of colors of the Gallai–Rado numbers studied in this paper is fixed at b, where the rainbow equation is $y = x+b$, the following problem can be considered.

Problem 2. For some equations $\mathcal{E}$, and the number of colors k satisfies $2 \le k \le b-1$, study the Gallai–Rado numbers $\mathrm{GR}_k(y = x+b : \mathcal{E})$.

Data availability

No data was used for the research described in the article.

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