On edge-primitive Cayley graphs

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Abstract

A graph is said to be edge-primitive if its automorphism group acts primitively on the edge set. In this paper, we investigate finite edge-primitive Cayley graphs of valency no less than 2. An explicit classification of such graphs is obtained in the case where the graphs admit an almost simple edge-primitive automorphism group which contains a regular subgroup on the vertices. This implies that the only edge-primitive Cayley graphs of valency at least 2 defined over simple groups are cycles with prime length and complete graphs. In addition, we also classify those edge-primitive Cayley graphs which are either 2-arc-transitive or of square-free order.

Keywords. Cayley graph, edge-primitive graph, automorphism group, simple group.

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1 Introduction

In algebraic graph theory, the study of graph symmetries, especially those of Cayley graphs, forms a major research theme. In this paper, we focus on the class of edge-primitive Cayley graphs. Throughout this paper, all graphs are finite, simple, undirected, and free of isolated vertices.

A graph Γ is called vertex-transitive, edge-transitive and arc-transitive if its automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on the vertex set, edge set and arc set, respectively. (Recall that an arc in a graph is an ordered pair of adjacent vertices.) An edge-transitive graph Γ is called edge-primitive if $\mathrm{Aut}(\Gamma)$ acts primitively on the edge set, that is, $\mathrm{Aut}(\Gamma)$ preserves no nontrivial partition of the edge set.

Since our graphs are free of isolated vertices, it can be easily deduced from the definition that an edge-primitive graph is either connected or a disjoint union of K_2 components, see also [11, Lemma 3.1]. In addition, by [11, Lemma 3.4], an edge-primitive graph is either arc-transitive or a star. Therefore, we always assume that all edge-primitive graphs in this paper have valency at least 2, which ensures the connectedness and arc-transitivity of the graphs under consideration.

Edge-primitive graphs were first studied by Weiss [28], who classified those graphs of valency three. Since then, however, little progress was made in this area for thirty years. In 2010, Giudici and Li [11] systematically studied the group-theoretic structures and constructions of edge-primitive graphs. They proved that edge-primitive graphs of valency at least three are necessarily arc-transitive, and that, with the exception of complete bipartite graphs, the automorphism groups acting on edges fall into precisely four primitive types, say AS, PA, SD and CD as defined in [27, Section 6], while exhibiting highly restricted behavior on vertices. Giudici and Li's work brought edge-primitive graphs back to the forefront, and spurred a great deal of work aimed at their classification and characterization under

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further constraints. See, for instance, [12, 13, 23, 24, 31] for edge-primitive graphs with given valency or order, and [10, 14, 20, 22] for those that are 2-arc-transitive.

While edge-primitive graphs have recently attracted attention for their rarity and algebraic significance, their systematic exploration within the framework of Cayley graphs remains largely unexplored. This motivates us to consider the intersection of these two concepts: Cayley graphs that are edge-primitive. Specifically, we aim to address the following problem.

Problem 1.1 ([25])**.** *Characterize edge-primitive Cayley graphs.*

An expression $G = HK$ of a group G as the product of subgroups H and K is called an exact factorization if $H \cap K = 1$. Recall that the automorphism group $\mathrm{Aut}(\Gamma)$ of a Cayley graph Γ contains a subgroup R that acts regularly on the vertex set. This implies that $\mathrm{Aut}(\Gamma)$ admits an exact factorization $\mathrm{Aut}(\Gamma) = RX$, where X is the stabilizer in $\mathrm{Aut}(\Gamma)$ of a vertex. Consequently, to effectively address and resolve Problem 1.1, a detailed characterization of the exact factorizations of primitive permutation groups may serve as a critical step. This task, however, appears to be quite challenging as the primitivity is imposed on the edges rather than on the vertices, and the factorization $\mathrm{Aut}(\Gamma) = RX$ exhibits only the action of $\mathrm{Aut}(\Gamma)$ on the vertices. Fortunately, as established by Giudici and Li in their seminal work [11], the edge-primitivity property of (Cayley) graphs imposes significant restrictions on the action of the automorphism group on the vertex set. This theoretical foundation provides us with a valuable starting point for addressing the problem within well-defined constraints.

In their work addressing Problem 1.1, Pan et al. [25] undertook the classification of edge-primitive Cayley graphs defined over abelian and dihedral groups. They proved that, with the exceptions of cycles of prime length and complete bipartite graphs, all such graphs are derived from almost simple groups. In this paper, appealing to the exact factorizations of almost simple groups, we present a classification of edge-primitive graphs with almost simple automorphism groups, which produces many interesting examples of edge-primitive Cayley graphs.

Theorem 1.2. *Let $\Gamma = (V, E)$ be an edge-primitive graph, let $A = \mathrm{Aut}(\Gamma)$ and $R \leqslant G \leqslant A$. Assume that G is an almost simple group acting primitively on E, and that R acts regularly on the vertex set. Then one of the following holds:*

(1) $A = \mathrm{S}_n$, $G = \mathrm{S}_n$ *or* A_n, $|R| = n$, *and* Γ *is the complete graph* K_n*;*

(2) $A = \mathrm{S}_{12} \wr \mathrm{S}_2$, $G = \mathrm{M}_{12}.2$, $|R| = 24$, *and* Γ *is the complete bipartite graph* $\mathsf{K}_{12,12}$*;*

(3) $A = \mathrm{S}_{12}$, $G = \mathrm{S}_{11}$ *or* S_{12}, $R \cong \mathrm{S}_7$ *or* $\mathrm{A}_7 \times 2$, *and* Γ *is isomorphic to the graph* Γ_1 *in Example* 2.3*;*

(4) $A = G = \mathrm{M}_{12}.2$, $R \cong 3^2{:}\mathrm{Q}_8.2$, *and* Γ *is isomorphic to the graph* Γ_2 *in Example* 2.3*;*

(5) $A = G = \mathrm{J}_2.2$, $R \cong 5^2{:}4$, *and* Γ *is isomorphic to the graph* Γ_3 *in Example* 2.3*;*

(6) $A = G = \mathrm{HS}.2$, $R \cong 5^2{:}4$, *and* Γ *is isomorphic to the graph* Γ_4 *in Example* 2.3*;*

(7) $A = G = \mathrm{He}.2$, $R \cong 7_+^{1+2}{:}6$, *and* Γ *is isomorphic to the graph* Γ_5 *in Example* 2.3*;*

(8) $A = G = \mathrm{PGL}_2(11)$, $R \cong \mathrm{D}_{22}$, *and* Γ *is isomorphic to one of the bipartite graphs* Γ_6 *and* $\Gamma_{6'}$ *in Example* 2.3*;*

(9) $A = G = \mathrm{PGL}_2(59)$, $R \cong 59{:}58$, *and* Γ *is isomorphic to the bipartite graph* Γ_7 *in* Example 2.3*;*

(10) $A = G = \mathrm{PSL}_5(2).2$, $R \cong 31{:}10$, *and* Γ *is isomorphic to one of the bipartite graphs* $\mathcal{D}(5, 2; 2)$ *and* $\mathcal{F}(5, 2; 2)$ *defined as in Example* 2.4*;*

(11) $A = \mathrm{P\Gamma L}_n(q).2$, $\mathrm{PSL}_n(q) < G \not\leqslant \mathrm{P\Gamma L}_n(q)$, *R is solvable and of order* $\frac{2(q^n-1)}{q-1}$, *and* Γ *is isomorphic to one of the bipartite graphs* $\mathcal{D}(n, q; 1)$ *and* $\mathcal{F}(n, q; 1)$ *defined as in Example* 2.4.

(12) $A = G = \mathrm{PSU}_4(3).(2^2)_{133}$, $|R| = 162$, *and* Γ *is isomorphic to one of the graphs* Γ_8 *and* $\Gamma_{8'}$ *in Example* 2.3*;*

(13) $A = \mathrm{P\Gamma U}_4(8)$, $|A : G| \leqslant 2$, $|R| = 4617$, *and* Γ *is isomorphic to the graph* $\mathcal{U}(4, 8)$ *defined as in Example* 2.5.

Theorem 1.2 is applied to classify the edge-primitive Cayley graphs under specific constraints. It is proved that the only edge-primitive Cayley graphs of valency at least 2 over a simple group are cycles with prime length and complete graphs.

Theorem 1.3. *Assume that* Γ *is an edge-primitive Cayley graph of valency at least* 2 *over a simple group. Then* Γ *is either a cycle of prime length or a complete graph.*

Recall that a vertex-transitive graph is 2-arc-transitive if the stabilizer of a vertex acts 2-transitively on the set of neighbors of the vertex. It was proved in [22] that, with the exceptions of cycles and complete bipartite graphs, 2-arc-transitive edge-primitive graphs have almost simple automorphism groups. Then, using Theorem 1.2, all 2-arc-transitive edge-primitive Cayley graphs are classified.

Theorem 1.4. *Let* Γ *be an edge-primitive Cayley graph of valency at least* 3*. Assume that* Γ *is* 2*-arc-transitive. Then one of the following holds:*

(1) Γ *is either a complete graph or a complete bipartite graph;*

(2) Γ *is bipartite and isomorphic to one of the graphs* Γ_4, Γ_6, $\Gamma_{6'}$ *and* Γ_7 *defined in Example* 2.3*, and the graphs* $\mathcal{F}(5, 2; 2)$, $\mathcal{D}(n, q; 1)$ *and* $\mathcal{F}(n, q; 1)$ *defined in Example* 2.4.

It will be proved in Section 8 that, with the exception of cycles and complete bipartite graphs, edge-primitive graphs of square-free order have almost simple automorphism groups. This observation suggests that a classification of the edge-primitive graphs of square-free order may be feasible. Thus, we propose the following problem, which will be addressed in detail in a subsequent work.

Problem 1.5. *Classify edge-primitive graphs of square-free order.*

Building on Theorem 1.2, we have a classification of the edge-primitive Cayley graphs defined over groups of square-free order.

Theorem 1.6. *Assume that* Γ *is an edge-primitive Cayley graph of square-free order and valency at least* 3*. Then one of the following holds:*

(1) Γ *is either a complete graph or a complete bipartite graph;*

(2) Γ *is bipartite and isomorphic to one of the graphs* Γ_6, $\Gamma_{6'}$ *and* Γ_7 *defined in Example* 2.3*, and the graphs* $\mathcal{D}(5, 2; 2)$, $\mathcal{F}(5, 2; 2)$, $\mathcal{D}(n, q; 1)$ *and* $\mathcal{F}(n, q; 1)$ *defined in Example* 2.4*, where* $\frac{q^n-1}{q-1}$ *is odd and square-free.*

It is not hard to observe that, with the exceptions of cycles and complete bipartite graphs, those edge-primitive Cayley graphs involved in Theorems 1.2-1.4 have almost simple automorphism groups. This assertion is not generally valid. Indeed, as demonstrated in [11, Construction 5.6 and Examples 5.8, 5.10], one can construct infinitely many edge-primitive Cayley graphs whose automorphism groups are not almost simple. It is particularly noteworthy that we have no example of edge-primitive Cayley graphs over solvable groups in which the automorphism group fails to be almost simple and the graph is not a cycle or a complete bipartite graph. This naturally gives rise to the following interesting problem.

Problem 1.7. *Construct or characterize, if they exist, edge-primitive Cayley graphs defined over solvable groups, excluding cycles and complete bipartite graphs, where the automorphism groups are not almost simple.*

This paper is structured as follows: Section 2 reviews some simple but useful facts about coset graphs that are edge-primitive, and constructs examples involved in the theorems. Sections 3-7 are devoted to analyze the edge-primitive Cayley graphs arising from exact factorizations of almost simple groups, where Section 3 reduces such exact factorizations to that of those groups having socle a simple alternating, sporadic, linear and unitary group, which are dealt with in Sections 4-7, respectively. Section 8 presents a proof of Theorem 1.2 and several applications of Theorem 1.2 to the classification of edge-primitive Cayley graphs under specific constraints.

The group-theoretic notation used in this paper is standard, see, for example, [6] and [30]. Following [6], for a positive integer n, the symbol $[n]$ sometimes denotes an (unspecified) group of order n for convenience; in particular, n denotes a cyclic group of order n, and p^f with p prime denotes an elementary abelian group of order p^f. For two groups K and H, denote by $K.H$ an extension of K by H, while $K{:}H$ stands for a split extension and $K^{\cdot}H$ indicates any case of $K.H$ which is a non-split extension. The notation $K \wr H$ stands for a wreath product of K by H.

2 Coset graphs

In this section, we assume that $G \neq 1$ is a finite group and X, $Y \leqslant G$ such that

(a) X is a core-free subgroup of G, that is, X contains no nontrivial normal subgroup of G; and

(b) Y is a maximal subgroup of G, written as $Y \lessdot G$ occasionally; and

(c) $|Y : (X \cap Y)|$, the index of $X \cap Y$ in Y, equals to 2, and $|X : (X \cap Y)| \geqslant 2$.

Lemma 2.1. *Let $y \in Y \setminus X$. Then $X \cap X^y = X \cap Y$.*

Proof. Since $|Y : (X \cap Y)| = 2$, we have $y^2 \in X \cap Y$ and $X \cap Y \trianglelefteq Y$. Then $X \cap Y = (X \cap Y)^y \leqslant X \cap X^y = (X^y \cap X)^y$. Noting that $Y = (X \cap Y)\langle y \rangle$, it follows that Y normalizes $X \cap X^y$. Thus $(X \cap X^y)Y$ is a subgroup of G and, since Y is maximal in G, either $X \cap X^y \leqslant Y$ or $(X \cap X^y)Y = G$. If $(X \cap X^y)Y = G$ then $X \cap X^y \trianglelefteq G$, and so $X \cap X^y = 1$ as X is core-free in G, yielding $G = Y$, a contradiction. Therefore, $X \cap X^y \leqslant Y$, and then $X \cap X^y = X \cap Y$. This completes the proof. $\square$

Denote by $[G : X]$ the set of right cosets of X in G. For $Xg_1, Xg_2 \in [G : X]$, we have $(Xg_2)(Xg_1)^{-1} = Xg_2g_1^{-1}X$, which is a double coset with a representative $g_2g_1^{-1}$. Pick $y \in Y \setminus X$. Then $XyX = Xy^{-1}X$. Define a graph on $[G : X]$ such that $Xg_1, Xg_2 \in [G : X]$ are adjacent if and only if $Xg_2g_1^{-1}X = XyX$, equivalently, $g_2g_1^{-1} \in XyX$. Noting that $XyX = XYX \setminus X$, the graph defined above is independent of the choice of $y \in Y \setminus X$. Thus we always denote this graph by $\mathsf{Cos}(G, X, Y)$. An arbitrary vertex Xg of $\mathsf{Cos}(G, X, Y)$ has neighborhood $\{Xyxg \mid x \in X\}$, which has size $|X : (X \cap X^y)|$. Thus, by Lemma 2.1, $\mathsf{Cos}(G, X, Y)$ is a regular graph of valency $|X : (X \cap Y)|$.

Since X is a core-free subgroup of G, we have a faithful permutation representation of G on $[G : X]$ by right multiplication. The resulting permutation group, denoted still by G, is a subgroup of the automorphism group of $\mathsf{Cos}(G, X, Y)$, in which the vertex-stabilizers, edge-stabilizers and arc-stabilizers are just the conjugates of X, Y and $X \cap Y$ in G, respectively. It follows from the maximality of Y in G that $\mathsf{Cos}(G, X, Y)$ is an edge-primitive graph, with valency given by $|X : (X \cap Y)|$. Of course, $\mathsf{Cos}(G, X, Y)$ is connected and, as observed in [11, Lemma 3.4], either $\mathsf{Cos}(G, X, Y)$ is a cycle of prime length or G acts transitively on the arc set of $\mathsf{Cos}(G,X,Y)$. We call $\mathsf{Cos}(G,X,Y)$ an edge-primitive coset graph. Combining [11, Proposition 2.5], we have the following lemma, which says that every edge-primitive graph of valency at least 2 is isomorphic to an edge-primitive coset graph.

Lemma 2.2. *Let Γ be a regular graph of valency $k \geqslant 2$. Then Γ is an edge-primitive if and only if $\Gamma \cong \mathsf{Cos}(G,X,Y)$ for some $G \leqslant \mathrm{Aut}(\Gamma)$ and $X, Y < G$ satisfying the conditions* (a)-(c) *with $k = |X : (X \cap Y)|$.*

2.1 Examples

The present subsection gives some examples of edge-primitive graphs involved in Theorem 1.2. For convenience, we call a graph G-edge-primitive if its automorphism group contains a subgroup G that acts primitively on the edge set. We begin with the following observation. Let $G \neq 1$ be a finite group, and let X be a core-free subgroup of G. By Lemma 2.2, a G-edge-primitive graph of valency $k > 2$ with a vertex-stabilizer X exists if and only if X has a subgroup Z of index k whose normalizer $Y := \mathbf{N}_G(Z)$ is a maximal subgroup of G satisfying $|Y : Z| = 2$ and $X \cap Y = Z$. Note, up to isomorphism of graphs, the subgroups X and $X \cap Y$ may be chosen under the G-conjugacy and X-conjugacy, respectively. The following example illustrates the construction of edge-primitive graphs from some groups G with small order.

Row	G	v	X	$Y = \mathbf{N}_G(Z)$	Z	k	Remarks
1	S_{12}	5040	M_{12}	$\mathrm{PGL}(2,11)$	$\mathrm{PSL}_2(11)$	144	$Z \lessdot X$
1’	S_{11}	5040	M_{11}	11:10	11:5	144	
2	$\mathrm{M}_{12}.2$	144	$\mathrm{PGL}_2(11)$	S_5	A_5	22	$X \cap \mathrm{M}_{12} \lessdot \mathrm{M}_{12}$
3	$\mathrm{J}_2.2$	100	$\mathrm{G}_2(2)$	$\mathrm{PSL}_3(2){:}2{\times}2$	$\mathrm{PSL}_3(2){:}2$	36	$Z \lessdot X \lessdot G$
4	HS.2	100	$\mathrm{M}_{22}.2$	$\mathrm{PSL}_3(4){:}2^2$	$\mathrm{P\Sigma L}_3(4)$	22	$Z \lessdot X \lessdot G$
5	He.2	2058	$\mathrm{Sp}_4(4).4$	$(\mathrm{S}_5{\times}\mathrm{S}_5){:}2$	$(\mathrm{A}_5{\times}\mathrm{A}_5){:}2^2$	272	$X \lessdot G$
6	$\mathrm{PGL}_2(11)$	22	A_5	D_{20}	D_{10}	6	
6’	$\mathrm{PGL}_2(11)$	22	A_5	S_4	A_4	5	
7	$\mathrm{PGL}_2(59)$	3422	A_5	S_4	A_4	5	
8	$\mathrm{PSU}_4(3).2^2$	162	$\mathrm{PSL}_3(4).2^2$	$(4^2{\times}2)(2{\times}\mathrm{S}_4)$	$2^4{:}\mathrm{S}_4.2$	105	$Z \lessdot X \lessdot G$
8’	$\mathrm{PSU}_4(3).2^2$	162	$\mathrm{PSL}_3(4).2^2$	$\mathrm{M}_{10}.2^2$	$\mathrm{A}_6.2^2$	56	$Z \lessdot X \lessdot G$

Table 1: Examples of edge-primitive coset graphs.

Example 2.3. Let G be one of the almost simple groups listed in Table 1. Note, for Rows 8 and 8′, $G = \mathrm{PSU}_4(3).(2^2)_{133}$ in the notation of the Atlas [6]. By inspecting the Atlas [6] and computation with GAP [8], the subgroup X of G and the subgroup Z of X (satisfying the conditions in the ‘Remarks’-column) are uniquely determined under the G-conjugacy and X-conjugacy, respectively, the normalizer Y of Z is a maximal subgroup of G, and $X \cap Y = Z$. Clearly, $|Y : (X \cap Y)| = 2$. Then, for each given group G in Table 1, we have a unique edge-primitive graph $\mathsf{Cos}(G,X,Y)$ of order v and valency k, where v and k are given as in Row i or i', corresponding to the group G under consideration. Denote the resulting graph by Γ_i or $\Gamma_{i'}$, respectively. The following are some further information of these graphs.

(1) The graphs Γ_1, $\Gamma_{1'}$, Γ_6, $\Gamma_{6'}$ and Γ_7 are bipartite; the union of Γ_6 and $\Gamma_{6'}$ is the complete bipartite graph $\mathsf{K}_{11,11}$; the union of Γ_8 and $\Gamma_{8'}$ is the complete graph K_{162}.

(2) Recall that for a finite transitive permutation group, the number of orbits of a point-stabilizer is defined as its rank, and the sizes of these orbits are referred to as its subdegrees. For each of the graphs Γ_3, Γ_4, Γ_8, and $\Gamma_{8'}$, inspection of the Atlas [6] and computation with GAP [8] show that the

corresponding group G is a rank three primitive group in its action on the vertices, with subdegrees: 1, 36, 63 for $\mathrm{J}_2.2$; 1, 22, 77 for HS.2; and 1, 56, 105 for $\mathrm{PSU}_4(3).2^2$. Computation with GAP [8] shows that the graph Γ_2 yields a rank 5 presentation of the Mathieu group M_{12} of degree 144 with subdegrees 1, 11, 11, 55 and 66, while the group $\mathrm{M}_{12}.2$ acts on the 144 points and fuses the suborbits of degree 11, see also [3, Section 10]. By inspecting the Atlas [6], the graph Γ_5 arises from a rank 5 presentation of the sporadic Held group He of degree 2058 with subdegrees 1, 136, 136, 425 and 1360, while the group He.2 acts on the 2058 points and fuses the suborbits of degree 136.

(3) The graphs Γ_1 and $\Gamma_{1'}$ are isomorphic. Note that Γ_1 and $\Gamma_{1'}$ are uniquely determined by the triples $(\mathrm{S}_{12}, \mathrm{M}_{12}, \mathrm{PGL}_2(11))$ and $(\mathrm{S}_{11}, \mathrm{M}_{11}, 11{:}10)$, respectively, where $\mathrm{PGL}_2(11)$ is a transitive subgroup of the symmetric group S_{12} in the natural action of degree 12. Dealing with S_{11} as a point-stabilizer of S_{12} in the natural action, we have $\mathrm{M}_{12} \cap \mathrm{S}_{11} \cong \mathrm{M}_{11}$ and $\mathrm{PGL}_2(11) \cap \mathrm{S}_{11} \cong 11{:}10$. Let E be the edge set of Γ_1. Then $|E| = |\mathrm{S}_{12} : \mathrm{PGL}_2(11)| = |\mathrm{S}_{11} : (\mathrm{PGL}_2(11) \cap \mathrm{S}_{11})|$, and so S_{11} acts transitively on E. By the Atlas [6], $\mathrm{PGL}_2(11) \cap \mathrm{S}_{11}$ is a maximal subgroup of S_{11}. It follows that S_{11} is an edge-primitive group of the graph Γ_1. Then $\Gamma_1 \cong \mathsf{Cos}(\mathrm{S}_{11}, \mathrm{M}_{12} \cap \mathrm{S}_{11}, \mathrm{PGL}_2(11) \cap \mathrm{S}_{11}) \cong \Gamma_{1'}$, as claimed.

Thus, up to isomorphism of graphs, we have ten distinct graphs: Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, Γ_6, $\Gamma_{6'}$, Γ_7, Γ_8 and $\Gamma_{8'}$. The automorphism groups of these graphs are precisely the groups A given in (3)-(9) and (12) of Theorem 1.2. $\square$

Example 2.4. Let $n \geqslant 3$ be an integer, and denote $V(n,q)$ the n-dimensional vector space over the finite field $\mathbb{F}_q$ of order q. Let m be an integer with $1 \leqslant m < \frac{n}{2}$. A 2-set $\{\alpha, \beta\}$ of subspaces is called an $\{m, n-m\}$-decomposition if $V(n,q) = \alpha \oplus \beta$ and α has dimension $\dim(\alpha) = m$. A chain $0 < \alpha < \beta < V(n,q)$ of subspaces is called an $(m, n-m)$-flag if $\dim(\alpha) = m$ and $\dim(\beta) = n-m$.

Let $G = \mathrm{P\Gamma L}_n(q){:}\langle\tau\rangle$, where τ is the inverse transpose automorphism of $\mathrm{PGL}_n(q)$. Then G acts primitively on both the set of $\{m, n-m\}$-decompositions and the set of $(m, n-m)$-flags. This fact leads to two edge-primitive bipartite graphs, say $\mathcal{D}(n,q;m) := \mathsf{Cos}(G, X, Y_D)$ and $\mathcal{F}(n,q;m) := \mathsf{Cos}(G, X, Y_F)$, where X is the stabilizer of an m-space α in $\mathrm{P\Gamma L}_n(q)$, Y_D and Y_F are the stabilizers of an $\{m, n-m\}$-decomposition and an $(m, n-m)$-flag, respectively, corresponding to the m-space α under consideration.

Denote V_m and V_{n-m} the sets of m-subspaces and $(n-m)$-subspaces of $V(n,q)$, respectively. Then

$$|V_m| = |V_{n-m}| = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-m+1}-1)}{(q^m-1)(q^{m-1}-1)\cdots(q-1)}.$$

Now the graphs $\mathcal{D}(n,q;m)$ and $\mathcal{F}(n,q;m)$ can be defined as the bipartite graphs with bipartition (V_m, V_{n-m}) such that

(1) $\alpha \in V_m$ and $\beta \in V_{n-m}$ are adjacent in $\mathcal{D}(n,q;m)$ if and only if $\{\alpha, \beta\}$ is an $\{m, n-m\}$-decomposition;

(2) $\alpha \in V_m$ and $\beta \in V_{n-m}$ are adjacent in $\mathcal{F}(n,q;m)$ if and only if $0 < \alpha < \beta < V(n,q)$ is an $(m, n-m)$-flag.

Clearly, both $\mathcal{D}(n,q;m)$ and $\mathcal{F}(n,q;m)$ have order $\frac{2(q^n-1)(q^{n-1}-1)\cdots(q^{n-m+1}-1)}{(q^m-1)(q^{m-1}-1)\cdots(q-1)}$, $\mathcal{D}(n,q;m)$ has valency $q^{m(n-m)}$, and $\mathcal{F}(n,q;m)$ has valency $\frac{(q^{n-m}-1)(q^{n-m-1}-1)\cdots(q^{n-2m+1}-1)}{(q^m-1)(q^{m-1}-1)\cdots(q-1)}$. We also observe that the complete bipartite graph $\mathsf{K}_{q^2+q+1, q^2+q+1}$ decomposes into the edge-primitive graphs $\mathcal{D}(3,q;1)$ and $\mathcal{F}(3,q;1)$. $\square$

Example 2.5. Let $m \geqslant 2$ be an integer, and denote $U(2m,q)$ the $2m$-dimensional unitary space over the finite field $\mathbb{F}_{q^2}$. Denote U_m the set of totally singular m-subspaces of $U(2m,q)$. Then $|U_m| = \prod_{i=1}^{m}(q^{2i-1}+1)$. A 2-subset $\{\alpha, \beta\}$ of U_m is called a totally singular m-decomposition if $U(2m,q) =$

$\alpha \oplus \beta$. Let $G = \mathrm{P\Gamma U}_{2m}(q)$. Then G acts primitively on both U_m and the set of totally singular m-decompositions. Thus we have an edge-primitive graph $\mathcal{U}(2m, q)$ with vertex set U_m such that α, $\beta \in U_m$ are adjacent if and only if $\{\alpha, \beta\}$ is a totally singular m-decomposition of $U(2m, q)$. This graph is of valency q^{m^2} and isomorphic to $\mathsf{Cos}(G, X, Y)$, where X and Y are the stabilizers of a totally singular m-subspaces and a totally singular m-decomposition, respectively. $\square$

Remark 2.6. Recall that a vertex-transitive graph is 2-arc-transitive if and only if a vertex-stabilizer induces a 2-transitive permutation group on the neighborhood of the vertex. Based on this, we investigate the 2-arc-transitivity of the graphs presented in Examples 2.3-2.5. Our analysis reveals that only the following graphs exhibit 2-arc-transitivity: $\mathsf{\Gamma}_4$, $\mathsf{\Gamma}_6$, $\mathsf{\Gamma}_{6'}$, $\mathsf{\Gamma}_7$, $\mathcal{D}(n, q; 1)$, $\mathcal{F}(n, q; 1)$, and $\mathcal{F}(2m+1, q; m)$. The detailed proofs and technical verifications are omitted for brevity. $\square$

2.2 Observations

This subsection collects some simple but useful observations derived from the conditions (a)-(c). Put $k = |X : (X \cap Y)|$ and $v = |G : X|$. In view of the condition (c), we have $|Y| = \frac{2|X|}{k} \leqslant |X|$. (Note, if $|X| = |Y|$ then $|X| = |Y| = 2 = k$; in this case, $\mathsf{Cos}(G, X, Y)$ is a cycle of prime length say p, and G is isomorphic to the dihedral group D_{2p}.) Since $\mathsf{Cos}(G, H, K)$ is a graph of order v and valency k, we have $k < v$, and so $|Y| > \frac{|2X|}{v}$. Then we have the following simple fact.

Lemma 2.7. *Let $v = |G : X|$. Then $|G : X| > |X : (X \cap Y)|$. Moreover, if $|X : (X \cap Y)| > 2$ then $\frac{2|X|}{v} < |Y| < |X|$, and $|Y|$ is a divisor of $2|X|$.*

Lemma 2.8. *The maximal subgroup Y is core-free in G. If $1 \neq N \trianglelefteq G$ then $G = NY$, $|G : (X\cap Y)N| = \imath \in \{1, 2\}$, $|(Y \cap N) : (X \cap Y \cap N)| = \frac{2}{\imath}$, and either*

(1) *$G = XN$, $|X : (X \cap Y)| = \imath|(X \cap N) : (X \cap Y \cap N)|$, and $|Y \cap N||X : (X \cap Y)| = 2|X \cap N|$; or*

(2) *$|G : XN| = 2 = \imath$, $|X : (X \cap Y)| = |(X \cap N) : (X \cap Y \cap N)|$, and $|Y \cap N||X : (X \cap Y)| = |X \cap N|$.*

Proof. Let $M \trianglelefteq G$ with $M \leqslant Y$. Recall that Y serves as an edge-stabilizer of the graph $\mathsf{Cos}(G, X, Y)$. It follows that M fixes every edge of $\mathsf{Cos}(G, X, Y)$. Since $\mathsf{Cos}(G, X, Y)$ is connected and of valency at least 2, we conclude that M fixes every vertex of the graph, yielding $M = 1$. Then the first part of this lemma follows.

Let $1 \neq N \trianglelefteq G$. Since $\mathsf{Cos}(G, X, Y)$ is G-edge-primitive, N acts transitively on the edges of $\mathsf{Cos}(G, X, Y)$. Note that $X \cap N$, $Y \cap N$ and $X \cap Y \cap N$ serve as the stabilizers in N of some vertex, edge and arc of the graph, respectively. Then $G = NY$, and $|G : Y| = |N : (Y \cap N)|$. In addition, by the arc-transitivity of G and the edge-transitivity of N, we conclude that N either acts transitively or has two orbits with equal length on the arc set of $\mathsf{Cos}(G, X, Y)$. We have $|G : (X\cap Y)| = \imath|N : (X\cap Y\cap N)|$, where $\imath$ is the number of N-orbits on the arc set. Then

$$|G| = \imath|(X \cap Y) : (X \cap Y \cap N)||N| = \imath|(X \cap Y)N|,$$

yielding $|G : (X \cap Y)N| = \imath$. Also,

$$\begin{aligned}|G : Y||Y : (X \cap Y)| &= |G : (X \cap Y)| = \imath|N : (X \cap Y \cap N)| \\ &= \imath|N : (Y \cap N)||(Y \cap N) : (X \cap Y \cap N)|.\end{aligned}$$

Since $|G : Y| = |N : (Y \cap N)|$, we get $|(Y \cap N) : (X \cap Y \cap N)| = \frac{2}{\imath}$.

The edge-transitivity of N implies that either N is transitive on the vertex set, or $\mathsf{Cos}(G, X, Y)$ is bipartite and N has two orbits with equal length on the vertex set. Then $|G : X| = |N : (X \cap N)|$ or

$2|N:(X\cap N)|$, and $|G:XN|=1$ or 2, respectively. In addition, by $|G:(X\cap Y)|=\imath|N:(X\cap Y\cap N)|$, we have

$$|G:X||X:(X\cap Y)|=\imath|N:(X\cap N)||(X\cap N):(X\cap Y\cap N)|.$$

If $|G:X|=|N:(X\cap N)|$ then $|X:(X\cap Y)|=\imath|(X\cap N):(X\cap Y\cap N)|$, and so

$$|X\cap N|=\frac{1}{\imath}|X\cap Y\cap N||X:(X\cap Y)|=\frac{1}{2}|Y\cap N||X:(X\cap Y)|,$$

as in (1) of the lemma. If $|G:X|=2|N:(X\cap N)|$ then $\imath=2$, and

$$|G:X||X:(X\cap Y)|=2|N:(X\cap N)||(X\cap N):(X\cap Y\cap N)|,$$

we have $|X:(X\cap Y)|=|(X\cap N):(X\cap Y\cap N)|$, and so

$$|X\cap N|=|X\cap Y\cap N||X:(X\cap Y)|=|Y\cap N||X:(X\cap Y)|,$$

as in (2) of the lemma. This completes the proof. □

Denote the intersection of terms appearing in the derived series of a group U by U^∞. In fact, U^∞ is the least normal subgroup N of U with solvable quotient group U/N.

Lemma 2.9. *Let $N=G^\infty$. Then $Y^\infty=(X\cap Y)^\infty=(Y\cap N)^\infty=(X\cap Y\cap N)^\infty$, and $X^\infty=(X\cap N)^\infty$. In particular, if X is insolvable then $Y^\infty<X^\infty$, and if one of Y, $X\cap Y$, $Y\cap N$ and $(X\cap Y\cap N)$ is solvable then the other three groups are solvable.*

Proof. Let $U\leqslant G$. Then $UN\leqslant G$, and $U/(U\cap N)\cong UN/N\leqslant G/N$. Since G/N is solvable, we have $U^\infty\leqslant U\cap N$, and so $U^\infty\leqslant (U\cap N)^\infty\leqslant U^\infty$. Then $U^\infty=(U\cap N)^\infty$. Choosing U from $\{Y, X\cap Y, Y\cap N, X\cap Y\cap N\}$ and $\{X, X\cap N\}$, we have $Y^\infty=(X\cap Y)^\infty=(Y\cap N)^\infty=(X\cap Y\cap N)^\infty$ and $X^\infty=(X\cap N)^\infty$, respectively. Clearly, $Y^\infty\leqslant X^\infty$. If $Y^\infty=X^\infty$ then $X^\infty\trianglelefteq\langle X,Y\rangle=G$, and so $X^\infty=1$, i.e., X is solvable, as X is core-free in G. Thus $Y^\infty<X^\infty$ when X is insolvable, and the lemma follows. □

We remark that the maximality of Y is not necessary for Lemma 2.9.

Lemma 2.10. *If the maximal subgroup Y is insolvable, then $Y=\mathbf{N}_G(Y^\infty)$.*

Proof. Assume that Y is insolvable, and so $Y^\infty\neq 1$. Since $Y^\infty\leqslant X\cap Y$ and X is core-free in G, we have $\mathbf{N}_G(Y^\infty)<G$. Noting that $Y\leqslant\mathbf{N}_G(Y^\infty)$, since Y is maximal in G, we have $Y=\mathbf{N}_G(Y^\infty)$, as desired. □

3 Graphs arising from exact factorizations

Recall that a graph is (isomorphic to) a Cayley graph if and only if its automorphism group contains a subgroup that acts regularly on the vertex set. By Lemma 2.2, we have following fact.

Lemma 3.1. *Let Γ be a regular graph of valency $k\geqslant 2$. Then Γ is an edge-primitive Cayley graph if and only if $\Gamma\cong\mathsf{Cos}(G,X,Y)$ for some $G\leqslant\mathrm{Aut}(\Gamma)$ and $R,X,Y<G$ such that $G=RX$, $R\cap X=1$, $k=|X:(X\cap Y)|$ and (G,X,Y) satisfies the conditions* (a)-(c) *at the beginning of Section* 2.

Recall that an expression $G=RX$ is called an exact factorization of a group G if R, $X\leqslant G$ with $R\cap X=1$. Occasionally, we call R a complement of X in G. This and the following four sections (Sections 4-7) are devoted to analyze the edge-primitive Cayley graphs arising from exact factorizations

$G = RX$ of almost simple groups G. Suppose that $\Gamma = (V, E)$ is a graph demonstrating such behavior. Then the insolvability of G yields that Γ is not a cycle, and so Γ has valency $k \geqslant 3$. Of course, the edge-primitivity of G implies that the socle $\mathrm{soc}(G)$ of G acts transitively on the edge set E. This forces that $k|R| = k|V| = 2|E| \leqslant 2|\mathrm{soc}(G)|$. In particular, $|R| < |\mathrm{soc}(G)|$, and so $\mathrm{soc}(G) \not\leq R$. In view of Lemma 3.1, our task is to determine all possible quintuples $(G, R, X, Y, X \cap Y)$ satisfying the following assumptions.

Hypothesis 3.2. *Assume that G is an almost simple group. Let $T = \mathrm{soc}(G)$, and let R, X, $Y < G$ be such that*

(I) *$G = RX$ is an exact factorization with $T \not\leq R$ and $T \not\leq X$; and*

(II) *$Y \lessdot G$, $k := |X : (X \cap Y)| > 2$, and $|Y : (X \cap Y)| = 2$.*

Note that the triple (G, X, Y) in Hypothesis 3.2 satisfies (a)-(c) defined at the beginning of Section 2. Under the hypothesis (I), by [4, 18], T is a simple alternating, sporadic, linear, unitary, symplectic or orthogonal group (up to isomorphism), and $\{R, X\} = \{H, K\}$ with H and K defined as in either [4, Theorem 3] or [18, Theorem 1.2].

Under the hypothesis (II), the symplectic groups of dimension at least 4 and orthogonal groups of dimension at least 8 are excluded in the following two lemmas, respectively.

Lemma 3.3. *Assume that (G, T, R, X, Y) is described as in Hypothesis 3.2. Then $T \neq \mathrm{PSp}_{2n}(q)$ with $n \geqslant 2$.*

Proof. Suppose that $\mathrm{soc}(G) = T = \mathrm{PSp}_{2n}(q)$ with $n \geqslant 2$. Let $\{H, K\} = \{R, X\}$. Then, by [4, Theorem 3] and [18, Theorem 1.2], one of the following occurs:

(s.1) $n \geqslant 3$, q is even, $H \cap T \lesssim q^n{:}(q^n - 1).n$, and $K \cap T \cong \Omega_{2n}^-(q)$;

(s.2) (G, H, K) is listed in Cases 31-34 of [4, Table 4] and Rows 14-16 and 18 of [18, Table 1];

(s.3) $G = \mathrm{Sp}_6(4).2$, $H \cong \mathrm{SL}_2(16).4$, and $K \cong \mathrm{G}_2(4).2$.

Recall that Y has a subgroup $X \cap Y$ of index 2, $|Y| < |X|$ and $|Y|$ is a divisor of $2|X|$, see Lemma 2.7. This allows us to exclude cases (s.1)-(s.3) by examining the maximal subgroups of G, and then a contradiction arises.

Suppose that case (s.2) occurs. Inspecting the orders of maximal subgroups of G, refer to the Atlas [6], we deduce that either $G = \mathrm{PSp}_6(3).2$, $Y \cong \mathrm{S}_5$ and $X = H \cong ([3^5]{:}[2^5].\mathrm{D}_{10}).2$, or $G = \mathrm{Sp}_8(2)$, $Y \cong \mathrm{S}_{10}$ and $X = K \cong \Omega_8^-(2)$. The former case implies that $X \cap Y$ is solvable, and so does Y, a contradiction. The latter case implies that $\Omega_8^-(2)$ has a subgroup isomorphic to A_{10}, which is impossible.

Suppose that case (s.3) occurs. Then $H^\infty \cong \mathrm{SL}_2(16)$ and $K^\infty \cong \mathrm{G}_2(4)$. By [20, Theorem 1.1 and Table 17], $G = \mathrm{Sp}_6(4).2$ has no solvable maximal subgroup, and so $Y^\infty \neq 1$. By Lemma 2.9, $Y^\infty < X^\infty = H^\infty$ or K^∞. Thus Y^∞ is isomorphic to one of the insolvable proper subgroups of $\mathrm{SL}_2(16)$ or $\mathrm{G}_2(4)$. It follows that Y^∞ is isomorphic to one of the following groups:

$$\mathrm{A}_5,\ \mathrm{PSL}_2(13),\ \mathrm{A}_5 \times \mathrm{A}_5,\ \mathrm{PSU}_3(3),\ \mathrm{PSL}_2(7),\ \mathrm{PSU}_3(4),\ \mathrm{SL}_3(4),\ \mathrm{J}_2,\ 3^{\cdot}\mathrm{A}_6,\ [2^s]{:}\mathrm{A}_5,$$

where $s \geqslant 4$. Inspecting the maximal subgroups of $G = \mathrm{Sp}_6(4).2$ in [2, Tables 8.28 and 8.29], it deduces that $Y \cong 2^{12}{:}\mathrm{GL}_3(4).2$. Then neither $2|H|$ nor $2|K|$ is divisible by $|Y|$, and so $|Y|$ is not a divisor of $2|X|$, a contradiction.

Finally, suppose that case (s.1) occurs, in particular, q is even and $n \geqslant 3$. Recall that $X \in \{H, K\}$. Assume that Y is solvable. Then, by [20, Theorem 1.1 and Table 17], we have $T = \mathrm{Sp}_6(2)$ with $Y \cap T \cong [2^7]{:}(\mathrm{S}_3 \times \mathrm{S}_3)$. Since $X \cap T \lesssim q^n{:}(q^n-1).n$ or $\Omega_{2n}^-(q)$, we conclude that $2|X \cap T|$ is not divisible by $|Y \cap T|$; however, by Lemma 2.8 $|(Y \cap T)| = |(X \cap Y \cap T)|$ or $2|(X \cap Y \cap T)$, a contradiction. Therefore, Y is insolvable, and so does $X \cap Y$. This implies that $X = K$ and $X \cap T \cong \Omega_{2n}^-(q)$. In addition, by Lemma 2.9, $Y^\infty < X^\infty \cong \Omega_{2n}^-(q)$.

Aschbacher's Theorem, see [1], says that either Y is one of the geometrically defined subgroups of G, or Y is included in a collection $\mathcal{S}$ of almost simple groups, refer to [16, p. 4] for the definition of $\mathcal{S}$. Noting that $Y^\infty < X^\infty \cong \Omega_{2n}^-(q)$, by the definition of $\mathcal{S}$ and [16, p. 5, Remark], we know that $Y \notin \mathcal{S}$. Thus $Y \cap T$ is one of those subgroups of G described as in [16, Table 3.5.C]. Since $Y^\infty < X^\infty \cong \Omega_{2n}^-(q)$ and Y is a maximal subgroup of G, combining [16, Table 3.5.H], we conclude that $Y \cap T$ is maximal in $T = \mathrm{Sp}_{2n}(q)$ and described as in [16, Propositions 4.1.3, 4.1.19, 4.2.10, 4.3.10 and 4.5.4]. Thus one of the following occurs:

(i) $Y \cap T \cong \mathrm{Sp}_{2m}(q) \times \mathrm{Sp}_{2(n-m)}(q)$, where $1 \leqslant m < \frac{n}{2}$;

(ii) $Y \cap T \cong [q^{\frac{m(4n+1-3m)}{2}}]{:}(\mathrm{GL}_{2m}(q) \times \mathrm{Sp}_{2(n-m)(q)})$, where $1 \leqslant m \leqslant n$;

(iii) $Y \cap T \cong \mathrm{Sp}_{2m}(q) \wr \mathrm{S}_t$, where $t \geqslant 2$ with $n = mt$, and if $q = 2$ then $m > 1$;

(iv) $Y \cap T \cong \mathrm{Sp}_{2m}(q^r).r$, where r is a prime with $n = mr$;

(v) $Y \cap T \cong \mathrm{Sp}_{2n}(\sqrt[r]{q})$, where r is a prime.

Since $Y \cap T$ is maximal in T, we have $Y \cap T \not\leqslant X \cap T$; otherwise, $\Omega_{2n}^-(q) \cong X \cap T = Y \cap T \trianglelefteq \langle X, Y \rangle = G$, a contradiction. Thus $Y \cap T > X \cap Y \cap T$, and so $|(Y \cap T) : (X \cap Y \cap T)| = 2$ by Lemma 2.8. Since $n \geqslant 3$ and q is even, $\mathrm{Sp}_{2n}(\sqrt[r]{q})$ is a nonabelian simple group, and so $\mathrm{Sp}_{2n}(\sqrt[r]{q})$ has no subgroup of index 2. Thus, case (v) is excluded. Again by Lemma 2.8, $|Y \cap T|$ is a divisor of $2|X \cap T|$. Note that

$$2|X \cap T| = 2q^{n(n-1)}(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1).$$

If case (ii) occurs then $|Y \cap T|$ has a divisor q^{n^2}, and so $2q^{n(n-1)}$ is divisible by q^{n^2}, which is impossible.

Recalling that $|(Y \cap T) : (X \cap Y \cap T)| = 2$, we have $G = XT = (X \cap Y)T$, and $|X : (X \cap Y)| = |(X \cap T) : (X \cap Y \cap T)|$ by Lemma 2.8. Then $|G : X| = |T : (X \cap T)| = q^n(q^n-1)$, and $|X : (X \cap Y)| = \frac{2|X \cap T|}{|Y \cap T|}$. By Lemma 2.7, $|G : X| > |X : (X \cap Y)|$, and so $q^n(q^n-1) > |X : (X \cap Y)| = \frac{2|X \cap T|}{|Y \cap T|}$. Then $\frac{|X \cap T|}{|Y \cap T|} < q^n(q^n-1)$.

Assume that case (iv) holds. Then $Y \cap T \cong \mathrm{Sp}_{2m}(q^r).r$ has a subgroup of index 2, which forces that $r = 2$ and $n = 2m$. Then $|Y \cap T| = 2q^{2m^2}\prod_{i=1}^{m}(q^{4i}-1)$. We have

$$\frac{|X \cap T|}{|Y \cap T|} = \frac{1}{2(q^{2m}-1)}q^{2m(m-1)}\prod_{i=1}^{m}(q^{2(2i-1)}-1).$$

If $m > 2$ then $\frac{|X \cap T|}{|Y \cap T|} > q^{2m}(q^{4m-2}-1) > q^n(q^n-1)$, a contradiction. Now let $m \leqslant 2$. Since $n = 2m \geqslant 3$, we have $n = 2m = 4$. Then

$$2|X \cap T| = 2q^{12}(q^4+1)\prod_{i=1}^{3}(q^{2i}-1), \ |Y \cap T| = 2q^8\prod_{i=1}^{2}(q^{4i}-1).$$

Recalling that $|Y \cap T|$ is a divisor of $2|X \cap T|$, it follows that q^2+1 is a divisor of q^6-1, which is impossible.

Assume that case (i) holds. Then $\mathrm{Sp}_{2m}(q) \times \mathrm{Sp}_{2(n-m)}(q)$ has a subgroup of index 2. Since q is even and $1 \leqslant m < \frac{n}{2}$, we have $m=2$, $q=2$, and $n > 2m = 4$. Then

$$|Y \cap T| = 2^{n^2-4n+8}45\prod_{i=1}^{n-2}(2^{2i}-1),\ |X \cap T| = 2^{n(n-1)}(2^n+1)\prod_{i=1}^{n-1}(2^{2i}-1).$$

Calculation shows that $\frac{|X\cap T|}{|Y\cap T|} > 2^n(2^n-1)$, a contradiction.

Finally, assume that case (iii) holds. Writing $t! = 2^a b$ for positive integers a and b with $\gcd(2,b)=1$, we have

$$|Y \cap T| = q^{m^2t}(\prod_{i=1}^{m}(q^{2i}-1))^t t! = 2^a q^{m^2t}(\prod_{i=1}^{m}(q^{2i}-1))^t b.$$

Let $\ell = \frac{(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)}{(\prod_{i=1}^{m}(q^{2i}-1))^t b}$. Recall that $|Y \cap T|$ is a divisor of $2|X \cap T| = 2q^{n(n-1)}(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)$. Since q is even, putting $q=2^f$, we know that ℓ is an odd integer, and $\frac{2|X\cap T|}{|Y\cap T|} = 2^{(n(n-1)-m^2t)f+1-a}\ell$. Then $\frac{|X\cap T|}{|Y\cap T|} = 2^{(n(n-1)-m^2t)f-a}\ell$ and, since $\frac{|X\cap T|}{|Y\cap T|} < q^n(q^n-1)$, we have

$$2^{2nf} = q^{2n} > q^n(q^n-1) > 2^{(n(n-1)-m^2t)f-a}\ell.$$

In particular, $(n(n-1)-m^2t)f - a < 2nf$. Since $t! = 2^a b$ with $\gcd(2,b)=1$, by Legendre's formula, $a = t - s_2(t)$, where $s_2(t)$ is the sum of digits of t in base 2. Clearly, $s_2(t) \geqslant 1$, which yields $a \leqslant t-1$, and so $(n(n-1)-m^2t)f - t + 1 < 2nf$. Recalling that $n = mt$ and $t \geqslant 2$, we have

$$\begin{aligned} 2nf > (n(n-1)-m^2t)f - t + 1 &= (n-1-m)nf - \frac{n}{m} + 1 \\ &\geqslant (n-1-m)nf - n + 1 > (n-1-m)nf - n. \end{aligned}$$

Then $2f > (n-1-m)f - 1 \geqslant (\frac{n}{2}-1)f - 1$. We have $\frac{n}{2}-1 < 3$, and if $m=1$ then $n \leqslant 4$. Since $n = mt \geqslant 3$ and $t \geqslant 2$, we deduce that

$$(n,m,t) = (3,1,3),\ (4,1,4),\ (4,2,2),\ (6,2,3),\ (6,3,2).$$

For each of these cases, calculation shows that ℓ is not an integer, a contradiction. This completes the proof. $\square$

Lemma 3.4. *Assume that (G,T,R,X,Y) is described as in Hypothesis 3.2. If T is a simple orthogonal group of dimension n, then $n < 7$.*

Proof. Suppose that $T = \mathrm{soc}(G)$ is a simple orthogonal group of dimension $n \geqslant 7$. Let $\{H,K\} = \{R,X\}$. Then, by [4, Theorem 3] and [18, Theorem 1.2], $T = \Omega_8^+(2)$ or $\Omega_8^+(4)$, and one of the following occurs:

(o.1) $G = \Omega_8^+(2).o$, $H \cong 2^6{:}15$ or $2^4{:}15{:}4$, and $K \cong \mathrm{A}_9.o$, where $o \in \{1,2\}$;

(o.2) $G = \Omega_8^+(2).o$, $H \cong 2^4{:}\mathrm{A}_5.o_1$, and $K \cong \mathrm{A}_9.o_2$, where $o = o_1o_2 \in \{1,2\}$;

(o.3) $G = \Omega_8^+(2).o$, $H \cong \mathrm{S}_5 \times o_1$, and $K \cong \mathrm{Sp}_6(2) \times o_2$, where $o = o_1o_2 \in \{1,2\}$;

(o.4) $G = \Omega_8^+(4).(2 \times o)$, $H \cong \mathrm{SL}_2(16).4 \times o_1$, and $K \cong \mathrm{Sp}_6(4).2 \times o_2$, where $o = o_1o_2 \in \{1,2\}$; in this case, K fixes a non-singular 1-space and H fixes a totally singular i-space with $i \in \{1,3,4\}$, refer to [18, Example 4.6].

By Lemma 2.7, $|Y| < |X|$ and $|Y|$ is a divisor of $2|X|$. Since $X \in \{H, K\}$, we have that $|Y| < |H|$ or $|K|$, and $|Y|$ is a divisor of $2|H|$ or $|2|K|$, respectively. In view of this, we next deduce the contradiction by examining the maximal subgroups of G. For cases (o.1)-(o.3), inspecting the maximal subgroups of G in the Atlas [6], we deduce that $G = \Omega_8^+(2).o$, $X = K \cong \mathrm{Sp}_6(2) \times o_2$, and $Y \cong \mathrm{A}_9.o$. Then $\mathrm{A}_9 \cong Y^\infty = (X \cap Y)^\infty \leqslant X^\infty \cong \mathrm{Sp}_6(2)$; however, $\mathrm{Sp}_6(2)$ has no subgroup isomorphic to A_9, a contradiction.

Now assume that case (o.4) occurs. It is easy to check that neither $2|H|$ nor $2|K|$ is divisible by 5^4. Since $X \in \{H, K\}$ and $|Y|$ is a divisor of $2|X|$, we know that $|Y|$ is not divisible by 5^4. Noting that $|G : T| = 2$ or 4, by [20, Theorem 1.1 and Table 19], every solvable maximal subgroup of G has order divisible by 5^4. It follows that Y is insolvable. Then, by Lemma 2.9, $Y^\infty < X^\infty \cong \mathrm{SL}_2(16)$ or $\mathrm{Sp}_6(4)$ with $X = H$ or K, respectively. Taking this into account, we inspect the maximal subgroups of G in [2, Table 8.50] whose order is a divisor of $2|H|$ or $2|K|$. As a result, the only possibility is that $Y \cap T \cong \mathrm{PSL}_3(4).3$, and $\mathrm{soc}(Y \cap T)$ is an absolutely irreducible subgroup of $T = \Omega_8^+(4)$. Moreover, since $(Y \cap T)^\infty = Y^\infty < X^\infty$, we have $X = K$. Noting that $\mathrm{soc}(Y \cap T) = (Y \cap T)^\infty$, it follows that $\mathrm{soc}(Y \cap T)$ fixes a non-singular 1-space, a contradiction. This completes the proof. □

Theorem 3.5. *Let $\Gamma = (V, E)$ be a G-edge-primitive graph. Assume that G is an almost simple group with socle T, and G contains a subgroup R that acts regularly on V. Then T is isomorphic to a simple alternating, sporadic, linear or unitary group. If $T \cong \mathrm{A}_n$ with $5 \leqslant n \leqslant 10$, then one of the following holds:*

(1) $G = \mathrm{A}_n$ *or* S_n, $|R| = n$, *and* $\Gamma \cong \mathsf{K}_n$;

(2) $G = \mathrm{PGL}_2(9)$ *or* $G = \mathrm{P\Gamma L}_2(9)$, $R \cong \mathbb{Z}_{10}$ *or* D_{10}, *and* $\Gamma \cong \mathsf{K}_{10}$;

(3) $G = \mathrm{PSL}_4(2).2$, $R \cong \mathrm{D}_{30}$ *or* $5 \times \mathrm{S}_3$, *and* $\Gamma \cong \mathcal{D}(4, 2; 1)$ *or* $\mathcal{F}(4, 2; 1)$.

Proof. Pick $\{\alpha, \beta\} \in E$, let $X = G_\alpha$ and $Y = G_{\{\alpha\beta\}}$, the stabilizers of vertex α and edge of $\{\alpha, \beta\}$, respectively. Then $(G, R, X, Y, X \cap Y)$ satisfies Hypothesis 3.2. Considering the exact factorization $G = RX$, by [4, 18], T is a simple alternating, sporadic, linear, unitary, symplectic or orthogonal group. Considering the isomorphisms among simple groups, the first part of this theorem follows from Lemmas 3.3 and 3.4.

Assume now that $T \cong \mathrm{A}_n$ with $5 \leqslant n \leqslant 10$. Note that Y is maximal in G and $|Y : (X \cap Y)| = 2$. Computation with GAP [8] shows that one of the following occurs:

(i) $G = \mathrm{A}_n$ or S_n, X is a point-stabilizer in the natural action of G on an n-set, while Y is the stabilizer of a 2-subset;

(ii) $G = \mathrm{PGL}_2(9)$, $X \cong \mathrm{A}_5$, and $Y \cong \mathrm{D}_{20}$;

(iii) $G = \mathrm{PGL}_2(9)$, $X \cong 3^2{:}8$ or S_4, and $Y \cong \mathrm{D}_{16}$;

(iv) $G = \mathrm{M}_{10}$, $X \cong 3^2{:}\mathrm{Q}_8$ or S_4, and $Y \cong 8{:}2$;

(v) $G = \mathrm{P\Gamma L}_2(9)$, $X \cong 3^2{:}[2^4]$ or $\mathrm{S}_4 \times 2$, and $Y \cong [2^5]$;

(vi) $G = \mathrm{S}_7 < \mathrm{PSL}_4(2).2$, $X \cong \mathrm{PSL}_2(7)$, and $Y \cong 7{:}3$;

(vii) $G = \mathrm{PSL}_4(2).2 \cong \mathrm{S}_8$, $X \cong 2^3{:}\mathrm{PSL}_3(2)$, and $Y \cong 2^4{:}\mathrm{S}_4$ or $\mathrm{PSL}_3(2).2$.

Case (i) implies that Γ is a complete graph, and (1) of the theorem follows. For case (ii), (iv) or (vi), computation with GAP [8] shows that X has no complement in G, which is not the case. Assume that one of (iii) and (v) occurs. Then Γ is isomorphic to either K_{10} or the Tutte-Coxeter graph. Since Γ is a Cayley graph, we have $\Gamma \cong \mathsf{K}_{10}$. Further computation shows that X has a complement $R \cong \mathbb{Z}_{10}$ or D_{10}, as in part (2) of the theorem. Finally, assume that case (vii) occurs. Then X is the stabilizer in G of a 1- or 3-space, while Y is the stabilizer in G of a $\{1,3\}$-decomposition or a $(1,3)$-flag. We have $\Gamma \cong \mathcal{D}(4,2;1)$ or $\mathcal{F}(4,2;1)$. Computation with GAP [8] shows that X has a complement $R \cong \mathrm{D}_{30}$ or $5 \times \mathrm{S}_3$, and so (3) of the theorem follows. This completes the proof. $\square$

4 Alternating groups

Let (G,T,R,X) be described as in (I) of Hypothesis 3.2. Assume that $T = \mathrm{soc}(G)$ is a simple alternating group of degree n. By Theorem 3.5, we may assume $n \geqslant 11$, and $G = \mathrm{A}_n$ or S_n. Let $\{H,K\} = \{R,X\}$, so that we have an exact factorization $G = HK$ of either A_n or S_n. The triple (G,H,K) is then determined by [29, Theorems A and S]. In particular, we may assume without loss of generality that H is a transitive subgroup of S_n acting on $\Omega := \{1,2,\ldots,n\}$. Combining [29, Theorems A and S] with [4, Theorem 3] and [18, Theorem 1.2], one of the following holds:

(a.1) $G = \mathrm{S}_n$ or A_n, $|H| \in \{n, 2n\}$ and $K \cong \mathrm{A}_{n-1}$ or S_{n-1}; in particular, K fixes a point in Ω;

(a.2) $G = \mathrm{S}_{p^a}$ or A_{p^a} with p prime and $p^a \geqslant 11$, $H \leqslant \mathrm{A\Gamma L}_1(p^a)$ is 2-homogeneous on Ω, and $\mathrm{A}_{p^a-2} \trianglelefteq K \leqslant \mathrm{S}_{p^a-2} \times \mathrm{S}_2$; in particular, K fixes a 2-subset of Ω.

(a.3) (G,H,K) is listed in Cases 3-11 of [4, Table 4] and Rows 2, 3 and 6-11 of [18, Table 1]; in this case, H is a transitive subgroup of S_n, and K fixes an m-subset but does not fix an $\frac{n}{2}$-subset of Ω, where $m \in \{2,3,4,5\}$.

Lemma 4.1. *Let $G = \mathrm{A}_n$ or S_n with $n \geqslant 11$. Assume that (G,T,R,X,Y) is described as in Hypothesis 3.2, and that Y is intransitive on Ω. Then $|R| = n$, and either*

(1) *$G = \mathrm{A}_n$, $X \cong \mathrm{A}_{n-1}$ and $Y \cong \mathrm{S}_{n-2}$; or*

(2) *$G = \mathrm{S}_n$, $X \cong \mathrm{S}_{n-1}$ and $Y \cong \mathrm{S}_{n-2} \times \mathrm{S}_2$.*

Proof. Since Y is an intransitive maximal subgroup of S_n, it deduces from (a)-(f) stated in [21] that $Y = (\mathrm{Sym}(\Delta) \times \mathrm{Sym}(\Omega \setminus \Delta)) \cap G$ is the stabilizer in G of a proper subset Δ of Ω, where $|\Delta| \neq \frac{n}{2}$. Without loss of generality, we let $|\Delta| < \frac{n}{2}$. Note that

$$|G:Y||Y:(X\cap Y)| = |G:(X\cap Y)| = |G:X||X:(X\cap Y)|.$$

Since $|Y:(X\cap Y)| = 2$ and $k = |X:(X\cap Y)| > 2$, we have $|G:X| = \frac{2}{k}|G:Y| < \binom{n}{m}$, where $m = |\Delta|$. Since $n \geqslant 11$, by [7, Theorems 5.2A and 5.2B], $\mathrm{Alt}(\Omega\setminus\Lambda) \leqslant X \leqslant (\mathrm{Sym}(\Lambda) \times \mathrm{Sym}(\Omega\setminus\Lambda)) \cap G$, where $\emptyset \neq \Lambda \subset \Omega$ with $|\Lambda| < |\Delta|$. Again since $|Y:(X\cap Y)| = 2$, we have $\mathrm{Alt}(\Delta) \times \mathrm{Alt}(\Omega\setminus\Delta) \leqslant X \cap Y$. Then

$$\mathrm{Alt}(\Delta) \times \mathrm{Alt}(\Omega\setminus\Delta) < X \leqslant \mathrm{Sym}(\Lambda) \times \mathrm{Sym}(\Omega\setminus\Lambda).$$

Note that $|\Omega\setminus\Lambda| > |\Omega\setminus\Delta| > \frac{n}{2}$. If $\Omega\setminus\Delta \not\subseteq \Omega\setminus\Lambda$, then $(\Omega\setminus\Lambda)\cap\Delta \neq \emptyset$ and $(\Omega\setminus\Lambda)\cap(\Omega\setminus\Delta) \neq \emptyset$, which forces that X acts transitively on Ω, a contradiction. Thus $\Omega\setminus\Delta \subset \Omega\setminus\Lambda$, and $\Lambda \subset \Delta$. It follows that $\mathrm{Alt}(\Delta) \times \mathrm{Alt}(\Omega\setminus\Delta)$ fixes $\Delta\setminus\Lambda$ set-wise. Then only possibility is that $|\Delta| = 2$ and $|\Lambda| = 1$. Recalling that $\{R,X\} = \{H,K\}$ with H and K described as in (a.1)-(a.3), it follows that $X = K \cong \mathrm{A}_{n-1}$ or S_{n-1}, and $|R| = |H| \in \{n, 2n\}$.

Suppose that $|R| = 2n$. Then $G = \mathrm{S}_n$, $Y \cong \mathrm{S}_{n-2} \times \mathrm{S}_2$, and $X \cong \mathrm{A}_{n-1}$ as $|G| = |R||X|$ and $\mathrm{A}_{n-1} \lesssim X$. This forces that $X \cap Y \cong \mathrm{A}_{n-2}$, and so $|Y : (X \cap Y)| = 4$, a contradiction. Then $|R| = n$, and the lemma follows. $\square$

Lemma 4.2. *Let $G = \mathrm{A}_n$ or S_n with $n \geqslant 11$. Assume that (G, T, R, X, Y) is described as in Hypothesis 3.2, and that Y is transitive on Ω. Then one of the following holds:*

(1) *$G = \mathrm{S}_{11}$, $R \cong \mathrm{S}_7$ or $\mathrm{A}_7 \times 2$, $X \cong \mathrm{M}_{11}$ and $Y \cong 11{:}10$;*

(2) *$G = \mathrm{S}_{12}$, $R \cong \mathrm{S}_7$ or $\mathrm{A}_7 \times 2$, $X \cong \mathrm{M}_{12}$ and $Y \cong \mathrm{PGL}_2(11)$.*

Proof. Note that $X \cap Y$ is a normal subgroup of Y of index 2. It follows that either $X \cap Y$ is transitive on Ω or $X \cap Y$ has two orbits on Ω with length $\frac{n}{2}$. This forces that either X is transitive on Ω or X fixes an $\frac{n}{2}$-subset of Ω. Recalling that $\{R, X\} = \{H, K\}$ with H and K described as in cases (a.1)-(a.3), it follows that $X = H$ is transitive on Ω. If case (a.1) occurs then $|Y| < |X| \leqslant 2n$, and n is a divisor of $|Y|$ as Y is transitive on Ω, yielding $|Y| = n$; however, it is easily observed from (a)-(f) stated in [21] that G has no maximal subgroup of order n, contrary to the maximality of Y. Thus case (a.2) or (a.3) occurs with $X = H$.

Assume first that $X = H$ is described as in case (a.2). Then X is solvable, and so does Y as $|Y : (X \cap Y)| = 2$. Recall that $n \geqslant 11$. By [20, Theorem 1.1], we may choose a normal subgroup G_0 of G such that $T \leqslant G_0$ and $Y \cap G_0$ is one of the maximal subgroups of G_0 listed as follows:

(i) $G_0 = \mathrm{A}_{16}$, and $Y \cap G_0 \cong (\mathrm{S}_4 \wr \mathrm{S}_4) \cap \mathrm{A}_{16}$; in this case, $X \lesssim 2^4{:}15{:}4$;

(ii) $G_0 = \mathrm{A}_p$, and $Y \cap G_0 \cong \mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$ with $p \notin \{11, 17, 23\}$; in this case, $X \leqslant \mathrm{AGL}_1(p)$;

(iii) $G_0 = \mathrm{S}_p$, and $Y \cap G_0 \cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ with $p \in \{11, 17, 23\}$; in this case, $X \leqslant \mathrm{AGL}_1(p)$.

Case (ii) or (iii) implies that X and Y have a common normal subgroup of order p, which is impossible as $\langle X, Y \rangle = G = \mathrm{A}_p$ or S_p has no normal subgroup of order p. If case (i) holds then $|Y|$ has a divisor 3^5, and so $|Y|$ is not a divisor of $2|X|$, contrary to Lemma 2.7.

We assume next that $X = H$ is described as in case (a.3).

Case 1. Suppose that H is given as in Cases 3-11 of [4, Table 4]. Then $X = H$ is solvable, and so does Y. In addition, $\mathrm{soc}(G) = \mathrm{A}_{32}$ or A_{p^2}, where $p \in \{5, 7, 11, 23\}$. In this case, it can be deduced from [20, Theorem 1.1 and Table 14] that G has no solvable maximal subgroup, contrary to the maximality of Y.

Case 2. Suppose that H is given as in Rows 6-8 of [18, Table 1]. Then either $n \in \{11, 12\}$ and $X = H \cong \mathrm{M}_n$, or $n = 33$ and $X = H \cong \mathrm{P\Gamma L}_2(32)$. Suppose first that $n = 33$. Then $G = \mathrm{A}_{33}$ or S_{33}. Checking the maximal subgroups of $\mathrm{P\Gamma L}_2(32)$ in the Atlas [6], it follows that X has no insolvable proper subgroup. Then $X \cap Y$ is solvable, and so does Y as $|Y : (X \cap Y)| = 2$; however, by inspecting [20, Table 14], neither A_{33} nor S_{33} has solvable maximal subgroups, a contradiction. Therefore, $n \in \{11, 12\}$. For $n = 11$, checking the maximal subgroups of G in the Atlas [6], since $|Y| < |X|$, we have $G = \mathrm{S}_{11}$ and $Y \cong 11{:}10$, as in (1) of the lemma. Similarly, for $n = 12$, since $|Y| < |X|$ and $|Y|$ is a divisor of $2|X|$, we have $G = \mathrm{S}_{12}$ and $Y \cong \mathrm{PSL}_2(11){:}2$, as in (2) of the lemma.

Case 3. Suppose that H is given as in Rows 9-11 of [18, Table 1]. Then $\mathrm{soc}(G) = \mathrm{A}_{p^2}$ with $p \in \{11, 29, 59\}$, and $X = H$ has a unique Sylow p-subgroup P. Recalling that every $(X \cap Y)$-orbit on Ω has length n or $\frac{n}{2}$, since $n = p^2$ is odd, it follows that $X \cap Y$ is transitive on Ω. Then $|X \cap Y|$ is divisible by $p^2 = |P|$, and thus $P \leqslant X \cap Y$. Now P is a characteristic subgroup of $X \cap Y$, since $X \cap Y \trianglelefteq Y$, we have $P \trianglelefteq Y$. Then $P \trianglelefteq \langle X, Y \rangle = G = \mathrm{A}_{p^2}$ or S_{p^2}, a contradiction.

Case 4. Suppose finally that H is given as in Rows 2 and 3 of [18, Table 1]. We have $T = \mathrm{soc}(G) = \mathrm{A}_{q+1}$, and $X \cap T = H \cap T \cong \mathrm{PGL}_2(q)$ or $\mathrm{PSL}_2(q)$, where q is a prime power. Recall that $q+1 = n \geqslant 11$. Assume that Y is solvable. It can be deduced from [20, Theorem 1.1] that $n = q+1$ is a prime, and $Y \cap T \cong \mathbb{Z}_{q+1}{:}\mathbb{Z}_{\frac{q}{2}}$. Inspecting the subgroups of $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$, refer to [15, II.8.27] and [5, Theorem 2] respectively, neither $\mathrm{PSL}_2(q)$ nor $\mathrm{PGL}_2(q)$ contains a subgroup which is isomorphic to a subgroup of $\mathbb{Z}_{q+1}{:}\mathbb{Z}_{\frac{q}{2}}$ of index 2 or 1; however, by Lemma 2.8, $|(Y \cap T) : (X \cap Y \cap T)| \leqslant 2$, a contradiction. Therefore, Y is insolvable.

By Lemmas 2.9 and 2.10, $Y^\infty < X^\infty$, and $Y = \mathbf{N}_G(Y^\infty)$. Clearly, $X^\infty = (X \cap T)^\infty \cong \mathrm{PSL}_2(q)$. Then Y^∞ is isomorphic to an insolvable proper subgroup of $\mathrm{PSL}_2(q)$. Inspecting the subgroups of $\mathrm{PSL}_2(q)$, we get $Y^\infty \cong \mathrm{A}_5$ or $\mathrm{PSL}_2(q_0)$, where $q = q_0^t$ for some integer $t \geqslant 2$. Since Y is a transitive maximal subgroup of G and $T \not\leqslant Y$, our observations together with (a)-(f) stated in [21] imply that Y is an almost simple primitive subgroup of S_{q+1}. Also, $Y^\infty = \mathrm{soc}(Y)$, which is a transitive subgroup of S_{q+1}. If $Y^\infty \cong \mathrm{A}_5$, so that $Y \cong \mathrm{A}_5$ or S_5 then, considering the primitive permutation representations of A_5 and S_5, we have $n = q+1 \leqslant 10$, a contradiction. Thus $Y^\infty \cong \mathrm{PSL}_2(q_0)$. Since Y^∞ is a transitive subgroup of S_{q+1}, it follows that $q+1$ is a divisor of $\frac{1}{d}q_0(q_0^2-1)$, and so $q+1$ is a divisor of $\frac{1}{d}(q_0^2-1)$, where $d = \gcd(2, q_0-1)$. Let $q_0 = p^e$ for some prime p, so that $q = p^{te}$. Then $p^{2et}-1$ has no primitive prime divisor. By a theorem of Zsigmondy [32], since $2et \geqslant 4$, we have $p = 2$ and $2et = 6$. This forces that $e = 1$ and $q_0 = 2$, which is impossible as $\mathrm{PSL}_2(q_0)$ is a nonabelian simple group. This completes the proof. □

The argument above, together with Lemma 3.1, leads to the following result.

Theorem 4.3. *Let $\Gamma = (V, E)$ be a G-edge-primitive graph of valency k. Assume that $G = \mathrm{A}_n$ or S_n with $n \geqslant 11$, and G contains a subgroup R that acts regularly on V. Then one of the following holds:*

(1) *$G = \mathrm{A}_n$ or S_n, $|R| = n$ and Γ is the complete graph K_n;*

(2) *$G = \mathrm{S}_n$ with $n \in \{11, 12\}$, $R \cong \mathrm{S}_7$ or $\mathrm{A}_7 \times 2$, and Γ is isomorphic to the graph Γ_1 in Example 2.3.*

Proof. Pick $\{\alpha, \beta\} \in E$, let $X = G_\alpha$ and $Y = G_{\{\alpha\beta\}}$. Then $(G, R, X, Y, X \cap Y)$ satisfies Hypothesis 3.2. By Lemmas 4.1 and 4.2, one of the following occurs:

(i) $G = \mathrm{A}_n$ or S_n, $X \cong \mathrm{A}_{n-1}$ or S_{n-1}, and $Y \cong \mathrm{S}_{n-2}$ or $Y \cong \mathrm{S}_{n-2} \times \mathrm{S}_2$, respectively;

(ii) $G = \mathrm{S}_{11}$, $R \cong \mathrm{S}_7$, $X \cong \mathrm{M}_{11}$ and $Y \cong 11{:}10$;

(iii) $G = \mathrm{S}_{12}$, $R \cong \mathrm{S}_7$, $X \cong \mathrm{M}_{12}$ and $Y \cong \mathrm{PGL}_2(11)$.

For case (i), we have $|V| = |G : X| = n$ and $k = |X : (X \cap Y)| = n-1$, and so Γ is the complete graph of order n, as in (1) of the theorem. Cases (ii) and (iii) produce the graphs $\Gamma_{1'}$ and Γ_1 in Example 2.3, respectively. As noted in Example 2.3 (3), $\Gamma_{1'}$ and Γ_1 are isomorphic. Then (2) of the theorem follows. This completes the proof. □

5 Sporadic groups

Let (G, T, R, X) satisfy (I) of Hypothesis 3.2. Assume that $T = \mathrm{soc}(G)$ is a simple sporadic group, and $\{H, K\} = \{R, X\}$. Then the triple (G, H, K) is described as in Cases 36-47 of [4, Table 4] and Rows 22 and 23 of [18, Table 1], see also [9, Theorem 1.3].

Lemma 5.1. *Assume that (G, T, R, X, Y) is described as in Hypothesis 3.2, and $T = \mathrm{soc}(G)$ is a simple sporadic group. Then one of the following holds:*

(1) $G = \mathrm{M}_{11}$, $R \cong \mathbb{Z}_{11}$, $X \cong \mathrm{M}_{10}$, $Y \cong \mathrm{M}_9{:}2$ *and* $X \cap Y \cong 3^2{:}\mathrm{Q}_8$*;*

(2) $G = \mathrm{M}_{12}$, $R \cong \mathbb{Z}_6 \times \mathbb{Z}_2$, A_4 *or* D_{12}, $X \cong \mathrm{M}_{11}$, $Y \cong \mathrm{M}_{10}{:}2$ *and* $X \cap Y \cong \mathrm{M}_{10}$*;*

(3) $G = \mathrm{M}_{22}.2$, $R \cong \mathrm{D}_{22}$, $X \cong \mathrm{P\Sigma L}_3(4)$, $Y \cong 2^5{:}\mathrm{S}_5$ *and* $X \cap Y \cong 2^4{:}\mathrm{S}_5$*;*

(4) $G = \mathrm{M}_{23}$, $R \cong \mathbb{Z}_{23}$, $X \cong \mathrm{M}_{22}$, $Y \cong \mathrm{P\Sigma L}_3(4)$ *and* $X \cap Y \cong \mathrm{PSL}_3(4)$*;*

(5) $G = \mathrm{M}_{24}$, $R \cong \mathrm{S}_4$, D_{24}, $3{:}\mathrm{D}_8$ *or* $\mathrm{A}_4 \times 2$, $X \cong \mathrm{M}_{23}$, $Y \cong \mathrm{M}_{22}{:}2$ *and* $X \cap Y \cong \mathrm{M}_{22}$*;*

(6) $G = \mathrm{M}_{12}.2$, $R \cong (3^2{:}\mathrm{Q}_8).2$, $X \cong \mathrm{PGL}_2(11)$, $Y \cong \mathrm{S}_5$ *and* $X \cap Y \cong \mathrm{A}_5$*; in this case,* $R < \mathrm{M}_{12}$ *and* $X \cap \mathrm{M}_{12}$ *is a maximal subgroup of* M_{12}*;*

(7) $G = \mathrm{M}_{12}.2$, $R \cong \mathrm{S}_4$, D_{24} *or* $3{:}\mathrm{D}_8$, $X \cong \mathrm{M}_{11}$, $Y \cong \mathrm{PGL}_2(11)$ *and* $X \cap Y \cong \mathrm{PSL}_2(11)$*;*

(8) $G = \mathrm{J}_2.2$, $R \cong 5^2{:}4$, $X \cong \mathrm{G}_2(2)$, $Y \cong \mathrm{PSL}_3(2){:}2 \times 2$ *and* $X \cap Y \cong \mathrm{PSL}_3(2){:}2$*;*

(9) $G = \mathrm{HS}.2$, $R \cong 5^2{:}4$, $X \cong \mathrm{M}_{22}.2$, $Y \cong \mathrm{PSL}_3(4){:}2^2$ *and* $X \cap Y \cong \mathrm{P\Sigma L}_3(4)$*;*

(10) $G = \mathrm{He}.2$, $R \cong 7_+^{1+2}{:}6$, $X \cong \mathrm{Sp}_4(4).4$, $Y \cong (\mathrm{S}_5 \times \mathrm{S}_5){:}2$ *and* $X \cap Y \cong (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2$.

Proof. We claim that $X = K$ is given as in Cases 36-47 of [4, Table 4]. Suppose first that (G, H, K) is given as in Rows 22 and 23 of [18, Table 1]. Then $G = \mathrm{M}_{24}$, $K \cong \mathrm{PSL}_2(23)$ and $H \cong 2^4{:}\mathrm{A}_7$ or $\mathrm{P\Sigma L}_3(4)$. Since $|Y| < |X|$, $X \in \{H, K\}$ and $Y \lessdot G$, by the Atlas [6], $Y \cong \mathrm{PSL}_2(7)$ or $\mathrm{PSL}_2(23)$, which is impossible as Y has a subgroup of index 2. Suppose now that $X = H$ is given as in Cases 36-47 of [4, Table 4]. Checking the maximal subgroups of G with order less than $|X|$, we conclude that $G = \mathrm{M}_{12}$, $X \cong 3^2{:}\mathrm{Q}_8.2$ and $Y \cong \mathrm{A}_4 \times \mathrm{S}_3$. Since $|Y : (X \cap Y)| = 2$, we have $|X \cap Y| = 36$. Then $X \cap Y$ contains the unique Sylow 3-subgroup of X. Thus $X \cap Y$ has a normal Sylow 3-subgroup, and so does Y; however, $\mathrm{A}_4 \times \mathrm{S}_3$ has no normal Sylow 3-subgroup, a contradiction. Therefore, $X = K$ is described as in Cases 36-47 of [4, Table 4].

Suppose that $X = K$ is given as in Cases 36, 39, 41, 42 and 44 of [4, Table 4]. Then $\mathrm{soc}(G) = \mathrm{M}_n$ with $n \in \{11, 12, 22, 23, 24\}$, $|R| = n$, and G is a 3-transitive permutation group of degree n on $[G : X]$. Recalling that $X \cap Y = X \cap X^y$ for some $y \in Y \setminus X$, it follows that $X \cap Y$ is the point-wise stabilizer of the 2-subset $\{X, Xy\}$ of $[G : X]$, while Y is the set-wise stabilizer of $\{X, Xy\}$. This gives (1)-(5) of the lemma.

Suppose that Case 37 of [4, Table 4] occurs for $X = K$. Then $G = \mathrm{M}_{11}$ and $X \cong 3^2{:}\mathrm{Q}_8.2$. Checking the maximal subgroups of M_{11} with order less than $|X|$, since $|Y|$ is a divisor of $2|X|$, we have $Y \cong \mathrm{M}_8.\mathrm{S}_3 = 2{}^{\cdot}\mathrm{S}_4$. Note that X has a normal Sylow 3-subgroup, so does $X \cap Y$. Since $|Y : (X \cap Y)| = 2$, we conclude that $Y \cong 2{}^{\cdot}\mathrm{S}_4$ has a normal Sylow 3-subgroup, which is impossible.

Suppose that Case 38 of [4, Table 4] occurs for $X = K$. Then $G = \mathrm{M}_{12}.o$, $R \cong 3^2{:}\mathrm{Q}_8.2$, and $X \cong \mathrm{PSL}_2(11).o$, where $o \in \{1, 2\}$. In this case, confirmed by using GAP [8], $X \cap \mathrm{M}_{12}$ is a maximal subgroup of M_{12}. Since $|Y| < |X|$ and $|Y|$ is a divisor of $2|X|$, inspecting the maximal subgroups of G in the Atlas [6], we conclude that either $G = \mathrm{M}_{12}.2$, $X \cong \mathrm{PGL}_2(11)$, $Y \cong \mathrm{S}_5$ and $X \cap Y \cong \mathrm{A}_5$, or $Y = (\mathrm{A}_4 \times \mathrm{S}_3).o$. Since $|Y : (X \cap Y)| = 2$, the latter case implies that $\mathrm{PSL}_2(11).o$ has a subgroup $\mathrm{A}_4 \times 3$, which is impossible. For the former case, computation with GAP [8] shows that the exact factorization $G = RX$ requires $R < T$ and the maximality of $X \cap T$ in T, and so (6) of the lemma follows.

Suppose that Case 40 of [4, Table 4] occurs for $X = K$. Then $G = \mathrm{M}_{12}.2$, $R \cong \mathrm{S}_4$, D_{24} or $3{:}\mathrm{D}_8$, and $X \cong \mathrm{M}_{11}$. Since $G = \langle X, Y \rangle$, we have $G = \langle X, y \rangle = \mathrm{M}_{12}\langle y \rangle$ for $y \in X \setminus Y$. Then, viewing X as a point-stabilizer of M_{12} in the natural action of degree 12, X^y is a 3-transitive subgroup of M_{12} with a point-stabilizer isomorphic to $\mathrm{PSL}_2(11)$, see [30, p. 201, 5.3.6] for example. Then $X \cap Y = X \cap X^y \cong \mathrm{PSL}_2(11)$, and $Y \cong \mathrm{PGL}_2(11)$, as in part (7) of the lemma.

Suppose that Case 43 of [4, Table 4] occurs for $X = K$. Then $G = \mathrm{M}_{23}$, $R \cong 23{:}11$, and $X \cong \mathrm{P\Sigma L}_3(4)$ or $2^4{:}\mathrm{A}_7$. By Lemma 2.7, $|Y| < |X|$. Since Y is maximal in G and has a subgroup of index 2, by the Atlas [6], we get $Y \cong 2^4{:}(3 \times \mathrm{A}_5){:}2$, and so $X \cap Y \cong 2^4{:}(3 \times \mathrm{A}_5)$. It is easily shown that neither $\mathrm{P\Sigma L}_3(4)$ nor $2^4{:}\mathrm{A}_7$ has a subgroup isomorphic to $2^4{:}(3 \times \mathrm{A}_5)$, a contradiction.

Suppose that Case 45 of [4, Table 4] occurs for $X = K$. Then $G = \mathrm{J}_2.2$, $X \cong \mathrm{G}_2(2) \cong \mathrm{PSU}_3(3).2$ and $R \cong 5^2{:}4$. Inspecting the maximal subgroups of G in the Atlas [6], since $|Y|$ is a divisor of $2|X|$, it can be deduced that $Y \cong \mathrm{PSL}_3(2){:}2 \times 2$. Since $|Y : (X \cap Y)| = 2$, we have $X \cap Y \cong \mathrm{PSL}_3(2){:}2$, a maximal subgroup of $\mathrm{PSU}_3(3)$. Then (8) of the lemma follows.

Suppose that Case 46 of [4, Table 4] occurs for $X = K$. Then $G = \mathrm{HS}.2$, $X \cong \mathrm{M}_{22}.2$ and $R \cong 5^2{:}4$. Checking the maximal subgroups of G in the Atlas [6], since $|Y| < |X|$ and $|Y|$ is a divisor of $2|X|$, we deduce that Y is isomorphic to one of $\mathrm{PSL}_3(4){:}2^2$, $\mathrm{S}_8 \times 2$ and $(2 \times \mathrm{A}_6^{\cdot}2^2).2$. Since $|Y : (X \cap Y)| = 2$, we have $X \cap Y \cong \mathrm{PSL}_3(4).2$, $\mathrm{A}_8.2$ or $\mathrm{A}_6.2^3$. Inspecting the maximal subgroups of $\mathrm{M}_{22}.2$ in the Atlas [6], it can be deduced that X has no subgroup isomorphic to $\mathrm{A}_8.2$ or $\mathrm{A}_6.2^3$. Then $Y \cong \mathrm{PSL}_3(4){:}2^2$ and $X \cap Y \cong \mathrm{PSL}_3(4).2 = \mathrm{P\Sigma L}_3(4)$, as in part (9) of the lemma.

Suppose that Case 47 of [4, Table 4] occurs for $X = K$. Then $G = \mathrm{He}.2$, $X \cong \mathrm{Sp}_4(4).4$ and $R \cong 7_+^{1+2}{:}6$. By Lemma 2.7, $\frac{2|X|}{|G:X|} < |Y| < |X|$. Inspecting the maximal subgroups of G in the Atlas [6], either Y is insolvable or $Y \cong 2^{4+4}.(\mathrm{S}_3 \times \mathrm{S}_3).2$. For the latter, since $|Y : (X \cap Y)| = 2$, we have $|X \cap Y| = 9216$; however, computation with GAP [8] shows that $\mathrm{Sp}_4(4).4$ has no subgroup of order 9216. Then Y is insolvable, and either $Y \cong (\mathrm{S}_5 \times \mathrm{S}_5){:}2$ or Y^∞ has a composition factor isomorphic to one of $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(2)$ and A_7. By Lemma 2.9, $Y^\infty < X^\infty \cong \mathrm{Sp}_4(4)$. Inspecting the maximal subgroups of $\mathrm{Sp}_4(4)$ in the Atlas [6], we deduce that $\mathrm{Sp}_4(4)$ has no maximal subgroup in which a subgroup has a quotient $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(2)$ or A_7. It follows that $Y \cong (\mathrm{S}_5 \times \mathrm{S}_5){:}2$, and $X \cap Y \cong (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2$, as in (10) of the lemma. Then the lemma follows. $\square$

Theorem 5.2. *Let $\Gamma = (V, E)$ be a G-edge-primitive graph. Assume that G is almost simple with socle a simple sporadic group, and G contains a subgroup R that acts regularly on V. Then one of the following holds:*

(1) $G = \mathrm{M}_n.o$, $|R| = n$ *and* $\Gamma \cong \mathsf{K}_n$ *with* $n \in \{11, 12, 22, 23, 24\}$, *where* $o = 2$ *if* $n = 22$, *and* $o = 1$ *otherwise;*

(2) $G = \mathrm{M}_{12}.2$, $|R| = 24$ *and* $\Gamma \cong \mathsf{K}_{12,12}$*;*

(3) $G = \mathrm{M}_{12}.2$, $R \cong 3^2{:}\mathrm{Q}_8.2$, *and* Γ *is isomorphic to the graph* Γ_2 *in Example* 2.3*;*

(4) $G = \mathrm{J}.2$, $R \cong 5^2{:}4$, *and* Γ *is isomorphic to the graph* Γ_3 *in Example* 2.3*;*

(5) $G = \mathrm{HS}.2$, $R \cong 5^2{:}4$, *and* Γ *is isomorphic to the graph* Γ_4 *in Example* 2.3*;*

(6) $G = \mathrm{He}.2$, $R \cong 7_+^{1+2}{:}6$, *and* Γ *is isomorphic to the graph* Γ_5 *in Example* 2.3.

Proof. Pick $\{\alpha, \beta\} \in E$, let $X = G_\alpha$ and $Y = G_{\{\alpha\beta\}}$. Then $(G, R, X, Y, X \cap Y)$ satisfies Hypothesis 3.2, and so one of (1)-(10) of Lemma 5.1 occurs. For (1)-(5) of Lemma 5.1, we get five complete graphs K_n with $n \in \{11, 12, 22, 23, 24\}$, and (1) of the theorem follows. For (7) of Lemma 5.1, we have $G = \mathrm{M}_{12}.2$, $R \cong \mathrm{S}_4$, D_{24} or $3{:}\mathrm{D}_8$, and $\Gamma \cong \mathsf{K}_{12,12}$, as in (2) of the theorem. Finally, (6), (8)-(10) of Lemma 5.1 produce the graphs Γ_2, Γ_3, Γ_4 and Γ_5 in Example 2.3, respectively. Then the theorem follows. $\square$

6 Linear groups

Let (G,T,R,X) be described as in (I) of Hypothesis 3.2. Assume that $\mathrm{soc}(G) = T = \mathrm{PSL}_n(q)$. In view of Theorem 3.5, we may let $(n,q) \neq (2,4)$, $(2,5)$, $(2,9)$ or $(4,2)$. Let $\{H,K\} = \{R,X\}$. By [4, Theorem 3 and Remark 4 (b)] and [18, Theorem 1.2], one of the following occurs:

(l.1) $H \cap T \lesssim \mathbb{Z}_{\frac{q^n-1}{d(q-1)}}{:}\mathbb{Z}_n$, and $K \cap T$ has a normal subgroup isomorphic to $q^{n-1}{:}\mathrm{SL}_{n-1}(q)$, where $d = \gcd(n, q-1)$;

(l.2) (G,H,K) is listed in Cases 12-23 of [4, Table 4];

(l.3) $G = \mathrm{SL}_4(4).(2 \times o)$, $H \cong (\mathrm{SL}_2(16).4) \times o_1$ and $K = \mathrm{SL}_3(4).(2 \times o_2)$ with $o = o_1 o_2 \in \{1,2\}$.

Lemma 6.1. *Assume that (G,T,R,X,Y) is described as in Hypothesis 3.2. Then case (l.3) does not occur.*

Proof. Suppose the contrary. Then $T = \mathrm{soc}(G) = \mathrm{SL}_4(4)$, and either $X^\infty = H^\infty \cong \mathrm{SL}_2(16)$ or $X^\infty = K^\infty \cong \mathrm{SL}_3(4)$. It follows from [20, Theorem 1.1 and Table 16] that G has no solvable maximal subgroup. Then Y is an insolvable maximal subgroup of G, and so $1 \neq Y^\infty < X^\infty \cong \mathrm{SL}_2(16)$ or $\mathrm{SL}_3(4)$, see Lemma 2.9. Noting that Y^∞ is contained in a maximal subgroup of X^∞, we conclude that either $X^\infty \cong \mathrm{SL}_2(16)$ and $Y^\infty \cong \mathrm{A}_5$, or $X^\infty \cong \mathrm{SL}_3(4)$ and Y^∞ is isomorphic to one of A_5, $2^4{:}\mathrm{A}_5$, $\mathrm{SL}_3(2)$ and $3^{\cdot}\mathrm{A}_6$. Inspecting the insolvable maximal subgroups of G in [2, Tables 8.8 and 8.9], we conclude that Y^∞ is isomorphic to one of $\mathrm{SL}_3(4)$, $\mathrm{A}_5 \times \mathrm{A}_5$, $\mathrm{SL}_2(16)$, A_8, $2^{\cdot}\mathrm{A}_6$, $\mathrm{Sp}_4(4)$ and $\mathrm{PSp}_4(3)$, contrary to the observation above made for Y^∞. This completes the proof. □

Lemma 6.2. *Assume that (G,T,R,X,Y) is described as in Hypothesis 3.2, and case (l.2) holds. Then one of the following holds:*

(1) $G = \mathrm{PSL}_2(11)$, $R \cong \mathbb{Z}_{11}$, $X \cong \mathrm{A}_5$, $Y \cong \mathrm{D}_{12}$ *and* $X \cap Y \cong \mathrm{S}_3$*;*

(2) $G = \mathrm{PGL}_2(11)$, $R \cong \mathrm{D}_{22}$, $X \cong \mathrm{A}_5$, *and either* $Y \cong \mathrm{D}_{20}$ *with* $X \cap Y \cong \mathrm{D}_{10}$ *or* $Y \cong \mathrm{S}_4$ *with* $X \cap Y \cong \mathrm{A}_4$*;*

(3) $G = \mathrm{PGL}_2(11)$, $R \cong \mathrm{A}_4$, $X \cong 11{:}10$, $Y \cong \mathrm{D}_{20}$ *and* $X \cap Y \cong \mathbb{Z}_{10}$*;*

(4) $G = \mathrm{PSL}_2(23)$ *or* $\mathrm{PGL}_2(23)$, $R \cong \mathrm{S}_4$, $X \cong 23{:}11$ *or* $23{:}22$, $Y \cong \mathrm{D}_{22}$ *or* D_{44}, *and* $X \cap Y \cong \mathbb{Z}_{11}$ *or* $\mathbb{Z}_{22}$, *respectively;*

(5) $G = \mathrm{PSL}_2(59)$, $R \cong \mathrm{A}_5$, $X \cong 59{:}29$, $Y \cong \mathrm{D}_{58}$ *and* $X \cap Y \cong \mathbb{Z}_{29}$*;*

(6) $G = \mathrm{PGL}_2(59)$, $X \cong \mathrm{A}_5$, $R \cong 59{:}58$, $Y \cong \mathrm{S}_4$ *and* $X \cap Y \cong \mathrm{A}_4$*;*

(7) $G = \mathrm{PSL}_5(2).2$, $R \cong 31{:}10$, $X \cong 2^6{:}(\mathrm{S}_3 \times \mathrm{PSL}_3(2))$, *and either* $Y \cong 2^{4+4}{:}(\mathrm{S}_3 \times \mathrm{S}_3){:}2$ *with* $X \cap Y \cong 2^{4+4}{:}(\mathrm{S}_3 \times \mathrm{S}_3)$ *or* $Y \cong \mathrm{S}_3 \times \mathrm{PSL}_3(2){:}2$ *with* $X \cap Y \cong \mathrm{S}_3 \times \mathrm{PSL}_3(2)$*; in this case, X is the stabilizer in* $\mathrm{PSL}_5(2)$ *of a 2- or 3-space.*

Proof. Suppose that (G,H,K) is given as in Case 22 of [4, Table 4]. Then $G = \mathrm{P\Gamma L}_4(4).o$, $K \cong ((5 \times \mathrm{PSL}_2(16)){:}2).2.o$, and H is solvable, where $o \in \{1,2\}$. It can be deduced from [20, Theorem 1.1 and Table 16] that G has no solvable maximal subgroup. Thus Y is insolvable, and so does $X \cap Y$. This forces that $X = K$. By Lemma 2.9, $Y^\infty < X^\infty \cong \mathrm{PSL}_2(16)$, and so $Y^\infty \cong \mathrm{A}_5$. An examination of the maximal subgroups of G listed in [2, Tables 8.8 and 8.9] shows that Y is not maximal in G, a contradiction.

The argument above says that (G,H,K) is listed in Cases 12-21 and 23 of [4, Table 4]. Note that $|Y|<|X|=|H|$ or $|K|$, and $|X:(X\cap Y)|=\frac{2|X|}{|Y|}>2$. We can use GAP [8] to handle these eleven cases. Computation with GAP [8] shows that (1) and (2) of the lemma arise from Case 12, (3) arises from Case 13, (4) of the lemma arises from Case 14, (5) and (6) of the lemma arise from Case 16, and (7) of the lemma arises from Case 23 of [4, Table 4], respectively. By the computation, Cases 18 and 19 of [4, Table 4] do not produce a desired Y. Besides, by the computation, we have five further cases:

(i) $G=\mathrm{PGL}_2(11)$, $X=K\cong\mathrm{A}_5$ and $Y\cong\mathrm{D}_{24}$;

(ii) $G=\mathrm{PSL}_2(29)$, $X=K\cong\mathrm{A}_5$ and $Y\cong\mathrm{D}_{30}$;

(iii) $G=\mathrm{PSL}_3(3).o$, $X=K\cong\mathrm{A\Gamma L}_1(9)$ and $Y\cong\mathrm{S}_4\times o$ with $o\in\{1,2\}$;

(iv) $G=\mathrm{PSL}_3(3).2$, $R\cong 13{:}6$, $X\cong\mathrm{A\Gamma L}_1(9)$ and $Y\cong 2\mathrm{S}_4{:}2$;

(v) $G=\mathrm{PGL}_4(3).o$, $X=K\cong((4\times\mathrm{PSL}_2(9)){:}2).(2\times o)$, and $Y\cong\mathrm{A}_6.2^2\times o$, $2.(\mathrm{S}_4\times\mathrm{S}_4).2\times o$ or $(\mathrm{S}_4\times\mathrm{S}_4){:}2\times o$ with $o\in\{1,2\}$;

where (i) arises from Case 12, (ii) arises form Case 15, (iii) and (iv) arise from Case 17, and (v) arises from Cases 20-21 of [4, Table 4], respectively

To complete the proof, it suffices to exclude cases (i)-(v). Note that X contains a subgroup which has index 2 in Y. If (i) or (ii) occurs then A_5 has a subgroup isomorphic to $\mathbb{Z}_{12}$, D_{12} or $\mathbb{Z}_{15}$, which is impossible. For case (iii), we have $X\cap Y\cong\mathrm{A}_4$ or S_4; however, computation with GAP [8] shows that $\mathrm{A\Gamma L}_1(9)$ has no subgroup isomorphic to A_4 or S_4, a contradiction. For case (iv), we have $|X\cap Y|=48$; however, computation with GAP [8] shows that $\mathrm{A\Gamma L}_1(9)$ has no subgroup of order 48, a contradiction. Finally, suppose that case (v) occurs. Then $T=\mathrm{soc}(G)=\mathrm{PSL}_4(3)$, $X\cap T\cong(4\times\mathrm{PSL}_2(9)){:}2$, a maximal subgroup of T, and $Y\cap T\cong\mathrm{A}_6.2$, $2.(\mathrm{A}_4\times\mathrm{A}_4).[2^2]$ or $(\mathrm{A}_4\times\mathrm{A}_4).[2^2]$. By Lemma 2.8, $|(Y\cap T):(X\cap Y\cap T)|\leqslant 2$, and so $|X\cap Y\cap T|\in\{288,360,576,720,1152\}$. Let $U=X\cap Y\cap T$. Computation with GAP [8] shows that, as a subgroup of $X\cap T$, either $|U|=288$ and U has a normal Sylow 3-subgroup, or U is isomorphic to one of A_6, $\mathrm{A}_6\times 2$ and S_6. The former case implies that $Y\cap T$ has a normal Sylow 3-subgroup, which is impossible. For the latter case, $Y\cap T\cong\mathrm{A}_6.2$ is in fact a maximal subgroup of T, a further computation shows that U is not conjugate to any subgroup of $Y\cap T$, a contradiction. This completes the proof. $\square$

Lemma 6.3. *Assume that* (G,T,R,X,Y) *is described as in Hypothesis* 3.2*, and case* (l.1) *occurs. Then one of the following holds:*

(1) $G=\mathrm{PGL}_2(7)$*,* $R\cong\mathrm{D}_{14}$*,* $X\cong\mathrm{S}_4$*, and either* $Y\cong\mathrm{D}_{16}$ *with* $X\cap Y\cong\mathrm{D}_8$ *or* $Y\cong\mathrm{D}_{12}$ *with* $X\cap Y\cong\mathrm{S}_3$*;*

(2) $G=XT$*,* $T=\mathrm{PSL}_2(q)$*,* $|R|=q+1$*,* $X\cap T\cong\mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{q-1}{d}}$*,* $|X:(X\cap Y)|=q$*,* $Y\cap T\cong\mathrm{D}_{\frac{2(q-1)}{d}}$ *and* $X\cap Y\cap T\cong\mathbb{Z}_{\frac{q-1}{d}}$*, where* $d=\gcd(2,q-1)$*;*

(3) $\mathrm{PSL}_n(q)<G\not\leqslant\mathrm{P\Gamma L}_n(q)$*,* R *is solvable and of order* $\frac{2(q^n-1)}{q-1}$*,* X *is the stabilizer in* $G\cap\mathrm{P\Gamma L}_n(q)$ *of some* 1*- or* $(n-1)$*-space, and* Y *is the stabilizer in* G *of a* $\{1,n-1\}$*-decomposition or a* $(1,n-1)$*-flag, where* $n\geqslant 3$.

Proof. We discuss in three cases: $X=H$; $X=K$ with $n=2$; and $X=K$ with $n>2$.

Case 1. Suppose that $X=H$, so that $R=K$. In this case, X is solvable. Recalling that $|X:(X\cap Y)|>2$, by Lemma 2.8, $|Y\cap T|$ is a divisor of $2|X\cap T|$. Thus, together with the assertions in case (l.1), $|Y\cap T|$ is a divisor of $\frac{(q^n-1)2n}{d(q-1)}$, where $d=\gcd(n,q-1)$.

Let G_0 be a normal subgroup of G such that $T = \mathrm{soc}(G) \leqslant G_0$, and put $X_0 = X \cap G_0$ and $R_0 = R \cap G_0$. Noting that $G/G_0 = (RG_0/G_0)(XG_0/G_0)$, it follows that $|G/G_0|$ is a divisor of $|RG_0/G_0||XG_0/G_0|$. Since $|G| = |R||X|$, we have

$$|RG_0/G_0||XG_0/G_0| = |R : (R \cap G_0)||X : (X \cap G_0)| = \frac{|G|}{|R_0||X_0|}.$$

Thus $|R_0||X_0|$ is a divisor of $|G_0|$, and so $|X_0|$ is a divisor of $|G_0 : R_0|$. Clearly, $R_0 \geqslant R \cap T = K \cap T$. We have

$$|G_0 : R_0||R_0 : (R \cap T)| = |G_0 : (R \cap T)| = |G_0 : T||T : (K \cap T)|.$$

Since $q^{n-1}\mathrm{SL}_{n-1}(q) \trianglelefteq K \cap T$, we conclude that $|T : (K \cap T)|$ is a divisor of $\frac{(q^n-1)}{d}$, and then $|X_0|$ is a divisor of $\frac{(q^n-1)|G_0:T|}{d}$. (In particular, if we choose $G_0 = G$ then $|X|$ is a divisor of $\frac{(q^n-1)|G:T|}{d}$.) Let $Y_0 = Y \cap G_0$. By Lemma 2.8, $|Y_0|$ is a divisor of $2|X_0|$. Then $|Y_0|$ is a divisor of $\frac{(q^n-1)2|G_0:T|}{d}$.

Recall that $X = H$ is solvable, and so is $X \cap Y$. Then Y is solvable as $|Y : (X \cap Y)| = 2$. Now, by [20, Theorem 1.1], we may choose the above group G_0 such that Y_0 is one of the maximal subgroups of G_0 defined as in [20, Table 16], in particular, $|G_0 : T| \in \{1, 2\}$. Clearly, $Y_0 \cap T = Y \cap T$. Recall that $(n, q) \neq (2, 4)$, $(2, 5)$, $(2, 9)$ or $(4, 2)$. Inspecting the orders of these Y_0's, since $|Y_0|$ is a divisor of $\frac{(q^n-1)2|G_0:T|}{d}$ and $|Y_0 \cap T|$ is a divisor of $\frac{(q^n-1)2n}{d(q-1)}$, only one of the following occurs:

(i) $G_0 = T = \mathrm{PSL}_2(q)$ with q odd, and $Y_0 = Y \cap T \cong \mathrm{A}_4$;

(ii) $G_0 = \mathrm{PGL}_2(q)$ with q a prime, and $Y_0 \cong \mathrm{S}_4$;

(iii) $G_0 = \mathrm{PGL}_2(7)$, and $Y_0 \cong \mathrm{D}_{16}$ or D_{12}.

In particular, $n = 2$ and $d = \gcd(n, q-1) = 2$.

Suppose that case (i) or (ii) occurs. Then $Y \cap T \cong \mathrm{A}_4$, and so $Y \cap T$ has no subgroup group of index 2. By Lemma 2.8, $|(Y \cap T) : (X \cap Y \cap T)| \leqslant 2$. It follows that $Y \cap T = X \cap Y \cap T$. Then $\mathrm{A}_4 \cong Y \cap T \leqslant X \cap T \lesssim \mathbb{Z}_{\frac{q+1}{2}}{:}\mathbb{Z}_2$, which is impossible.

Suppose that case (iii) occurs. Then $G = G_0 = \mathrm{PGL}_2(7)$, and $Y \cong \mathrm{D}_{16}$ or D_{12}. Since $|Y : (X \cap Y)| = 2$, we have $|X \cap Y| = 8$ or 6, respectively. Inspecting the solvable subgroups of $\mathrm{PGL}_2(7)$, since $|X| > |Y|$ and $|X|$ is divisible by 8 or 6, we have $X \cong \mathrm{S}_4$. Noting that $G = RX$ is an exact factorization of $\mathrm{PGL}_2(7)$, we have $|R| = 14$, and thus $R \cong \mathrm{D}_{14}$. Then (1) of the lemma follows.

Case 2. Suppose that $X = K$ and $n = 2$. Then X is solvable, and $X \cap T$ has a normal subgroup of order q. Set $q = p^f$ for a prime p and an integer $f \geqslant 1$. Then $X \lesssim \mathbb{Z}_p^f{:}\mathbb{Z}_{p^f-1}{:}\mathbb{Z}_f$, and $\mathbb{Z}_p^f \lesssim X \cap T \lesssim \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{d}}$, where $d = \gcd(2, q-1)$. Since $|X : (X \cap Y)| > 2$, by Lemma 2.8, $|Y \cap T| < |X \cap T|$, and $|Y \cap T|$ is a divisor of $2|X \cap T|$. Then $|Y \cap T|$ is a divisor of $\frac{2q(q-1)}{d}$.

Since X is solvable, so does Y as $|Y : (X \cap Y)| = 2$. Then, by [20, Theorem 1.1], we choose a normal subgroup G_0 of G such that $T = \mathrm{PSL}_2(q) \leqslant G_0$ and $Y_0 := Y \cap G_0$ is one of the maximal subgroups of G_0 defined as in [20, Table 16]. Recall that $(n, q) \neq (2, 4)$, $(2, 5)$ or $(2, 9)$. Noting that $Y \cap T = Y_0 \cap T$, since $|Y \cap T| < |X \cap T|$ and $|Y \cap T|$ is a divisor of $\frac{2q(q-1)}{d}$, it follows that one of the following holds:

(iv) $G_0 = \mathrm{PSL}_2(p)$ and $Y_0 \cong \mathrm{A}_4$, or $G_0 = \mathrm{PGL}_2(p)$ and $Y_0 \cong \mathrm{S}_4$;

(v) $G_0 = \mathrm{PSL}_2(q)$, and $Y_0 \cong \mathrm{D}_{\frac{2(q-1)}{d}}$ with $q \notin \{7, 11\}$;

(vi) $G_0 = \mathrm{PGL}_2(q)$ and $Y_0 \cong \mathrm{D}_{2(q-1)}$ with $q \in \{7, 11\}$.

Suppose that case (iv) occurs. Then $Y \cap T \cong \mathrm{A}_4$. Noting that A_4 has no subgroup of index 2, since $|(Y \cap T) : (X \cap Y \cap T)| \leqslant 2$ by Lemma 2.8, we have $Y \cap T = X \cap Y \cap T$. Thus $\mathrm{A}_4 \cong Y \cap T \leqslant X \cap T \lesssim \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{d}}$. It follows that $q = p^f = 4$, and $\mathrm{A}_4 \cong Y \cap T = X \cap T$, which is incompatible with $|Y \cap T| < |X \cap T|$.

Suppose that case (v) occurs. Then $T = G_0$ and $Y \cap T = Y_0 \cong \mathrm{D}_{\frac{2(q-1)}{d}}$. By Lemma 2.8, $|(Y \cap T) : (X \cap Y \cap T)| \leqslant 2$. It follows that $\frac{(q-1)}{d}$ is a divisor of $|X \cap T|$. Recalling that $\mathbb{Z}_p^f \lesssim X \cap T \lesssim \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{d}}$, we have $X \cap T \cong \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{d}}$, $X \cap Y \cap T \cong \mathbb{Z}_{\frac{(q-1)}{d}}$, and $|(Y \cap T) : (X \cap Y \cap T)| = 2$. Again by Lemma 2.8, $G = XT$, and $|X : (X \cap Y)| = |(X \cap T) : (X \cap Y \cap T)| = q$. Now, $G = XT = RX$, yielding that $|R| = |T : (X \cap T)| = q + 1$. Then (2) of the lemma holds with $q \notin \{7, 11\}$.

Finally, for case (vi), noting that $Y \cap T = Y_0 \cap T \cong \mathrm{D}_{\frac{2(q-1)}{d}}$, a similar argument as above implies that (2) of the lemma holds with $q \in \{7, 11\}$.

Case 3. Suppose that $X = K$ and $n > 2$. Then $X \cap T$ has a normal subgroup isomorphic to $q^{n-1}{:}\mathrm{SL}_{n-1}(q)$. Let $G_0 = G \cap \mathrm{P\Gamma L}_n(q)$. Then $|G : G_0| \leqslant 2$, and $X \cap G_0$ is contained in the stabilizer M in $\mathrm{P\Gamma L}_n(q)$ of some 1-space or $(n-1)$-space. In particular,

$$M \cap T \cong q^{n-1}{:}(\frac{q-1}{d}.\mathrm{PGL}_{n-1}(q)),\ G_0 \cap M = (T \cap M).\mathcal{O},$$

and $|(M \cap T) : (X \cap T)|$ is a divisor of $\frac{q-1}{d}$, where $d = \gcd(n, q-1)$, and $\mathcal{O} = G_0/T$. Without loss of generality, we may assume that M is the stabilizer of some 1-space.

By [19, Remark 1.2 (i)], $\mathbf{N}_T(X \cap T)$ is maximal in T, and so $\mathbf{N}_T(X \cap T) = M \cap T$. Assume that $X \not\leqslant G_0$, and let $x \in X \setminus G_0$. Then M^x is the stabilizer in $\mathrm{P\Gamma L}_n(q)$ of some $(n-1)$-space, and we have

$$M \cap M^x \cap T \cong [q^{2n-3}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q) \text{ or } \frac{q-1}{d}.\mathrm{PGL}_{n-1}(q).$$

It is easily shown that $|M \cap M^x \cap T|$ is not divisible by $q^{n-1}|\mathrm{SL}_{n-1}(q)|$. Then $X \cap T \not\leqslant M \cap M^x \cap T$ as $X \cap T = K \cap T$ has a subgroup $q^{n-1}{:}\mathrm{SL}_{n-1}(q)$. On the other hand, since $\mathbf{N}_T(X \cap T) = \mathbf{N}_T(X \cap T)^x = M^x \cap T$, we have $X \cap T \leqslant \mathbf{N}_T(X \cap T) = M \cap M^x \cap T$, a contradiction. Therefore, $X \leqslant G_0$, $TX \leqslant G_0$, and $X = X \cap G_0 \leqslant M$.

Pick $y \in Y \setminus X$. We claim that $y \notin G_0$. Suppose the contrary. Then $G = G_0$, and M^y is the stabilizer in $\mathrm{P\Gamma L}_n(q)$ of some 1-space. If $M^y = M$ then $y \in M$ as M is maximal in $\mathrm{P\Gamma L}_n(q)$, and so $G = \langle X, Y \rangle \leqslant M \cap G$, a contradiction. Thus $M^y \neq M$, and so

$$M \cap M^y \cap G \cong [q^{2n-4}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q).\mathcal{O}.$$

Since $|Y{:}(X \cap Y)| = 2$, we have $y^2 \in X \cap Y \leqslant M \cap G_0$. Then $(X \cap Y)^y = X \cap Y \leqslant M \cap M^y \cap G = (M \cap M^y \cap G)^y$, and so

$$Y = (X \cap Y)\langle y \rangle \leqslant (M \cap M^y \cap G)\langle y \rangle \cong [q^{2n-4}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q).\mathcal{O}.2.$$

Since Y is maximal in G, we have $Y \cong [q^{2n-4}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q).\mathcal{O}.2$. On the other hand, noting that Y fixes a 2-space, Y is contained in (and hence equal to) the stabilizer in G of some 2-space. We have $|Y| = \frac{|G|}{m}$, where $m = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}$. Then

$$|T||\mathcal{O}| = |G| = m|Y| = 2m|\mathcal{O}||[q^{2n-4}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q)|,$$

yielding that $q = \frac{2}{q+1}$, which is impossible. Thus $y \notin G_0$, and then M^y is the stabilizer in $\mathrm{P\Gamma L}_n(q)$ of some $(n-1)$-space.

Since $y \notin G_0$ and T, $X \leqslant G_0$, we have $G \neq XT$. By Lemma 2.8, $|G : XT| = 2$. Recalling that $|G : G_0| \leqslant 2$, we have $G_0 = XT$ and $G = G_0\langle y\rangle$. By Lemma 2.1, $X \cap Y = X \cap X^y \leqslant M \cap M^y \cap G_0$. Then $Y = (X \cap Y)\langle y\rangle \leqslant (M \cap M^y \cap G_0)\langle y\rangle$. Since Y is maximal in G, we have $Y = (M \cap M^y \cap G_0)\langle y\rangle$. Considering the orders of $X \cap Y$ and $M \cap M^y \cap G_0$, it follows that

$$X \cap Y = M \cap M^y \cap G_0 \cong \frac{q-1}{d}.\mathrm{PGL}_{n-1}(q).\mathcal{O} \text{ or } [q^{2n-3}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q).\mathcal{O}.$$

Since $G_0 = XT$, we have $X/(X \cap T) \cong G_0/T = \mathcal{O}$. Then $X = (X \cap T).\mathcal{O}$. Thus, if $X \cap T = M \cap T$ then (3) of the lemma follows.

To complete the proof, we next show that $X \cap T = M \cap T$. Suppose the contrary. Then $X \cap T$ is a proper subgroup of $M \cap T$. Let $U = X \cap X^y \cap T$. Then

$$U \cong \frac{q-1}{d}.\mathrm{PGL}_{n-1}(q) \text{ or } [q^{2n-3}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q).$$

Note that

$$|(M \cap T) : (X \cap T)||(X \cap T) : U| = |(M \cap T) : U| = q^{n-1} \text{ or } \frac{q^{n-1}-1}{q-1}.$$

Then $|(M \cap T) : (X \cap T)| \neq 1$ is a divisor of q^{n-1} or $\frac{q^{n-1}-1}{q-1}$. Recall that $|(M \cap T) : (X \cap T)|$ is a divisor of $\frac{q-1}{d}$. It follows that

$$|(M \cap T) : (X \cap T)||(X \cap T) : U| = |(M \cap T) : U| = \frac{q^{n-1}-1}{q-1}.$$

Then $|(M \cap T) : (X \cap T)|$ is a divisor of $\gcd(\frac{q^{n-1}-1}{q-1}, \frac{q-1}{d})$, and so $q > 2$. In addition, combining $M \cap T \cong q^{n-1}{:}(\frac{q-1}{d}.\mathrm{PGL}_{n-1}(q))$, we have $U \cong [q^{2n-3}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q)$.

Assume that $n = 3$ and $q \in \{3, 5, 7, 9, 11\}$. Then $\gcd(\frac{q^{n-1}-1}{q-1}, \frac{q-1}{d}) = 2$, and so $|(M \cap T) : (X \cap T)| = 2$. This implies that $X \cap T \cong q^2{:}\mathrm{SL}_2(q).\frac{q-1}{2}$ for $q \neq 7$, and $X \cap T \cong 7^2{:}\mathrm{SL}_2(7)$ for $q = 7$. As $U \cong [q^{2n-3}]{:}[\frac{(q-1)^2}{d}].\mathrm{PGL}_{n-2}(q)$, we have $|U| = \frac{q^3(q-1)^2}{2}$ for $q \neq 7$, and $|U| = 7^3 \cdot 12$ otherwise. Then $|(X \cap T) : U| = q + 1$ for $q \neq 7$, and $|(X \cap T) : U| = 4$ for $q = 7$. It can be easily shown that $7^2{:}\mathrm{SL}_2(7)$ has no subgroup of index 4. Thus $q \in \{3, 5, 9, 11\}$ and $|(X \cap T) : U| = q + 1$. On the other hand, since $|(M \cap T) : (X \cap T)||(X \cap T) : U| = \frac{q^2-1}{q-1}$, we have $|(X \cap T) : U| = \frac{q+1}{2}$, a contradiction.

Now we may assume that the pair (n, q) is not one of $(3, 2)$, $(3, 3)$, $(3, 5)$, $(3, 7)$, $(3, 9)$ and $(3, 11)$, and $(n, q) \neq (5, 2)$ as $q > 2$. Then $\mathrm{SL}_{n-1}(q)$ is insolvable, and it follows from [16, Theorem 5.2.2] that $\mathrm{SL}_{n-1}(q)$ has no proper subgroup of index less than $\frac{q^{n-1}-1}{q-1}$. Let

$$q^{n-1}{:}\mathrm{SL}_{n-1}(q) \cong N \trianglelefteq X \cap T,\ s = |(X \cap T) : UN| \text{ and } t = |(M \cap T) : (X \cap T)|.$$

Then $t > 1$, and

$$\frac{q^{n-1}-1}{q-1} = t|(X \cap T) : U| = t|(X \cap T) : UN||UN : U| = ts|N{:}(U \cap N)|,$$

and so

$$|N{:}(U \cap N)| = \frac{q^{n-1}-1}{ts(q-1)} < \frac{q^{n-1}-1}{q-1}.$$

Let q be a power of some prime p, and P be the largest normal p-subgroup of N. Then $|P| = q^{n-1}$ and $N/P \cong \mathrm{SL}_{n-1}(q)$. Clearly, $|N{:}(U \cap N)|$ and $|P|$ are coprime, so we have $P \leqslant U \cap N$. Then

$$|N/P{:}(U \cap N)/P| = |N{:}(U \cap N)| = \frac{q^{n-1}-1}{ts(q-1)} < \frac{q^{n-1}-1}{q-1}.$$

Recalling that $\mathrm{SL}_{n-1}(q)$ has no proper subgroup of index less than $\frac{q^{n-1}-1}{q-1}$, which yields $\frac{q^{n-1}-1}{ts(q-1)} = 1$, and so $\frac{q^{n-1}-1}{q-1} = st$. Note that

$$\frac{q-1}{d} = |(M \cap T) : N| = t|(X \cap T) : N| = t|(X \cap T) : UN||UN : N| = ts|UN : N|.$$

Then ts is a divisor of $q-1$, and so $\frac{q^{n-1}-1}{q-1}$ is a divisor of $q-1$, which is impossible. This completes the proof. $\square$

Theorem 6.4. *Let $\Gamma = (V, E)$ be a G-edge-primitive graph. Assume that G is an almost simple group with socle $T = \mathrm{PSL}_n(q)$, and G contains a subgroup R that acts regularly on V, where $(n, q) \neq (2, 4)$, $(2, 5)$, $(2, 9)$ or $(4, 2)$. Then one of the following holds:*

(1) *$G = \mathrm{PSL}_2(11)$, $R \cong \mathbb{Z}_{11}$, and $\Gamma \cong \mathsf{K}_{11}$;*

(2) *$\mathrm{soc}(G) = \mathrm{PSL}_2(q)$, $\Gamma \cong \mathsf{K}_{q+1}$, and either $R \cong \mathrm{D}_{q+1}$ or one of the following occurs: $R \cong \mathrm{A}_4$ with $q = 11$, $R \cong \mathrm{S}_4$ with $q = 23$, or $R \cong \mathrm{A}_5$ with $q = 59$;*

(3) *$G = \mathrm{PGL}_2(11)$, $R \cong \mathrm{D}_{22}$, and Γ is isomorphic to one of the graphs Γ_6 and $\Gamma_{6'}$ in Example 2.3;*

(4) *$G = \mathrm{PGL}_2(59)$, $R \cong 59{:}58$, and Γ is isomorphic to the graph Γ_7 in Example 2.3;*

(5) *$G = \mathrm{PSL}_5(2).2$, $R \cong 31{:}10$, and Γ is isomorphic to one of the graphs $\mathcal{D}(5, 2; 2)$ and $\mathcal{F}(5, 2; 2)$ defined as in Example 2.4;*

(6) *$\mathrm{PSL}_n(q) < G \not\leqslant \mathrm{P\Gamma L}_n(q)$, R is solvable and of order $\frac{2(q^n-1)}{q-1}$, and Γ is isomorphic to one of the graphs $\mathcal{D}(n, q; 1)$ and $\mathcal{F}(n, q; 1)$ defined as in Example 2.4.*

Proof. Pick $\{\alpha, \beta\} \in E$, let $X = G_\alpha$ and $Y = G_{\{\alpha\beta\}}$. Then the quintuple $(G, R, X, Y, X \cap Y)$ satisfies Hypothesis 3.2, and so it is determined by Lemmas 6.2 and 6.3. If (1) of Lemma 6.2 occurs then $G = \mathrm{PSL}_2(11)$, $R \cong \mathbb{Z}_{11}$ and Γ has valency 10, and thus $\Gamma \cong \mathsf{K}_{11}$ as in (1) of the theorem. For (3)-(5) of Lemma 6.2 and (2) of Lemma 6.3, $\mathrm{soc}(G) = \mathrm{PSL}_2(q)$, the graph Γ has order $q + 1$ and valency q, and so $\Gamma \cong \mathsf{K}_{q+1}$ as in (2) of the theorem. For (2) and (6) of Lemma 6.2, Γ is isomorphic to one of the three graphs Γ_6, $\Gamma_{6'}$ and Γ_7 constructed in Example 2.3, and so we have (3) and (4) of the theorem.

Assume that (7) of Lemma 6.2 occurs. Then $G = \mathrm{PSL}_5(2).2$, X is the stabilizer a 2- or 3-space, while Y is the stabilizer of some $\{2,3\}$-decomposition or $(2,3)$-flag. It follows that Γ is isomorphic to one of the graphs $\mathcal{D}(5, 2; 2)$ and $\mathcal{F}(5, 2; 2)$ defined as in Example 2.4, and so (5) of the theorem follows. Assume that (3) of Lemma 6.3 occurs. Then X is the stabilizer of a 1- or $(n-1)$-space, while Y is the stabilizer of some $\{1, n-1\}$-decomposition or $(1, n-1)$-flag. Thus Γ is isomorphic to one of the graphs $\mathcal{D}(n, q; 1)$ and $\mathcal{F}(n, q; 1)$ defined as in Example 2.4, and so (6) of the theorem follows. Finally, noting that $\mathrm{PGL}_2(7) \cong \mathrm{PSL}_3(2).2$, if (1) of Lemma 6.3 occurs then it not hard to see that $\Gamma \cong \mathcal{D}(3, 2; 1)$ or $\mathcal{F}(3, 2; 1)$, which is a special case of (6) of the theorem. Then the theorem follows. $\square$

7 Unitary groups

Let (G,T,R,X) be described as in Hypothesis 3.2. Assume that $T=\mathrm{soc}(G)=\mathrm{PSU}_n(q)$ with $n\geqslant 3$, and let $\{H,K\}=\{R,X\}$. By [4, Theorem 3] and [18, Theorem 1.2], (G,H,K) is listed in Cases 24-30 of [4, Table 4] and Row 13 of [18, Table 1]. In particular, (n,q) is one of $(3,8)$, $(4,3)$, $(4,4)$ and $(4,8)$.

Lemma 7.1. *Assume that (G,T,R,X,Y) is described as in Hypothesis 3.2. Let $T=\mathrm{PSU}_n(q)$ with $n\geqslant 3$. Then one of the following holds:*

(1) $G=\mathrm{PSU}_4(3).2^2$, $|R|=162$, $X\cong\mathrm{PSL}_3(4).2^2$, $Y\cong(4^2\times 2)(2\times\mathrm{S}_4)$, *and* $X\cap Y\cong 2^4{:}\mathrm{S}_4.2$*;*

(2) $G=\mathrm{PSU}_4(3).2^2$, $|R|=162$, $X\cong\mathrm{PSL}_3(4).2^2$, $Y\cong\mathrm{M}_{10}.2^2$ *and* $X\cap Y\cong\mathrm{A}_6.2^2$*;*

(3) $G=\mathrm{PSU}_4(8).3.o$, $R\cong(513{:}3).3$, $X\cong(2^{12}{:}\mathrm{SL}_2(64){:}7).3.o$, $Y\cong(\mathrm{SL}_2(64){:}7.2).3.o$, *and* $X\cap Y\cong\mathrm{SL}_2(64){:}7.3.o$, *where* $o\in\{1,2\}$*; in this case, X and Y are the stabilizers of some totally singular 2-space and 2-decomposition, respectively.*

Proof. We discuss in two cases according to the solvability of Y.

Case 1. Suppose that Y is solvable. Then, by [20, Theorem 1.1 and Table 18], G has a normal subgroup G_0 such that $T\leqslant G_0$, $Y_0:=Y\cap G_0$ is maximal in G_0, and

(i) $G_0=T=\mathrm{PSU}_3(8)$, and $Y_0=Y\cap T\cong 3^2{:}\mathrm{Q}_8.3$, $[2^9]{:}\mathbb{Z}_{21}$ or $[3^4]{:}2$; or

(ii) $G_0=T=\mathrm{PSU}_4(3)$, and $Y_0=Y\cap T\cong[3^5]{:}2\mathrm{S}_4$; or

(iii) $G_0=T=\mathrm{PSU}_4(3)$, and $Y_0=Y\cap T\cong 2.(\mathrm{A}_4\times\mathrm{A}_4).4$; or

(iv) $G_0=\mathrm{PSU}_4(3).4$, and $Y_0\cong 4^3\mathrm{S}_4$; or

(v) $G_0=\mathrm{PSU}_4(3).2_3$, and $Y_0\cong(4^2\times 2)\mathrm{S}_4$.

In particular, (G,H,K) is listed in Cases 24-29 of [4, Table 4]. By Lemma 2.8, $|Y_0|$ is a divisor of $2|H\cap G_0|$ or $2|K\cap G_0|$ according to $X=H$ or $X=K$, respectively. In view of this, inspecting $2|H\cap G_0|$ and $2|K\cap G_0|$ with H and K listed in Cases 24-29 of [4, Table 4], we conclude that $G_0=\mathrm{PSU}_4(3).2_3$, $X=K\lesssim\mathrm{PSL}_3(4).2^2$, and $X\cap T\cong\mathrm{PSL}_3(4)$; in particular, only case (v) occurs.

Inspecting the solvable maximal subgroups of G in the Atlas [6], it can be deduced that one of the following three cases occurs: $G=\mathrm{PSU}_4(3).2_3$; $G_0<G=\mathrm{PSU}_4(3).(2^2)_{133}$ with $Y\cong(4^2\times 2)(2\times\mathrm{S}_4)$; $G=T.\mathrm{D}_8$ with $Y\cong 4^3(2\times\mathrm{S}_4)$. The last case implies that $|Y|$ is not a divisor of $2|X|$, contrary to Lemma 2.7. For the first case, we have $G=\mathrm{PSU}_4(3).2_3$, and $X=K\cong\mathrm{PSL}_3(4).2_1$ or $\mathrm{PSL}_3(4).2_3$ by the Atlas [6] and Case 25 of [4, Table 4], and so $|R|=|G:X|=162$; however, computation with GAP [8] shows that neither $\mathrm{PSL}_3(4).2_1$ nor $\mathrm{PSL}_3(4).2_3$ has a complement of order 162 in $\mathrm{PSU}_4(3).2_3$, a contradiction. (This says that the group $\mathrm{PSU}_4(3).2$ in Case 25 of [4, Table 4] is $\mathrm{PSU}_4(3).2_1$.) Then only the second case is left, that is, $G=\mathrm{PSU}_4(3).(2^2)_{133}$ and $Y\cong(4^2\times 2)(2\times\mathrm{S}_4)$. Noting that $|Y|$ is divisible by 2^8, since $|Y|$ is a divisor of $2|X|$, we have $X=K\cong\mathrm{PSL}_3(4).2^2$. By the Atlas [6], X has two choices up to the conjugacy in G, and these two choices are conjugate under $T.\mathrm{D}_8$. Computation with GAP [8] shows that, for each choice, X has a complement of order 162 in G. Then (1) of the lemma follows.

Case 2. Suppose that Y is insolvable. Then X is insolvable, and so either $X=K$ with K listed in Cases 25-30 of [4, Table 4], or $X\in\{H,K\}$ with H and K listed in Row 13 of [18, Table 1].

Assume that $X=K$ is listed in Cases 25-29 of [4, Table 4]. Note that Y is a maximal subgroup of G and $|Y:(X\cap Y)|=2$. Computation with GAP [8] shows that either $G=\mathrm{PSU}_4(3).(2^2)_{133}$,

$X = K \cong \mathrm{PSL}_3(4).2^2$, $Y \cong \mathrm{M}_{10}.2^2$ and $X \cap Y \cong \mathrm{A}_6.2^2$, or $G = \mathrm{PSU}_4(3).2_3$ and X is isomorphic to one of $\mathrm{PSL}_3(4).2_1$ and $\mathrm{PSL}_3(4).2_3$. The latter case is excluded as neither $\mathrm{PSL}_3(4).2_1$ nor $\mathrm{PSL}_3(4).2_3$ has a complement of order 162 in $\mathrm{PSU}_4(3).2_3$. Then (2) of the lemma follows.

Assume that $X = K$ is given as in Case 30 of [4, Table 4]. Then $G = \mathrm{PSU}_4(8).3.o$, $R \cong (513{:}3).3.o_1$ and $X \cong (2^{12}{:}\mathrm{SL}_2(64){:}7)3.o_2$, where $o = o_1o_2 \in \{1, 2\}$. In particular, X has index o_1 in the stabilizer in G of some totally singular 2-space. Since Y is maximal in G and $|Y|$ is a divisor of $2|X|$, inspecting the maximal subgroups of G in [2, Tables 8.10 and 8.11], we conclude that $Y \cong (\mathrm{SL}_2(64){:}7.2).3.o$, which is the stabilizer in G of some totally singular 2-decomposition. Note that $Y \cap T \cong \mathrm{SL}_2(64){:}7.2$, and $X \cap T \cong 2^{12}{:}\mathrm{SL}_2(64){:}7$ (see row 21 of [19, Table 1.2]). It is easily shown that $Y \cap T \not\leqslant X \cap T$, and so $|(Y \cap T) : (X \cap Y \cap T)| > 1$. By Lemma 2.8, we have $|(Y \cap T) : (X \cap Y \cap T)| = 2$, and then $X \cap Y \cap T \cong \mathrm{SL}_2(64){:}7$. This in turn implies that $G = XT$ and $|X : (X \cap Y)| = |(X \cap T) : (X \cap Y \cap T)| = 2^{12}$. Thus $|X \cap Y| = 2^{-12}|X|$, and so $|Y| = 2|X \cap Y| = 2^{-11}|X|$. Then $3o = |Y : (Y \cap T)| = \frac{2^{-11}|X|}{|Y \cap T|} = 3o_2$, yielding $o = o_2$, and $o_1 = 1$. This leads to (3) of the lemma.

Finally, let $X = H$ or K be given as in Row 13 of [18, Table 1]. Since Y is insolvable, by Lemma 2.9, $Y^\infty = (X \cap Y)^\infty < X^\infty \cong \mathrm{SL}_2(16)$ or $\mathrm{SU}_3(4)$. Then Y^∞ is isomorphic to an insolvable proper subgroup of $\mathrm{SL}_2(16)$ or $\mathrm{SU}_3(4)$. We have $Y^\infty \cong \mathrm{A}_5$. Inspecting the maximal subgroups of $G = \mathrm{SU}_4(4).4$ in [2, Tables 8.10 and 8.11], we conclude that Y is not a maximal subgroup of G, a contradiction. This completes the proof. □

The graphs Γ_8 and $\Gamma_{8'}$ in Example 2.3 arise from (1) and (2) of Lemma 7.1, respectively; and the graph $\mathcal{U}(4, 8)$ defined as in Example 2.5 arises from (3) of Lemma 7.1. Then we have the following theorem.

Theorem 7.2. *Let $\Gamma = (V, E)$ be a G-edge-primitive graph. Assume that G is an almost simple group with socle $T = \mathrm{PSU}_n(q)$ for some integer $n \geqslant 3$, and G contains a subgroup R that acts regularly on V. Then one of the following holds:*

(1) *$G = \mathrm{PSU}_4(3).(2^2)_{133}$, $|R| = 162$, and Γ is isomorphic to one of the graphs Γ_8 and $\Gamma_{8'}$ constructed in Example* 2.3*;*

(2) *$\mathrm{PSU}_4(8){:}3 \leqslant G \leqslant \mathrm{P\Gamma U}_4(8)$, $|R| = 4617$, and Γ is isomorphic to the graph $\mathcal{U}(4, 8)$ defined as in Example* 2.5.

8 The conclusions

In this section, $\Gamma = (V, E)$ is assumed to be a G-edge-primitive graph of valency $k \geqslant 2$.

Lemma 8.1. *Let $R < G \leqslant A \leqslant \mathrm{Aut}(\Gamma)$. Assume that G is an almost simple group with socle T, and R acts regularly on V. Then one of the following holds:*

(1) *Γ is a complete graph;*

(2) *$\mathrm{soc}(A) = T$;*

(3) *$G = \mathrm{S}_{11}$, $A = \mathrm{S}_{12}$ and Γ is isomorphic to the graph Γ_1 in Example* 2.3*;*

(4) *$G = \mathrm{M}_{12}.2$, $\mathrm{A}_{12} \wr \mathrm{S}_2 \leqslant A \leqslant \mathrm{S}_{12} \wr \mathrm{S}_2$ and $\Gamma \cong \mathsf{K}_{12,12}$.*

Proof. Note that G and Γ are determined by Theorems 3.5, 4.3, 5.2, 6.4 and 7.2. We suppose that Γ is not a complete graph. If A is almost simple, then Theorems 4.3, 5.2, 6.4 and 7.2 (with G replaced by

A) imply that either $\mathrm{soc}(A) = \mathrm{soc}(G) = T$, or $G = \mathrm{S}_{11}$, $A = \mathrm{S}_{12}$ and $\Gamma \cong \Gamma_1$. This gives (2) and (3) of the lemma.

Now suppose further that A is not an almost simple group. Note that $\mathrm{soc}(G) = T$ is isomorphic to one of A_{11}, A_{12}, M_{12}, J_2, HS, He, $\mathrm{PSU}_4(3)$, $\mathrm{PSU}_4(8)$ and $\mathrm{PSL}_n(q)$ (with $n \geqslant 2$). Since both G and A are primitive groups on E, it follows from [26, Proposition 6.1] that one of the following occurs:

(i) $T \cong \mathrm{PSL}_2(7)$, and $|E| = 8$;

(ii) $T \cong \mathrm{A}_6$, and $|E| = 36$;

(iii) $G = \mathrm{M}_{12}.2$, $\mathrm{A}_{12} \wr \mathrm{S}_2 \leqslant A \leqslant \mathrm{S}_{12} \wr \mathrm{S}_2$ and $|E| = 144$.

Since G is insolvable, Γ has order $v \geqslant 5$ and valency $k \geqslant 3$. By $2|E| = vk$, we have $|E| \geqslant 9$. Thus case (i) does not occur.

Assume that case (ii) occurs. Then an edge-stabilizer Y in G has index 36. Since Y is a maximal subgroup of G, we conclude that $G = \mathrm{PGL}_2(9)$ with $Y \cong \mathrm{D}_{20}$, $G = \mathrm{M}_{10}$ with $Y \cong 5{:}4$, or $G = \mathrm{P\Gamma L}_2(9)$ with $Y \cong 5{:}4 \times 2$. Let X be a vertex-stabilizer in G with $|Y : (X \cap Y)| = 2$. By Lemma 2.7, $|Y| < |X|$ and $|Y|$ is a divisor of $2|X|$. It follows that $|X|$ is divisible by 30. Computation with GAP [8] shows that such an X has no complement in G, contrary to the fact that $G = RX$ is an exact factorization.

It remains to deal with case (iii). Since $G = \mathrm{M}_{12}.2$, by Theorem 5.2, $\Gamma \cong \mathsf{K}_{12}$, $\mathsf{K}_{12,12}$ or Γ_2. Then (4) of the lemma follows as $|E| = 144$. This completes the proof. □

Now we are ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that G is an almost simple group, $\Gamma = (V, E)$ is a G-edge-primitive graph, and G contains a regular subgroup R on V. By Theorems 3.5, 4.3, 5.2, 6.4 and 7.2, Γ is isomorphic to one of the edge-primitive graphs involved in the theorem. Now, Lemma 8.1 gives the desired conclusion on the automorphism group $\mathrm{Aut}(\Gamma)$. □

Lemma 8.2. *Assume that G contains a subgroup R that acts regularly on V, and R is a nonabelian simple group. Then G is almost simple.*

Proof. Clearly, Γ is a regular graph of valency $k \geqslant 3$ due to the transitivity (on V) and the insolvability of R. Moreover, Γ is not bipartite; otherwise, R has a subgroup which fixes the bipartition of Γ, and so R has a subgroup of index 2, contrary to the simplicity of R. Then, by [11, Theorem 6.12 and Proposition 6.13], $\mathrm{soc}(G)$ is the unique minimal normal subgroup of G, and $\mathrm{soc}(G)$ acts transitively on both V and E.

Let $\alpha \in V$. Then $G = RG_\alpha$ is an exact factorization of G. By [18, Theorem 1.5], one of the following three cases occurs:

(i) $\mathrm{soc}(G) = R$;

(ii) $\mathrm{soc}(G) \cong R \times R$, and the stabilizer $\mathrm{soc}(G)_\alpha$ is a full diagonal subgroup of $\mathrm{soc}(G)$;

(iii) $R < \mathrm{soc}(G)$, and $\mathrm{soc}(G)$ is simple.

Noting that $|E| = \frac{1}{2}k|V| = \frac{1}{2}k|R|$, since $\mathrm{soc}(G)$ is transitive on E, it follows that $\frac{1}{2}k|R|$ is a divisor of $|\mathrm{soc}(G)|$. If case (i) holds then $|\mathrm{soc}(G)| = |R|$, and thus $k = 2$, a contradiction.

Suppose that case (ii) holds. Since Γ is not bipartite, by [11, Corollaries 6.6, 6.7 and Theorem 6.12], G acts on E in the product action. Then [11, Lemma 6.14] implies that G acts on V in the product action, and then there exists a proper subgroup S of R such that $\mathrm{soc}(G)_\alpha$ is isomorphic to a subgroup of $S \times S$, which is impossible as $\mathrm{soc}(G)_\alpha \cong R$. Thus only case (iii) is left, and the lemma follows. $\square$

Proof of Theorem 1.3. Assume that $\Gamma = (V, E)$ is a G-edge-primitive graph of valency $k \geqslant 2$, and G contains a simple subgroup R that acts regularly on V. Clearly, $|R| \geqslant 3$. Suppose that R is nonabelian simple. Then G is almost simple by Lemma 8.2, and so Γ is determined by Theorem 1.2. Inspecting the simplicity of those groups R involved in Theorem 1.2, we deduce that Γ is a complete graph.

Now let $R \cong \mathbb{Z}_p$ for some prime $p \geqslant 3$. Then either G is an almost simple 2-transitive group on V, or $G \leqslant \mathrm{AGL}_1(p)$, see [7, p. 99, Corollary 3.5B] for example. The former case implies that $\Gamma \cong \mathsf{K}_p$ with $p \geqslant 5$. Assume that $G \leqslant \mathrm{AGL}_1(p)$. Then G is a Frobenius group, and so a vertex-stabilizer G_α acts semiregularly on $V \setminus \{\alpha\}$. Pick a neighbor β of α. We have $G_{\alpha\beta} = 1$, and so $G_{\{\alpha,\beta\}} \cong \mathbb{Z}_2$. Since $G_{\{\alpha,\beta\}}$ is maximal in G, we have $RG_\alpha = G = RG_{\{\alpha,\beta\}}$. This forces that $G_\alpha \cong \mathbb{Z}_2$. Thus $k = |G_\alpha : G_{\alpha\beta}| = 2$, and so Γ is a cycle of length p. Then Theorem 1.3 follows. $\square$

Proof of Theorem 1.4. Let $\Gamma = (V, E)$ be an edge-primitive Cayley graph of valency $k \geqslant 3$. Assume that Γ is 2-arc-transitive. By [22], either Γ is a complete bipartite graph or $\mathrm{Aut}(\Gamma)$ is an almost simple group. Then Theorem 1.4 follows from inspecting the 2-arc-transitivity of those graphs involved in Theorem 1.2, see also Remark 2.6. $\square$

Lemma 8.3. *Assume that Γ has square-free order and valency at least* 3. *Then either Γ is a complete bipartite graph, or G is almost simple.*

Proof. We suppose that Γ is not a complete bipartite graph, and we aim to prove that G is almost simple. By [11, Lemma 3.5], every minimal normal subgroup of G has at most two orbits on V. Since $|V|$ is square-free and Γ is not a complete bipartite graph, by [17, Lemma 13], either G is almost simple, or $G_\alpha \cong \mathbb{Z}_k$, $\mathbb{Z}_p \lesssim G \leqslant \mathrm{AGL}_1(p)$ and $|V| \in \{p, 2p\}$, where $\alpha \in V$ and $p \geqslant 3$ is a prime. Suppose that the latter case occurs. Then, for a neighbor β of α, we have $G_{\alpha\beta} = 1$, and so $G_{\{\alpha,\beta\}} \cong \mathbb{Z}_2$. Recalling that $G_{\{\alpha,\beta\}}$ is maximal in G, since $\mathbb{Z}_p \lesssim G \leqslant \mathrm{AGL}_1(p)$, we have $G \cong \mathbb{Z}_p{:}\mathbb{Z}_2$. Since G is transitive on V, we have $G_\alpha \cong \mathbb{Z}_2$, $|V| = p$, and $k = 2$, contrary to the hypothesis that $k \geqslant 3$. Therefore, G is almost simple, and the lemma follows. $\square$

Proof of Theorem 1.6. Assume that $\Gamma = (V, E)$ is an edge-primitive Cayley graph of valency $k \geqslant 3$, and $|V|$ is square-free. By Lemma 8.3, either Γ is a complete bipartite graph or G is almost simple. Then Theorem 1.6 follows from inspecting the orders of those groups R involved in Theorem 1.2. $\square$

Declaration of competing interest

The authors have no conflicts of interest to declare.

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Data availability

No data was used for the research described in the article.

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