LINKED PARTITION IDEALS AND OVERPARTITIONS

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Abstract. Linked partition ideals which were first introduced by Andrews have recently appeared in a series of works to study generating functions for partitions. Recently, Andrews found some relations between a certain kind of overpartitions and 4-regular partitions into distinct parts. Then with the aid of linked partition ideals for overpartitions, Andrews and Chern established a general relation between these two sets of partitions. Motivated by their work, we consider the overpatitions denoted by $\mathscr{A}^k$ satisfying the following conditions: (1) Only odd parts may be overlined; (2) The difference between any two parts is $\geqslant 2k$ where the inequality is strict if the larger one is overlined. Let S be a set of given parts. Then $\mathscr{A}_S^k$ denotes the subset of overpartitions in $\mathscr{A}^k$ where parts from S are forbidden. Combining linked partition ideals and a recurrence relation for a family of multiple series given by Chern, we study the generating functions for $\mathscr{A}_S^k$ for some given S. Furthermore, by establishing a q-series identity, we find a relation between $\mathscr{A}_{\{\overline{1}\}}^1$ and distinct partitions. Meanwhile, some statistics on partitions are discussed.

1. Introduction

Here and throughout the paper, we adopt the standard q-series notation [24]. Assume that q is a complex number such that $|q| < 1$. For any positive integer n, the q-shifted factorials are defined by

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

A partition of a positive integer n is a finite weakly decreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ such that $\sum_{i=1}^{\ell} \lambda_i = n$. Let λ be a partition of n. Then $|\lambda| = n$. The λ_i $(1 \leqslant i \leqslant \ell)$ are called the parts of λ, and the number of parts denoted by $\sharp(\lambda)$ is called the length of λ. The generating function for $p(n)$ which denotes the number of partitions of n is stated as

$$\sum_{n\geqslant 0} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

In the development of the theory of partitions, many different kinds of partitions have been studied. For example, Corteel and Lovejoy [20] introduced the definition of overpartitions. An overpartition of n is a partition of n where the first occurrence of a number may

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be overlined. Let $\overline{p}(n)$ be the total number of overpartitions of n. Then

$$\sum_{n\geqslant 0}\overline{p}(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Another kind of partitions that is often concerned about is ℓ-regular partitions. For a positive integer $\ell>1$, a partition is called ℓ-regular if none of its parts is divisible by ℓ. The number of ℓ-regular partitions of n is usually denoted by $b_\ell(n)$ and its arithmetic properties were investigated extensively. See, for example, [12, 19, 21–23]. The generating function for $b_\ell(n)$ is given by

$$\sum_{n\geqslant 0}b_\ell(n)q^n=\frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

Dating back to 1740s, Euler found the first partition identity:

$$(-q;q)_\infty=\frac{1}{(q;q^2)_\infty},$$

which means that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts. From another point of view, we may state the above identity as the number of partitions of n in which the difference between any two parts is at least 1 equals the number of partitions of n into parts congruent to $1 \pmod 2$.

In the literature, the relations between partitions with gap conditions and those with modular conditions have received a great deal of attention. For example, the following theorem was found by Schur in 1926.

Theorem 1.1. *(Schur [29]). Let $H_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$. Let $F_1(n)$ denote the number of partitions of n into distinct parts congruent to $\pm 1 \pmod 3$. Let $G_1(n)$ denote the number of partitions of n of the form $n=b_1+b_2+\cdots+b_s$, where $b_i-b_{i+1}\geqslant 3$ with strict inequality if $3|b_i$. Then $H_1(n)=F_1(n)=G_1(n)$.*

Schur's proof was based on a lemma concerning recurrence relations for certain polynomials. Then Andrews [3] gave a new proof by utilizing recurrent sequences. Furthermore, Andrews [2,4] found two generalizations of Schur's theorem. In 1980, Bressoud [11] achieved a more general result related to distinct partitions with parts congruent to $\pm r \pmod m$. In addition, Gleißberg [25] found a refinement of Schur's result. By the method of weighted words, Alladi and Gordon [1] discovered some other companion partition functions which are equal to $F_1(n)$.

Recall the celebrated Rogers–Ramanujan identities [27]:

$$\sum_{n\geqslant 0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty}\qquad\text{and}\qquad\sum_{n\geqslant 0}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}.$$

For $i\in\{1,2\}$, these two identities can be interpreted as the number of partitions of n in which part 1 occurs less than i times and the difference between any two parts is at least 2 equals the number of partitions of n into parts $\not\equiv 0, \pm i \pmod 5$. Then Gordon [26] generalized the combinatorial forms of the Rogers–Ramanujan identities to arbitrary odd modulus $2k+1$.

Theorem 1.2. *(Gordon [26]). Let $B_{k,i}(n)$ denote the number of partitions of n of the form $(b_1 b_2 \cdots b_s)$, where $b_j - b_{j+k-1} \geqslant 2$, and at most $i-1$ of the b_j equal 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Then $A_{k,i}(n) = B_{k,i}(n)$ for all n.*

Subsequently, Andrews [5] discovered the following identity corresponding to the above theorem.

Theorem 1.3. *(Andrews [5]). For $1 \leqslant i \leqslant k-1$ and $k \geqslant 2$,*

$$\sum_{n_1,\ldots,n_{k-1}\geqslant 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q;q)_\infty}, \tag{1.1}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

The identity (1.1) is now commonly referred to as the Andrews–Gordon identity. Bressoud [10] succeeded in finding a generalization of (1.1) for any given integer modulus by using an algebraic approach in the spirit of Andrews. Then Lovejoy proved the analogues of the cases $i = k$ and $i = 1$ of Gordon's theorem for overpartitions [28]. For example,

Theorem 1.4. *(Lovejoy [28]). Let $\overline{B}_k(n)$ denote the number of overpartitions of n of the form $y_1 + y_2 + \cdots + y_s$, where $y_j - y_{j+k-1} \geqslant 1$ if y_{j+k-1} is overlined and $y_j - y_{j+k-1} \geqslant 2$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of n into parts not divisible by k. Then $\overline{A}_k(n) = \overline{B}_k(n)$.*

In 2013, Chen, Sang and Shi [13] derived the overpartition version of Theorem 1.2. Meanwhile, an identity similar to (1.1) was established.

The discovery of (1.1) has sparked great interest in the study of identities involving multiple series. In particular, the following family of q-multi-summations which is usually called *the series of Andrews–Gordon type*,

$$\sum_{n_1,\ldots,n_r\geq 0} \frac{(-1)^{L_1(n_1,\ldots,n_r)} q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}},$$

has attracted a lot of attention. Here L_1 and L_2 are linear forms and Q is a quadratic form in the indices $n_1, \ldots, n_r$. In recent years, linked partition ideals which were initially introduced by Andrews [6, Chapter 8] have experienced a resurgence in a series of recent projects [8, 9, 14–18]. It is shown that this tool plays a very important role in finding relations between partitions and multiple series.

Very recently, Andrews and Chern [8] extended the concept of linked partition ideals to overpartitions, and considered a specific set of overpartitions denoted by $\mathscr{A}^{\vee}_{\{\bar{1}\}}$ satisfying the following properties:

(1) Only odd parts larger than 1 may be overlined;

(2) The difference between any two parts is at least 4, and the inequality is strict if the larger part is overlined or divisible by 4, with the exception that both $\overline{5}$ and 1 may simultaneously appear as parts.

For a given overpartition λ, let $\sharp_{a,M}(\lambda)$ be the number of parts in λ which are congruent to a modulo M, and $O(\lambda)$ denotes the number of overlined parts. In view of generating functions, Andrews and Chern [8] proved the following theorem which extends two relations given by Andrews [7].

Theorem 1.5. *(Andrews–Chern [8]). Let $A(n;m,\ell)$ count the number of overpartitions λ of n in $\mathscr{A}^{\vee}_{\{\overline{1}\}}$ such that $\sharp_{1,2}(\lambda)+2\sharp_{0,4}(\lambda)=m$ and $\sharp_{2,4}(\lambda)+O(\lambda)=\ell$. Further, let $B(n;m,\ell)$ count the number of 4-regular partitions of n into distinct parts with m odd parts and ℓ even parts. Then $A(n;m,\ell)=B(n;m,\ell)$.*

Inspired by the work of Andrews and Chern [8], the goal of this paper is to find more partition identities related to overpartitions. Our first discovery is the following identity.

Theorem 1.6. *We have*

$$\begin{aligned}&\sum_{n_1,n_2,n_3\geqslant 0}\frac{x^{n_1+n_2}y^{n_2+n_3}q^{2\binom{n_1}{2}+4\binom{n_2}{2}+2\binom{n_3}{2}+2n_1n_2+2n_1n_3+2n_2n_3+n_1+3n_2+2n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_{n_3}}\\&\quad=(-xq;q^2)_\infty(-yq^2;q^2)_\infty.\end{aligned}\tag{1.2}$$

Definition 1.7. For any integer $k\geqslant 1$, let $\mathscr{A}^k$ denote the set of overpartitions which satisfy the following conditions:

(1) Only odd parts may be overlined;

(2) The difference between any two parts is $\geqslant 2k$ where the inequality is strict if the larger one is overlined.

Let S be a set of given parts, and let $\mathscr{A}^k_S$ denote the subset of overpartitions in $\mathscr{A}^k$ where parts from S are not allowed. With the aid of linked partition ideals for overpartitions and a recurrence relation for a type of q-multi-summations due to Chern [16], we study the generating functions for $\mathscr{A}^k_S$ with some given S. In particular, the generating function for $\mathscr{A}^k_{\{\overline{1}\}}$ is stated in the following theorem.

Theorem 1.8. *We have*

$$\begin{aligned}\sum_{\lambda\in\mathscr{A}^k_{\{\overline{1}\}}}x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}z^{O(\lambda)}q^{|\lambda|}&=\sum_{n_1,n_2,n_3\geqslant 0}\frac{x^{n_1+n_2+n_3}y^{n_3}z^{n_2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_{n_3}}\\&\quad\times q^{2k\binom{n_1}{2}+(2k+2)\binom{n_2}{2}+2k\binom{n_3}{2}+2kn_1n_2+2kn_1n_3+2kn_2n_3+n_1+3n_2+2n_3}.\end{aligned}\tag{1.3}$$

Then combining Theorem 1.6 and Theorem 1.8 with $k=1$, we find a relation between $\mathscr{A}^1_{\{\overline{1}\}}$ and distinct partitions.

Corollary 1.9. *Let $C(n;m,\ell)$ denote the number of overpartitions λ with $|\lambda|=n$ in $\mathscr{A}^1_{\{\overline{1}\}}$ where $\sharp_{1,2}(\lambda)=m$ and $\sharp_{0,2}(\lambda)+O(\lambda)=\ell$. Additionally, let $D(n;m,\ell)$ count the* *number of partitions of n into distinct parts with m odd parts and ℓ even parts. Then $C(n;m,\ell) = D(n;m,\ell)$.*

Moreover, by giving a combinatorial proof of Corollary 1.9, we find the following fact.

Corollary 1.10. *The number of overlined parts in overpatitions of n in $\mathscr{A}^1_{\{1\}}$ equals the number of consecutive integer pairs (each part using only one time) in distinct partitions of n.*

The remainder of this paper proceeds as follows. In Section 2, we provide the definition of linked partition ideals and some other preliminaries. Section 3 is devoted to showing the proofs of the main results. In Section 4, we conclude the paper by listing some other generating functions for $\mathscr{A}^k_S$ with some given S.

2. Preliminaries

In this section, we present some preliminaries.

Lemma 2.1. *[24, Equation (1.3.2)] (the q-binomial theorem) For $|z| < 1$,*

$$\sum_{n\geqslant 0} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.1}$$

Setting $a = 0$ and $a \to \infty$ in (2.1), respectively, we have

$$\sum_{n\geqslant 0} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} \tag{2.2}$$

and

$$\sum_{n\geqslant 0} \frac{z^n q^{\binom{n}{2}}}{(q;q)_n} = (-z;q)_\infty. \tag{2.3}$$

We also need the following functional operator $\mathcal{B}$ defined on $\mathbb{C}(q)[[x,y]]$:

$$\mathcal{B}\left(\sum_{m,n\geqslant 0} c_{m,n} x^m y^n\right) := \sum_{m,n\geqslant 0} c_{m,n} q^{2\binom{m}{2}+2\binom{n}{2}} x^m y^n, \tag{2.4}$$

where the coefficients $c_{m,n}$ are in $\mathbb{C}(q)$. This operator can be considered as a specialization of the q-Borel operators, and for more applications, one can see [8, 15].

Next, recall the definition of linked partition ideals for overpartitions given by Andrews and Chern [8]. Let $\phi^m(\mu)$ denote the overpartition given by adding m to each part of the overpartition μ with overlines preserved. For two overpartitions μ and ν that do not have any overlapping overlined parts, the operation $\mu \oplus \nu$ yields an overpartition including all the parts in μ and ν.

Definition 2.2. [8, Definition 4.1] Assume that

(1) $\Pi = \{\pi_1, \pi_2, \ldots, \pi_K\}$ is a finite set of overpartitions, where $\pi_1 = \varnothing$;

(2) for each $\pi_a \in \Pi$, there exists a corresponding linking set $\mathcal{L}(\pi_a) \subset \Pi$, with especially, $\mathcal{L}(\pi_1) = \mathcal{L}(\varnothing) = \Pi$ and $\pi_1 = \varnothing \in \mathcal{L}(\pi_k)$ for any $1 \leqslant k \leqslant K$;
(3) and there is a positive integer T, referred to as the *modulus*, which is greater than or equal to the largest part among all overpartitions in Π.

We say a *span one linked partition ideal* $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, T)$ is the collection of all overpartitions of the form

$$\begin{aligned}\lambda &= \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \cdots \oplus \phi^{NT}(\lambda_N) \oplus \phi^{(N+1)T}(\pi_1) \oplus \phi^{(N+2)T}(\pi_1) \oplus \cdots \\ &= \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \cdots \oplus \phi^{NT}(\lambda_N), \end{aligned} \tag{2.5}$$

where $\lambda_i \in \mathcal{L}(\lambda_{i-1})$ for each i and λ_N is not the empty partition. Notice that $\mathscr{I}$ includes the empty partition which corresponds to $\phi^0(\pi_1) \oplus \phi^T(\pi_1) \oplus \cdots$.

It is obvious that each summand $\phi^{iT}(\lambda_i)$ consists of parts ranging in size from $iT+1$ to $iT+T$, indicating that no part appears in two different summands simultaneously.

Based on the above definition, we derive the following lemma related to $\mathscr{A}^k$.

Lemma 2.3. *For any positive integer k, $\mathscr{A}^k$ is equinumerous with the span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, 2k)$, where $\Pi = \{\pi_1 = \varnothing, \pi_2 = (1), \pi_3 = (\overline{1}), \pi_4 = (2), \ldots, \pi_{3k-1} = (2k-1), \pi_{3k} = (\overline{2k-1}), \pi_{3k+1} = (2k)\}$ and*

$$\begin{cases} \mathcal{L}(\pi_1) = \{\pi_1, \pi_2, \pi_3, \pi_4, \ldots, \pi_{3k-1}, \pi_{3k}, \pi_{3k+1}\}, \\ \mathcal{L}(\pi_2) = \mathcal{L}(\pi_3) = \{\pi_1, \pi_2, \pi_4, \pi_5, \ldots, \pi_{3k-1}, \pi_{3k}, \pi_{3k+1}\}, \\ \mathcal{L}(\pi_4) = \{\pi_1, \pi_4, \pi_5, \pi_6, \ldots, \pi_{3k-1}, \pi_{3k}, \pi_{3k+1}\}, \\ \quad\vdots \\ \mathcal{L}(\pi_{3j-1}) = \mathcal{L}(\pi_{3j}) = \{\pi_1, \pi_{3j-1}, \pi_{3j+1}, \pi_{3j+2}, \ldots, \pi_{3k-1}, \pi_{3k}, \pi_{3k+1}\}, \\ \mathcal{L}(\pi_{3j+1}) = \{\pi_1, \pi_{3j+1}, \pi_{3j+2}, \pi_{3j+3}, \ldots, \pi_{3k-1}, \pi_{3k}, \pi_{3k+1}\}, \\ \quad\vdots \\ \mathcal{L}(\pi_{3k-1}) = \mathcal{L}(\pi_{3k}) = \{\pi_1, \pi_{3k-1}, \pi_{3k+1}\}, \\ \mathcal{L}(\pi_{3k+1}) = \{\pi_1, \pi_{3k+1}\}. \end{cases}$$

Proof. It can be easily verified that all overpartitions in $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, 2k)$ satisfy the conditions for $\mathscr{A}^k$.

On the other hand, for a given positive integer k, decompose each overpartition in $\mathscr{A}^k$ into blocks $\mathbf{B}_0, \mathbf{B}_1, \ldots$, such that all parts (including those that are overlined) between $2ki+1$ and $2ki+2k$ belong to the block $\mathbf{B}_i$. It is evident that $\phi^{-2ki}(\mathbf{B}_i)$ exclusively belongs to Π. Furthermore, if $\phi^{-2ki}(\mathbf{B}_i)$ is equal to π_1 (i.e., $\mathbf{B}_i$ is empty), then $\phi^{-2k(i+1)}(\mathbf{B}_{i+1})$ can be any element from Π. If $\phi^{-2ki}(\mathbf{B}_i)$ is equal to π_2 or π_3 (i.e., $\mathbf{B}_i$ is either $(2ki+1)$ or $(\overline{2ki+1})$), then $\mathbf{B}_{i+1}$ cannot be $(\overline{2ki+2k+1})$ due to the second condition for $\mathscr{A}^k$. Consequently, $\phi^{-2k(i+1)}(\mathbf{B}_{i+1})$ cannot be π_3. Since similar arguments can be applied to other possibilities of $\phi^{-2ki}(\mathbf{B}_i)$, the details are omitted. Therefore, we complete the proof. ∎

Chern [14] introduced a crucial recurrence relation for a family of q-multi-summations, and later he gave a refinement of the recurrence in [16]. Let R and J be positive integers. Then fix a symmetric matrix $\underline{\boldsymbol{\alpha}} = (\alpha_{i,j}) \in \mathbf{Mat}_{R\times R}(\mathbb{N})$, a vector $\underline{\boldsymbol{A}} = (A_r) \in \mathbb{N}^R_{>0}$ and J vectors $\underline{\boldsymbol{\gamma_j}} = (\gamma_{j,r}) \in \mathbb{N}^R_{\geqslant 0}$ for $j = 1, 2, \ldots, J$. Let $x_1, x_2, \ldots, x_J$ and q be indeterminates such that the following q-multi-summation $H(\underline{\boldsymbol{\beta}}) = H(\beta_1, \ldots, \beta_R)$ for $\underline{\boldsymbol{\beta}} \in \mathbb{Z}^R$ converges:

$$H(\underline{\boldsymbol{\beta}}) := \sum_{n_1,\ldots,n_R\geqslant 0} \frac{x_1^{\sum_{r=1}^R \gamma_{1,r}n_r} \cdots x_J^{\sum_{r=1}^R \gamma_{J,r}n_r} q^{\sum_{r=1}^R \alpha_{r,r}\binom{n_r}{2} + \sum_{1\leqslant i<j\leqslant R} \alpha_{i,j}n_in_j + \sum_{r=1}^R \beta_r n_r}}{(q^{A_1};q^{A_1})_{n_1} \cdots (q^{A_R};q^{A_R})_{n_R}}. \tag{2.6}$$

Then Chern [16] established the following recurrence relation.

Lemma 2.4. *[16, Lemma 2.1] For* $1 \leqslant r \leqslant R$,

$$\begin{aligned} H(\beta_1, \ldots, \beta_r, \ldots, \beta_R) =& H(\beta_1, \ldots, \beta_r + A_r, \ldots, \beta_R) \\ &+ x_1^{\gamma_{1,r}} \cdots x_J^{\gamma_{J,r}} q^{\beta_r} H(\beta_1 + \alpha_{r,1}, \ldots, \beta_r + \alpha_{r,r}, \ldots, \beta_R + \alpha_{r,R}). \end{aligned}$$

In [16], the recurrence is illustrated with a binary tree, in which the coordinate β_r is displayed in boldface. See Figure 1.

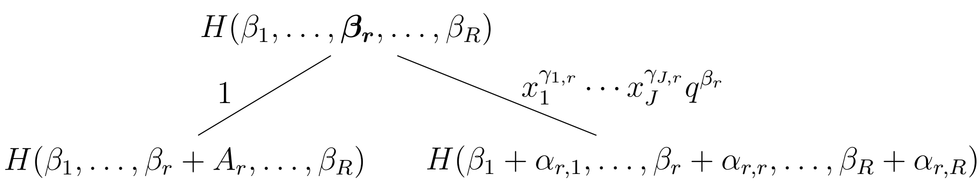

FIGURE 1. Node $H(\beta_1, \ldots, \beta_r, \ldots, \beta_R)$ and its children

3. Proofs of the main results

In this section, we prove Theorems 1.6 and 1.8. Then two proofs of Corollary 1.9 are provided. Finally, Corollary 1.10 follows from the combinatorial proof of Corollary 1.9.

Proof of Theorem 1.6. First, we prove the following equivalent identity of (1.2).

$$\sum_{n_1,n_2,n_3\geqslant 0} \frac{x^{n_1+n_2} y^{n_2+n_3} q^{2n_1n_3+n_1+3n_2+2n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_{n_3}} = \frac{1}{(xq;q^2)_\infty (yq^2;q^2)_\infty}. \tag{3.1}$$

From (2.2), it follows that

$$\begin{aligned} LHS(3.1) &= \sum_{n_1,n_3\geqslant 0} \frac{x^{n_1} y^{n_3} q^{2n_1n_3+n_1+2n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_3}} \times \frac{1}{(xyq^3;q^2)_\infty} \\ &= \frac{1}{(xyq^3;q^2)_\infty} \sum_{n_1\geqslant 0} \frac{x^{n_1}q^{n_1}}{(q^2;q^2)_{n_1}} \sum_{n_3\geqslant 0} \frac{(yq^{2n_1+2})^{n_3}}{(q^2;q^2)_{n_3}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(xyq^3;q^2)_\infty (yq^2;q^2)_\infty} \sum_{n_1\geqslant 0} \frac{x^{n_1}q^{n_1}(yq^2;q^2)_{n_1}}{(q^2;q^2)_{n_1}} \\
&= \frac{1}{(yq^2;q^2)_\infty (xq;q^2)_\infty} \\
&= RHS(3.1),
\end{aligned}
$$

where we use (2.2) to derive the third equality, and the penultimate step follows from (2.1). Then using the operator $\mathcal{B}$ defined in (2.4), we obtain that

$$
\begin{aligned}
LHS(1.2) = \mathcal{B}\left(LHS(3.1)\right) &= \mathcal{B}\left(RHS(3.1)\right) \\
&= \mathcal{B}\left(\sum_{n_1,n_2\geqslant 0} \frac{(xq)^{n_1}(yq^2)^{n_2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}}\right) \\
&= \sum_{n_1\geqslant 0} \frac{(xq)^{n_1} q^{2\binom{n_1}{2}}}{(q^2;q^2)_{n_1}} \sum_{n_2\geqslant 0} \frac{(yq^2)^{n_2} q^{2\binom{n_2}{2}}}{(q^2;q^2)_{n_2}} \\
&= RHS(1.2),
\end{aligned}
$$

where we obtain the third equality by using (2.2), and the last step follows from (2.3). Therefore, we complete the proof. $\square$

Proof of Theorem 1.8. We decompose overpartitions $\lambda \in \mathscr{A}^k = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, 2k)$ as in (2.5). Then for $1 \leqslant i \leqslant 3k+1$, define the generating function for overpartitions in $\mathscr{A}^k$ according to the first decomposed block:

$$
G_i(x) = G_i(x,y,z,q) := \sum_{\substack{\lambda\in\mathscr{A}^k \\ \lambda_0=\pi_i}} x^{\sharp(\lambda)} y^{\sharp_{0,2}(\lambda)} z^{O(\lambda)} q^{|\lambda|}.
$$

It is plain that

$$
G_i(x) = x^{\sharp(\pi_i)} y^{\sharp_{0,2}(\pi_i)} z^{O(\pi_i)} q^{|\pi_i|} \sum_{j:\pi_j\in\mathcal{L}(\pi_i)} G_j(xq^{2k}).
$$

Hence,

$$
\begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_{3k+1}(x) \end{pmatrix} = W \cdot A \cdot \begin{pmatrix} G_1(xq^{2k}) \\ G_2(xq^{2k}) \\ \vdots \\ G_{3k+1}(xq^{2k}) \end{pmatrix}, \tag{3.2}
$$

where

$$
W = \operatorname{diag}(1, xq, xzq, xyq^2, xq^3, xzq^3, xyq^4, \ldots, xq^{2k-1}, xzq^{2k-1}, xyq^{2k}) \tag{3.3}
$$

and

$$A=\begin{pmatrix}1&1&1&1&1&\cdots&1&1&1&1\\1&1&0&1&1&\cdots&1&1&1&1\\1&1&0&1&1&\cdots&1&1&1&1\\1&0&0&1&1&\cdots&1&1&1&1\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\1&0&0&0&0&\cdots&0&1&0&1\\1&0&0&0&0&\cdots&0&1&0&1\\1&0&0&0&0&\cdots&0&0&0&1\end{pmatrix}. \tag{3.4}$$

More precisely, W and A are both $(3k+1)\times(3k+1)$ matrices. Meanwhile, the elements in the first row of matrix A are all 1, and for $1\leqslant i\leqslant k$,

$$A_{3i-1,j}=A_{3i,j}=\begin{cases}0, & \textit{if } 1<j<3i-1, \textit{ or } j=3i,\\ 1, & \textit{otherwise},\end{cases}$$
$$A_{3i+1,j}=\begin{cases}0, & \textit{if } 1<j<3i+1,\\ 1, & \textit{otherwise}.\end{cases}$$

Recall that $\mathscr{A}_S^k$ denotes the subset of overpartitions in $\mathscr{A}^k$ such that parts from S are forbidden. Define

$$F_1(x):=\sum_{\lambda\in\mathscr{A}^k}x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}z^{O(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,2,3,4,\ldots,3k-1,3k,3k+1\}}G_i(x), \tag{3.5}$$

$$F_2(x)=F_3(x):=\sum_{\lambda\in\mathscr{A}^k_{\{\bar{1}\}}}x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}z^{O(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,2,4,5,\ldots,3k-1,3k,3k+1\}}G_i(x), \tag{3.6}$$

$$F_4(x):=\sum_{\lambda\in\mathscr{A}^k_{\{1,\bar{1}\}}}x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}z^{O(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,4,5,6,\ldots,3k-1,3k,3k+1\}}G_i(x), \tag{3.7}$$

$$\vdots$$

$$F_{3j-1}(x)=F_{3j}(x):=\sum_{\lambda\in\mathscr{A}^k_{\{1,\bar{1},2,\ldots,2j-2,\overline{2j-1}\}}}x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}z^{O(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,3j-1,3j+1,3j+2,\ldots,3k+1\}}G_i(x), \tag{3.8}$$

$$F_{3j+1}(x):=\sum_{\lambda\in\mathscr{A}^k_{\{1,\bar{1},2,\ldots,2j-2,2j-1,\overline{2j-1}\}}}x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}z^{O(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,3j+1,3j+2,\ldots,3k+1\}}G_i(x), \tag{3.9}$$

$$\vdots$$

$$F_{3k-1}(x)=F_{3k}(x):=\sum_{\lambda\in\mathscr{A}^k_{\{1,\bar{1},2,\ldots,2k-2,\overline{2k-1}\}}}x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}z^{O(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,3k-1,3k+1\}}G_i(x), \tag{3.10}$$

$$F_{3k+1}(x):=\sum_{\lambda\in\mathscr{A}^k_{\{1,\bar{1},2,\ldots,2k-2,2k-1,\overline{2k-1}\}}} x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}z^{O(\lambda)}q^{|\lambda|}=\sum_{i\in\{1,3k+1\}}G_i(x). \tag{3.11}$$

It is obvious that

$$\begin{pmatrix}F_1(x)\\F_2(x)\\\vdots\\F_{3k+1}(x)\end{pmatrix}=A\cdot\begin{pmatrix}G_1(x)\\G_2(x)\\\vdots\\G_{3k+1}(x)\end{pmatrix}=A\cdot W\cdot A\cdot\begin{pmatrix}G_1(xq^{2k})\\G_2(xq^{2k})\\\vdots\\G_{3k+1}(xq^{2k})\end{pmatrix},$$

where the last equality follows from (3.2). Thus, we have

$$\begin{pmatrix}F_1(x)\\F_2(x)\\\vdots\\F_{3k+1}(x)\end{pmatrix}=A\cdot W\cdot\begin{pmatrix}F_1(xq^{2k})\\F_2(xq^{2k})\\\vdots\\F_{3k+1}(xq^{2k})\end{pmatrix}. \tag{3.12}$$

Meanwhile, note that $G_i(0)=1$ when $i=1$ and $G_i(0)=0$ otherwise. So,

$$F_1(0)=F_2(0)=\cdots=F_{3k+1}(0)=1.$$

Next, based on the definition of $H(\underline{\boldsymbol{\beta}})$ in (2.6), set $R=3$ and $J=3$. Then choose

$$\underline{\alpha}=\begin{pmatrix}2k&2k&2k\\2k&2k+2&2k\\2k&2k&2k\end{pmatrix},\qquad \underline{A}=(2,2,2),\qquad \begin{matrix}x_1=x, & \gamma_1=(1,1,1),\\ x_2=y, & \gamma_2=(0,0,1),\\ x_3=z, & \gamma_3=(0,1,0).\end{matrix}$$

So, we obtain

$$\begin{aligned}H(\beta_1,\beta_2,\beta_3)=&\sum_{n_1,n_2,n_3\geqslant 0}\frac{x^{n_1+n_2+n_3}y^{n_3}z^{n_2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_{n_3}}\\&\times q^{2k\binom{n_1}{2}+(2k+2)\binom{n_2}{2}+2k\binom{n_3}{2}+2kn_1n_2+2kn_1n_3+2kn_2n_3+\beta_1n_1+\beta_2n_2+\beta_3n_3}.\end{aligned} \tag{3.13}$$

Then starting from $H(1,1,2)$, we repeatedly apply Lemma 2.4 in (213) order, namely, for $H(\beta_1,\beta_2,\beta_3)$, we apply Lemma 2.4 with respect to β_2, β_1 and β_3 in sequence. The whole binary tree is shown in Figure 2. Hence, based on the process, we derive the following relation.

$$\begin{pmatrix} H(1,1,2) \\ H(1,3,2) \\ H(1,3,2) \\ H(3,3,2) \\ \vdots \\ H(2i-1,2i+1,2i) \\ H(2i-1,2i+1,2i) \\ H(2i+1,2i+1,2i) \\ \vdots \\ H(2k-1,2k+1,2k) \\ H(2k-1,2k+1,2k) \\ H(2k+1,2k+1,2k) \end{pmatrix} = A \cdot W \cdot \begin{pmatrix} H(1+2k,1+2k,2+2k) \\ H(1+2k,3+2k,2+2k) \\ H(1+2k,3+2k,2+2k) \\ H(3+2k,3+2k,2+2k) \\ \vdots \\ H(2k+2i-1,2k+2i+1,2k+2i) \\ H(2k+2i-1,2k+2i+1,2k+2i) \\ H(2k+2i+1,2k+2i+1,2k+2i) \\ \vdots \\ H(4k-1,4k+1,4k) \\ H(4k-1,4k+1,4k) \\ H(4k+1,4k+1,4k) \end{pmatrix}, \tag{3.14}$$

in which W and A are the same as (3.3) and (3.4), respectively. Therefore, the vector on the left-hand side of (3.12) and that on the left-hand side of (3.14) satisfy the same recurrence relation. Furthermore, taking $x = 0$ in the H-vector on the left-hand side of (3.14) gives $(1, 1, 1, \ldots, 1, 1)^T$. So, these two vectors also have the same initial condition. Thus, we derive that

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ \vdots \\ F_{3k-1}(x) \\ F_{3k}(x) \\ F_{3k+1}(x) \end{pmatrix} = \begin{pmatrix} H(1,1,2) \\ H(1,3,2) \\ H(1,3,2) \\ H(3,3,2) \\ \vdots \\ H(2k-1,2k+1,2k) \\ H(2k-1,2k+1,2k) \\ H(2k+1,2k+1,2k) \end{pmatrix}. \tag{3.15}$$

To consider $\mathscr{A}^k_{\{\bar{1}\}}$, we need $F_2(x) = H(1, 3, 2)$ derived from the above equation. Then combining (3.6) and (3.13) yields (1.3). Hence, we complete the proof. $\square$

Next, combining Theorems 1.6 and 1.8, we provide an analytic proof of Corollary 1.9.

First proof of Corollary 1.9. It is plain that

$$\begin{aligned} \sum_{l,m,n\geqslant 0} C(n;m,l)x^m y^l q^n &= \sum_{\lambda\in\mathscr{A}^1_{\{\bar{1}\}}} x^{\sharp_{1,2}(\lambda)} y^{\sharp_{0,2}(\lambda)+O(\lambda)} q^{|\lambda|} \\ &= \sum_{\lambda\in\mathscr{A}^1_{\{\bar{1}\}}} x^{\sharp(\lambda)} (x^{-1}y)^{\sharp_{0,2}(\lambda)} y^{O(\lambda)} q^{|\lambda|} \\ &= \sum_{n_1,n_2,n_3\geqslant 0} \frac{x^{n_1+n_2} y^{n_2+n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_{n_3}} \end{aligned}$$

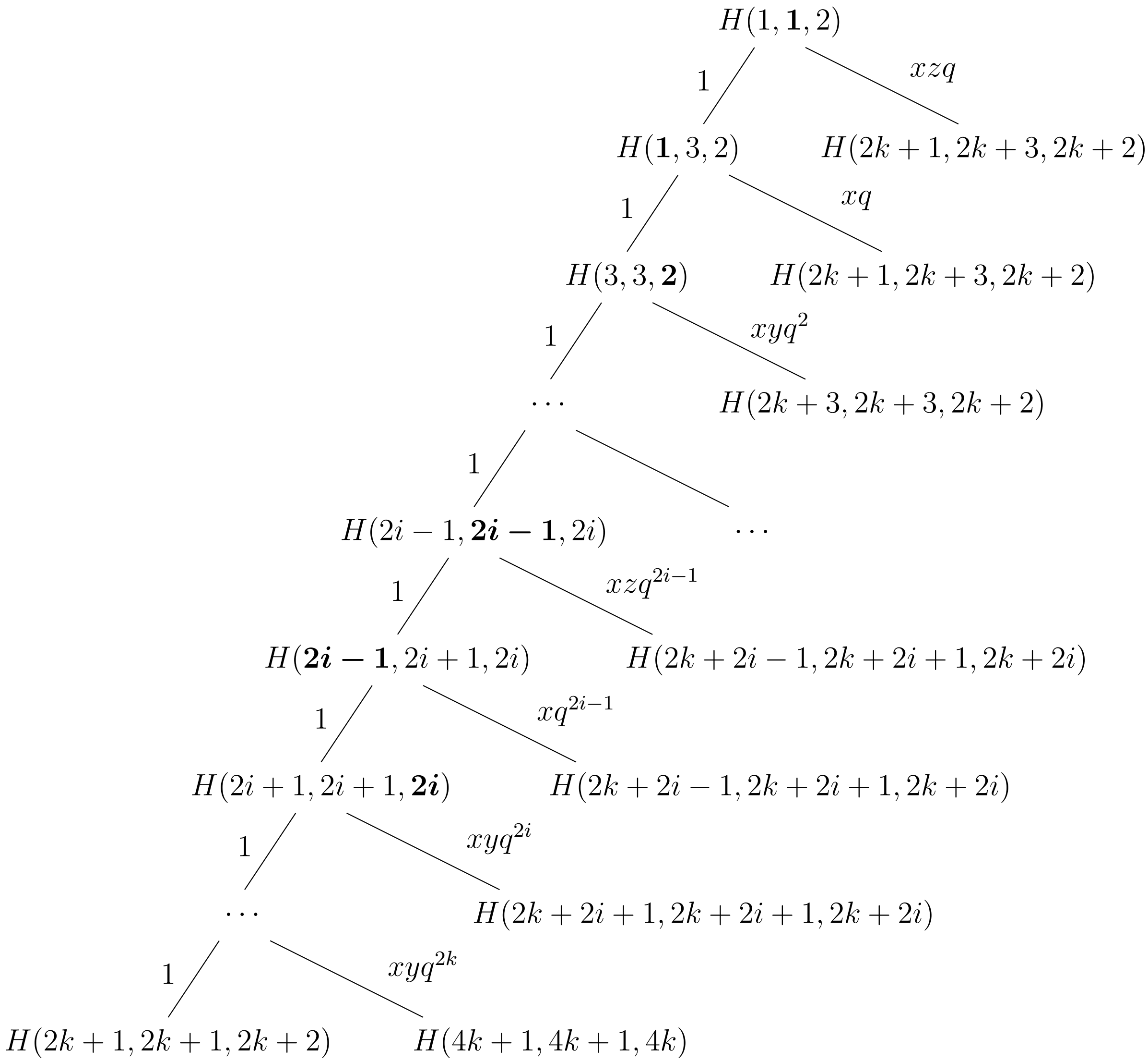

Figure 2. The binary tree for arbitrary k

$$
\begin{aligned}
&\times q^{2\binom{n_1}{2}+4\binom{n_2}{2}+2\binom{n_3}{2}+2n_1n_2+2n_1n_3+2n_2n_3+n_1+3n_2+2n_3} \\
&= (-xq;q^2)_\infty(-yq^2;q^2)_\infty \\
&= \sum_{l,m,n\geqslant 0} D(n;m,l)x^my^lq^n,
\end{aligned}
$$

where the third equality follows from (1.3) with $k=1$, $y\to x^{-1}y$, $z\to y$, and we obtain the penultimate step by using (1.2). Therefore, we complete the proof. $\square$

In the following, inspired by the work of Bressoud [11], we provide a bijective proof of Corollary 1.9.

Second proof of Corollary 1.9. We first describe the map ξ from overpartitions in $\mathscr{A}^1_{\{\bar{1}\}}$ to distinct partitions. For a given overpartition $\lambda \in \mathscr{A}^1_{\{\bar{1}\}}$, we list all parts of λ in a single column for convenience. Since only odd parts larger than 1 may be overlined, we split each overlined part into two consecutive integers which we call pairs remaining in the same row as before. So the number of pairs is equal to the number of overlined parts in λ. In the following, if we call a pair larger, it means that the sum of the two integers in the pair is larger.

Next, we start the map ξ from the smallest pair, and deal with the pairs in increasing order. Let $(a+1, a)$ denote a pair under consideration, and b is the part below it. If $a < b+2$, then subtract two from the larger part of the pair; add two to the part below; and switch their positions. The operation is shown as follows.

$$\begin{array}{ll} a+1 & a \\ b & \end{array} \quad \xrightarrow{a<b+2} \quad \begin{array}{ll} b+2 & \\ a & a-1. \end{array}$$

For the new pair $(a, a-1)$, repeat the above operation until the smaller part of the pair is greater than or equal to the part below with two up, or another pair is just under this pair, or there is nothing under it. Then we continue to deal with the next larger pair. Clearly, the process ends in a partition $\xi(\lambda)$ whose parts are distinct, where from the smallest part, the consecutive parts are paired up. Therefore, we derive the desired distinct partition, and the number of odd (resp. even) parts of $\xi(\lambda)$ is $\sharp_{1,2}(\lambda)$ (resp. $\sharp_{0,2}(\lambda) + O(\lambda)$).

Conversely, for a given partition μ with distinct parts, put all the parts in a column. Then starting from the smallest part, let two consecutive integers be a pair and put them in a row. Next, we treat these pairs in decreasing order. Let $(c, c-1)$ be a pair under consideration, and let d be the part above it. Then if the sum of the pair is larger than $d-2$, we subtract two from the part above; add two to the smaller part of the pair; and switch their positions. The operation is stated as follows.

$$\begin{array}{ll} d & \\ c & c-1 \end{array} \quad \xrightarrow{2c-1>d-2} \quad \begin{array}{ll} c+1 & c \\ d-2. & \end{array}$$

For the pair $(c+1, c)$, repeat the operation until the sum of the pair is less than or equal to the part above reduced two, or there is a pair just above this pair, or there is nothing above it. Then we move to a smaller pair. Once we complete the process for all pairs, merge the pairs together and overline their sums. As a result, we obtain an overpartition belonging to $\mathscr{A}^1_{\{\bar{1}\}}$. Hence, we complete the proof. $\square$

Example 3.1. For $\lambda = 25+\overline{23}+\overline{17}+14+\overline{11}+4+2 \in \mathscr{A}^1_{\{\bar{1}\}}$, we obtain $\xi(\lambda) = \mu = 25+18+11+10+8+7+6+5+4+2$.

$$\begin{array}{cccccccccccccccccc}
 & 25 & & 25 & & & 25 & & & 25 & & & 25 & & & 25 & & \\
 & \overline{23} & & 12 & 11 & & 12 & 11 & & 12 & 11 & & 12 & 11 & & 18 & & \\
 & \overline{17} & & 9 & 8 & & 9 & 8 & & 16 & & & 16 & & & 11 & 10 & \\
\lambda = & 14 & \mapsto & 14 & & \mapsto & 14 & & \mapsto & 8 & 7 & \mapsto & 8 & & \mapsto & 8 & & = \mu. \\
 & \overline{11} & & 6 & 5 & & 6 & & & 6 & & & 7 & 6 & & 7 & 6 & \\
 & 4 & & 4 & & & 5 & 4 & & 5 & 4 & & 5 & 4 & & 5 & 4 & \\
 & 2 & & 2 & & & 2 & & & 2 & & & 2 & & & 2 & &
\end{array}$$

We establish λ from μ according to the above transformation in reverse order.

Proof of Corollary 1.10. Based on the bijection in the above proof, the corollary follows immediately. $\square$

4. Concluding remarks

From (3.5)-(3.11), (3.13) and (3.15), in addition to Theorem 1.8, we can also establish some other generating functions for $\mathscr{A}^k_S$ for some given S. For example, the generating functions for $\mathscr{A}^k$ and $\mathscr{A}^k_{\{1,\bar{1}\}}$ are given below.

$$\sum_{\lambda\in\mathscr{A}^k} x^{\sharp(\lambda)} y^{\sharp_{0,2}(\lambda)} z^{O(\lambda)} q^{|\lambda|} = \sum_{n_1,n_2,n_3\geqslant 0} \frac{x^{n_1+n_2+n_3} y^{n_3} z^{n_2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_{n_3}} \times q^{2k\binom{n_1}{2}+(2k+2)\binom{n_2}{2}+2k\binom{n_3}{2}+2kn_1n_2+2kn_1n_3+2kn_2n_3+n_1+n_2+2n_3},$$

$$\sum_{\lambda\in\mathscr{A}^k_{\{1,\bar{1}\}}} x^{\sharp(\lambda)} y^{\sharp_{0,2}(\lambda)} z^{O(\lambda)} q^{|\lambda|} = \sum_{n_1,n_2,n_3\geqslant 0} \frac{x^{n_1+n_2+n_3} y^{n_3} z^{n_2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_{n_3}} \times q^{2k\binom{n_1}{2}+(2k+2)\binom{n_2}{2}+2k\binom{n_3}{2}+2kn_1n_2+2kn_1n_3+2kn_2n_3+3n_1+3n_2+2n_3}.$$

So it would be interesting to find some other q-series identities like (1.2) to establish more relations among partitions.

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