The maximum spectral radius of theta-free graphs with given size

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Abstract

A graph G is said to be F-free if it does not contain F as a subgraph. Let $\mathcal{G}(m,F)$ denote the set of F-free graphs with m edges having no isolated vertices. A theta graph, denoted by θ_{l_1,l_2,l_3}, is the graph obtained by connecting two distinct vertices with three internally disjoint paths of length l_1, l_2, l_3, where $l_1 \leq l_2 \leq l_3$ and $l_2 \geq 2$. Recently, Li, Zhao and Zou (2025) characterized the $\theta_{1,p,q}$-free graph of size m having the largest spectral radius, where $q \geq p \geq 3$ and $p+q \geq 2k+1 \geq 7$. Up to now, for all $\theta_{1,p,q}$-free graphs with $q \geq p \geq 2$, except for the case $q = p = 3$, the graphs in $\mathcal{G}(m, \theta_{1,p,q})$ with the largest spectral radius have been determined. So they proposed a problem on characterizing the graphs with the maximum spectral radius among $\theta_{1,3,3}$-free graphs. In this paper, we consider this problem and determine the maximum spectral radius of $\theta_{1,3,3}$-free graphs with size m and characterize the extremal graph.

Keywords: Spectral radius; $\mathcal{F}$-free graphs; Theta graphs; Extremal graph
AMS subject classification 2020: 05C35, 05C50.

1 Introduction

For a simple graph $G = (V(G), E(G))$, we use $n := |G| = |V(G)|$ and $m := e(G)$ to denote the order and the size of G, respectively. Since isolated vertices do not have an effect on the spectral radius of a graph, throughout this paper we consider graphs without isolated vertices. Let $N(v)$ or $N_G(v)$ be the set of neighbors of v, and $d(v)$ or $d_G(v)$ be the degree of a vertex v in G. Denote $N[v] = N(v) \cup \{v\}$. For a subset $U \subseteq V(G)$, we denote by $N_U(u)$ the set of vertices of U that are adjacent to u, that is, $N_U(u) = N_G(u) \cap U$, and let $d_U(u)$ be the number of vertices of $N_U(u)$. For subsets X, Y of $V(G)$, we write $E(X, Y)$ for the set of edges with one end in X and the other in Y. Let $e(X, Y) = |E(X, Y)|$. If $Y = X$, we simply write $e(X)$ for $e(X, X)$. The distance between two distinct vertices $u, v \in V(G)$ is the length of a shortest path from u to v in G. The diameter $diam(G)$ of a graph G is the greatest distance between any two vertices of G. The join of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G \cup H$ by adding the edges to join every vertex of G with every vertex of H. For graph notation and terminology undefined here, readers are referred to [1].

Let $A(G)$ be the adjacency matrix of a connected graph G. The maximum of modulus of all eigenvalues of $A(G)$ is the spectral radius of G and denoted by $\lambda(G)$. Since $A(G)$ is irreducible and nonnegative for a connected graph G, by the Perron-Frobenius theorem there exists a unique positive unit eigenvector $\mathbf{x}$ corresponding to $\lambda(G)$, which is called Perron vector of G.

As usual, let P_n, C_n and $K_{1,n-1}$ be the path, the cycle, and the star on n vertices, respectively. Let $K_{1,n-1} + e$ be the graph obtained from $K_{1,n-1}$ by adding one edge within its independent set. A theta graph, say θ_{l_1,l_2,l_3}, is the graph obtained by connecting two distinct vertices with three internally disjoint paths of length l_1, l_2, l_3, where $l_1 \le l_2 \le l_3$ and $l_2 \ge 2$.

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if it does not contain any element in $\mathcal{F}$ as a subgraph. When the forbidden set $\mathcal{F}$ is a singleton, say F, then we write F-free for $\mathcal{F}$-free. Let $\mathcal{G}(m, \mathcal{F})$ denote the set of $\mathcal{F}$-free graphs with m edges having no isolated vertices. If $\mathcal{F} = \{F\}$, then we write $\mathcal{G}(m, F)$ for $\mathcal{G}(m, \mathcal{F})$.

The classic Turán type problem asks what is the maximum number of edges in an $\mathcal{F}$-free graph of order n. In spectral graph theory, Nikiforov [13] proposed a spectral Turán type problem which asks to determine the maximum spectral radius of an $\mathcal{F}$-free graph with n vertices, which is known as the Brualdi-Solheid-Turán type problem. In the past few decades, this problem has been studied for many classes of graphs, see [11, 12, 13, 18, 20]. In addition, Brualdi and Hoffman [2] raised another spectral Turán type problem: What is the maximal spectral radius of an $\mathcal{F}$-free graph with given size m? This problem is called the Brualdi-Hoffman-Turán type problem. Up to now, much attention has been paid to the Brualdi-Hoffman-Turán type problem for various families of graphs. For example, [15] for K_3-free graphs, [7] for non-bipartite K_3-free graphs, [9, 10] for K_{r+1}-free graphs, [12] for C_4-free graphs, [19] for $K_{2,r+1}$-free graphs, [3] for non-star $K_{2,r+1}$-free graphs, [5] for C_k^+-free graphs where C_k^+ is a graph on k vertices obtained from C_k by adding a chord between two vertices with distance two, [14] for B_k-free graphs where B_k is obtained from k triangles by sharing an edge, [21] for F_5-free graphs, [4] for F_{2k+2}-free graphs where

$F_k = K_1 \vee P_{k-1}$, [6] for $F_{2,3}$-free graphs, [4] for $F_{k,3}$-free graphs where $F_{k,3}$ is the friendship graph obtained from k triangles by sharing a common vertex.

For theta graphs, Sun et al. [16] established sharp upper bounds on spectral radius for G in $\mathcal{G}(m, \theta_{1,2,3})$ and $\mathcal{G}(m, \theta_{1,2,4})$, respectively. Subsequently, Lu et al. [8] determined the graph among $\mathcal{G}(m, \theta_{1,2,5})$ having the largest spectral radius. Generally, Li et al. [5] confirmed the following theorem.

Theorem 1.1 *[5] Let $k \geq 3$ and $m \geq 4\left(k^2+3k+1\right)^2$. If $G \in \mathcal{G}(m, \theta_{1,2,2k-1}) \cup \mathcal{G}(m, \theta_{1,2,2k})$, then*

$$\lambda(G) \leq \frac{k-1+\sqrt{4m-k^2+1}}{2},$$

and equality holds if and only if $G \cong K_k \vee \left(\frac{m}{k} - \frac{k-1}{2}\right) K_1$.

Recently, Li et al. [4] determined the largest spectral radius of $\theta_{1,p,q}$-free graph with size m for $q \geq p \geq 3$ and $p+q \geq 7$.

Theorem 1.2 *[4] Let $k \geq 3$ and $m \geq \frac{9}{4}k^6+6k^5+46k^4+56k^3+196k^2$. If $G \in \mathcal{G}(m, \theta_{1,p,q}) \cup \mathcal{G}(m, \theta_{1,r,s})$ with $q \geq p \geq 3, s \geq r \geq 3, p+q = 2k+1$ and $r+s = 2k+2$, then*

$$\lambda(G) \leq \frac{k-1+\sqrt{4m-k^2+1}}{2},$$

and equality holds if and only if $G \cong K_k \vee \left(\frac{m}{k} - \frac{k-1}{2}\right) K_1$.

At the same time, they proposed the following problem in [4].

Problem 1.3 *[4] How can we characterize the graphs among $\mathcal{G}(m, \theta_{1,3,3})$ having the largest spectral radius?*

In this paper, we consider the above problem and characterize the unique graph with the maximum spectral radius among $\mathcal{G}(m, \theta_{1,3,3})$.

Theorem 1.4 *Let $G \in \mathcal{G}(m, \theta_{1,3,3})$ with $m \geq 43$. Then $\lambda(G) \leq \frac{1+\sqrt{4m-3}}{2}$ and equality holds if and only if $G \cong K_2 \vee \frac{m-1}{2} K_1$.*

2 Preliminaries

In this section, we introduce some basic lemmas which are useful in the subsequent sections.

Lemma 2.1 *[9, 15] If $G \in \mathcal{G}(m, K_3)$, then $\lambda(G) \leq \sqrt{m}$. Equality holds if and only if G is a complete bipartite graph.*

If G is a bipartite graph with m edges, then $G \in \mathcal{G}(m, K_3)$. By Lemma 2.1, it follows that $\lambda(G) \leq \sqrt{m}$.

Wu et al. [17] obtained a relationship between spectral radius of two graphs under graph operation, which plays an important role in our proofs.

Lemma 2.2 *[17] Let u and v be two vertices of the connected graph G with order n. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \backslash N_G(u)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of G, where x_i corresponds to the vertex $v_i (1 \leq i \leq n)$. Let $G' = G - \{vv_i | 1 \leq i \leq s\} + \{uv_i | 1 \leq i \leq s\}$. If $x_u \geq x_v$, then $\lambda(G) < \lambda(G')$.*

A cut vertex of a graph is a vertex whose deletion increases the number of components. A graph is called 2-connected, if it is a connected graph without cut vertices. Let $\mathbf{x}$ be the Perron vector of G with coordinate x_v corresponding to the vertex $v \in V(G)$. A vertex u^* is said to be an extremal vertex if $x_{u^*} = \max\{x_u | u \in V(G)\}$.

Lemma 2.3 *[19] Let G be a graph in $\mathcal{G}(m, F)$ with the maximum spectral radius. If F is a 2-connected graph and u^* is an extremal vertex of G, then G is connected and $d(u) \geq 2$ for any $u \in V(G) \setminus N[u^*]$.*

3 Proof of Theorem 1.4

Let G^* be a graph in $\mathcal{G}(m, \theta_{1,3,3})$ with the maximum spectral radius. Note that $\lambda\left(K_2 \vee \frac{m-1}{2} K_1\right) = \frac{1+\sqrt{4m-3}}{2}$ and $K_2 \vee \frac{m-1}{2} K_1$ is $\theta_{1,3,3}$-free, we have

$$\lambda(G^*) \geq \lambda\left(K_2 \vee \frac{m-1}{2} K_1\right) = \frac{1+\sqrt{4m-3}}{2}.$$

By Lemma 2.3, we have G^* is connected. Let $\lambda = \lambda(G^*)$ and $\mathbf{x}$ be the Perron vector of G^* with coordinate x_v corresponding to the vertex $v \in V(G^*)$. Assume that u^* is the extremal vertex of G^*. That is, $x_{u^*} \geq x_u$ for any $u \in V(G^*) \setminus \{u^*\}$. Set $U = N_{G^*}(u^*)$ and $W = V(G^*) \setminus N_{G^*}[u^*]$. Let $W_H = N_W(V(H))$ for any component H of $G^*[U]$. Since G^* is $\theta_{1,3,3}$-free, $G^*[U]$ does not contain any path of length four and any cycle of length more than four.

Lemma 3.1 *For any non-trivial component H in $G^*[U]$, if H contains a cycle of length four, then $N_W(u) \cap N_W(v) = \emptyset$ for any vertices u and v in the cycle of length four.*

Proof. Let the cycle C_4 in H be $u_1u_2u_3u_4u_1$. Suppose on the contrary that $N_W(u_i) \cap N_W(u_j) \neq \emptyset$ for some vertices u_i and u_j, $1 \leq i \neq j \leq 4$. It follows that $N_W(u_i) \neq \emptyset$ and $N_W(u_j) \neq \emptyset$. Let $w \in N_W(u_i) \cap N_W(u_j)$. If u_i and u_j are adjacent in C_4, without loss of generality, we assume that $u_i = u_1$ and $u_j = u_2$. Then u^*u_1, $u^*u_3u_4u_1$ and $u^*u_2wu_1$ are three internally disjoint paths of length 1,3,3 between u^* and u_1. So G^* contains $\theta_{1,3,3}$ as a subgraph, a contradiction. Hence, u_i and u_j are not adjacent in C_4. Assume that $u_i = u_1$ and $u_j = u_3$. It is easy to get that u_1u_2, $u_1u_4u^*u_2$ and $u_1wu_3u_2$ are three internally disjoint paths of length 1,3,3 between u_1 and u_2. Clearly, G^* contains $\theta_{1,3,3}$ as a subgraph, a contradiction. This completes the proof. $\square$

For convenience, we divide U into two subsets U_0 and U_+ where U_0 is the set of isolated vertices of $G^*[U]$ and $U_+ = U \setminus U_0$. It is easy to see that $m = |U| + e(U_+) + e(U, W) + e(W)$. Since $\lambda(G^*)\mathbf{x} = A(G^*)\mathbf{x}$, we have

$$\lambda x_{u^*} = \sum_{u\in U} x_u = \sum_{u\in U_+} x_u + \sum_{u\in U_0} x_u.$$

Furthermore, we can get

$$\begin{aligned}\lambda^2 x_{u^*} &= \lambda(\lambda x_{u^*}) = \lambda \sum_{u\in U} x_u \\ &= \sum_{u\in U} \sum_{v\in N_G(u)} x_v = \sum_{v\in V(G)} d_U(v) x_v \\ &= |U| x_{u^*} + \sum_{u\in U_+} d_U(u) x_u + \sum_{w\in W} d_U(w) x_w.\end{aligned}$$

Therefore,

$$\begin{aligned}(\lambda^2 - \lambda) x_{u^*} &= |U| x_{u^*} + \sum_{u\in U_+} (d_U(u) - 1) x_u + \sum_{w\in W} d_U(w) x_w - \sum_{u\in U_0} x_u \\ &\leq |U| x_{u^*} + \sum_{u\in U_+} (d_U(u) - 1) x_u + e(U, W) x_{u^*} - \sum_{u\in U_0} x_u.\end{aligned}$$

Recall that $\lambda \geq \frac{1+\sqrt{4m-3}}{2}$, that is, $\lambda^2 - \lambda \geq m - 1 = |U| + e(U_+) + e(U, W) + e(W) - 1$. Hence

$$\sum_{u\in U_+} (d_U(u) - 1) x_u \geq \left(e(U_+) + e(W) + \sum_{u\in U_0} \frac{x_u}{x_{u^*}} - 1 \right) x_{u^*}.$$

Let $\mathcal{H}$ be the set of all non-trivial components in $G^*[U]$. For each non-trivial component H of $\mathcal{H}$, we denote $\eta(H) := \sum_{u\in V(H)} (d_H(u) - 1) x_u$. Clearly,

$$\sum_{H\in\mathcal{H}} \eta(H) \geq \left(e(U_+) + e(W) + \sum_{u\in U_0} \frac{x_u}{x_{u^*}} - 1 \right) x_{u^*}, \tag{1}$$

with equality if and only if $\lambda^2 - \lambda = m - 1$ and $x_w = x_{u^*}$ for any $w \in W$ with $d_U(w) \geq 1$.

Claim 3.2 $G^*[U]$ *contains no any cycle of length four.*

Proof. Suppose to the contrary that $G^*[U]$ contains C_4. Since $G^*[U]$ is P_5-free, we can get that $G^*[U]$ contains a component $H \in \{C_4, C_4+e, K_4\}$ where C_4+e is the graph obtained from C_4 by adding one edge to two nonadjacent vertices. Let $\mathcal{H}'$ be the family of components of $G^*[U]$ each of which contains C_4 as a subgraph, that is, $H \in \{C_4, C_4+e, K_4\}$ for any $H \in \mathcal{H}'$, then $\mathcal{H} \setminus \mathcal{H}'$ is the family of other components of $G^*[U]$ each of which is a tree with diameter at most 3 or a unicyclic graph containing a triangle. Therefore, for each $H \in \mathcal{H} \setminus \mathcal{H}'$, we have

$$\eta(H) = \sum_{u \in V(H)} (d_H(u) - 1)x_u \leq (2e(H) - |H|)x_{u^*} \leq e(H)x_{u^*}.$$

Next we show that

$$\eta(H) < (e(H) - 1)x_{u^*} + \frac{2\sum_{w \in W_H} x_w}{\lambda - 3}$$

for each $H \in \mathcal{H}'$. Let $H^* \in \mathcal{H}'$ with $V(H^*) = \{u_1, u_2, u_3, u_4\}$ and the cycle of length four be $u_1u_2u_3u_4u_1$.

First, we consider the case $W_{H^*} = \emptyset$. Let $x_{u_1} = \max\{x_{u_i} | 1 \leq i \leq 4\}$. Then

$$\lambda x_{u_1} = \sum_{u \in N(u_1)} x_u \leq x_{u^*} + x_{u_2} + x_{u_3} + x_{u_4} \leq x_{u^*} + 3x_{u_1}.$$

Hence, $x_{u_1} \leq \frac{1}{\lambda - 3} x_{u^*}$. Since $m \geq 43$, we have $\lambda \geq \frac{1+\sqrt{4m-3}}{2} \geq 7$. Thus, $x_{u_1} < \frac{1}{2} x_{u^*}$ and

$$\eta(H^*) \leq (2e(H^*) - |H^*|)x_{u_1} < (e(H^*) - 2)x_{u^*} < (e(H^*) - 1)x_{u^*} + \frac{2\sum_{w \in W_{H^*}} x_w}{\lambda - 3},$$

as desired.

In the following, we assume that $W_{H^*} \neq \emptyset$. We consider the following two cases.

Case 1. All vertices in W_{H^*} have a unique common neighbor in $V(H^*)$.

Without loss of generality, let the common neighbor be u_1. It follows that $N_W(u_i) = \emptyset$ for $i \in \{2, 3, 4\}$. Let $x_{u_2} = \max\{x_{u_i} | 2 \leq i \leq 4\}$. Then

$$\lambda x_{u_2} \leq x_{u_1} + x_{u_3} + x_{u_4} + x_{u^*} \leq 2x_{u_2} + 2x_{u^*}.$$

Thus, $x_{u_2} \leq \frac{2}{\lambda - 2} x_{u^*} \leq \frac{2}{5} x_{u^*}$ since $\lambda \geq 7$. Therefore, we have

$$\begin{aligned}
\eta(H^*) &= \sum_{u \in V(H^*)} (d_{H^*}(u) - 1)x_u \\
&\leq (d_{H^*}(u_1) - 1)x_{u_1} + (2e(H^*) - d_{H^*}(u_1) - 3)x_{u_2} \\
&\leq \left(d_{H^*}(u_1) - 1 + \frac{4}{5}e(H^*) - \frac{2}{5}d_{H^*}(u_1) - \frac{6}{5}\right) x_{u^*} \\
&= \left(\frac{4}{5}e(H^*) + \frac{3}{5}d_{H^*}(u_1) - \frac{11}{5}\right) x_{u^*}.
\end{aligned}$$

Since $d_{H^*}(u_1) \leq 3$, the above inequality becomes

$$\begin{aligned}\eta(H^*) &\leq \left(\frac{4}{5}e(H^*) - \frac{2}{5}\right) x_{u^*} \\ &< (e(H^*) - 1)x_{u^*} \\ &< (e(H^*) - 1)x_{u^*} + \frac{2\sum_{w \in W_{H^*}} x_w}{\lambda - 3}.\end{aligned}$$

Case 2. There are at least two vertices of W_{H^*} such that they have distinct neighbors in $V(H^*)$.

Since

$$\begin{cases} \lambda x_{u_1} \leq x_{u_2} + x_{u_3} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_{H^*}}(u_1)} x_w, \\ \lambda x_{u_2} \leq x_{u_1} + x_{u_3} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_{H^*}}(u_2)} x_w, \\ \lambda x_{u_3} \leq x_{u_1} + x_{u_2} + x_{u_4} + x_{u^*} + \sum_{w \in N_{W_{H^*}}(u_3)} x_w, \\ \lambda x_{u_4} \leq x_{u_1} + x_{u_2} + x_{u_3} + x_{u^*} + \sum_{w \in N_{W_{H^*}}(u_4)} x_w, \end{cases}$$

we obtain

$$\lambda(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) \leq 3(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) + 4x_{u^*} + \sum_{i=1}^{4} \sum_{w \in N_{W_{H^*}}(u_i)} x_w.$$

By Lemma 3.1, we get that $N_{W_{H^*}}(u_i) \cap N_{W_{H^*}}(u_j) = \emptyset$ for any vertices $u_i \neq u_j \in V(H^*)$. Thus, $\sum_{w \in W_{H^*}} x_w = \sum_{w \in N_W(V(H^*))} x_w = \sum_{i=1}^{4} \sum_{w \in N_{W_{H^*}}(u_i)} x_w$. Therefore, by $\lambda \geq 7$, we obtain

$$\begin{aligned}x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4} &\leq \frac{4x_{u^*}}{\lambda - 3} + \frac{\sum_{w \in W_{H^*}} x_w}{\lambda - 3} \\ &\leq x_{u^*} + \frac{\sum_{w \in W_{H^*}} x_w}{\lambda - 3}.\end{aligned}$$

Since $H^* \in \mathcal{H}'$, it follows that $H^* \in \{C_4, C_4 + e, K_4\}$. Then $d_{H^*}(u) \leq 3$ for any $u \in V(H^*)$. Hence, by the definition of $\eta(H^*)$,

$$\begin{aligned}\eta(H^*) &\leq 2(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) \\ &\leq 2x_{u^*} + \frac{2\sum_{w \in W_{H^*}} x_w}{\lambda - 3} \\ &< (e(H^*) - 1)x_{u^*} + \frac{2\sum_{w \in W_{H^*}} x_w}{\lambda - 3}.\end{aligned}$$

Therefore, we conclude that $\eta(H) < (e(H) - 1)x_{u^*} + \frac{2\sum_{w \in W_H} x_w}{\lambda - 3}$ for each $H \in \mathcal{H}'$. Recall that $\eta(H) \leq e(H)x_{u^*}$ for each $H \in \mathcal{H} \setminus \mathcal{H}'$. Thus,

$$\sum_{H \in \mathcal{H}} \eta(H) = \sum_{H \in \mathcal{H}'} \eta(H) + \sum_{H \in \mathcal{H} \setminus \mathcal{H}'} \eta(H)$$

$$< \sum_{H\in\mathcal{H}'}(e(H)-1)x_{u^*} + \sum_{H\in\mathcal{H}'}\frac{2\sum_{w\in W_H}x_w}{\lambda-3} + \sum_{H\in\mathcal{H}\setminus\mathcal{H}'}e(H)x_{u^*}$$
$$= e(U_+)x_{u^*} - \sum_{H\in\mathcal{H}'}x_{u^*} + \sum_{H\in\mathcal{H}'}\frac{2\sum_{w\in W_H}x_w}{\lambda-3}.$$

For any $H\in\mathcal{H}'$ satisfying $W_H=\emptyset$, we have $\sum_{w\in W_H}x_w=0$. For any $H\in\mathcal{H}'$ satisfying $W_H\neq\emptyset$ and any $w\in W_H$, since G^* is $\theta_{1,3,3}$-free, we obtain $W_H\cap W_{G^*[U]\setminus H}=\emptyset$. Then $d_{U\setminus V(H)}(w)=0$. By Lemma 3.1, we have $d_H(w)=1$. It follows that $d_U(w)=1$. As $d(w)\geq 2$ by Lemma 2.3, it is easy to get that $d_W(w)\geq 1$. Thus, $\sum_{H\in\mathcal{H}'}\sum_{w\in W_H}x_w\leq \sum_{H\in\mathcal{H}'}\sum_{w\in W_H}d_W(w)x_w \leq \sum_{H\in\mathcal{H}'}\sum_{w\in W_H}d_W(w)x_{u^*}\leq 2e(W)x_{u^*}$. Note that $\lambda\geq 7$. Therefore,

$$\begin{aligned}\sum_{H\in\mathcal{H}}\eta(H) &< e(U_+)x_{u^*} - \sum_{H\in\mathcal{H}'}x_{u^*} + \sum_{H\in\mathcal{H}'}\frac{2\sum_{w\in W_H}x_w}{\lambda-3}\\ &\leq e(U_+)x_{u^*} - \sum_{H\in\mathcal{H}'}x_{u^*} + \frac{4e(W)}{\lambda-3}x_{u^*}\\ &\leq \left(e(U_+)+e(W)-\sum_{H\in\mathcal{H}'}1\right)x_{u^*},\end{aligned}$$

which contradicts with (1). Hence, $G^*[U]$ contains no C_4. This completes the proof. $\square$

By Claim 3.2, we know that each non-trivial component of $G^*[U]$ is either a tree with diameter at most 3 or a unicyclic graph $K_{1,r}+e$ with $r\geq 2$. Let c be the number of non-trivial tree-components of $G^*[U]$. Then

$$\sum_{H\in\mathcal{H}}\eta(H)\leq \sum_{H\in\mathcal{H}}\sum_{u\in V(H)}(d_H(u)-1)x_{u^*} = \sum_{H\in\mathcal{H}}(2e(H)-|H|)x_{u^*} = (e(U_+)-c)x_{u^*}.$$

Combining with (1), we get

$$e(W)\leq 1-c-\sum_{u\in U_0}\frac{x_u}{x_{u^*}}. \tag{2}$$

Thus, $e(W)\leq 1$ and $c\leq 1$. In addition, if $e(W)=1$, then $c=0$, $U_0=\emptyset$, $\lambda^2-\lambda=m-1$, $x_w=x_{u^*}$ for any $w\in W$ with $d_U(w)\geq 1$ and $x_u=x_{u^*}$ for any $u\in V(H)$ with $d_H(u)\geq 2$.

Claim 3.3 $e(W)=0$.

Proof. Suppose on the contrary that $e(W)=1$. Let w_1w_2 be the unique edge in $G^*[W]$. Note that $c=0$ and $U_0=\emptyset$, it follows that each component of $G^*[U]$ is isomorphic to a unicyclic graph $K_{1,r}+e$ with $r\geq 2$. That is, each component of $G^*[U]$ contains a triangle. Let H be a component of $G^*[U]$ and $u_1u_2u_3u_1$ be the triangle C_3 of H. Since

G^* is $\theta_{1,3,3}$-free, we get that $d_{C_3}(w_1) + d_{C_3}(w_2) \leq 3$. Otherwise, there are two cases. One case is $d_{C_3}(w_i) = 3$ and $d_{C_3}(w_j) \geq 1$ for $i \neq j \in \{1, 2\}$. Without loss of generality, we suppose that $d_{C_3}(w_1) = 3$ and $d_{C_3}(w_2) \geq 1$. Let $u_1 \in N_{C_3}(w_2)$. It is easy to find that u_1u_2, $u_1u^*u_3u_2$ and $u_1w_2w_1u_2$ are three internally disjoint paths of length 1,3,3 between u_1 and u_2. It is a contradiction. The other case is $d_{C_3}(w_i) \geq 2$ for $i \in \{1, 2\}$. Since $|C_3| = 3$, we obtain $|N_{C_3}(w_1) \cap N_{C_3}(w_2)| \geq 1$. Assume $u_1, u_p \in N_{C_3}(w_1)$ and $u_1, u_q \in N_{C_3}(w_2)$ with $p, q \in \{2, 3\}$. If $p \neq q$, then u_1u_q, $u_1w_1w_2u_q$ and $u_1u^*u_pu_q$ are three internally disjoint paths of length 1,3,3 between u_1 and u_q. If $p = q$, for convenience, suppose $p = q = 2$, then u_1u_q, $u_1w_1w_2u_q$ and $u_1u^*u_3u_q$ are three internally disjoint paths of length 1,3,3 between u_1 and u_q. It is also a contradiction. Since $d_{C_3}(w_1) + d_{C_3}(w_2) \leq 3$, we have $\sum_{u\in N_{C_3}(w_1)} x_u + \sum_{u\in N_{C_3}(w_2)} x_u \leq (d_{C_3}(w_1) + d_{C_3}(w_2))x_{u^*} \leq 3x_{u^*}$. According to $\lambda x_{u^*} = \sum_{u\in N(u^*)} x_u = \sum_{u\in C_3} x_u + \sum_{u\in U\setminus C_3} x_u$, we obtain $\sum_{u\in U\setminus C_3} x_u = \lambda x_{u^*} - x_{u_1} - x_{u_2} - x_{u_3}$. By Lemma 2.3 and $e(W) = 1$, we get $d_U(w_i) \geq 1$ for $i \in \{1, 2\}$. Recall that $x_w = x_{u^*}$ for any $w \in W$ with $d_U(w) \geq 1$. We get $x_{w_1} = x_{w_2} = x_{u^*}$. As $d_H(u_i) \geq 2$, we have $x_{u_i} = x_{u^*}$ for $i \in \{1, 2, 3\}$. This implies that

$$\begin{aligned}
2\lambda x_{u*} &= \lambda x_{w_1} + \lambda x_{w_2} \\
&= x_{w_2} + \sum_{u\in N_U(w_1)} x_u + x_{w_1} + \sum_{u\in N_U(w_2)} x_u \\
&\leq x_{w_2} + x_{w_1} + \sum_{u\in N_{C_3}(w_1)} x_u + \sum_{u\in N_{C_3}(w_2)} x_u + 2\sum_{u\in U\setminus C_3} x_u \\
&\leq x_{w_2} + x_{w_1} + 3x_{u^*} + 2(\lambda x_{u^*} - x_{u_1} - x_{u_2} - x_{u_3}) \\
&= 2x_{u^*} + 3x_{u^*} + 2(\lambda x_{u^*} - 3x_{u^*}) \\
&= 2\lambda x_{u^*} - x_{u^*}.
\end{aligned}$$

It is a contradiction for $x_{u^*} > 0$. The proof is complete. □

Claim 3.4 *$G^*[U]$ contains no triangle.*

Proof. Suppose on the contrary that $G^*[U]$ contains triangles. Then $G^*[U]$ contains a component which is isomorphic to $K_{1,r} + e$ with $r \geq 2$. Let $H^* \cong K_{1,r} + e$ be a component of $G^*[U]$. It follows that $e(H^*) = r + 1$. Suppose $u_1u_2u_3u_1$ is the triangle of H^* and $d_{H^*}(u_1) = d_{H^*}(u_2) = 2$.

If $W_{H^*} = \emptyset$, then $x_{u_1} = x_{u_2}$. Hence,

$$\lambda x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} \leq x_{u_1} + 2x_{u^*}.$$

This implies that $x_{u_1} \leq \frac{2}{\lambda-1} x_{u^*}$. Therefore,

$$\eta(H^*) = x_{u_1} + x_{u_2} + (r - 1)x_{u_3} \leq \frac{4}{\lambda - 1} x_{u^*} + (r - 1)x_{u^*}.$$

Since $m \geq 43$ and $\lambda \geq 7$, we get $\eta(H^*) < rx_{u^*} = (e(H^*)-1)x_{u^*}$. Note that $\eta(H) \leq e(H)x_{u^*}$ for any other component $H \in \mathcal{H} \setminus \{H^*\}$ of $G^*[U]$. Hence,

$$\begin{aligned}\sum_{H\in\mathcal{H}} \eta(H) = \eta(H^*) + \sum_{H\in\mathcal{H}\setminus\{H^*\}} \eta(H) \\ &< (e(H^*)-1)x_{u^*} + \sum_{H\in\mathcal{H}\setminus\{H^*\}} e(H)x_{u^*} \\ &= (e(U_+)-1)x_{u^*},\end{aligned}$$

which contradicts with (1). Thus, $W_{H^*} \neq \emptyset$.

Because $e(W) = 0$, by Lemma 2.3, we have $d_U(w) \geq 2$ for any $w \in W_{H^*}$. Suppose $r \geq 3$, let $u_4, \ldots, u_{r+1}$ be the neighbors of u_3. For $w \in W_{H^*}$, if $\{u_1, u_2\} \subseteq N_U(w)$, then u_1u_3, $u_1u^*u_4u_3$ and $u_1wu_2u_3$ are three internally disjoint paths of length 1,3,3 between u_1 and u_3, a contradiction. If $\{u_i, u_3\} \subseteq N_U(w)$, then u_ju_3, $u_ju^*u_4u_3$ and $u_ju_iwu_3$ are three internally disjoint paths of length 1,3,3 between u_j and u_3 where $i \neq j \in \{1,2\}$, a contradiction. If $\{u_i, u_j\} \subseteq N_U(w)$, then u_iu_3, $u_iu^*u_{\{1,2\}\setminus\{i\}}u_3$ and $u_iwu_ju_3$ are three internally disjoint paths of length 1,3,3 between u_i and u_3 where $i \in \{1,2\}$ and $j \in \{4, \ldots, r+1\}$, a contradiction. If $\{u_3, u_j\} \subseteq N_U(w)$, then u^*u_3, $u^*u_jwu_3$ and $u^*u_1u_2u_3$ are three internally disjoint paths of length 1,3,3 between u^* and u_3 where $j \in \{4, \ldots, r+1\}$, a contradiction. If $\{u_i, u_j\} \subseteq N_U(w)$, then u^*u_j, $u^*u_iwu_j$ and $u^*u_1u_3u_j$ are three internally disjoint paths of length 1,3,3 between u^* and u_j where $i \neq j \in \{4, \ldots, r+1\}$, a contradiction. If $\{u_i, v\} \subseteq N_U(w)$, then u^*u_i, u^*vwu_i and $u^*u_4u_3u_i$ are three internally disjoint paths of length 1,3,3 between u^* and u_i where $i \in \{1,2\}$ and $v \in U \setminus V(H^*)$, a contradiction. If $\{u_3, v\} \subseteq N_U(w)$, then u^*u_3, u^*vwu_3 and $u^*u_1u_2u_3$ are three internally disjoint paths of length 1,3,3 between u^* and u_3 where $v \in U \setminus V(H^*)$, a contradiction. If $\{u_i, v\} \subseteq N_U(w)$, then u^*u_i, u^*vwu_i and $u^*u_1u_3u_i$ are three internally disjoint paths of length 1,3,3 between u^* and u_i where $i \in \{4, \ldots, r+1\}$ and $v \in U \setminus V(H^*)$, a contradiction. Therefore, $r = 2$. That is, H^* is a triangle $u_1u_2u_3u_1$.

First, we assume that $|W_{H^*}| = 1$. Let $W_{H^*} = \{w\}$. Since G^* is $\theta_{1,3,3}$-free, it follows that $d_{U\setminus V(H^*)}(w) = 0$. Therefore, $d(w) = d_{H^*}(w)$. As H^* is a triangle, we obtain $d(w) = 2$ or $d(w) = 3$. If $d(w) = 2$, without loss of generality, we suppose $N(w) = \{u_1, u_2\}$. Then $x_{u_1} = x_{u_2}$. Since

$$\lambda x_{u_3} = x_{u_1} + x_{u_2} + x_{u^*} \leq 3x_{u^*},$$

we obtain $x_{u_3} \leq \frac{3}{\lambda}x_{u^*}$. Furthermore,

$$\lambda x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + x_w \leq x_{u_1} + \frac{3}{\lambda}x_{u^*} + 2x_{u^*}.$$

This implies that $x_{u_1} \leq \frac{3+2\lambda}{\lambda(\lambda-1)}x_{u^*}$. Thus,

$$\eta(H^*) = x_{u_1} + x_{u_2} + x_{u_3} \leq \frac{7\lambda+3}{\lambda(\lambda-1)}x_{u^*}.$$

Since $\frac{7x+3}{x(x-1)}$ is decreasing in variable $x > 1$ and $\lambda \geq 7$, we get

$$\eta(H^*) \leq \frac{7 \times 7 + 3}{7 \times 6} x_{u^*} < 2x_{u^*} = (e(H^*) - 1)x_{u^*}.$$

Recall that $\eta(H) \leq e(H)x_{u^*}$ for any other component $H \in \mathcal{H} \setminus \{H^*\}$ of $G^*[U]$. Hence,

$$\begin{aligned}\sum_{H \in \mathcal{H}} \eta(H) &= \eta(H^*) + \sum_{H \in \mathcal{H} \setminus \{H^*\}} \eta(H) \\ &< (e(H^*) - 1)x_{u^*} + \sum_{H \in \mathcal{H} \setminus \{H^*\}} e(H)x_{u^*} \\ &= (e(U_+) - 1)x_{u^*},\end{aligned}$$

which contradicts with (1). If $d(w) = 3$, that is, $N(w) = \{u_1, u_2, u_3\}$, then $x_{u_1} = x_{u_2} = x_{u_3}$. By

$$\lambda x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + x_w \leq 2x_{u_1} + 2x_{u^*},$$

we obtain $x_{u_1} \leq \frac{2}{\lambda - 2} x_{u^*}$. Therefore, by $\lambda \geq 7$,

$$\eta(H^*) = x_{u_1} + x_{u_2} + x_{u_3} \leq \frac{6}{\lambda - 2} x_{u^*} < 2x_{u^*} = (e(H^*) - 1)x_{u^*}.$$

Thus, $\sum_{H \in \mathcal{H}} \eta(H) < (e(U_+) - 1)x_{u^*}$, a contradiction. So $|W_{H^*}| \geq 2$.

Similarly, since G^* is $\theta_{1,3,3}$-free, we have $d_{U \setminus V(H^*)}(w) = 0$ for any $w \in W_{H^*}$. Therefore, $2 \leq d(w) = d_{H^*}(w) \leq 3$. If there is a vertex $w' \in W_{H^*}$ such that $d(w') = 3$, then $N(w') = \{u_1, u_2, u_3\}$. Note that $|W_{H^*}| \geq 2$, there exists a vertex $w'' \neq w'$ in W_{H^*} satisfying $d(w'') \geq 2$. Suppose that $u_1, u_2 \in N(w'')$. Then u^*u_1, $u^*u_3w'u_1$ and $u^*u_2w''u_1$ are three internally disjoint paths of length 1,3,3 between u^* and u_1, a contradiction. Hence, $d(w) = d_{H^*}(w) = 2$ for any $w \in W_{H^*}$. This implies that $1 \leq |N(w) \cap N(w')| \leq 2$ for any two vertices $w, w' \in W_{H^*}$. If $|N(w) \cap N(w')| = 1$, without loss of generality, we assume that $N(w) = \{u_1, u_2\}$ and $N(w') = \{u_1, u_3\}$. It is easy to see that u^*u_1, $u^*u_3w'u_1$ and $u^*u_2wu_1$ are three internally disjoint paths of length 1,3,3 between u^* and u_1, a contradiction. Therefore, $|N(w) \cap N(w')| = 2$. That is, $N(w) = N(w')$ for any $w, w' \in W_{H^*}$. Without loss of generality, we suppose that $N(w) = \{u_1, u_2\}$ for any $w \in W_{H^*}$. Let G' be a graph such that $V(G') = V(G^*)$ and $E(G') = E(G^*) - \{u_1w | w \in N_W(u_1)\} + \{u^*w | w \in N_W(u_1)\}$. One can verify that G' is $\theta_{1,3,3}$-free. By Lemma 2.2, we have $\lambda(G') > \lambda$. It is a contradiction with the maximality of G^*. We complete the proof. $\square$

Proof of Theorem 1.4. By Claims 3.2 and 3.4, we have that each component of $G^*[U]$ is a non-trivial tree or an isolated vertex. By inequality (2), the number c of non-trivial tree-components is at most 1. If $c = 0$, then G^* is bipartite. By Lemma 2.1, $\lambda \leq \sqrt{m} < \frac{1+\sqrt{4m-3}}{2}$, a contradiction. Hence $c = 1$. It follows that $U_0 = \emptyset$. Let H be the unique component of $G^*[U]$. That is, $G^*[U] \cong H$ and $W = W_H$. Since G^* is $\theta_{1,3,3}$-free, $diam(H) \leq 3$.

If $diam(H) = 3$, then H is a double star. Denote the two centers of H by u_1 and u_2. If $W_H = \emptyset$, then $G^* = \{u^*\} \vee H$. Without loss of generality, suppose that $x_{u_1} \geq x_{u_2}$. Let G' be a graph such that $V(G') = V(G^*)$ and $E(G') = E(G^*) - \{u_2v | v \in N_H(u_2) \setminus \{u_1\}\} + \{u_1v | v \in N_H(u_2) \setminus \{u_1\}\}$. One can verify that G' is $\theta_{1,3,3}$-free. By Lemma 2.2, we have $\lambda(G') > \lambda$, a contradiction. If $W_H \neq \emptyset$, then $N(w) = \{u_1, u_2\}$ for any $w \in W_H$. Otherwise, G^* contains $\theta_{1,3,3}$ as a subgraph, a contradiction. Let G'' be a graph such that $V(G'') = V(G^*)$ and $E(G'') = E(G^*) - \{u_2w | w \in W_H\} + \{u^*w | w \in W_H\}$. Obviously, G'' is $\theta_{1,3,3}$-free. By Lemma 2.2, we have $\lambda(G'') > \lambda$, a contradiction. Hence, $diam(H) \leq 2$. That is, $H \cong K_{1,r}$ with $r \geq 1$.

Let $V(H) = \{u_0, u_1, \ldots, u_r\}$ and u_0 be the center of H with $r \geq 1$. Since

$$\lambda x_{u_0} = x_{u_1} + x_{u_2} + \cdots + x_{u_r} + x_{u^*} + \sum_{w \in N_W(u_0)} x_w,$$

and

$$\lambda x_{u^*} = x_{u_0} + x_{u_1} + x_{u_2} + \cdots + x_{u_r},$$

we obtain $\lambda(x_{u_0} - x_{u^*}) = x_{u^*} + \sum_{w \in N_W(u_0)} x_w - x_{u_0}$. Note that $x_{u_0} \leq x_{u^*}$ and $x_v > 0$ for any $v \in V(G^*)$. Thus, $N_W(u_0) = \emptyset$ and $x_{u_0} = x_{u^*}$. If $W \neq \emptyset$, by Lemma 2.3, we have $d(w) \geq 2$ for $w \in W_H$. Note that $e(W) = 0$. Let $w_0 \in W$. Suppose $u_1, u_2 \in N_H(w_0)$. If $r \geq 3$, then u^*u_2, $u^*u_1w_0u_2$ and $u^*u_ru_0u_2$ are three internally disjoint paths of length 1,3,3 between u^* and u_2, a contradiction. So $r \leq 2$. Since $d(w) \geq 2$ and $N_W(u_0) = \emptyset$, we obtain $r \neq 1$. Therefore, $r = 2$ and $N(w) = \{u_1, u_2\}$ for any $w \in W$. Hence, $x_{u_0} = x_{u^*}$ and $x_{u_1} = x_{u_2}$. As

$$\lambda x_{u^*} = x_{u_0} + x_{u_1} + x_{u_2} = x_{u^*} + 2x_{u_1},$$

it follows that $x_{u_1} = \frac{\lambda-1}{2} x_{u^*}$. Note that $x_{u^*} \geq x_{u_1}$. We can get $\lambda \leq 3$, it is a contradiction with $\lambda \geq 7$. Thus $W = \emptyset$. Equivalently, $G^* \cong K_1 \vee K_{1,r}$ with $2r + 1 = m$. Hence $G^* \cong K_2 \vee \frac{m-1}{2} K_1$. This completes the proof. $\square$

Acknowledgments

The authors would like to thank the anonymous referees for their helpful comments on improving the presentation of the paper. This work was supported by National Natural Science Foundation of China (Nos.12131013 and 12161141006).

Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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