Every signed planar graph of girth 5 has circular chromatic number strictly less than 4

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Abstract

For a real number $r \geq 2$, a circular r-colouring of a signed graph (G, σ) is a mapping $c : V(G) \to [0, r)$ such that $|c(x) - c(y)| \in [1, r - 1]$ for each positive edge xy and $|c(x) - c(y)| \in [0, r/2 - 1] \cup [r/2 + 1, r)$ for each negative edge xy. This concept is recently introduced by Naserasr, Wang, and Zhu in 2021, and they show that for any $\varepsilon > 0$, there exist signed planar bipartite graphs (of girth 4) which are not circular $(4 - \varepsilon)$-colourable. In this paper, we prove that for each signed planar graph (G, σ) of girth at least 5, there exists a real number $\varepsilon = \varepsilon(G, \sigma) > 0$ such that (G, σ) is circular $(4 - \varepsilon)$-colorable. Our proof utilizes a Thomassen-type inductive argument on the dual version in terms of circular flows, which is motivated by a result of Richter, Thomassen, and Younger (2016) on group connectivity of 5-edge-connected planar graphs.

Keywords: circular coloring, signed graphs, circular flow, group connectivity.

1 Introduction

A *signed graph* is a pair (G, σ), where G is a graph and σ: $E(G) \to \{+, -\}$ is a signature which assigns to each edge of G a sign. Let $r \geq 2$ be a real number. As introduced by Naserasr, Wang, and Zhu in [6], a circular r-colouring of a signed graph (G, σ) is a mapping $c : V(G) \to [0, r)$ such that $|c(x) - c(y)| \in [1, r - 1]$ for each positive edge xy and $|c(x) - c(y)| \in [0, r/2 - 1] \cup [r/2 + 1, r)$ for each negative edge xy. The *circular chromatic number* of a signed graph (G, σ) is defined as

$$\chi_c(G, \sigma) = \inf\{r : G \text{ admits a circular } r\text{-colouring}\}.$$

It is shown in [6] that $\chi_c(G, \sigma)$ is well-defined and must be a rational number. This concept is a refinement of 0-free $2k$-coloring of signed graphs and extends the circular coloring concept introduced by Vince [8] from graphs to signed graphs.

The classical Grötzsch's theorem states that every triangle-free planar graph is circular 3-colorable. This is no longer true for signed graphs, as it is observed in [6] that $\chi_c(G, \sigma) \leq 4$ for any signed bipartite graph (G, σ), and there exists a sequence of signed bipartite planar graphs whose circular chromatic numbers are tending to 4. On the other hand, the circular chromatic number of any signed bipartite planar graph cannot be equal to 4 as proved in [2].

Theorem 1.1. *(Kardos, Narboni, Naserasr, and Wang [2]) For every signed bipartite planar graph (G, σ), $\chi_c(G, \sigma) < 4$.*

In [2], it is also observed that every 2-degenerate signed graph has circular chromatic number strictly less than 4. By Euler's formula, it is straightforward to obtain that every signed planar graph of girth at least 6 is 2-degenerate, and thus has circular chromatic number strictly less than 4. Our main result of this paper shows that this property is still valid for signed planar graphs of girth 5.

Theorem 1.2. *For every signed planar graph (G, σ) of girth at least 5, $\chi_c(G, \sigma) < 4$.*

We conjecture that this conclusion is still true for signed planar graphs of girth 4, which, if true, would be best possible as can be evidenced by K_4 with all positive signs.

Conjecture 1.3. *For every signed planar graph (G, σ) of girth 4, $\chi_c(G, \sigma) < 4$.*

Note that applying some standard arguments (see Proposition 22 in [6] or Theorem 2.2 in [2]), we may further extend Theorem 1.2 to show that every n-vertex signed planar graph (G, σ) of girth at least 5 satisfies $\chi_c(G, \sigma) \leq 4 - \frac{4}{n+1}$. But this upper bound still relies on the order of the given signed graph. We propose below a stronger conjecture to suggest a universal upper bound smaller than 4.

Conjecture 1.4. *There exists a constant $\varepsilon_0 > 0$ such that every signed planar graph (G, σ) of girth at least 5 satisfies $\chi_c(G, \sigma) \leq 4 - \varepsilon_0$.*

It is proved in [5] that every signed planar graph (G, σ) of girth at least 7 satisfies $\chi_c(G, \sigma) \leq 3$, and so Conjecture 1.4 is true for that subclass of signed graphs.

2 Duality between circular coloring and circular flows in signed graphs

Our proof of Theorem 1.2 actually uses the dual concept about circular flows of signed graphs recently introduced in [3], which is a natural extension of the same concept on graphs [1].

Definition 2.1. *([3]) Given a signed graph (G, σ) and a real number $r \geq 2$, a circular r-flow is a pair (D, f) where D is an orientation and $f : E(G) \to (-r, r)$ satisfies the following three conditions:*

- *for each positive edge e, $|f(e)| \in [1, r-1]$;*
- *for each negative edge e, $|f(e)| \in [0, \frac{r}{2}-1] \cup [\frac{r}{2}+1, r)$;*
- *for each vertex v, $\partial_D f(v) = \sum\limits_{e \in \overrightarrow{E}(v)} f(e) - \sum\limits_{e \in \overleftarrow{E}(v)} f(e) = 0$,*

where $\overrightarrow{E}(v)$ is the set of arcs that v is the tail, and $\overleftarrow{E}(v)$ is the set of arcs that v is the head.

The *circular flow number* of a signed graph (G, σ) is defined as

$$\Phi_c(G, \sigma) = \inf\{r : G \text{ admits a circular } r\text{-flow}\}.$$

Theorem 2.2. *([3]) A signed plane graph (G, σ) admits a circular r-coloring if and only if its dual signed graph (G^*, σ^*) admits a circular r-flow, and thus $\chi_c(G, \sigma) = \Phi_c(G^*, \sigma^*)$.*

In the study of circular coloring and circular flows, we usually use the discrete form for $r = \frac{p}{q} \geq 2$, where we have the following equivalent definition: a circular $\frac{p}{q}$-flow is a pair (D, f) where D is an orientation and $f : E(G) \to \{0, \pm 1, \dots, \pm(p-1)\}$ such that for each edge $e \in E(G)$, $|f(e)| \in \{q, \dots, p-q\}$ if $\sigma(e) = +$ and $|f(e)| \in \{0, \dots, \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, \dots, p-1\}$ if $\sigma(e) = -$, and moreover, for each vertex v,

$$\partial_D f(v) = \sum_{e \in \overrightarrow{E}(v)} f(e) - \sum_{e \in \overleftarrow{E}(v)} f(e) = 0.$$

For convenience, we shall sometimes also use modular flows, whose definition is almost the same except the equality above is taken modulo p. Some relations between circular flow number and strongly connected orientation are established in [4].

Theorem 2.3. *(Li, Thomassen, Wu, and Zhang [4]) A connected graph has circular flow number strictly less than $\frac{p}{q}$ if and only if it admits a modular circular $\frac{p}{q}$-flow (D, f) such that $f : E(G) \to \{q, q+1, \dots, p-q-1\}$ and D is strongly connected.*

Our result for signed graphs has a similar flavor. In fact, we prove the following result concerning strongly connected orientations, which implies Theorem 1.2 as a corollary. We shall need a few more definitions before presenting the following result. Let $k \geq 2$ be an integer. A mapping $\alpha : V(G) \to \mathbb{Z}_k$ is called a $\mathbb{Z}_k$-boundary if $\sum\limits_{v \in V(G)} \alpha(v) \equiv 0 \pmod{k}$. For a $\mathbb{Z}_k$-boundary α, a $(\mathbb{Z}_k, \alpha)$-flow on a graph G is a pair (D, f) where D is an orientation and $f : E(G) \to \{\pm 1, \pm 2, \dots, \pm(k-1)\}$ such that for each vertex $v \in V(G)$, $\partial_D f(v) = \sum\limits_{e \in \overrightarrow{E}(v)} f(e) - \sum\limits_{e \in \overleftarrow{E}(v)} f(e) \equiv \alpha(v) \pmod{k}$.

Theorem 2.4. *Let G be a 5-edge-connected planar graph. Then for any $\mathbb{Z}_4$-boundary α, there exists a $(\mathbb{Z}_4, \alpha)$-flow (D, f) on G such that $f : E(G) \to \{1, 2\}$ and D is strongly connected.*

Proof of Theorem 1.2 assuming Theorem 2.4: Let (G, σ) be a signed planar graph of girth at least 5, and let (H, σ^*) be its dual signed graph. Then H is 5-edge-connected. For each vertex $v \in V(H)$, denote $d^-(v)$ to be the number of negative edges incident to v. Define $\alpha(v) \equiv 2d^-(v) \pmod 4$ for every $v \in V(H)$. Then

$$\sum_{v \in V(H)} \alpha(v) \equiv 2 \sum_{v \in V(H)} d^-(v) = 4|\{e : \sigma^*(e) = -\}| \equiv 0 \pmod 4,$$

and so α is a $\mathbb{Z}_4$-boundary of H. By Theorem 2.4, H has a $(\mathbb{Z}_4, \alpha)$-flow (D, f_1) such that $f_1 : E(G) \to \{1, 2\}$ and D is strongly connected.

Define another mapping $f_2 : E(H) \to \mathbb{Z}_4$ such that for each $e \in E(H)$,

$$f_2(e) = \begin{cases} 2 & \text{if } \sigma^*(e) = -, \\ 0 & \text{if } \sigma^*(e) = +. \end{cases}$$

Let $f_3 = f_1 + f_2$. Consider the pair (D, f_3). For each edge $e \in E(H)$, we have $f_3(e) \in \{1, 2\}$ if e is positive and $f_3(e) \in \{0, 3\}$ otherwise. Moreover, for any vertex $v \in V(H)$, we have

$$\partial_D f_3(v) = \partial_D f_1(v) + \partial_D f_2(v) \equiv \alpha(v) + 2d^-(v) \equiv 0 \pmod 4.$$

Hence (D, f_3) is a $(\mathbb{Z}_4, 0)$-flow on H.

Since D is strongly connected, every arc a in $D(H)$ is contained in a directed cycle, say C_a. For every arc $a \in A(D(H))$, define f_a: $E(H) \to \{0, 1\}$ by setting

$$f_a(e) = \begin{cases} 1 & e \in E(C_a), \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = |E(H)| + 1$. Define $f = Mf_3 + \sum_{a \in A(D(H))} f_a$. Clearly, $\partial_D f(v) \equiv 0 \pmod{4M}$ for each $v \in V(H)$ by definition. Furthermore, for any edge $e \in E(H)$, since $\sum\limits_{a \in A(D(H))} f_a(e) \in \{1, \ldots, M-1\}$, we have

$$f(e) \in \{M+1, \ldots, 3M-1\} \text{ if } e \text{ is positive}$$

and

$$f(e) \in \{1, \ldots, M-1\} \cup \{3M-1, \ldots, 4M-1\} \text{ if } e \text{ is negative.}$$

Therefore, (D, f) is a modular circular $\frac{4M}{M+1}$-flow on the signed graph (H, σ^*). By duality from Theorem 2.2, we conclude that (G, σ) admits a circular $\frac{4M}{M+1}$-coloring, i.e., $\chi_c(G, \sigma) < 4$. This completes the proof of Theorem 1.2. ■

Next, we shall prove Theorem 2.4 in the rest of this paper. In fact, we utilize a Thomassen-type induction to prove a stronger theorem, which implies Theorem 2.4. Our technical theorem and proof ideas are mainly motivated by a result of Richter, Thomassen, and Younger [7] on group connectivity of 5-edge-connected planar graphs.

However, we have to make certain modifications for our purpose of searching strongly connected orientations.

For a $\mathbb{Z}_4$-boundary β of a graph G and a subset $A \subseteq V(G)$, we define

$$\beta(A) \equiv \sum_{v \in A} \beta(v) \pmod 4.$$

For a vertex $v \in V(G)$, we use $\delta(v)$ to denote the set of edges incident to v, and similarly we use $\delta(A)$ to denote the set of edges with exactly one end in A for a vertex subset A. We use $deg(v)$ to denote the degree of a vertex v, and we say v is a k-vertex if $deg(v) = k$. For a vertex subset $A \subseteq V(G)$, we use A^c to denote the complement of A in $V(G)$, while $\bar{A}$ is the complement of A in the vertex set of a certain subgraph of G. A k-cut is an edge cut of size k. A 2-cut $[A, A^c]$ is said to be *bad* if $\beta(A) \equiv \beta(A^c) \equiv 2 \pmod 4$.

Now we are ready to state our main theorem below:

Theorem 2.5. *Let G be a 3-edge-connected planar graph embedded in the plane. Let β be a given $\mathbb{Z}_4$-boundary of G. Suppose that G has at most two specified vertices d and t such that:*

(i) if d exists, then it is in the boundary of the unbounded face, has degree 3, 4, or 5, and has its incident edges oriented and labelled with 1 or 2 satisfying boundary $\beta(d)$ (i.e., at vertex d the outflow minus inflow is congruent to $\beta(d)$ modulo 4);

(ii) if t exists, then it has degree 3 and is in the boundary of the unbounded face;

(iii) except for possibly $\delta(d)$ and $\delta(t)$, every edge-cut of G is of size at least 4;

(iv) if d has degree 5, then t does not exist;

(v) every vertex not in the boundary of the unbounded face has five edge-disjoint paths to the boundary of the unbounded face;

(vi) $G - d$ has no bad 2-cut (Note that $G - d = G$ when d does not exist).

Then the prescription at d can be extended to a $(\mathbb{Z}_4, \beta)$-flow (D, f) on G such that:

(a) $\partial_D f \equiv \beta \pmod 4$, that is, $\sum_{e \in \overrightarrow{E}(v)} f(e) - \sum_{e \in \overleftarrow{E}(v)} f(e) \equiv \beta(v) \pmod 4$ for every $v \in V(G)$;

(b) $f(e) \in \{1, 2\}$ for every $e \in E(G)$;

(c) $D(G - d)$ is strongly connected.

Theorem 2.4 follows from it when the first specified vertex d does not exist. The rest of this paper is devoted to a proof of Theorem 2.5. We first investigate the properties of a potential minimal counterexample in Section 3, especially those about small cuts, which will be frequently used later. In Section 4, we apply some more structural results to find certain local configurations to complete the proof. We will divide our proof into three different cases, determined by whether there exists a copy of contractible structure containing t, and whether the set of boundary vertices of the unbounded face is sparse enough, and deal with them using proper methods accordingly.

3 Properties of the Minimal Counterexample to Theorem 2.5

With a little abuse of the usage of symbols, in the remaining part of this paper, we always use G (with a $\mathbb{Z}_4$-boundary β) to denote the minimal counterexample of Theorem 2.5 in the sense of the lexicographical order $(|V(G)| + |E(G)|, |E(G - d)|)$. It is a trivial job to check that $|V(G)| > 3$.

We will deal with plenty of cuts and edges in the proof, for the sake of clarity and readability, it is necessary to make a statement about notations at first. When multiple cuts occur in a part of the proof, we name them after B, F, K and Q in sequence, and in this section, we always use A to represent a cut when stating the content of a theorem. When we need to operate on a single edge, if it is unoriented, we note it e; if it is an arc, we note it a; the two end vertices are chosen to be x and y. When we need to operate on a vertex, we note it z. These notations are independent between different theorems, propositions and sections, and we suggest the readers to keep this statement in mind.

In this section, we study the necessary properties of G and list them by a series of propositions. *Prescribing a vertex* means orienting all edges incident to it and labelling each of them with a value. We say the orientation of a vertex is *proper*, or the vertex is *properly oriented* if it is neither a sink nor a source; in other words, the vertex has both indegree and outdegree nonzero.

We start with a lemma in [4], and we provide a proof here for completeness. For convenience, a strongly connected orientation is called *strong* for short in the rest of the paper.

Lemma 3.1. *([4]) Let H be a 2-edge-connected graph and $e = xy$ be an edge of H. If H/e has a strong orientation D, then D can be extended to a strong orientation of H.*

Proof. If D is strong on $H - e$, just orient e arbitrarily; if D is not strong on $H - e$, then there exists an arc $e' \in H - e$ not on a directed cycle, but e' is on some directed cycle in H/e. It can be deduced that there is a directed path between x and y in $H - e$, and that all such paths are directed the same direction, say from x to y. Then orient e from y to x. It is now easy to check that there exists a directed path between any pair of vertices of H, and D is extended to a strong orientation of H. ∎

Now we begin to present some properties of the minimal counterexample G. The first one says that G is essentially a simple graph, which is the basis of the whole proof of Theorem 2.5.

Proposition 3.2. *$G - d$ contains no multiple edges (which are unoriented).*

Proof. On the contrary, assume two vertices x and y form an unoriented sK_2 ($s \geq 2$) in $G - d$. By the induction, $G/\{x, y\}$ admits a $(\mathbb{Z}_4, \beta')$-flow (D', f') with a strong orientation $D'(G/\{x, y\} - d)$, where β' is the $\mathbb{Z}_4$-boundary induced from β, with the contraction vertex receiving value $\beta(x) + \beta(y)$. As for sK_2 itself, f' induces a $\mathbb{Z}_4$-boundary γ on it, where $\gamma(x)$ ($\gamma(y)$) is the difference of $\beta(x)$ ($\beta(y)$) and the total value x (y) receives from f'.

We claim that there always exists a prescription of sK_2 with a strong orientation to realize $\gamma(x)$ and $\gamma(y)$ at x and y, except the only case $s = 2$ and $\gamma(x) = \gamma(y) = 2$. This is clear when $s \geq 3$; as for $s = 2$, if $\gamma(x) = \gamma(y) = 0$, then we label the two edges with 1 and orient them oppositely; if $\{\gamma(x), \gamma(y)\} = \{1, 3\}$, then we label the two edges with 1 and 2, and orient them oppositely. In these cases, sK_2 receives a strong orientation, and thus (D', f') can be extended to a flow (D, f) on G naturally such that $D(G - d)$ is strong, resulting in a contradiction.

When $s = 2$ and $\gamma(x) = \gamma(y) = 2$, the two parallel edges between x and y must be oriented to the same direction and labeled with 1. Now we apply Lemma 3.1, there exists a certain orientation of the two edges to extend D' to a strong orientation of $G - d$, and f' is extended to a function f on $E(G)$ with the desired boundary β by labelling the two edges with 1. A contradiction is obtained too. ∎

Proposition 3.3. *G is 2-connected.*

Proof. Let z be a cut vertex separating G into two subgraphs G_1 and G_2 with $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = \{z\}$, and $d \in V(G_1)$. We use $deg_i(v)$ to denote the degree of a vertex v in G_i ($i = 1, 2$). Notice that $deg_i(z) = |\delta(G_i \setminus \{z\})|$, by assumption (iii) of Theorem 2.5, $deg_i(z) \geq 3$ holds for $i = 1, 2$.

If $deg_i(z) = 3$ for some i, then $\delta(G_i \setminus \{z\})$ is a 3-cut, $G_i \setminus \{z\}$ can only be d or t, and by Proposition 3.2, $G_i \setminus \{z\} = d$. Now d is a pendent vertex of G whose only neighbour is z, we can apply the induction on $G - d$ (setting the prescribed special vertex non-existent) and obtain a flow with a strong orientation, which is also a flow on G satisfying the conditions in Theorem 2.5.

If both $deg_1(z) \geq 4$ and $deg_2(z) \geq 4$, then by the minimality of G, G_1 admits a $(\mathbb{Z}_4, \beta_1)$-flow (D_1, f_1) with a strong orientation $D_1(G_1 - d)$, and G_2 admits a $(\mathbb{Z}_4, \beta_2)$-flow with a strong orientation $D_2(G_2)$ (setting the prescribed special vertex non-existent). $(D_1 \cup D_2, f_1 \cup f_2)$ is a desired $(\mathbb{Z}_4, \beta)$-flow on G with a strong orientation of $G - d$, a contradiction. ∎

From Proposition 3.3, we know that the boundary of the unbounded face of G is a cycle, use C to represent it. We say a cut $[A, A^c]$ of G is *essential* if $\min\{|A|, |A^c|\} \geq 2$, and *peripheral* if one of $|A|$ and $|A^c|$ is 1.

The next several propositions are concerned with essential 4-cuts and 5-cuts. We use a contraction method in their proofs, that is, contracting the vertex set on one side of an edge cut to a single vertex. Suppose vertex set X is contracted; the contraction naturally induces a $\mathbb{Z}_4$-boundary of G/X from β, by choosing $\beta(X)$ as the boundary value of the vertex contracted from X. In the remaining text, the phrase "the induced boundary" is used to refer to it. At first we introduce a fact from straightforward observation of the conditions (i)-(vi) in Theorem 2.5, which implies that contraction always preserves planarity. Recall that a *bond* of a graph H is a minimal non-empty edge cut. If $[X, X^c]$ is a bond, then both $H[X]$ and $H[X^c]$ are connected induced subgraphs.

Fact 3.4. *In G, cuts of size at most 5 are bonds; an essential 6-cut is a bond unless it is $\delta(\{d,t\})$, on the premise that $deg(d) = 3$, and d and t are not adjacent.*

Proposition 3.5. *There does not exist an essential 4-cut $[A, A^c]$ such that $d \in A$ and $G[A^c]$ is 3-edge-connected.*

Proof. By contracting A^c to a single vertex and applying the induction on G/A^c, we obtain a $(\mathbb{Z}_4, \beta_1)$-flow (D_1, f_1) with a strong orientation $D_1(G/A^c - d)$. Next, contract A to a vertex z in G, prescribe z according to f_1, and then apply the induction on G/A by viewing z as the new "d". There is another $(\mathbb{Z}_4, \beta_1)$-flow (D_2, f_2) with a strong orientation $D_2(G[A^c] - z)$. Combining the two flows together, we obtain a flow on G with a strong orientation of $G - d$, a contradiction. ∎

Proposition 3.6. *There does not exist an essential 5-cut $[A, A^c]$ such that $\{d, t\} \subseteq A$, and $G[A^c]$ is 3-edge-connected.*

Proof. The proof is very similar to the proof of Proposition 3.5. Note that, with the assumption $\{d, t\} \subseteq A$, we can apply the induction on G/A^c and G/A appropriately, as the conditions (i)-(vi) in Theorem 2.5 can be justified for both G/A^c and G/A. By applying the same method as in the proof of Proposition 3.5, we combine the two flows on G/A^c and G/A together to construct a flow on G with a strong orientation of $G - d$. This leads to a contradiction, hence verifying the proposition. ∎

We are now able to strengthen the two propositions above by deleting the restriction "3-edge-connected".

Proposition 3.7. *There is no essential 4-cuts in G.*

Proof. On the contrary, suppose that there exists an essential 4-cut $[A, A^c]$ with $d \in A$, we take the one with $|A^c|$ minimized. Since G is 3-edge-connected and the only possible 3-cuts are $\delta(d)$ or $\delta(t)$, $G[A^c]$ is 2-edge-connected, and by Proposition 3.5 $G[A^c]$ must contain a 2-cut. From Proposition 3.2, it can be deduced that $|A^c| \geq 3$, and by the minimality of $|A^c|$, this 2-cut can only be $[t, A^c \backslash \{t\}]$. As a result, $\delta(A \cup \{t\})$ is an essential 5-cut.

Among all essential 5-cuts $[B, B^c]$ with $A \cup \{t\} \subseteq B$, choose the one with $|B^c|$ minimized. $G[B^c]$ is 2-edge-connected, and by Proposition 3.6, $G[B^c]$ has a 2-cut

$[F, \bar{F}]$, with $|[B, F]| = 2$ and $|[B, \bar{F}]| = 3$, so $|\delta(F)| = 4$ and $|\delta(\bar{F})| = 5$. By minimality of $|A^c|$ and $|B^c|$, both F and $\bar{F}$ are singletons linked by parallel double edges, a contradiction to Proposition 3.2. ∎

Proposition 3.8. *There does not exist an essential 5-cut $[A, A^c]$ such that $\{d, t\} \subseteq A$.*

Proof. Suppose that there exists such an essential 5-cut, take the one with $|A^c|$ minimized. Since G is 3-edge-connected and the only possible 3-cuts are $\delta(d)$ or $\delta(t)$, $G[A^c]$ is 2-edge-connected, and by Proposition 3.6 $G[A^c]$ must contain a 2-cut $[B, \bar{B}]$, $|[A, B]| = 2$ and $|[A, \bar{B}]| = 3$, so $|\delta(B)| = 4$ and $|\delta(\bar{B})| = 5$. From Proposition 3.7 and the minimality of $|A^c|$, both B and $\bar{B}$ are singletons linked by parallel double edges, a contradiction to Proposition 3.2. ∎

From Proposition 3.7 and Proposition 3.8, two corollaries are obtained, which will be helpful in later proofs.

Corollary 3.9. *$G - d$ is 3-edge-connected.*

Proof. Suppose $[B, \bar{B}]$ is a cut of $G - d$ of size at most 2. First we point out that its size is exactly 2. Assume $|[B, \bar{B}]| \leq 1$, since G is 3-edge-connected and $deg(d) \leq 5$, $|[d, B]| \geq 2$, $|[d, \bar{B}]| \leq 3$, and both $|\delta(B)|$ and $|\delta(\bar{B})|$ are at most 4. From Proposition 3.7, both B and $\bar{B}$ are singletons, but we have mentioned that $|V(G)| > 3$ at the beginning of this section.

So $|[B, \bar{B}]| = 2$, and $|[d, B]| \geq 1$, $|[d, \bar{B}]| \geq 1$. If $|[d, B]|$ or $|[d, \bar{B}]|$ equals 1 (assume it is the former), then $B = \{t\}$, from condition (iv), $deg(d) \leq 4$, hence $|[d, \bar{B}]| \leq 3$ and $|\delta(\bar{B})| \leq 5$. By Proposition 3.8, $\bar{B}$ is a singleton as well, limiting G to 3 vertices.

So $|[d, B]| \geq 2$, $|[d, \bar{B}]| \geq 2$, and since $deg(d) \leq 5$, at least one of the equalities holds (still assume it is B). Thus $\delta(B) = 4$, by Proposition 3.7, B is a singleton. $|\delta(\bar{B})| = 4$ when $deg(d) = 4$ and 5 when $deg(d) = 5$, by Proposition 3.7 or 3.8 respectively, $\bar{B}$ is a singleton too, contradicting Proposition 3.2. ∎

Corollary 3.10. *An essential 7-cut of G is a bond unless it has the form $\delta(\{x_1, x_2\})$, where x_1, x_2 are two vertices, $deg(x_1) = 3$, $deg(x_2) = 4$, and they are not adjacent in G.*

Proof. Since G is 3-edge-connected, an essential 7-cut which is not a bond can only be a union of two cuts, whose size are 3 and 4 respectively. By Proposition 3.7, both of them are formed by a single vertex. ∎

In the following three propositions, we show that the two special vertices d and t both exist, and they are not adjacent.

Proposition 3.11. *d exists in G, and $deg(d) = 3$ or 4.*

Proof. If d does not exist, then G belongs to one of the following cases, and in each of them we can apply the induction on a smaller graph with less unoriented edges.

(1) All vertices in the boundary have degree at least 5. Take an edge e in the boundary and delete it, then essential cuts in $G-e$ have size at least 4 by Proposition 3.7. Apply the induction on $G-e$.

(2) There is no 3-vertex but a 4-vertex z in the boundary. $G-z$ is 2-edge-connected; assume there exists a 2-cut $[B,\bar{B}]$. In this case G has no cut of size at most 3, so $|[z,B]| = |[z,\bar{B}]| = 2$. By Proposition 3.7, both B and $\bar{B}$ are singletons with parallel double edges in between, violating Proposition 3.2, so $G-z$ is 3-edge-connected. Prescribe z properly and appoint it as "d" in Theorem 2.5, then we can apply the induction on $G-z$.

(3) There exists a 3-vertex z' in the boundary. Assume there exists a 2-cut $[F,\bar{F}]$ in $G-z'$, setting $|[z',F]| = 2$, $|[z',\bar{F}]| = 1$ without loss of generality. By Proposition 3.7, F is a singleton linked to z' by parallel double edges, violating Proposition 3.2, so $G-z'$ is 3-edge-connected. Prescribe z' properly and appoint it as "d" in Theorem 2.5, then we can apply the induction on $G-z'$.

Now we have proved the existence of d. If $deg(d) = 5$, choose an arc a in the boundary incident to d, and apply the induction on $G-a$. ∎

Proposition 3.12. *t exists.*

Proof. Suppose on the contrary that t does not exist. Then all boundary vertices except d have degree at least 4.

When $deg(d) = 4$, choose an arc a in the boundary incident to d, and apply the induction on $G-a$.

When $deg(d) = 3$, by Proposition 3.8, an essential cut formed by d and one of its boundary neighbours has size at least 6, and it can be deduced that the minimum degree of $G-d$ is at least 4. Moreover, $G-d$ contains no essential cut of size at most 3, since such a cut yields an essential cut of size at most 4 in G. We apply the induction on $G-d$. ∎

Proposition 3.13. *d and t are not adjacent.*

Proof. If d and t are adjacent, $\delta(\{d,t\})$ is an essential cut of size 4 or 5, contradicting Proposition 3.7 or Proposition 3.8. ∎

In the following propositions, we make some preliminary descriptions of the structure of G, which is the basis of our analyses in Section 4. We end this section with a more detailed depiction about essential 5-cuts as a corollary of these descriptions.

Proposition 3.14. *t is not incident to a chord.*

Proof. On the contrary, suppose tx is a chord separating G into two subgraphs G_1 and G_2 with $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{t,x\}$, and $d \in V(G_1)$. We use deg_i to denote the degree of a vertex in G_i ($i = 1,2$). In G_1, d and t are not adjacent; in G_2, there exists a vertex in the boundary other than $\{t,x\}$, with degree at least 4. By Proposition 3.2, each of G_i has at least four vertices, and it can be deduced from Proposition 3.7 that $deg_i(x) \geq 5$.

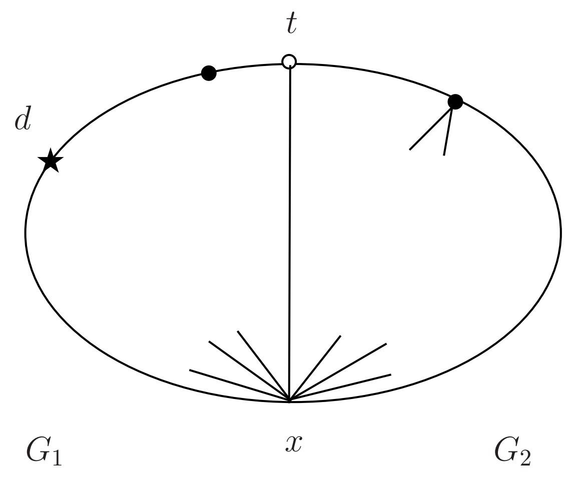

Figure 1

In G_1, contract tx to a vertex z. Denote the $\mathbb{Z}_4$-boundary of G_1/tx by β_1, $\beta_1(v) = \beta(v)$ for $v \in V(G_1) \setminus \{t, x\}$ and set the boundary value $\beta_1(z)$ so as to make the sum $\sum_{w \in V(G_1/tx)} \beta_1(w) = 0$. $G_1/tx - d$ contains no bad 2-cut under β_1, otherwise yielding a bad 2-cut under β in $G - d$. Now apply the induction on G_1/tx, and we obtain a $(\mathbb{Z}_4, \beta_1)$-flow (D_1, f_1) with a strong orientation $D_1(G_1/tx - d)$. We are going to construct a flow on G_1 stemming from it. From the proof of Proposition 3.2, we can orient tx to extend D_1 to a strong orientation of $G_1 - d$. We choose this orientation, but do not label tx right now.

Prescribe t to achieve $\beta(t)$ in G with the two unlabelled edges in G_2, hence tx has been definitely oriented and labelled so far, a flow on G_1 is obtained. Add an extra arc $a = tx$ labelled with a fixed value 1 or 2. $G_2 + a$ has no essential cut of size at most 3, since such a cut must separate t and x, and because $|[t, V(G_1) \setminus \{t, x\}]| = 1$, it will yield an essential cut of size at most 4 in G. Furthermore, $G_2 - t$ has no 2-cut, otherwise a cut of size at most 3 is induced in G, which is neither $\delta(d)$ nor $\delta(t)$. Denote the $\mathbb{Z}_4$-boundary of G_2 by β_2. Determine $\beta_2(t)$ according to its prescription in $G_2 + a$, $\beta_2(v) = \beta(v)$ for $v \in V(G_2) \setminus \{t, x\}$, and set $\beta_2(x)$ so as to make the sum $\sum_{w \in V(G_2)} \beta_2(w) = 0$. Now we can apply the induction on $G_2 + a$, a $(\mathbb{Z}_4, \beta_2)$-flow (D_2, f_2) on $G_2 + a$ (and thus on G_2) is obtained, with a strong orientation $D_2(G_2 - t)$. Combining the prescription of t, there arises a $(\mathbb{Z}_4, \beta)$-flow on G with a strong orientation of $G - d$, a contradiction. ∎

Proposition 3.15. *The degree of boundary neighbours of t is exactly 4.*

Proof. Assume one boundary neighbour of t has degree at least 5, then $G - t$ satisfies conditions (i), (ii) and (v) in Theorem 2.5, and our goal is to check (iii) and (vi) to apply the induction on $G - t$. First, since $deg(t) = 3$, $G - t$ has no essential cut of size at most 3, otherwise it induces an essential cut of size at most 4 in G. We will show that (vi) holds if t has been wisely prescribed by a series of Lemmas and Claims.

Say a *composition* of $b \in \mathbb{Z}_4$ is a way of writing b as the sum of an ordered sequence of values in $\mathbb{Z}_4$. We start with a preparing lemma:

Lemma 3.16. *Let L_1, L_2 be two subsets of $L_3 = \{1, -1, 2\} \subseteq \mathbb{Z}_4$, where $|L_1| = |L_2| = 2$. For every value $b \in \mathbb{Z}_4$, b has a composition $b = b_1 + b_2 + b_3$, $b_i \in L_i$ $(i = 1, 2, 3)$, where either $2 \in \{b_1, b_2, b_3\}$ or $\{1, -1\} \subseteq \{b_1, b_2, b_3\}$.*

Proof of Lemma 3.16. Both L_1 and L_2 can be regarded to be the remaining part after deleting one value from L_3. We make a table of all possible values of $L_1 + L_2$:

Deleted values	$L_1 + L_2$
1, 1	0, 1, 2
-1, -1	0, -1, 2
2, 2	0, 2
1, -1	0, 1, -1
1, 2	0, 1, -1, 2
-1, 2	0, 1, -1, 2

Anyway, $L_1 + L_2 + L_3 = \{0, 1, -1, 2\}$. We discuss the composition of b according to its parity.

- When b is even, there exists a 2 in any composition of b.
- When $b = \pm 1$, then if none of 2 is deleted, $b = 2 + 2 + b$; if one of 2 is deleted, $b = b + 2 + 2$ or $b = 2 + b + 2$; if two of 2 are deleted, $b = 1 + (-1) + b$.

This completes the proof of Lemma 3.16. ∎

Claim 3.17. *t can be prescribed properly without leaving any bad 2-cut in $G - d - t$, under the induced $\mathbb{Z}_4$-boundary.*

Proof. We show that the 2-cuts in $G - d - t$ are too special and rare to make great influence on the prescription of t. Let $[B, \bar{B}]$ be a 2-cut in $G - d - t$.

Subclaim 3.17.1. *B or $\bar{B}$ is a single vertex in $N(t)$.*

Proof of Subclaim 3.17.1. $[B \bigcup t, \bar{B}]$ or $[B, \bar{B} \bigcup t]$ is a 3-cut of $G - d$. We may assume it is $[B \bigcup t, \bar{B}]$, and $|[t, \bar{B}]| = 1$.

In G, $|\delta(B \bigcup t)| > 4$, $|\delta(\bar{B})| > 3$, and $deg(d) = 3$ or 4. It can be deduced from these facts that $|[d, \bar{B}]| \leq 2$, $|\delta(\bar{B})| = 4$ or 5. By Proposition 3.7 or 3.8, $\bar{B}$ is a single vertex adjacent to t.

Subclaim 3.17.2. *There are at most two 2-cuts in $G - d - t$.*

Proof of Subclaim 3.17.2. We write z the single vertex in a 2-cut for convenience. If $deg(d) = 3$, $deg(z) = 4$, z is a boundary neighbour of t and the subclaim holds. If $deg(d) = 4$, we presume that all three possible 2-cuts coexist, and the three corresponding z's are z_1, z_2 and z_3. The configuration of G is shown in Figure 2.

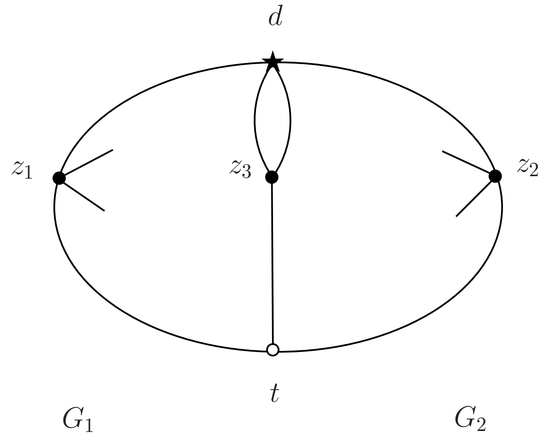

Figure 2

G is separated into two subgraphs G_1 and G_2, with $V(G_1)\bigcap V(G_2) = \{d, z_3, t\}$, $deg(z_1) = deg(z_2) = 4$ and $deg(z_3) = 5$. It can be deduced from Proposition 3.2 that the internal vertex set of G_i is nonempty, and we use U_i to represent it ($i = 1, 2$). From condition (v) of Theorem 2.5, $|\delta(U_i)| \geq 5$, but $|\delta(U_i)| = |\delta(U_i) \cap \delta(z_i)| + |\delta(U_i) \cap \delta(z_3)| \leq 2 + 2 = 4$, a contradiction.

Proof of Claim 3.17: To find a proper prescription of t, equivalently, we may make the three edges incident to t all out-arcs, each of which is equipped with a value list $\{1, -1, 2\} \in \mathbb{Z}_4$. Making at most two 2-cuts not bad is equivalent to selecting at most two of the lists, and delete one value from each of them. By Lemma 3.16, there exists a scheme of labelling containing 2 or $\{1, -1\}$. Since the edge labelled with 2 can be oriented arbitrarily, it corresponds a proper prescription of t. ∎

At last, we have proved that t has a proper prescription making (vi) hold in $G - t$. We can apply the induction on $G-t$, and gain a flow on $G-t$ with a strong orientation of $G-d-t$, which can be naturally extended to a flow on G with a strong orientation of $G - d$. The proof of Proposition 3.15 is finished. ∎

Proposition 3.18. *A vertex of degree* 4 *is not incident to an unoriented chord.*

Proof. Similarly, let x be the 4-vertex, and xy is a chord separating G into two subgraphs G_1 and G_2 with $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{x, y\}$, and $d \in G_1$. We use deg_i to denote the degree of a vertex in G_i ($i = 1, 2$). In G_2, there exists a vertex in the boundary other than $\{x, y\}$, with degree at least 3, so G_2 has at least four vertices, and by Proposition 3.7, $deg_2(y) \geq 4$.

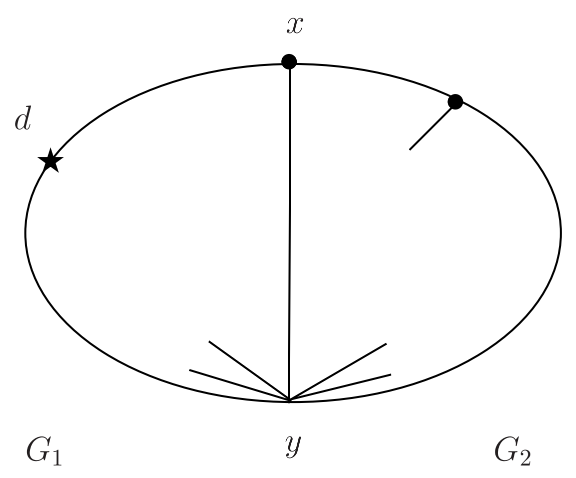

Figure 3

Case 1. $V(G_1) \setminus \{x, y\} = \{d\}$.

In this case $deg_2(x) = 3$, otherwise there are parallel double arcs between d and x, and thus $\delta(\{d, x\})$ is an essential 3 or 4-cut of G.

If there is an essential cut of size at most 3 in G_2, then it separates x and y, or the cut induces an essential cut of size at most 3 in G. Assume it is $[B, \bar{B}]$ and $x \in B$. d is connected with B by a single arc dx, and $\delta(B)$ is an essential cut of size at most 4 in G. So essential cuts in G_2 have size at least 4.

Still in G_2, assume $[F, \bar{F}]$ is a 2-cut of $G_2 - x$ and $y \in F$. $|[x, \bar{F}]| = 1$ or 2, $|\delta(\bar{F})| \leq 4$, so $\bar{F}$ is a singleton. If so, $|[x, \bar{F}]| = 1$ and $deg(\bar{F}) = 3$, hence $\bar{F} = \{t\}$, and the possible 2-cut in $G_2 - x$ is unique. Denote the $\mathbb{Z}_4$-boundary in G_2 by β_2, $\beta_2(x)$ and $\beta_2(y)$ are set to achieve $\beta(x)$ and $\beta(y)$ under the restriction of d, and $\beta_2(w) = \beta(w)$ for $w \in V(G_2) \setminus \{x, y\}$. The same as the proof of Claim 3.17, x can be prescribed properly so that there is no bad 2-cut in $G_2 - x$ under the induced boundary from β_2. Applying the induction on G_2, we obtain a flow (D_2, f_2), where $D_2(G_2 - x)$ is strong, and thus $D_2(G_2)$ is strong.

Case 2. $V(G_1) \setminus \{x, y\}$ has at least two vertices.

Copy the proof of Claim 3.14. G_1/xy admits a flow (D_1, f_1), where $D_1(G_1/xy - d)$ is strong. Add an extra arc $a = xy$ labelled with a fixed value 1 or 2, $G_2 + a$ contains no essential cut of size at most 3. Assume $[K, \bar{K}]$ is a 2-cut of $G_2 - x$, and $y \in K$. $|\delta(\bar{K})| \leq 4$, so $\bar{K}$ must be a single vertex adjacent to x. The only possible 2-cut of $G_2 - x$ is $[t, V(G_2) \setminus \{x, t\}]$.

(1) If x and t are not adjacent, prescribe x so that D_1 is extended to a strong orientation of $G_1 - d$; apply the induction on $G_2 + a$, G_2 admits a flow (D_2, f_2) where $D_2(G_2 - x)$ is strong. Combining the two parts together, we obtain a flow with a strong orientation of $G - d$.

(2) If x and t are adjacent and $deg_2(x) = 3$, prescribe x so that D_1 is extended to a strong orientation of $G_1 - d$ and $[t, V(G_2) \setminus \{x, t\}]$ is not bad under the induced

$\mathbb{Z}_4$-boundary β_2: the orientation of xy is fixed, and xt has at most one forbidden value, so this is equivalent to composing a certain value in $\mathbb{Z}_4$ with three copies of list $\{1, -1, 2\}$, at most two of which having one value deleted. By Lemma 3.16, there exists such a prescription, then apply the induction on $G_2 + a$ as we have done in (1).

(3) If x and t are adjacent and $deg_2(x) = 2$, prescribe x and t so that: (i) D_1 is extended to a strong orientation of $G_1 - d$; (ii) the orientation of t is proper.

$G_2 - x - t$ is 3-edge-connected, and the sizes of essential cuts are at least 4: Let $[Q, \bar{Q}]$ be a cut of size at most 3 with $y \in Q$, t is linked to $\bar{Q}$ by at most two edges, so $4 \leq |\delta(\bar{Q})| \leq 5$. The only possible case is that $\bar{Q}$ is a single vertex in $N(t) \setminus \{x\}$ of degree 4 in G, and $|[Q, \bar{Q}]| = 3$.

We apply the induction on $G_2 - x - t$: by Proposition 3.7, the condition $deg_2(x) = 2$ implies $deg_2(y) \geq 5$; remember t is not incident to a chord, $G_2 - x - t$ contains at most one 3-vertex. By the induction on $G_2 - x - t$ (setting the prescribed special vertex in Theorem 2.5 non-existent), we obtain a flow (D_2, f_2) where $D_2(G_2 - x - t)$ is strong. By prescriptions (i) and (ii), we get a flow with boundary β and a strong orientation of $G - d$ formed by D_1, D_2 and $\{t\}$. This completes the proof. ∎

Proposition 3.19. *There does not exist an essential 5-cut* $[A, A^c]$ *such that* $\delta(A) \cap \delta(t)$ *is a single edge in the boundary.*

Proof. Assume there exists such an $[A, A^c]$. By Proposition 3.8, d and t belong to different sides of the cut, say $d \in A$ and $t \in A^c$. Assume, furthermore that among all such cuts $|A^c|$ is choosen to be minimum. Prescribe t in some way and delete it. Denote the boundary edge of $\delta(A) \cap \delta(t)$ by e. First contract $A^c \setminus \{t\}$ and apply the induction on $(G - t)/(A^c - t)$ (equivalently $G/A^c - e$), then contract A and apply the induction on $G/A - t$. It will be shown that there exists a proper prescription of t to guarantee the proceeding of the two inductions.

We shall check the edge connectivity: essential cuts in $G/A^c - e$ have size at least 4 because of Proposition 3.7; $G/A - t$ contains no essential cut of size at most 3, otherwise yielding an essential cut of size at most 4 in G/A and thus in G.

Claim 3.20. *There is at most one* 2*-cut in* $G/A^c - e - d$.

Proof of Claim 3.20: By Corollary 3.9, $G - d$ is 3-edge-connected; $G/A^c - e - d$ is 2-edge-connected, and any 2-cut contains edge e in G. Let $[B, \bar{B}]$ be such a 2-cut, and suppose the vertex contracted from A^c is in $\bar{B}$. By Proposition 3.7, in G, $|[d, \bar{B}]| \geq 2$, $|[d, B]| \leq 2$, $|\delta(B)| \leq 5$, so B is a singleton from Proposition 3.8, which can only be the boundary neighbour of t contained in A.

Claim 3.21. *There is at most one* 2*-cut in* $G[A^c \setminus \{t\}]$.

Proof of Claim 3.21: Assume $[F, \bar{F}]$ is a 2-cut of $G[A^c \setminus \{t\}]$.

(1) If $|[A, F]|$ or $|[A, \bar{F}]| = 4$, we may suppose it is the former without loss of generality (for convenience, we will always choose F as the special one later in this proof). $|\delta(\bar{F})| \leq 4$, by Proposition 3.7, $\bar{F}$ is a singleton incident to t, resulting in a 3-vertex $w \notin \{d, t\}$ or multiple edges.

(2) If $|[A,F]|=3$, and $|[t,F]|$ or $|[t,\bar{F}]|=2$, then $|\delta(\bar{F})|=3$ or 5, $\bar{F}$ is a singleton incident to t, resulting in a 3-vertex $w\notin\{d,t\}$ or multiple edges too. So $|[t,F]|=|[t,\bar{F}]|=1$, $|\delta(\bar{F})|=4$, and $\bar{F}$ is the boundary neighbour of t not in A, see Figure 4.

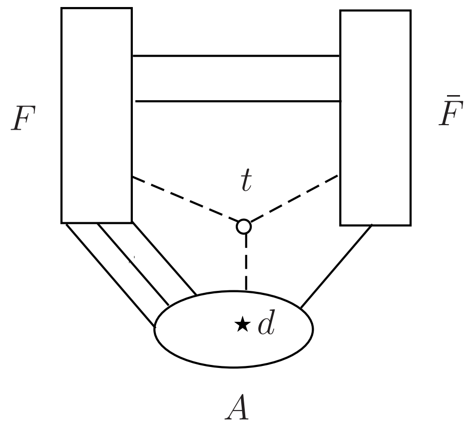

Figure 4

(3) If $|[A,F]|=|[A,\bar{F}]|=2$, and $|[t,F]|=|[t,\bar{F}]|=1$, then $|\delta(F)|=|\delta(\bar{F})|=5$, by Proposition 3.8, both F and $\bar{F}$ are singletons with parallel double edges in between. So t is linked to one of F and $\bar{F}$ by two edges (See Figure 5, where we take F as the special one as usual). $|\delta(F\cup\{t\})|=5$, and $\delta(t)\cap\delta(F\cup\{t\})$ is a single edge in the boundary of G, a contradiction to the minimality of $|A^c|$.

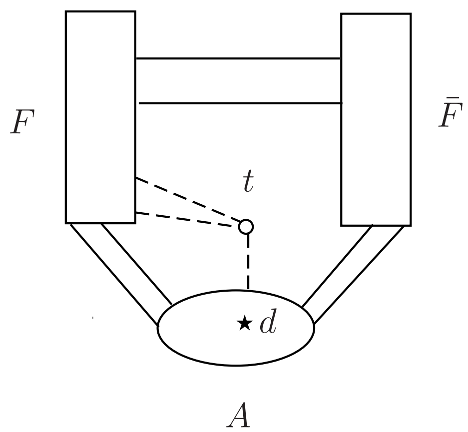

Figure 5

With the two claims above, we make a scheme to prescribe t to satisfy condition (vi) in Theorem 2.5, as we have mentioned at the beginning of this proposition.

There are three ways to orient and label e, of which we choose the one so that: (**I**) the only possible 2-cut of $G/A^c - e - d$ is not bad under the induced $\mathbb{Z}_4$-boundary; (**II**) the induced boundary value of the remaining two edges of t is not 2. We can apply the induction on $G/A^c - e$, and obtain a flow (D_1, f_1), where $D_1(G/A^c - e - d)$ is strong. Transfer this flow to $G/A - t$.

If the only possible 2-cut in $G/A - t$ exists, orient and label the remaining two edges of t to achieve $\beta(t)$, so that: **(1)** the 2-cut is not bad under the induced $\mathbb{Z}_4$-boundary, with the restriction of (D_1, f_1); **(2)** the orientation of t is proper. Because of (**II**), these can be realized, see the illustration in the proof of Proposition 3.2. We can apply the induction on $G/A - t$, and obtain a flow (D_2, f_2), where $D_2(G[A^c \setminus \{t\}])$ is strong. Combining (D_1, f_1), (D_2, f_2) as well as $\{t\}$, we get a flow with a strong orientation of $G - d$. This proves Proposition 3.19. ∎

4 Proof of Theorem 2.5

In this section, we denote the two boundary neighbours of t by u and v, while the internal neighbour is w. The neighbours of v are v_1, v_2, v_3 and t in cyclic order, where v_1 is in the boundary. Let $e_1 = vv_1$ and $e_2 = vv_2$. d and t cut the boundary cycle C (We named it in **Section 3** when finishing the proof of Proposition 3.3) into two segments separating u and v. Denote the two boundary neighbours of d by u' and v', where u and u' are on the same side and v and v' are on the same side.

With v specialized, we will perform different operations on the neighbours of t to create turning points for induction. As a result, a flow with a strong orientation of $G - d$ will be obtained.

4.1 Case 1: $w \in N(u) \cap N(v)$, and v and d are not adjacent.

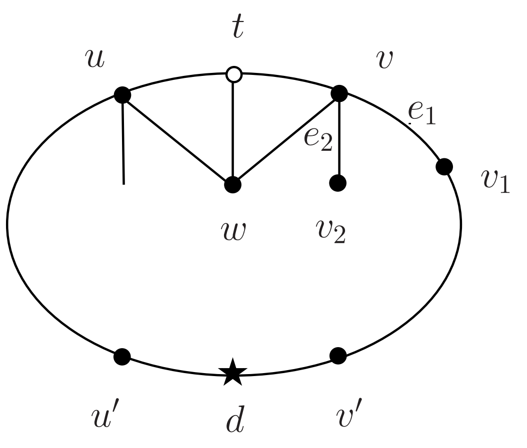

Figure 6: Case I, notice v_1 and v' may be identical.

Orient and label e_1 and e_2 to realize $\beta(v)$. Let $D^* = \{u, t, v, w\}$, and denote by d^* the vertex to which D^* is contracted. Delete e_1 and e_2 and let $G' = G/D^* - e_1 - e_2$. The $\mathbb{Z}_4$-boundary β' is induced from β by the contraction and deletion.

Lemma 4.1. *A flow on $G' - d$ with a strong orientation can be extended to a flow on $G - d$, whose orientation is strong too.*

Proof. This can be done by prescribing u and t wisely within the diamond D^*. Let (D, f) be such a flow on $G' - d$. First, by Lemma 3.1, there exists a certain orientation of uw so that D is extended to a strong orientation of $G - d - t - v$, then we prescribe u with this orientation of uw to achieve $\beta(u)$. The value and orientation of ut may be determined but still, we can prescribe t properly and since the $\mathbb{Z}_4$-boundary value of v in D^* is 0, we prescribe vw with the same value and orientation as tv. As the result, $utvw$ is a directed path, or utw, tvw are two directed paths. Anyway, (D, f) is extended to a flow on $G - d - \{e_1, e_2\}$ with a strong orientation, and thus on $G - d$. ∎

Lemma 4.2. *There is no essential cut of size at most 3 in G'.*

Proof. Assume $[B, B^c]$ is such a cut with $d^* \in B$. The only possible case is that $|\delta(B)| = 3$ and $v_1, v_2 \in B^c$. But if so, then B^c together with v induces an essential 5-cut in G intersecting $\delta(t)$ at tv, a contradiction to Proposition 3.8 or Proposition 3.19. ∎

We now present the last lemma stating that there is at most one 2-cut in $G' - d$. With the three lemmas, we can apply the induction on G' by making it not bad in the prescription of e_1 and e_2.

Lemma 4.3. *There is at most one possible 2-cut in $G' - d$, and if there is one, it can only be $\delta(v_1) \cap E(G' - d)$.*

Proof of Lemma 4.3: Let $[F, \bar{F}]$ be a 2-cut of $G' - d$ with $d^* \in F$, where $\bar{F} \neq \{v_1\}$. We will show that such a cut actually cannot exist.

- If $|[d, \bar{F}]| \leq 1$, then $|\delta(\bar{F})| \leq |[d, \bar{F}]| + |\delta(\bar{F}) \cap \{e_1, e_2\}| + 2 \leq 5$, so $\bar{F}$ is a single vertex by Proposition 3.7 or 3.8. The only possible case is that $|\delta(\bar{F})| = 4$ and thus $\bar{F} = \{v_1\}$.
- If $|[d, F]| \leq 1$, then $|\delta(F)| \leq 5$. While by Proposition 3.7, $|\delta(F)| \geq 5$, the only possible case is that $|\delta(F)| = 5$, and $v_1, v_2 \in \bar{F}$. $\delta(F \setminus \{v\})$ is an essential 5-cut in G intersecting $\delta(t)$ at tv, a contradiction to Proposition 3.19.

So $|[d, F]| = |[d, \bar{F}]| = 2$, and at least one of v_i $(i = 1, 2)$ must be in $\bar{F}$, since if not, $|\delta(F)| = 4$, violating Proposition 3.7.

If exactly one v_i is in $\bar{F}$, $|\delta(\bar{F})| = 5$, so it is v_2 and $\bar{F} = \{v_2\}$. Since $d \neq v_1$, from planarity, it can be shown from Figure 6 that $\{d, v, v_2\}$ forms an essential 7-cut which is not a bond, since $G - \{d, v, v_2\}$ is divided into two non-empty components. By

Corollary 3.10, this will not happen. So both v_1 and v_2 are in $\bar{F}$. This last case, see Figure 7, is more subtle than the ones above, requiring further research on the structures of F and $\bar{F}$ to eliminate it. Denote the two edges of $[F, \bar{F}]$ by e'_1 and e'_2. Contract $F \cup \{d\}$ to a vertex d_1, and contract $\bar{F} \cup \{d\}$ to a vertex d_2 (but not at the same time).

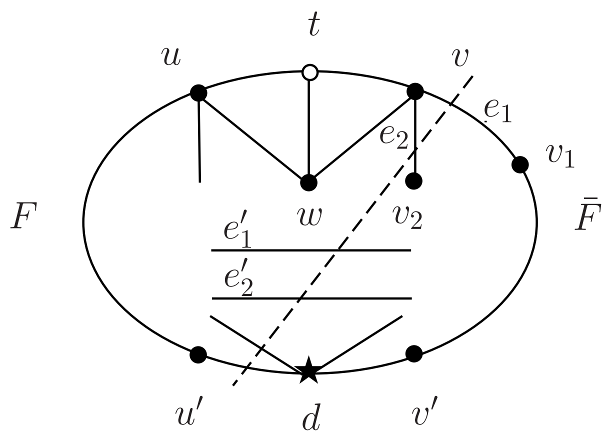

Figure 7: $|[d, F]| = |[d, \bar{F}]| = 2$, and $\{v_1, v_2\} \subseteq \bar{F}$.

Claim 4.4. *$G[\bar{F}]$ has at most one possible 2-cut, and if there is one, it can only be $\delta(v_1) \cap E(G[\bar{F}])$; $G[F]$ has no 2-cut.*

Proof. First concentrate on $G[\bar{F}]$. Suppose $[K, \bar{K}]$ is a 2-cut of $G[\bar{F}]$. It is obvious that both K and $\bar{K}$ are connected with d_1 by at least one edge from $E(G')$. For convenience, we call a path P an *inner path* if internal vertices of P do not lie on C.

Subclaim 4.4.1. *Vertices on C from v' to v_1 in cyclic order are contained in $\bar{F}$.*

Proof of Subclaim 4.4.1: By Fact 3.4, both $\delta(F)$ and $\delta(\bar{F})$ are bonds of G, so the boundary vertices belonging to F and $\bar{F}$ are consecutive respectively, otherwise $G[F]$ and $G[\bar{F}]$ cannot be connected at the same time by planarity. $F \cup \bar{F} \cup \{d\} = V(G)$, $v \in F$ and $v_1 \in \bar{F}$, where it can be deduced that the boundary vertices in $\bar{F}$ are vertices from v' to v_1 in cyclic order.

Subclaim 4.4.2. *In G', $|[d_1, K]| = |[d_1, \bar{K}]| = 2$ is impossible.*

Proof of Subclaim 4.4.2: If so, v_1 and v_2 cannot be separated by the cut, because otherwise $|\delta(K)| = |\delta(\bar{K})| = 5$ in G, by Proposition 3.8, K and $\bar{K}$ are singletons with parallel double edges in between. Assume v_1, $v_2 \in K$ without loss of generality, then $\bar{K}$ is a boundary 4-vertex of G. In fact, $\bar{K} = \{v'\}$, since $\delta(K)$ is an essential 6-bond, the boundary vertices within $K^c = F \cup \{d\} \cup \bar{K}$ are consecutive on C. What is more, $|\delta(v') \cap \{e'_1, e'_2\}| = 1$ because there is exactly one edge between d and v'.

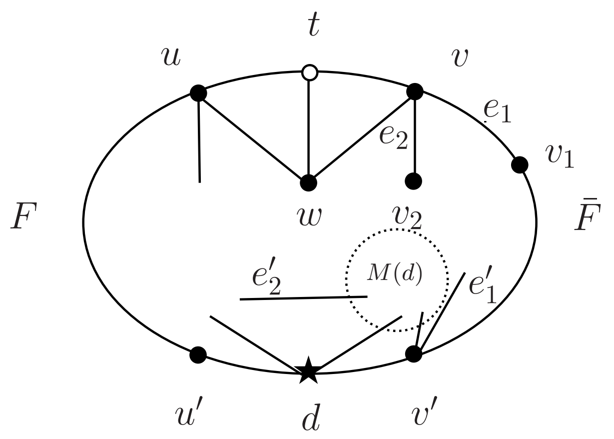

Figure 8: If v' forms a 2-cut of $G[\bar{F}]$.

$\delta(F \setminus \{v\})$ and $\delta(\bar{F} \cup \{v\})$ are essential 6-bonds, so both $G[F \setminus \{v\}]$ and $G[\bar{F} \cup \{v\}]$ are connected. Define $M(d)$ to be the internal vertex set which d can reach through an inner path in $G[\bar{F} \cup \{d, v\}]$. By planarity, d does not form a chord with any vertex in $\{v', \ldots, v_1, v\}$, because one of e_1', e_2' is incident to v'. But d contributes another edge to $\delta(\bar{F})$ besides dv', so $M(d) \neq \emptyset$. Still by planarity, there is no inner path within $G[\bar{F} \cup \{d, v\}]$ connecting d with a boundary vertex in $\bar{F} \cup \{v\}$ except v'. These are shown in figure 8. As a result, $|\delta(M(d))| = |\delta(M(d)) \cap \delta(d)| + |\delta(M(d)) \cap \delta(v')| + |\delta(M(d)) \cap \{e_1', e_2'\}| \leq 1 + 1 + 1 = 3$, a contradiction.

Subclaim 4.4.3. *In G', $|[d_1, K]| = 1$ or 3 is impossible unless $v' = v_1$ and $deg(v_1) = 4$ in G.*

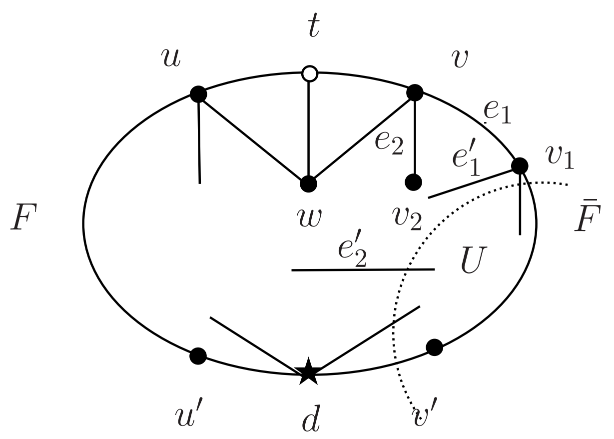

Figure 9: If v_1 forms a 2-cut of $G[\bar{F}]$.

Proof of Subclaim 4.4.3: Let $|[d_1, K]| = 3$ without loss of generality. $|\delta(\bar{K})| \leq 5$, so $\bar{K}$ is a singleton. Actually $\bar{K}$ can only be v_1 with degree 4 in G. If $v' \neq v_1$, then there is no chord between d and v_1, because otherwise there exists an essential 4-cut in G. So $|\delta(v_1) \cap \{e_1', e_2'\}| = 1$. Consider the 6-bond $\delta(\bar{F} \cup \{v\})$, by planarity, in $G[\bar{F} \cup \{v\}]$, no boundary vertex between d and v_1 admits an inner path to v. We define U to be these boundary vertices as well as the internal vertices they can reach through an inner path in $G[\bar{F}]$, see Figure 9. $\delta(U)$ is a cut not separating d and t, while $|\delta(U)| = |\delta(U) \cap \delta(v_1)| + |\delta(U) \cap \delta(d)| + |\delta(U) \cap \{e_1', e_2'\}| \leq 2 + 2 + 1 = 5$. So $U = \{v'\}$ with degree 4 or 5 in G, as the result, at least two inequalities of the three terms in the sum hold, causing parallel double edges between v' and d or v_1, neither of which is possible.

The former part of Claim 4.4 can be deduced from the above three subclaims, and now we turn to $G[F]$. Assume $[Q, \bar{Q}]$ is a 2-cut of $G[F]$ with $d^* \in Q$. In G', $|[d_2, \bar{Q}]| \geq 2$ and by Proposition 3.7, $|[d_2, Q]| \geq 1$. If $|[d_2, Q]| = 1$, then $|\delta(Q)| = 5$, and $\delta(Q \setminus \{v\})$ is an essential 5-cut of G intersecting $\delta(t)$ at tv, violating Proposition 3.19.

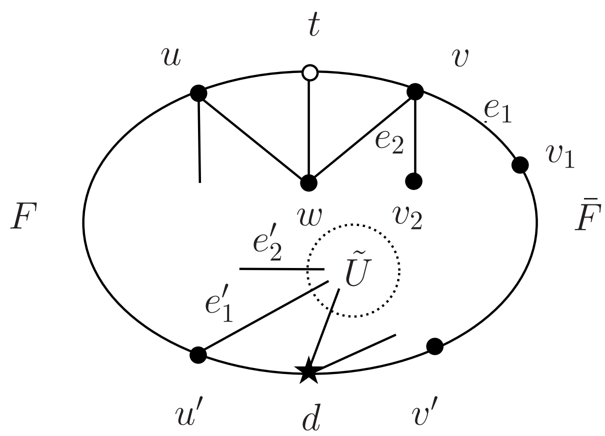

Figure 10: If u' forms a 2-cut in $G[F]$.

So $|[d_2, Q]| = |[d_2, \bar{Q}]| = 2$, and $\bar{Q}$ is a 4-vertex of G. $\delta(\bar{Q} \cup \bar{F} \cup \{d\})$ is a 6-bond, so $\bar{Q}$ can only be u' with degree 4 in G. There is exactly one edge between d and u', so $|\delta(u') \cap \{e_1', e_2'\}| = 1$. Consider the 6-bond $\delta(\bar{F} \cup \{d, v, u'\})$, by planarity, there is no chord nor an inner path in $G[\bar{F} \cup \{d, v, u'\}]$ connecting u' with boundary vertices in $\{v', \ldots, v_1, v\}$ because $|\delta(d) \cap \delta(F \setminus \{v, u'\})| = 1$. Define $\tilde{U}$ to be the set of internal vertices cannot be reached by $\{v', \ldots, v_1, v\}$ through an inner path in $G[\bar{F} \cup \{v\}]$, see Figure 10. It can be deduced that $\tilde{U} \neq \emptyset$ and u' is linked to $\tilde{U}$ by one of e_1' and e_2'. $|\delta(\tilde{U})| = |\delta(\tilde{U}) \cap \delta(d)| + |\delta(\tilde{U}) \cap \{e_1', e_2'\}| \leq 1 + 2 = 3$, a contradiction. The proof of Claim 4.4 is completed. ∎

In the end, we are able to verify Lemma 4.3 by constructing a flow on $G[F]$ and $G[\bar{F}]$ respectively. Remember we have prescribed e_1 and e_2 to achieve $\beta(v)$. Now prescribe e_1' and e_2' to achieve $\beta(F)$ and make sure that the directions of $\{e_1, e_2, e_1', e_2'\}$ between F and $\bar{F}$ are not identical. $G[F]$ has no 2-cut, applying the induction on $F \cup \{d_2\}$ in G', $G[F]$ admits a flow with a strong orientation. $G[\bar{F}]$ has one possible 2-cut $\delta(v_1) \cap E(G[\bar{F}])$ only when $v' = v_1$ and the degree of v_1 is 4 in G. When this case arises, we can guarantee the cut not bad in the prescription of e_1 and e_2: there are at least two ways to orient and label e_1 to make the cut not bad, of which there exists one so that e_2 is not labelled with 0 to achieve $\beta(v)$. So we can apply the induction on $\bar{F} \cup \{d_1\}$ and obtain a flow on $G[\bar{F}]$ with a strong orientation too. Combining them together, there is a flow with a strong orientation of $G' - d$. By Lemma 4.1, a flow with a strong orientation of $G - d$ can be obtained. This completes the proof of Lemma 4.3.

4.2 Case 2: $N(v) \cap N(t) = \emptyset$.

In this case we adopt the lifting method. Consider a vertex z with two neighbours x, y. The *lifting* of two edges xz and yz means deleting them and adding a new edge xy (even if one already exists).

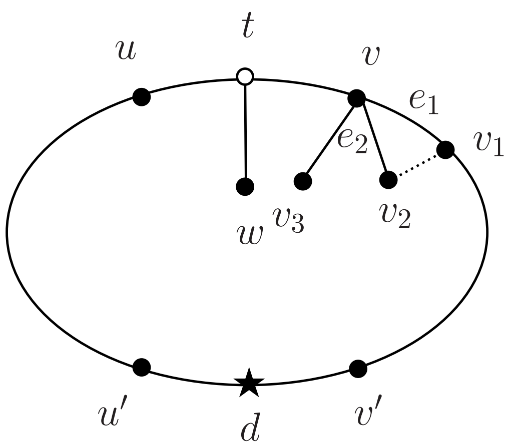

Figure 11: Case II, v_1 and d may be identical.

In G, lift e_1 and e_2 at v, orient and label the remaining two edges to realize $\beta(v)$, and properly prescribe t with the other two edges to realize $\beta(t)$. Then delete $\{v, t\}$ and denote the resulted graph by G''. Let β'' be the induced $\mathbb{Z}_4$-boundary of G''. β'' and β differ only at the three vertices u, w and v_3, according to the deletion of t and v, while for any other vertex $z \in V(G'')$, $\beta''(z) = \beta(z)$.

Lemma 4.5. *A flow on $G'' - d$ with a strong orientation can be extended to a flow on $G - d$, whose orientation is strong too.*

Proof. If a graph has a flow with a strong orientation, then there is also one if we subdivide an edge of the graph. In G'', we subdivide the edge v_1v_2 from the lifting with v; since t is properly prescribed, either utw or utv is a directed path. So there is a flow on $G - d$ with a strong orientation. ∎

Lemma 4.6. *There is no essential cut of size at most 3 in G''.*

Proof. Let $[B, \bar{B}]$ be an essential cut of size $s' \leq 3$ in G'', with $d \in B$. We use this cut to restore an essential cut of G by adding v and t to either B or $\bar{B}$. The rule is to add v (t) to the side containing the larger number of its neighbours. Since three vertices in $N(v)$ and two vertices in $N(t)$ are concerned, the resulted cut has size $s \leq s' + 3 \leq 6$ (do not forget the edge tv), which must be a bond. Moreover, by Proposition 3.7, $s \geq 5$, so $s = s' + 2$ or $s' + 3$.

If $s = s' + 2$, then $s' = 3$, $s = 5$, and thus t is added to $\bar{B}$. Notice that if in the restoration, both t and vt contribute an edge to the cut at the same time, we can move t to B and obtain an essential cut of size at most 4 in G, so there are only two possible situations: (1) v is added to B, v contributes one edge and t contributes no edge; (2) v is added to $\bar{B}$, and both of them contribute one edge to the cut. As for (1), we will restore an essential cut of size at most 5 intersecting $\delta(t)$ at tv, a contradiction to Proposition 3.19; as for (2), v_1 and v_2 cannot be separated, and $\delta(t) \cap \delta(B)$ is not in the boundary, so $\{v_3, w\} \subseteq B$ and $\{v_1, v_2, u\} \subseteq \bar{B}$. $\delta(B \cup \{t\})$ is an essential 6-cut of G which is not a bond, because by planarity, $G[B \cup \{t\}]$ and $G[\bar{B} \cup \{v\}]$ cannot be connected at the same time, which is a contradiction too.

If $s = s' + 3$, then v, t and vt contribute one edge respectively, and moreover, v_1 and v_2 are not separated. t can be transferred to v's side, resulting in an essential cut of size $s - 1$ in G, so $s' = 3$, $s = 6$; v is added to $\bar{B}$, and t is added to B. What is more, by Proposition 3.19, it is implied that $\{v_3, w\} \subseteq B$ and $\{v_1, v_2, u\} \subseteq \bar{B}$, which is just the content of (2). ∎

Lemma 4.7. *There is at most one 2-cut in $G'' - d$, and if there is one, it can only be $\delta(u) \cap E(G'' - d)$.*

Proof of Lemma 4.7: Let $[F, \bar{F}]$ be a 2-cut of $G'' - d$.

(1) If $|[d, F]|$ or $|[d, \bar{F}]| \leq 1$, since essential cuts in G'' have size at least 4 and the only 3-vertex except d is u, F or $\bar{F} = \{u\}$. This is just the exception we mentioned in the statement of the lemma.

(2) If $|[d, F]| = |[d, \bar{F}]| = 2$, write $T = \{v, t\}$ for convenience. The first thing we need to point out is that $|[d, F]| = |[d, \bar{F}]| = 2$ implies $d \neq v_1$. When $d = v_1$ happens, $|[T, F]| + |[T, \bar{F}]| = 3$, so $|\delta(F)| \leq 5$ or $|\delta(\bar{F})| \leq 5$ holds in G, whence F or $\bar{F}$ must be a singleton. Let $v_2 \in F$ without loss of generality. If the singleton is $F = \{v_2\}$, then $deg(v_2) = 5$ and $v_2 \in N(t)$, contradicting the premise of **Case 2**; if the singleton is $\bar{F}$, it can only be v_3 or w with degree 5 in G, as well as double arcs between d. As the result, $\delta(\{d, w, t\})$ is a bond of size 6, or $\delta(\{d, v_3, v\})$ is a bond of size 5 in G. However, G is separated by three vertices into two disjoint components in both situations.

Moreover, $|[T, F]| \geq 1$ and $|[T, \bar{F}]| \geq 1$, otherwise $\delta(F \cup T)$ or $\delta(\bar{F} \cup T)$ is an essential 4-cut in G, violating Proposition 3.7. We analyse the possible cuts between T and the two parts, with $\bar{F}$ specialized to be the side gaining less edges.

- $|[T, \bar{F}]| = 1$. $\bar{F}$ is a singleton with degree 4 or 5 in G, and degree 4 in G'', so $\bar{F} \in \{w, v_1, v_3\}$. If $\bar{F} = \{w\}$, then $\delta(\{t, w, d\})$ is an essential 5 or 6-cut which cannot be a bond by planarity, a contradiction to Fact 3.4; if $\bar{F} = \{v_1\}$,

$\delta(\{d, v_1\})$ is an essential 4-cut of G, a contradiction to Proposition 3.7; if $\bar{F} = \{v_3\}$, $deg(v_3) = 5$ and $\delta(\{v, v_3, d\})$ is an essential 7-cut which is not a bond by planarity, a contradiction to Corollary 3.10.

- $|[T, \bar{F}]|$ is 2. If $\{w, v\} \subseteq \bar{F}$, then $\delta(\bar{F} \cup \{d, t\})$ is an essential 5-cut in G, a contradiction to Proposition 3.8.

  If two neighbours of v are in $\bar{F}$: $\delta(\bar{F} \cup \{d, v\})$ is an essential 5-cut intersecting $\delta(t)$ at tv if $v_3 \in \bar{F}$, violating Proposition 3.19, so $\{v_1, v_2\} \subseteq \bar{F}$ and $v_3 \in F$.

  If one of $\{w, u\}$ and one of $\{v_1, v_2, v_3\}$ are in $\bar{F}$: $\delta(\bar{F} \cup \{d, t\})$ is an essential 6 or 7-cut, which must be a bond, so the boundary vertices of G contained in $\bar{F} \cup \{d, t\}$ are consecutive. $v \notin \bar{F}$, so $u \in \bar{F}$ and $w \in F$. If v_1 or $v_2 \in \bar{F}$, $\delta(\bar{F})$ is an essential 5-cut in G not separating d and t, violating Proposition 3.8. Above all, the only possible case is $\{u, v_3\} \subseteq \bar{F}$.

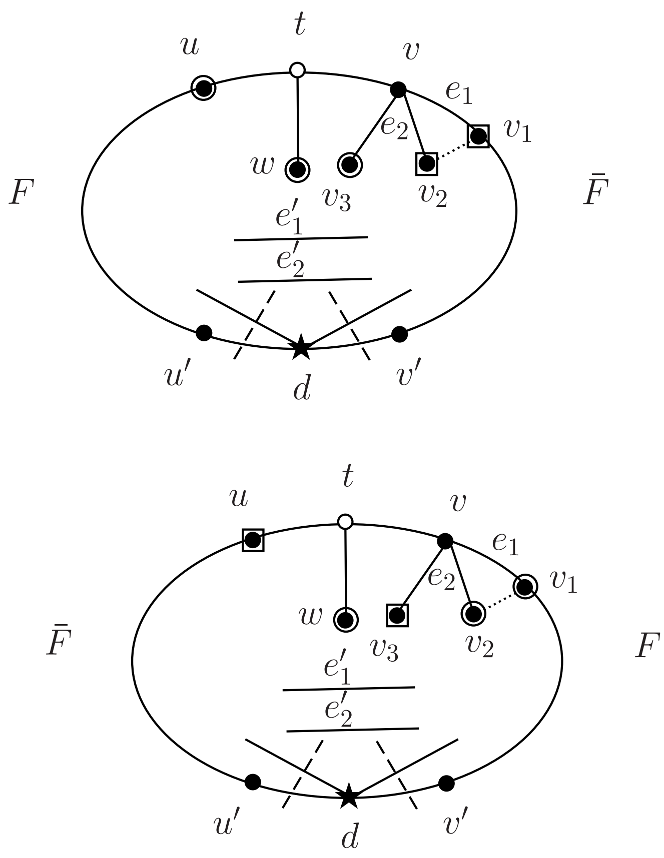

Figure 12: Two bad situations.

Figure 12 shows the configurations of the two possible situations, and for the five concerning vertices $\{u, w, v_1, v_2, v_3\}$, we marked each of them with a small circle if it is in F, or a small square if it is in $\bar{F}$. The same as the proof of **Case 1**, to exclude them, we will work on F and $\bar{F}$ respectively. We inherit the notations e'_1, e'_2, d_1, d_2 and $M(d)$ from **Case 1**. We will discuss the two situations separately with two claims. Actually there is little difference between the proofs of them, and some notations are shared by both proofs.

4.2.1 $\{v_1, v_2\} \subseteq \bar{F}$.

Claim 4.8. *$G[\bar{F}]$ has no 2-cut; $G[F]$ has at most one possible 2-cut, and if there is one, it can only be $\delta(u) \cap E(G[F])$.*

Proof of Claim 4.8: We shall prove Claim 4.8 by a series of subclaims.

Subclaim 4.8.1. *Boundary vertices from v' to v_1 in cyclic order are contained in $\bar{F}$.*

Proof. The proof is quite similar to that of Subclaim 4.4.1. By Fact 3.4 and Corollary 3.10, both $\delta(F)$ and $\delta(\bar{F})$ are bonds of G, so the boundary vertices belonging to F and $\bar{F}$ are consecutive on C respectively, otherwise $G[F]$ and $G[\bar{F}]$ cannot be connected at the same time by planarity. $F \cup \bar{F} \cup \{d, t, v\} = V(G)$, $v_1 \in \bar{F}$ and $u \in F$, so it can be deduced that the boundary vertices in $\bar{F}$ are vertices from v' to v_1 in cyclic order. ∎

Suppose $[K, \bar{K}]$ is a 2-cut of $G[\bar{F}]$. By Proposition 3.7, both K and $\bar{K}$ are connected with d_1 by at least one edge.

Subclaim 4.8.2. *$|[d_1, K]| = |[d_1, \bar{K}]| = 2$ is impossible.*

Proof. v_1 and v_2 cannot be separated by such a cut, otherwise $|\delta(K)| = |\delta(\bar{K})| = 4$ in G, by Proposition 3.7, K and $\bar{K}$ are just the two vertices v_1 and v_2, whence $deg(v_2) = 4$ is ridiculous. So assume $\{v_1, v_2\} \subseteq \bar{K}$ without loss of generality, then $|\delta(K)| = 4$, K is a single 4-vertex in the boundary; $|\delta(\bar{K})| = 6$, $\delta(\bar{K})$ is an essential 6-bond, and the boundary vertices in $\bar{K}$ are consecutive. It can be deduced that $K = \{v'\}$ distinct from v_1. There is exactly one edge between d and v', so $|\delta(v') \cap \{e'_1, e'_2\}| = 1$.

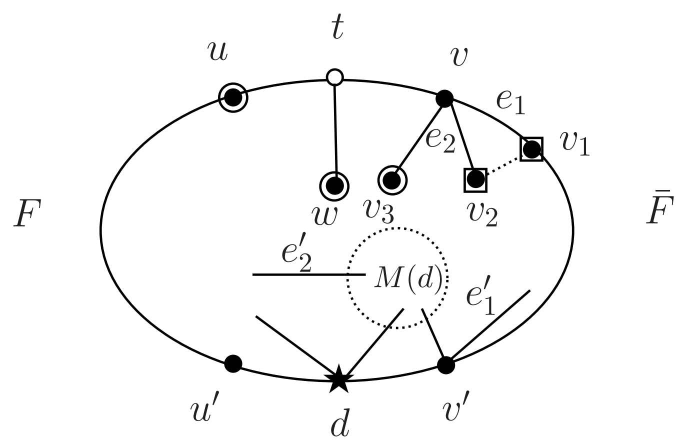

Figure 13: If $|[d_1, K]| = |[d_1, \bar{K}]| = 2$, and v' forms a 2-cut in $G[\bar{F}]$.

$\delta(F \cup \{t\})$ and $\delta(\bar{F} \cup \{v\})$ are essential 6-bonds, so both $G[F \cup \{t\}]$ and $G[\bar{F} \cup \{v\}]$ are connected. Define $M(d)$ to be the internal vertex set which d can reach through an inner path in $G[\bar{F} \cup \{d, v\}]$. By planarity, d does not form a chord with any vertex in $\{v', \ldots, v_1, v\}$, because one of e_1', e_2' is incident to v'. But d contributes another edge to $\delta(\bar{F})$ besides dv', so $M(d) \neq \emptyset$. Still by planarity, there is no inner path within $G[\bar{F} \cup \{d, v\}]$ connecting d with a boundary vertex in $\bar{F} \cup \{v\}$ except v'. These are shown in Figure 13. As a result, $|\delta(M(d))| = |\delta(M(d)) \cap \delta(d)| + |\delta(M(d)) \cap \delta(v')| + |\delta(M(d)) \cap \{e_1', e_2'\}| \leq 1 + 1 + 1 = 3$, a contradiction. ∎

Subclaim 4.8.3. *$|[d_1, K]| = 1$ or 3 is impossible.*

Proof. By symmetry, we only need to prove $|[d_1, K]| = 1$ does not hold. If $|[d_1, K]| = 1$, then $|\delta(K)| \leq 5$ in G, by Proposition 3.8, it is a single vertex with degree at least 4, so $|[T, K]| \geq 1$, and $K = \{v_1\}$ or $\{v_2\}$. Remember we added an extra edge between v_1 and v_2 in the lifting, it is deduced that the degree of K is actually 3 in G, which is impossible. ∎

We have finished the former part of the claim, now we turn to $G[F]$. Assume $[Q, \bar{Q}]$ is a 2-cut of $G[F]$. By Proposition 3.7 and 3.8, both Q and $\bar{Q}$ are connected with d_2 by at least one edge.

Subclaim 4.8.4. *$|[d_2, Q]| = |[d_2, \bar{Q}]| = 2$ is impossible.*

Proof. We prove this fact by analysing the edges between T and the two parts, and suppose $|[T, Q]| < |[T, \bar{Q}]|$ without loss of generality. Then there are two possibilities: $|[T, Q]| = 0$ or $|[T, Q]| = 1$.

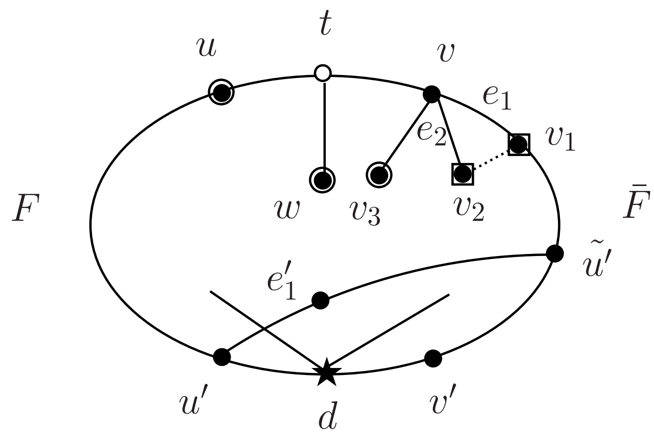

Figure 14: If $|[T, Q]| = 0$, and u' forms a 2-cut in $G[F]$.

If $|[T, Q]| = 0$, $|\delta(Q)| = 4$ in G, then Q is a 4-vertex by Proposition 3.7. Moreover, $\delta(\bar{F} \cup Q \cup \{d\})$ is an essential 6-cut which must be a bond, so the boundary vertices contained in $\bar{F} \cup Q \cup \{d\}$ are consecutive in cyclic order. It can be deduced that

$Q = \{u'\}$, and $u' \neq u$. Moreover, $|\delta(u') \cap \{e'_1, e'_2\}| = 1$ because there is exactly one edge between d and u'.

Since $G[\bar{F}]$ is connected, there is an inner path within $G[\bar{F} \cup \{d, u'\}]$ linking u' to a boundary vertex of $\{v', \ldots, v_1\}$ (note it $\tilde{u'}$, see Figure 14), while d contributes another edge to $\delta(\bar{F} \cup \{d, u'\})$, violating planarity.

If $|[T, Q]| = 1$, $|\delta(Q)| = 5$ in G, so Q is a 5-vertex by Proposition 3.8. Since $deg(u) = 4$, $Q = \{v_3\}$ or $\{w\}$. $Q = \{v_3\}$ is impossible: if so, $\delta(\bar{Q} \cup \{t\})$ is an essential 5-cut intersecting $\delta(t)$ at tv, a contradiction to Proposition 3.19. So $Q = \{w\}$.

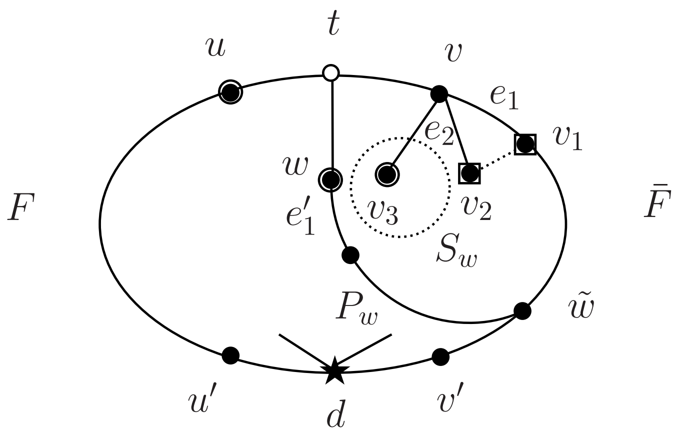

Figure 15: If $|[T, Q]| = 1$, and w forms a 2-cut in $G[F]$.

There are two edges in $\delta(d) \cap \delta(F)$, one of which is the boundary edge du', so $|\delta(w) \cap \{e'_1, e'_2\}| \geq 1$. Because $G[\bar{F}]$ is connected, there is an inner path P_w linking w to a boundary vertex of $\{v', \ldots, v_1\}$ (note it $\tilde{w}$), whose internal vertices lie in $\bar{F}$. $wP_w\tilde{w}Ctw$ is a cycle enclosing a non-empty set of internal vertices not in $\bar{F}$ (v_3 is enclosed), see Figure 15. Use S_w to denote this set, $|\delta(S_w)| = |\delta(S_w) \cap \delta(v)| + |\delta(S_w) \cap \delta(w)| + |\delta(S_w) \cap \{e'_1, e'_2\}| \leq 1 + 2 + 1 = 4$, a contradiction. ∎

Subclaim 4.8.5. *$|[d_2, Q]| = 1$ or 3 is impossible unless $u' = u$ and the degree of u is 4 in G.*

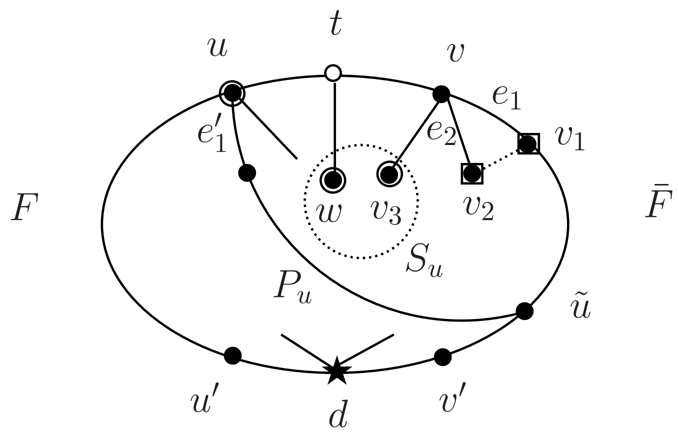

Figure 16: If $|[d_2, Q]| = 1$, and u forms a 2-cut in $G[F]$.

Proof. By symmetry, it is enough to prove the subclaim holds when $|[d_2, Q]| = 1$. obviously $|[T, Q]| > 0$. If $|[T, Q]| = 3$, $\delta(Q \cup \{t\})$ is an essential 5-cut intersecting $\delta(t)$ at tv, a contradiction to 3.19. So $|[T, Q] \leq 2$, $|\delta(Q)| \leq 5$, and thus Q is a single vertex by Proposition 3.7 and 3.8. The only possible case is that $Q = \{u\}$, and $deg(u) = 4$.

When $u' \neq u$, there is no chord between d and u, because otherwise there exists an essential 4-cut in G. So the edge d_2u is one of e'_1, e'_2. There is an inner path P_u linking u to some vertex in $\{v', \ldots, v_1\}$ (note it $\tilde{u}$), whose internal vertices lie in $\bar{F}$, and $uP_u\tilde{u}Ctu$ is a cycle enclosing a non-empty set of internal vertices not in $\bar{F}$ (v_3 and w are enclosed), see Figure 16. Use S_u to denote this set,, $|\delta(S_u)| = |\delta(S_u) \cap \delta(v)| + |\delta(S_u) \cap \delta(t)| + |\delta(S_u) \cap \delta(u)| + |\delta(S_u) \cap \{e'_1, e'_2\}| \leq 1 + 1 + 1 + 1 = 4$, a contradiction. So the only possible case of this configuration is $u = u'$, and the only 2-cut is $\delta(u) \cap E(G[F])$. This completes the proof of Claim 4.8. ∎

4.2.2 $\{u, v_3\} \subseteq \bar{F}$.

Claim 4.9. *$G[\bar{F}]$ has at most one possible 2-cut, and if there exists one, it can only be $\delta(u) \cap E(G[\bar{F}])$; $G[F]$ has no 2-cut.*

Proof of Claim 4.9: We shall present the proof of Claim 4.9 by a series of five subclaims.

Subclaim 4.9.1. *Boundary vertices from v' to v_1 in cyclic order are contained in F.*

Proof. By Fact 3.4 and Corollary 3.10, both $\delta(F)$ and $\delta(\bar{F})$ are bonds of G, so the boundary vertices belonging to F and $\bar{F}$ are consecutive on C respectively, otherwise $G[F]$ and $G[\bar{F}]$ cannot be connected at the same time by planarity. Because $F \cup \bar{F} \cup \{d, t, v\} = V(G)$, $v_1 \in F$ and $u \in \bar{F}$, it can be deduced from these facts that the boundary vertices in F are vertices from v' to v_1 in cyclic order. ∎

Suppose $[K, \bar{K}]$ is a 2-cut of $G[\bar{F}]$. $|[T, K]| + |T, \bar{K}| = 2$, both K and $\bar{K}$ are connected with d_1 by at least one edge because otherwise there exists an essential 4-cut in G.

Subclaim 4.9.2. *$|[d_1, K]| = |[d_1, \bar{K}]| = 2$ is impossible.*

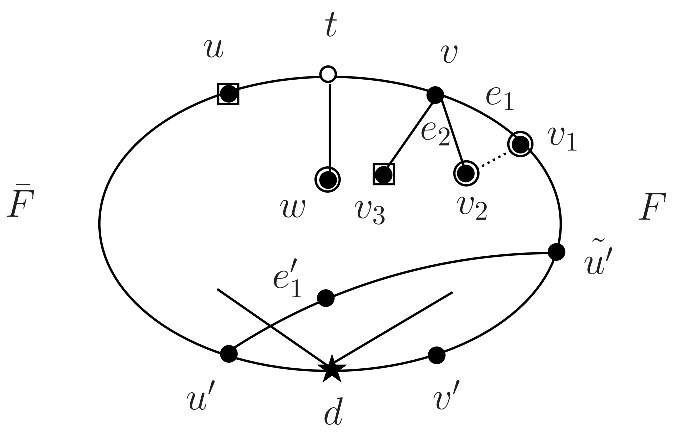

Figure 17: If $|[d_1, K]| = |[d_1, \bar{K}]| = 2$, and u' forms a 2-cut in $G[\bar{F}]$.

Proof. If so, then u and v_3 cannot be separated by the cut, because otherwise $|\delta(K)| = |\delta(\bar{K})| = 5$ in G, and K and $\bar{K}$ are singletons with parallel double edges in between. So assume u, $v_3 \in K$ without loss of generality, then $\bar{K}$ is a boundary 4-vertex of G. In fact, $\bar{K} = \{u'\}$, since $\delta(F \cup \{d, u'\})$ is an essential 7-bond, the boundary vertices in $F \cup \{d, u'\}$ must be consecutive. Moreover, $|\delta(u') \cap \{e_1', e_2'\}| = 1$ because there is exactly one edge between d and u'.

As a result, consider the 7-cut $\delta(F \cup \{d, u'\})$, there is an inner path within $G[F \cup \{d, u'\}]$ linking u' to a boundary vertex of $\{v', \ldots, v_1\}$ (note it $\tilde{u'}$, see Figure 17), while d contributes an edge to $\delta(F \cup \{d, u'\})$, violating planarity. ∎

Subclaim 4.9.3. *$|[d_1, \bar{K}]| = 1$ or 3 is impossible unless $u' = u$ and the degree of u is 4 in G.*

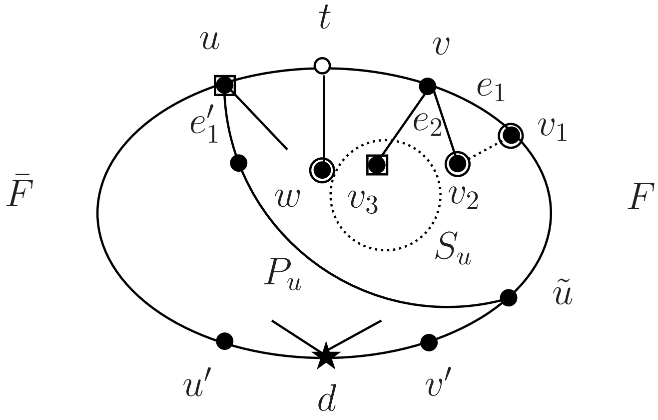

Figure 18: If $u' \neq u$, and u forms a 2-cut in $G[\bar{F}]$.

Proof. By symmetry, we only need to prove the subclaim holds when $|[d_1, \bar{K}]| = 1$. $|\delta(\bar{K})| \leq 5$ in G, so $\bar{K}$ is a singleton. Actually $|\delta(\bar{K})| = 4$ and $\bar{K}$ can only be $\{u\}$. If $u' \neq u$, then there is no chord between d and u, because otherwise there exists an essential 4-cut in G. So the edge d_1u is one of e_1', e_2'. There is an inner path P_u linking u to some vertex in $\{v', \ldots, v_1\}$ (note it $\tilde{u}$), whose internal vertices lie in F, and $uP_u\tilde{u}Ctu$ is a cycle enclosing a non-empty set of internal vertices not in F (v_3 is enclosed), see Figure 18. Use S_u to denote this set, $|\delta(S_u)| = |\delta(S_u) \cap \delta(v)| + |\delta(S_u) \cap \delta(u)| + |\delta(S_u) \cap \{e_1', e_2'\}| \leq 1 + 1 + 1 = 3$, a contradiction. So $[K, \bar{K}]$ does not exist on the premise that $|[d_1, \bar{K}]| = 1$ unless $u = u'$, whence the only possible 2-cut is $\delta(u) \cap E(G[\bar{F}])$. ∎

Now we deal with $G[F]$. Suppose $[Q, \bar{Q}]$ is a 2-cut of $G[F]$, $|[T, Q]| + |T, \bar{Q}| = 3$, so Q and $\bar{Q}$ are connected with d_2 by at least one edge for otherwise in G, there exists an essential 4-cut, or an essential 5-cut not separating d and t.

Subclaim 4.9.4. *$|[d_2, Q]| = |[d_2, \bar{Q}]| = 2$ is impossible.*

Proof. v_1 and v_2 cannot be separated by such a cut, for otherwise Q or $\bar{Q}$ containing w induces an essential 5-cut not separating d and t. There are two possibilities: w and $\{v_1, v_2\}$ are separated; or all three vertices are on the same side. We now eliminate them respectively with $\bar{Q}$ specialized.

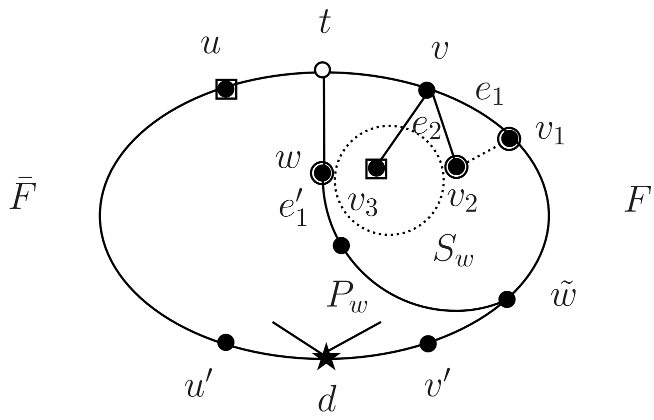

Figure 19: If $|[d_2, Q]| = |[d_2, \bar{Q}]| = 2$, and w forms a 2-cut in $G[F]$.

If $w \in Q$ and $\{v_1, v_2\} \subseteq \bar{Q}$, then $|\delta(Q)| = 5$ in G and $Q = \{w\}$. $\delta(d) \cap \delta(F)$ contains one boundary edge dv', so $|\delta(w) \cap \{e_1', e_2'\}| \geq 1$. Because $G[F]$ is connected, there is an inner path P_w linking w to a boundary vertex of $\{v', \ldots, v_1\}$ (note it $\tilde{w}$), whose internal vertices lie in F. $wP_w\tilde{w}Ctw$ is a cycle enclosing a non-empty set of internal vertices not in F (v_3 is enclosed), see Figure 19. Use S_w to denote this set, $|\delta(S_w)| = |\delta(S_w) \cap \delta(v)| + |\delta(S_w) \cap \{e_1', e_2'\}| \leq 1 + 2 = 3$, a contradiction.

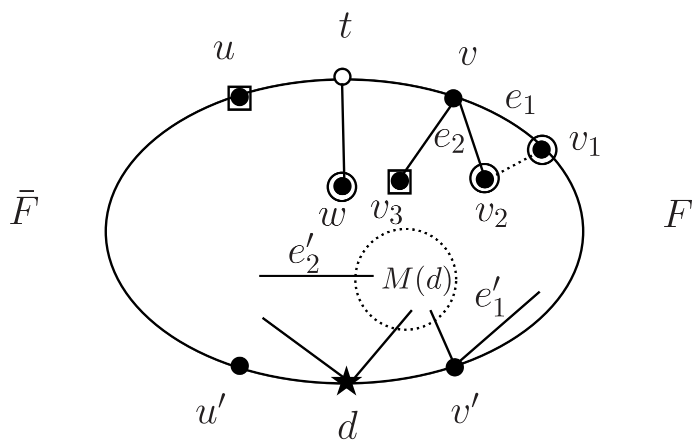

Figure 20: If $|[d_2, Q]| = |[d_2, \bar{Q}]| = 2$, and v' forms a 2-cut in $G[F]$.

If $\{v_1, v_2, w\} \subseteq \bar{Q}$, then Q is a boundary 4-vertex of G in F. In fact, $Q = \{v'\}$, since $\delta(\bar{F} \cup \{d, Q\})$ is an essential 6-bond, the boundary vertices in $\delta(\bar{F} \cup \{d, Q\})$ must be consecutive. There is exactly one edge between d and v', so $|\delta(v') \cap \{e_1', e_2'\}| = 1$. Define $M(d)$ to be the set of internal vertices which d can reach through an inner path in $G[F \cup \{d\}]$. By planarity, d does not form a chord with any vertex in $\{v', \ldots, v_1\}$, while d contributes another edge to $\delta(F)$ besides dv', so $M(d) \neq \emptyset$. Still by planarity, there is no inner path within $G[F \cup \{d\}]$ connecting d with w

or v_2, nor a boundary vertex in F except v'. These are shown in Figure 20. So $|\delta(M(d))| = |\delta(M(d)) \cap \delta(d)| + |\delta(M(d)) \cap \delta(v')| + |\delta(M(d)) \cap \{e'_1, e'_2\}| \leq 1+1+1 = 3$, a contradiction. ∎

Subclaim 4.9.5. *$|[d_2, Q]| = 1$ or 3 is impossible.*

Proof. By symmetry, it is enough to prove the subclaim holds when $|[d_2, Q]| = 1$. $3 \leq |\delta(Q)| \leq 3 + |[T, Q]| \leq 6$, by Proposition 3.7 and 3.8, $|[T, Q]| = 1$ or 3 are possible, whence $|\delta(Q)| = 4$ or 6. If $|\delta(Q)| = 4$, Q can only be $\{v_1\}$, however v_1 and v_2 are separated and thus $deg(v_1) = 3$ in G, which is ridiculous. So $|\delta(Q)| = 6$ and $\{v_1, v_2, w\} \subseteq Q$, then $\delta(Q \cup \{t, v\})$ is an essential 5-cut intersecting $\delta(t)$ at tu, which is a contradiction to Proposition 3.19. This completes the proof of Claim 4.9. ∎

Having finished the proofs of Claim 4.8 and Claim 4.9, now we can eliminate the two situations we mentioned at the beginning of Lemma 4.7.

By Lemma 3.16, t can be prescribed properly so that: when d and u are adjacent, $\delta(u) \cap E(G[F])$ (or $\delta(u) \cap E(G[\bar{F}])$) is not a bad 2-cut under the induced $\mathbb{Z}_4-$ boundary; the value of arc vt is not $\beta(v)$ so that the value of vv_3 is non-zero. Remember we lifted the two edges vv_1 and vv_2. In G, orient the 3-path v_1vv_2 as the orientation of edge v_1v_2, and label the two edges with the same value. Then in G there is a directed 3-path through v connecting F with $\bar{F}$, and prescribe $\{e'_1, e'_2\}$ to make sure that at least one of them has an opposite direction to this path. By the induction, both $G[F]$ and $G[\bar{F}]$ admit a flow with a strong orientation respectively, from which we manage to construct a flow with a strong orientation of $G - d$. This completes the proof of Lemma 4.7.

Having proved Lemma 4.6 and Lemma 4.7, we can prescribe t properly so that the only possible 2-cut in Lemma 4.7 is not bad under β'', as we just mentioned. Then we can apply the induction on G'' to obtain a flow with a strong orientation of $G'' - d$. By Lemma 4.5, it can be naturally extended to a flow with a strong orientation of $G - d$.

4.3 Case 3: $w, d \in N(u) \cap N(v)$.

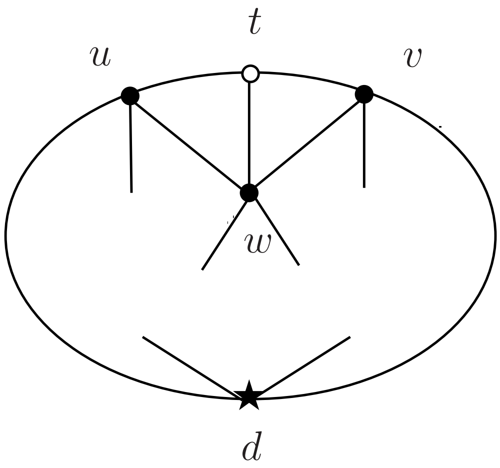

Figure 21

Finally we show that this last case is not possible either. When $deg(w) > 5$, we can still lift e_1 and e_2, prescribe t and v wisely to achieve $\beta(t)$ and $\beta(v)$, then delete them as we have done in **Case 2**. This is actually a special case of **Case 2** with $v_3 = w$, so we can repeat the analyses about essential cuts and bad 2-cuts before, and the restriction $v_3 = w$ will only make things simpler. The method of **Case 2** is still available when $deg(w) > 5$, so $deg(w) = 5$, and this last configuration is shown in Figure 21. Write $Y = \{d, u, v, w, t\}$, d and w cannot be adjacent, otherwise G is separated into two parts by the path dwt. A cut of size at most 3 would be formed by a set of internal vertices. Obviously $|Y^c| \geq 2$. First contract Y^c, there is a flow (D_1, f_1) on G/Y^c and $D_1(G/Y^c - d)$ is strong; then contract Y to a vertex y and delete an edge in $E(y)$. y has degree 5 in the resulted graph $\tilde{G}$, $\tilde{G}$ neither contains a vertex of degree less than 4 nor an essential cut of size less than 5. What is more, $\tilde{G} - y$ has no 2-cut, otherwise this will yield double edges or an essential cut of size at most 5 not separating d and t in G. By the induction, G/Y admits a flow (D_2, f_2) and $D_2(Y^c)$ is strong. Combining (D_1, f_1) and (D_2, f_2) together, we can obtain a flow (D, f) of G and $D(G - d)$ is strong. This completes the proof of Theorem 2.5.

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References

[1] L. Goddyn, M. Tarsi, C.-Q. Zhang, On (k, d)-colourings and fractional nowhere-zero flows, J. Graph Theory 28 (1998) 155-161.

[2] F. Kardos, J. Narboni, R. Naserasr, Z. Wang, Circular $(4-\varepsilon)$-colouring of some classes of signed graphs, arXiv:2107.12126, to appear in SIAM J. Discrete Math.

[3] J. Li, R. Naserasr, Z. Wang, X. Zhu, Circular flows in mono-directed signed graphs, arXiv:2212.10757, 2022.

[4] J. Li, C. Thomassen, Y. Wu, C.-Q. Zhang, The flow index and strongly connected orientations, European J. Combin., 70 (2018) 164-177.

[5] R. Naserasr, R. Škrekovski, Z. Wang, R. Xu, Mapping sparse signed graphs to (K_{2k}, M), arXiv:2101.08619, to appear in J. Graph Theory.

[6] R. Naserasr, Z. Wang, X. Zhu, Circular chromatic number of signed graphs, Electron. J. Combin., 28 (2021) no. 2, Paper No. 2.44, 40 pp.

[7] R. B. Richter, C. Thomassen, D. Younger, Group-colouring, group-connectivity, claw-decompositions, and orientations in 5-edge-connected planar graphs, Journal of Combinatorics, 7 (2016), 219-232.

[8] A. Vince, Star chromatic number, J. Graph Theory, 12 (1988) 551-559.