The minimal size of a graph with generalized connectivity $\kappa_3 = 2^*$

Shasha Li, Xueliang Li, Yongtang Shi
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China.
Email: lss@cfc.nankai.edu.cn, lxl@nankai.edu.cn, shi@nankai.edu.cn

Abstract

Let $G$ be a nontrivial connected graph of order $n$ and $k$ an integer with $2 \leq k \leq n$. For a set $S$ of $k$ vertices of $G$, let $\kappa(S)$ denote the maximum number $\ell$ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in $G$ such that $V(T_i) \cap V(T_j) = S$ for every pair $i, j$ of distinct integers with $1 \leq i, j \leq \ell$. Chartrand et al. generalized the concept of connectivity as follows: The $k$-connectivity, denoted by $\kappa_k(G)$, of $G$ is defined by $\kappa_k(G) = \min \{ \kappa(S) \}$, where the minimum is taken over all $k$-subsets $S$ of $V(G)$. Thus $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of $G$.

This paper mainly determines the minimal number of edges of a graph of order $n$ with $\kappa_3 = 2$, i.e., for a graph $G$ of order $n$ and size $e(G)$ with $\kappa_3(G) = 2$, it is proved that $e(G) \geq \lceil \frac{6}{5}n \rceil$, and the lower bound is sharp for all $n \geq 4$ but $n = 9, 10$, whereas for $n = 9, 10$ examples are given to show that $\lceil \frac{6}{5}n \rceil + 1$ is the best possible lower bound. This gives a clear picture on the minimal size of a graph of order $n$ with generalized connectivity $\kappa_3 = 2$.

Keywords: $k$-connectivity; internally disjoint trees

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1 Introduction

We follow the terminology and notations of [1] and all graphs considered here are always finite and simple. As usual, we denote the numbers of vertices and edges in $G$ by $n(G)$ and $e(G)$ (or simply $n$ and $e$), and these two basic parameters are called the order and size of $G$, respectively. Let $X$ be a set of vertices of $G$ and $G[X]$ the subgraph of $G$ whose vertex set is $X$ and whose edge set consists of all edges of $G$ which have both ends in $X$. A stable set in a graph is a set of vertices no two of which are adjacent. A vertex with degree one in a tree is called a leaf. The connectivity $\kappa(G)$ of a graph $G$ is defined as the minimum cardinality of a set $Q$ of vertices of $G$ such that $G - Q$ is disconnected or trivial. A well-known theorem of Whitney [4] provides an equivalent definition of the connectivity. For each 2-subset $S = \{u, v\}$ of vertices of $G$, let $\kappa(S)$ denote the maximum number of internally disjoint $uv$-paths in $G$. Then $\kappa(G) = \min\{\kappa(S)\}$, where the minimum is taken over all 2-subsets $S$ of $V(G)$.

In [2], the authors generalized the concept of connectivity. Let $G$ be a nontrivial connected graph of order $n$ and $k$ an integer with $2 \leq k \leq n$. For a set $S$ of $k$ vertices of $G$, let $\kappa_k(S)$ denote the maximum number $\ell$ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in $G$ such that $V(T_i) \cap V(T_j) = S$ for every pair $i, j$ of distinct integers with $1 \leq i, j \leq \ell$ (note that the trees are vertex-disjoint in $G \setminus S$). A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in $G$ with this property is called an internally disjoint set of trees connecting $S$. The $k$-connectivity, denoted by $\kappa_k(G)$, of $G$ is then defined by $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all $k$-subsets $S$ of $V(G)$. Obviously, $\kappa_2(G) = \kappa(G)$.

In [3], we focused on the investigation of $\kappa_3(G)$ and mainly studied the relationship between the 2-connectivity and the 3-connectivity of a graph. We gave sharp upper and lower bounds for $\kappa_3(G)$ for general graphs $G$, and showed that if $G$ is a connected planar graph, then $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$. Moreover, we studied the algorithmic aspects for $\kappa_3(G)$ and gave an algorithm to determine $\kappa_3(G)$ for a general graph $G$.

In this paper, we determine the minimal number of edges of a graph with $\kappa_3 = 2$, i.e., for a graph $G$ of order $n$ and size $e(G)$ with $\kappa_3(G) = 2$, we obtain that $e(G) \geq \lceil \frac{6}{5}n \rceil$, and the lower bound is sharp for all $n \geq 4$ but $n = 9, 10$, whereas for $n = 9, 10$ we give examples to show that $\lceil \frac{6}{5}n \rceil + 1$ is the best possible lower bound. This gives a clear picture on the minimal size of a graph of order $n$ with generalized connectivity $\kappa_3 = 2$. Note that for a graph $G$ of order $n$ and size $e(G)$ with $\kappa(G) = 2$, we have $e(G) \geq n$, and a cycle of this order attains the lower bound.
2 Lower bound

Before proceeding, we recall a result in [3], which will be used frequently in the sequel.

Lemma 2.1. If $G$ is a connected graph with minimum degree $\delta$, then $\kappa_3(G) \leq \delta$. In particular, if there are two adjacent vertices of degree $\delta$, then $\kappa_3(G) \leq \delta - 1$.

Now we give the lower bound.

Proposition 2.1. Every graph $G$ of order $n$ with $\kappa_3(G) = 2$ has at least $\lceil \frac{6}{5}n \rceil$ edges.

Proof. Since $\kappa_3(G) = 2$, by Lemma 2.1, we know that $\delta(G) \geq 2$ and any two vertices of degree 2 are not adjacent. Denote by $X$ the set of vertices of degree 2. We have that $X$ is a stable set. Put $Y = V(G) - X$ and obviously there are $2|X|$ edges joining $X$ to $Y$. Assume that $m'$ is the number of edges joining two vertices belonging to $Y$. It is clear that

$$e = 2|X| + m'. \quad (1)$$

Since every vertex of $Y$ has degree at least 3 in $G$, then $\sum_{v \in Y} d(v) = 2|X| + 2m' \geq 3|Y| = 3(n - |X|)$, namely,

$$5|X| + 2m' \geq 3n. \quad (2)$$

Combining (1) with (2), we have $\frac{5}{2}e = \frac{5}{2}(2|X| + m') = 5|X| + \frac{5}{2}m' \geq 5|X| + 2m' \geq 3n$, namely, $e \geq \frac{6}{5}n$. Since the number of edges is an integer, it follows that $e \geq \lceil \frac{6}{5}n \rceil$. The proof is complete.

Remark 2.1: Furthermore, when $n$ is a multiple of 5, in Proposition 2.1 equality holds if and only if $5|X| + \frac{5}{2}m' = 5|X| + 2m' = 3n$, namely, if and only if

(A) $m' = 0$, that is, $Y$ is a stable set and

(B) the maximum degree $\Delta$ is 3.

Moreover, in this case, inequality (2) becomes $5|X| = 3n$, that is, $|X| = \frac{3}{5}n$.

Remark 2.2: Obviously, for any graph $G$ with $e(G) = \lceil \frac{6}{5}n(G) \rceil$, $\kappa_3(G) \leq 2$. The next two lemmas show that the number $e(G) = \lceil \frac{6}{5}n(G) \rceil$ cannot guarantee that $\kappa_3(G) = 2$.

Lemma 2.2. For any connected graph $G$ of order 10 and size 12, $\kappa_3(G) = 1$.

Proof. Note that $e(G) = \lceil \frac{6}{5}n(G) \rceil$ and so $\kappa_3(G) \leq 2$. Assume, to the contrary, that there is a connected graph $G$ of order 10 and size 12 with $\kappa_3(G) = 2$. Therefore by Remark 2.1,
both $X$ and $Y$ are stable sets, $|X| = \frac{3}{5}n = 6$ and $|Y| = 4$, where $X$ and $Y$ are the sets of vertices of degrees 2 and 3, respectively. Let $X = \{x_1, \ldots, x_6\}$ and $Y = \{y_1, \ldots, y_4\}$.

**Case 1:** For every 2-subset $\{y_i, y_j\}$ of $Y$, there is a vertex in $X$ that is adjacent to both $y_i$ and $y_j$, where $1 \leq i \neq j \leq 4$.

Note that every vertex in $X$ has degree 2, and there are exactly six vertices in $X$ and six 2-subsets of $Y$, namely

\[
\{y_1, y_2\}, \{y_1, y_3\}, \{y_1, y_4\}, \{y_2, y_3\}, \{y_2, y_4\}, \{y_3, y_4\}.
\]

Thus we may assume that $G$ is isomorphic to the graph as shown in Figure 1. Then observe that it is impossible to find two internally-disjoint trees connecting the vertices $x_1$, $x_2$ and $x_4$, contrary to our assumption.

![Figure 1: The graph for Case 1 of Lemma 2.2](image1)

**Case 2:** There exists a 2-subset of $Y$ such that no vertex in $X$ is adjacent to both of the vertices in that subset.

For this case, there must exist some 2-subset $\{y_i, y_j\}$ such that at least two vertices in $X$ are adjacent to both $y_i$ and $y_j$, where $1 \leq i \neq j \leq 4$. Without loss of generality, we may assume that $\{y_i, y_j\} = \{y_1, y_2\}$. Since $G$ is connected, we can get that only two vertices in $X$ are adjacent to both $y_1$ and $y_2$. Then we may assume that $G$ is isomorphic to the graph as shown in Figure 2. Now consider the three vertices $x_1$, $x_3$ and $x_5$ and we can get $\kappa_3(G) = 1$, contrary to our assumption.

![Figure 2: The graph for Case 2 of Lemma 2.2](image2)

The proof is complete. $\blacksquare$

**Remark 2.3:** Note that there exists a graph $G$ such that $n = 10$, $e(G) = 13$ and $\kappa_3(G) = 2$, see Figure 3.
Now we turn to the graphs of order 9 and size 11.

Lemma 2.3. For any connected graph $G$ of order 9 and size 11, $\kappa_3(G) = 1$.

Proof. Assume, to the contrary, that there is a connected graph $G$ of order $n = 9$ and size $e = 11$ with $\kappa_3(G) \geq 2$. By Lemma 2.1, we have the minimum degree $\delta(G) \geq 2$. Denote by $X$ the set of vertices of degree 2 in $G$. It follows that $2e = \sum_{v \in V(G)} d(v) \geq 2|X| + 3(n - |X|)$, namely, $|X| \geq 3n - 2e = 5$. On the other hand, by Lemma 2.1 again, we get that $X$ is a stable set. Let $m'$ be the number of edges joining two vertices belonging to $Y$, where $Y = V(G) - X$. It is clear that $e = 2|X| + m'$. So $|X| \leq \frac{e}{2} = 5.5$. Now we can conclude that $|X| = 5$, $|Y| = 4$, $m' = 1$ and every vertex in $Y$ has degree exactly 3. Set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. Since $m' = 1$, without loss of generality, suppose that $y_1 y_2$ is the only edge in $G[Y]$.

Case 1: There is a vertex in $X$ that is adjacent to both $y_1$ and $y_2$.

Note that $G$ is a simple connected graph and every vertex in $X$ has degree 2. It is not hard to get that $G$ is isomorphic to the graph as shown in Figure 4. Then observe that it is impossible to find two internally-disjoint trees connecting the vertices $x_1$, $x_2$ and $x_4$, contrary to our assumption.

Case 2: There is no vertex in $X$ that is adjacent to both $y_1$ and $y_2$.

Subcase 2.1: For every 2-subset $\{y_i, y_j\}$ of $Y$ other than $\{y_1, y_2\}$, there is a vertex in $X$ that is adjacent to both $y_i$ and $y_j$, where $1 \leq i \neq j \leq 4$.

Note that there are exactly five vertices in $X$ and five 2-subsets of $Y$ other than $\{y_1, y_2\}$. Thus, we may assume that $G$ is isomorphic to the graph as shown in Figure 4.
5. Consider the three vertices $x_1$, $x_2$ and $x_5$, and we can get $\kappa_3(G) = 1$, contrary to our assumption.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{The graph for Subcase 2.1 of Lemma 2.3}
\end{figure}

**Subcase 2.2:** Except $\{y_1, y_2\}$, there exists another 2-subset such that no vertex in $X$ is adjacent to both of the vertices in that subset.

In such a situation, there must exist some 2-subset $\{y_i, y_j\}$ such that at least two vertices in $X$ are adjacent to both $y_i$ and $y_j$, where $1 \leq i \neq j \leq 4$. If $\{y_i, y_j\} = \{y_3, y_4\}$, it is not hard to get that there must exist a vertex in $X$ that is adjacent to both $y_1$ and $y_2$, contrary to the case. So without loss of generality, we may assume that $\{y_i, y_j\} = \{y_1, y_3\}$. Then we can get $G$ is isomorphic to the graph as shown in Figure 6. Observe that it is impossible to find two internally-disjoint trees connecting the vertices $x_1$, $x_4$ and $x_5$, contrary to our assumption.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6.png}
\caption{The graph for Subcase 2.2 of Lemma 2.3}
\end{figure}

The proof is complete.

**Remark 2.4:** Notice that there exists a graph $G$ such that $n = 9$, $e(G) = 12$ and $\kappa_3(G) = 2$, see Figure 7.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7.png}
\caption{The graph $G$ of order 9 and size 12 with $\kappa_3(G) = 2$.}
\end{figure}

In view of Lemmas 2.2 and 2.3 and Remarks 2.3 and 2.4, we can see that for $n = 9, 10$, $\left\lceil \frac{6}{5}n \right\rceil + 1$ is the best possible lower bound. Naturally, for any positive integer $n$ but
For a positive integer \( n = 9, 10 \), we want to know whether there is a graph of order \( n \) attaining the lower bound \( \left\lceil \frac{n}{5} \right\rceil \) in Proposition 2.1. For this purpose, we first construct a class of graphs.

Before constructing, we want to give some notions. For any two integers \( a \) and \( k \geq 1 \), denote by \([a]_k\) an integer such that \( 1 \leq [a]_k \leq k \) and \( a \equiv [a]_k \pmod{k} \). For a cycle \( C = x_1x_2x_3 \ldots x_{k-1}x_kx_1 \), we denote three special segments of \( C \) by \( \hat{x}_a, x_a, \hat{x}_b \). For a cycle \( C = x_1x_2x_3 \ldots x_{k-1}x_kx_1 \), we denote three special segments of \( C \) by \( \hat{x}_a, x_a, \hat{x}_b \).

### Lemma 2.4
For a positive integer \( k \neq 2 \), let \( C = x_1y_1x_2y_2 \ldots x_2y_2x_1 \) be a cycle of length \( 4k \). Add \( k \) new vertices \( z_1, z_2, \ldots, z_k \) to \( C \), and join \( z_i \) to \( x_i \) and \( x_{i+k} \), for \( 1 \leq i \leq k \). The resulting graph is denoted by \( H \). Then, the 3-connectivity of \( H \) is 2, namely, \( \kappa_3(H) = 2 \).

**Proof.** Since \( \delta(H) = 2 \), by Lemma 2.1 we can get \( \kappa_3(H) \leq 2 \). So the task is to show \( \kappa_3(H) \geq 2 \). By the definition of the generalized connectivity, it suffices to prove that \( \kappa(S) \geq 2 \), for every 3-subset \( S \) of \( V(H) \).

Firstly, partition \( V(H) \) into three types: \( V_1 = \{x_1, x_2, \ldots, x_{2k}\}, V_2 = \{z_1, z_2, \ldots, z_k\} \) and \( V_3 = \{y_1, y_2, \ldots, y_{2k}\} \). We proceed by considering all cases of \( S \).

**Case 1:** \( S = \{x_a, x_b, x_c\} \), where \( 1 \leq a < b < c \leq 2k \).

The three vertices divide the cycle \( C \) into three segments, at least one of which has length at most \( |C|/3 \). Without loss of generality, we may assume that \( |x_aCx_b| \leq |C|/3 \), namely, \( |x_bCx_a| \geq 2|C|/3 \). Let \( \nu = [b + k]_{2k} \). Note that \( |x_bCx_\nu| = |C|/2 \), and so \( x_\nu \in V(\hat{x}_bC\hat{x}_a) \).

**Subcase 1.1:** \( x_\nu \in V(x_cC\hat{x}_a) \). In this case, \( T_1 = x_aCx_bCx_c \) and \( T_2 = x_cCx_\nuCx_a \cup x_\nu z_{[a]_k}x_b \) are two internally disjoint trees connecting \( S \).

**Subcase 1.2:** \( x_\nu \in V(\hat{x}_bC\hat{x}_c) \). Let \( a' = [a + k]_{2k} \). We can get \( x_{a'} \in V(\hat{x}_bC\hat{x}_c) \), since \( 1 \leq |x_aCx_b| \leq |C|/3, |x_aCx_{a'}| = |C|/2 \) and \( |x_bCx_\nu| = |C|/2 \). Therefore, \( x_{a'} \in V(\hat{x}_cC\hat{x}_a) \), and then \( T_1 = x_cCx_aCx_b \) and \( T_2 = x_bCx_{a'}Cx_c \cup x_{a'}z_{[a]_k}x_a \) are two internally disjoint trees connecting \( S \).

**Case 2:** \( S = \{z_a, z_b, z_c\} \), where \( 1 \leq a < b < c \leq k \).

Since \( 1 \leq a < b < c \leq k \), \( a + k < b + k < c + k \leq 2k \), \( x_aCx_bCx_c \) and \( x_{a+k}Cx_{b+k}Cx_{c+k} \) are two disjoint segments of \( C \). It is easy to find two internally disjoint trees connecting \( S: T_1 = z_a x_aCx_bCx_cz_c \cup x_bz_b \) and \( T_2 = z_a x_{a+k}Cx_{b+k}Cx_{c+k}z_c \cup x_{b+k}z_b \).

**Case 3:** \( S = \{x_a, x_b, z_c\} \), where \( 1 \leq a < b \leq 2k \) and \( 1 \leq c \leq k \).
Observe that the two neighbors $x_c$ and $x_{c+k}$ of $z_k$ divide the cycle into two segments $x_cCx_{c+k}$ and $x_{c+k}Cx_c$.

**Subcase 3.1:** $x_a$ and $x_b$ lie in distinct segments. Without loss of generality, we may assume that $x_a \in V(x_cCx_{c+k})$ and $x_b \in V(x_{c+k}Cx_c)$. Now $T_1 = x_aCx_{c+k}Cx_b \cup x_{c+k}z_c$ and $T_2 = x_bCx_cCx_a \cup x_cz_c$ are two trees we want. Note that the subcase contains the situation that either $x_c$ or $x_{c+k}$ is exactly $x_a$ or $x_b$.

**Subcase 3.2:** $x_a$ and $x_b$ lie in the same segment. Without loss of generality, suppose that $x_a, x_b \in V(\hat{x}_cC\hat{x}_{c+k})$. Let $b' = [b + k]_{2k}$. Since $|x_cCx_{c+k}| = |C|/2$, $|x_bCx_{c+k}| = |C|/2$ and $x_b \in V(\hat{x}_cC\hat{x}_{c+k})$, we have $x_{b'} \in V(\hat{x}_{c+k}C\hat{x}_c)$ and $T_1 = x_aCx_{c+k}Cx_{c+k}z_c$ and $T_2 = x_bz_{[b+k]}x_{b'}Cx_aCx_c \cup x_cz_c$ are two internally disjoint trees connecting $S$.

**Case 4:** $S = \{x_a, z_b, z_c\}$, where $1 \leq a \leq 2k$ and $1 \leq b < c \leq k$.

Since $1 \leq b < c \leq k < b + k < c + k \leq 2k$, the two neighbors $x_b, x_{b+k}$ of $z_b$, together with two neighbors $x_c, x_{c+k}$ of $z_c$ divide the cycle into four segments $x_bCx_c, x_cCx_{b+k}, x_{b+k}Cx_{c+k}$ and $x_{c+k}Cx_b$. Actually, it is easy to see that no matter which segment $x_a$ lies in, the situations are equivalent. Therefore, without loss of generality, we may assume that $x_a \in V(x_bCx_c)$. We have $T_1 = x_aCx_cCx_{b+k}z_b \cup x_cz_c$ and $T_2 = z_cx_{c+k}Cx_bCx_a \cup x_bz_b$ are two internally disjoint trees connecting $S$. Note that this case includes the situation that $x_a$ is exactly $x_b$ or $x_c$.

Next we consider the cases in which $S$ contains the vertices in $V_3$.

**Case 5:** $S = \{y_a, y_b, y_c\}$, where $1 \leq a < b < c \leq 2k$.

Clearly, in this case, $k$ is a positive integer at least 3. Among the three segments $y_aCy_b, y_bCy_c$ and $y_cCy_a$ of $C$, at least one of them has length not more than $|C|/3$. We may assume that $|y_aCy_b| \leq |C|/3 = 4k/3$. Moreover, observe that $x_{a+1}$ lies between $y_a$ and $y_b$. We have $y_b \in V(\hat{x}_{a+1}C\hat{x}_{[a+1]_[2k]})$, since $|x_{a+1}Cy_b| < |y_aCy_b| \leq 4k/3$ and $|x_{a+1}Cx_{[a+1]_[2k]}| = |C|/2 = 2k$.

**Subcase 5.1:** $y_c \in V(\hat{y}_bC\hat{x}_{[a+1]_[2k]})$. There is at least one vertex $x_{b+1}$ between $y_b$ and $y_c$. Since $x_{b+1} \in V(\hat{x}_{a+1}C\hat{x}_{[a+1]_[2k]})$, it is clear that $x_{[b+1]_[2k]} \in V(\hat{x}_{[a+1]_[2k]}C\hat{x}_{a+1})$, namely, $x_{[b+1]_[2k]} \in V(\hat{x}_{[a+1]_[2k]}C\hat{y}_a)$. We can find two internally disjoint trees connecting $S$: $T_1 = y_ax_{a+1}Cy_b \cup y_aCx_{[a+1]_[2k]} \cup x_{a+1}z_{[a+1]}x_{[a+1]_[2k]}$ and $T_2 = y_bx_{b+1}Cy_c \cup x_{b+1}z_{[b+1]}x_{[b+1]_[2k]}Cy_a$.

**Subcase 5.2:** $y_c \in V(\hat{x}_{[a+1]_[2k]}C\hat{y}_a)$. There is at least one vertex $x_a$ between $y_c$ and $y_a$. Obviously, $x_{[a+k]_[2k]} \in V(\hat{x}_{a+1}C\hat{x}_{[a+1]_[2k]})$. Moreover, $x_aCy_b = |y_aCy_b| + 1 \leq |C|/3 + 1 = 4k/3 + 1$ and $x_aCx_{[a+k]_[2k]} = |C|/2 = 2k$, where $k \geq 3$. So $y_b \in V(\hat{x}_aC\hat{x}_{[a+k]_[2k]})$. 

8
Now \( T_1 = y_a x_{a+1} C y_b \cup x_{a+1} z_{[a+1]} x_{[a+1+k]} z_{[a+1+k]} \cup y_a C y_b \) and \( T_2 = y_b C x_{[a+k]} C y_a \cup y_b C x_a y_a \) are two internally disjoint trees connecting \( S \).

**Case 6:** \( S = \{ y_a, y_b, x_c \} \), where \( 1 \leq a < b \leq 2k \) and \( 1 \leq c \leq 2k \).

Notice that \( y_a \) and \( y_b \) divide \( C \) into two segments \( y_a C y_b \) and \( y_b C y_a \). Let \( c' = [c + k] \), and then two subcases arise.

**Subcase 6.1:** \( x_c \) and \( x_{c'} \) lie in distinct segments. We may assume that \( x_c \in V(y_a C y_b) \) and \( x_{c'} \in V(y_b C y_a) \). Thus, \( T_1 = y_a C x_c C y_b \) and \( T_2 = y_b C x_{c'} C y_a \cup x_c z_{[c]} x_{c'} \) are exactly two trees we want.

**Subcase 6.2:** \( x_c \) and \( x_{c'} \) lie in the same segment. Without loss of generality, we may assume that \( x_c, x_{c'} \in V(y_a C y_b) \) and they occur in cyclic order \( y_a, y_b, x_c, x_{c'} \) on \( C \). The segment \( y_a C y_b \) must contain a vertex \( x_{a+1} \) in \( V_1 \). Since \( x_{a+1} \in V(\hat{x}_c C \hat{x}_c) \), \( x_{[a+1+k]} \in V(\hat{x}_c C \hat{x}_c) \). So we can find two internally disjoint trees connecting \( S \): \( T_1 = y_a x_{a+1} C y_b \cup x_{a+1} z_{[a+1]} x_{[a+1+k]} \cup x_{c} C x_{[a+1+k]} \cup y_a C y_b \) and \( T_2 = y_b C x_{c'} z_{[c]} x_{c'} C y_a \).

**Case 7:** \( S = \{ y_a, y_b, z_c \} \), where \( 1 \leq a < b \leq 2k \) and \( 1 \leq c \leq k \).

If \( k = 1 \), then \( C = x_1 y_1 x_2 y_2 x_1 \) and \( H = C \cup x_1 z_1 x_2 \). So \( y_a, y_b \) and \( z_c \) are exactly \( y_1, y_2 \) and \( z_1 \), respectively. Now \( T_1 = y_2 x_1 y_1 \cup x_1 z_1 \) and \( T_2 = y_1 x_2 y_2 \cup x_2 z_1 \) are two internally disjoint trees connecting \( S \).

Otherwise, \( k \geq 3 \), since \( k \neq 2 \). We know that \( y_a, y_b \) divide \( C \) into two segments \( y_a C y_b, y_b C y_a \), and \( z_c \) has two neighbors \( x_c \) and \( x_{c+k} \).

**Subcase 7.1:** \( x_c \) and \( x_{c+k} \) lie in distinct segments. Suppose that \( x_c \in V(y_a C y_b) \) and \( x_{c+k} \in V(y_b C y_a) \). Clearly \( T_1 = y_a C x_c C y_b \cup x_{c} z_{c} \) and \( T_2 = y_b C x_{c+k} C y_a \cup x_{c+k} z_{c} \) are two internally disjoint trees connecting \( S \).

**Subcase 7.2:** \( x_c \) and \( x_{c+k} \) lie in the same segment. Without loss of generality, we may assume that \( x_c, x_{c+k} \in V(y_b C y_a) \) and they occur in cyclic order \( y_a, y_b, x_c, x_{c+k} \) on \( C \).

**Subsubcase 7.2.1:** Between \( y_a \) and \( y_b \), there are at least two vertices in \( V_1 \). Clearly \( x_{a+1} \neq x_b \), and \( y_a, x_{a+1}, x_b, y_b, x_c, x_{a+1+k+2k}, x_{b+k} \) and \( x_{c+k} \) are the cyclic order in which they occur on \( C \). So we can find two internally disjoint trees connecting \( S \): \( T_1 = y_a x_{a+1} z_{[a+1]} x_{[a+1+k+2k]} \cup y_b C x_c C x_{[a+1+k+2k]} \cup x_{c} z_{c} \) and \( T_2 = y_b C x_{c+k} C y_a \cup x_{c+k} z_{c} \).

**Subsubcase 7.2.2:** Between \( y_a \) and \( y_b \), there is only one vertex in \( V_1 \), i.e. \( x_{a+1} = x_b \). Let \( b' = [b + k] \) and clearly \( x_{b'} \in V(\hat{x}_c C \hat{x}_{c+k}) \). Since \( k \geq 3 \), \( V(\hat{x}_c C \hat{x}_{c+k}) \) contains at least two vertices \( x_{c+1}, x_{c+k-1} \) in \( V_1 \). If \( x_{c+1} \neq x_{b'} \), then \( x_{c+1+k} \neq x_{c+k-1} \neq x_c \in V(\hat{x}_{c+k} C y_a) \). So \( T_1 = y_a x_{b} y_b \cup x_{b} z_{b} x_{b'} C x_{c+k} z_{c} \) and \( T_2 = y_b C x_c y_c x_{c+1} z_{[c+1]} x_{[c+k+1]} C y_a \cup x_{c} z_{c} \) are two internally disjoint trees connecting \( S \). Otherwise, \( x_{c+k-1} \neq x_{b'} \), i.e. \( x_{c+k-1} \neq x_{c+k} \).
We have $x_{|c-1|2k} \in V(\hat{y}_bCx_c)$. So $T_1 = y_ax_by_b \cup x_by_bz_{[b]}y_{[b]} \cup z_{c}Cx_{[c]}$ and $T_2 = y_bC_{[c-1]}2kz_{[c-1]}x_{c+k}\{-1}y_{c+k-1}x_{c+k}Cy_a \cup x_{c+k}z_c$ are two internally disjoint trees connecting $S$.

**Case 8:** $S = \{y_a, x_b, x_c\}$, where $1 \leq a \leq 2k$ and $1 \leq b < c \leq 2k$.

Let $b' = [b + k]2k$ and $c' = [c + k]2k$. If $b' = c$, i.e., $c = [b + k]2k$, then without loss of generality, we may assume that $y_a \in V(x_bC{x_c})$. We have $T_1 = y_ax_cz_{[c]}x_b$ and $T_2 = x_cCx_bCy_a$ are two internally disjoint trees connecting $S$. Otherwise, $b' \neq c$. Without loss of generality, suppose $x_b, x_c, x_{b'}$ and $x_{c'}$ are the cyclic order in which they occur on $C$, and then they divide $C$ into four segments $x_bCx_c, x_cCx_{b'}, x_{b'}Cx_{c'}$ and $x_{c'}Cx_b$.

**Subcase 8.1:** $y_a \in V(x_bC{x_c})$. We can find two internally disjoint trees connecting $S$: $T_1 = x_bCy_a \cup x_cC_{b'}z_{[b]}x_b$ and $T_2 = y_ax_cz_{[c]}x_{c'}Cx_{b}$.

**Subcase 8.2:** $y_a \in V(x_cCy_b)$ or $y_a \in V(x_{c'}Cy_b)$. It is easy to see that the two situations are actually equivalent. So we only consider the former. We can find two internally disjoint trees connecting $S$: $T_1 = x_bcCx_cy_a$ and $T_2 = y_ax_cy_bCy_a \cup x_cz_{[c]}x_{b'} \cup x_{b'}z_{[b]}x_b$.

**Subcase 8.3:** $y_a \in V(x_{b}Cy_{c'})$. We can find two internally disjoint trees connecting $S$: $T_1 = x_bCy_c \cup x_by_bz_{[b]}x_{b'}y_a$ and $T_2 = y_ax_cy_cCy_b \cup x_cz_{[c]}x_{c'}$.

**Case 9:** $S = \{y_a, z_b, z_c\}$, where $1 \leq a \leq 2k$ and $1 \leq b < c \leq k$.

Observe that $x_b, x_c, x_{b+k}$ and $x_{c+k}$ divide the cycle into four segments $x_bCx_c, x_cCx_{b+k}, x_{b+k}Cx_{c+k}$ and $x_{c+k}Cx_b$. Actually, no matter which segment $y_a$ lies in, the situations are equivalent. So without loss of generality, we may assume that $y_a \in V(x_bCx_c)$. Now $T_1 = y_ax_cy_cx_{b+k}z_b \cup x_cz_c$ and $T_2 = z_cx_{c+k}Cy_b \cup x_by_b$ are two internally disjoint trees connecting $S$.

**Case 10:** $S = \{y_a, x_b, z_c\}$, where $1 \leq a \leq 2k$, $1 \leq b \leq 2k$ and $1 \leq c \leq k$.

**Subcase 10.1:** $b = c$ or $b = c + k$. Without loss of generality, we may assume that $b = c$ and $y_a \in V(x_{c+k}Cy_b)$. Therefore, $T_1 = y_ax_cy_cz_c$ and $T_2 = x_bCy_cz_{[c]}x_{c+k}z_{c}$ are two internally disjoint trees connecting $S$.

**Subcase 10.2:** $b \neq c$ and $b \neq c + k$. Let $b' = [b + k]2k$. We may assume that $x_b, x_c, x_{b'}$ and $x_{c+k}$ are the cyclic order in which they occur on $C$. Moreover, they divide $C$ into four segments $x_bCy_c, x_cCy_{b'}, x_{b'}Cy_{c+k}$ and $x_{c+k}Cx_b$.

If $y_a \in V(x_bCy_c)$, then $T_1 = y_ax_cy_cy_bz_{[b]}x_b \cup x_cz_c$ and $T_2 = z_cx_{c+k}Cy_bCy_a$ are two internally disjoint trees connecting $S$.  

10
If \( y_a \in V(x_cCx_bC_{x+c+k}) \), then \( T_1 = x_bCx_cCy_a \cup x_cz_c \) and \( T_2 = y_aCx_{c+k}Cx_b \cup x_{c+k}z_c \) are two internally disjoint trees connecting \( G \).

If \( y_a \in V(x_{c+k}Cx_b) \), then \( T_1 = y_aCx_bCx_{c+z_c} \) and \( T_2 = x_bz[y_h]Cx_{c+k}Cy_a \cup x_{c+k}z_c \) are two internally disjoint trees connecting \( S \).

The proof is complete.

**Remark 2.5:** Clearly, the order \( n(H) \) of the graph \( H \) is 5\( k \) and the size \( e(H) \) is \( 4k+2k = 6k \). If \( k = 2 \), then \( H \) is a connected graph of order 10 and size 12. By Lemma 2.2, we can get \( \kappa_3(H) = 1 \). This is the reason why we add the condition \( k \neq 2 \) to Lemma 2.4. Moreover, no graphs of order 10 can attain the lower bound.

Next we describe an operation on a vertex of degree 2 in a graph. For a vertex \( u \) of degree 2, to *smooth* \( u \) is to delete \( u \) and then add an edge between its neighbors. Obviously, performing such an operation, the numbers of vertices and edges decrease by one, respectively. Moreover, the degrees of the remaining vertices are not changed.

**Lemma 2.5.** Let \( G \) be a graph such that the set \( X \) of vertices of degree 2 is nonempty. Denote by \( G' \) the new graph obtained by smoothing a vertex in \( X \), and then we have \( \kappa_3(G') \geq \kappa_3(G) \).

**Proof.** Let \( u \) be a vertex in \( X \) and \( \{w_1, w_2\} \) the neighbor set of \( u \). Suppose that \( G' \) is obtained by smoothing \( u \). Clearly, \( V(G') = V(G) - u \). For any three vertices \( v_1, v_2 \) and \( v_3 \) of \( G' \), let \( S = \{v_1, v_2, v_3\} \). Obviously, \( S \subseteq V(G) \). Let \( T \) be a tree connecting \( S \) in \( G \). Note that if \( v \) is a leaf of \( T \), we can assume that \( v \in S \). Otherwise, \( T' = T - v \) is still a tree connecting \( S \) and uses less vertices. Now if \( u \in V(T) \), then we can see that \( T' = T - u + w_1w_2 \) is exactly a tree connecting \( S \) in \( G' \). If \( u \notin V(T) \), the operation of smoothing \( u \) has nothing to do with \( T \) and so \( T \) is still a tree connecting \( S \) in \( G' \). Therefore, it is not hard to get that \( \kappa_G(S) \geq \kappa_G(S) \). From the definition of \( \kappa_3 \), the conclusion that \( \kappa_3(G') \geq \kappa_3(G) \) follows.

**Remark 2.6:** For a given \( G \), if we successively do the operation of smoothing a vertex of degree 2 more than once, and the resulting graph is denoted by \( G' \), then we can also get \( \kappa_3(G') \geq \kappa_3(G) \).

Now, we can get our main result.

**Theorem 2.2.** If \( G \) is a graph of order \( n \) with \( \kappa_3(G) = 2 \), then \( e(G) \geq \lceil \frac{6}{5}n \rceil \). Moreover, the lower bound is sharp for all \( n \geq 4 \) but \( n = 9, 10 \), whereas the best lower bound for \( n = 9, 10 \) is \( \lceil \frac{6}{5}n \rceil + 1 \).
Proof. The lower bound $\lceil \frac{6}{5}n \rceil$ is clear from Proposition 2.1. The best lower bound $\lceil \frac{6}{5}n \rceil + 1$ for $n = 9, 10$ is given in Remarks 2.3 and 2.4. Note that all graphs considered here are always simple. Therefore, any graph attaining the lower bound must have at least four vertices. Next, we will show that the lower bound $\lceil \frac{6}{5}n \rceil$ is best possible for all $n \geq 4$ but $n = 9, 10$.

For $n = 8$, there is a graph $G'$ of order $n$ such that $\kappa_3(G') = 2$ as shown in Figure 8. Moreover, $e(G') = 10 = \lceil \frac{6}{5} \times 8 \rceil$, which means that $G'$ attains the lower bound for $n = 8$.

![Figure 8: The graph $G'$ attaining the lower bound for $n = 8$](image)

Now, smooth a vertex of degree 2 in $G'$. Clearly, the resulting graph $G''$ is simple and $\delta(G'') = 2$. By Lemma 2.5, we can get $\kappa_3(G'') \geq (\kappa(G') = 2)$ and so clearly $\kappa_3(G'') = 2$. Moreover, $n = 8 - 1 = 7$ and $e = 10 - 1 = 9 = \lceil \frac{6}{5} \times 7 \rceil$. The graph $G''$ is what we want to find for $n = 7$. Similarly, the graph obtained from $G''$ by smoothing any one vertex of degree 2 attains the lower bound for $n = 6$.

Next, consider the graph $H$ in Lemma 2.4. We know that $\kappa_3(H) = 2$, $n(H) = 5k$ and $e(H) = 6k = \frac{6}{5}n(H)$, for a positive integer $k \neq 2$. So $H$ is exactly the graph of order $n = 5k$ which attains the lower bound.

For $k \geq 3$, let $k' = k - 1$ and then $n(H) = 5k' + 5$ and $e(H) = 6k' + 6$. Let $X$ be the set of vertices of degree 2. Clearly $|X| = 3k' + 3 > 4$, where $k' \geq 2$. Now for the graph $H$, smooth successively any $t$ vertices in $X$, for $1 \leq t \leq 4$. For any $t$, it is easy to check that no parallel edge can arise. Moreover, since $|X| > 4$, the minimum degree of the resulting graph $H'$ is still 2. Combining Lemma 2.1 and Remark 2.6, we can get the 3-connectivity of the resulting graph $H'$ is 2. Now let us consider the numbers of vertices and edges of $H'$.

When $t = 1$, $n(H') = 5k' + 4$ and $e(H') = 6k' + 5 = \lceil \frac{6}{5}(5k' + 4) \rceil$;

When $t = 2$, $n(H') = 5k' + 3$ and $e(H') = 6k' + 4 = \lceil \frac{6}{5}(5k' + 3) \rceil$;

When $t = 3$, $n(H') = 5k' + 2$ and $e(H') = 6k' + 3 = \lceil \frac{6}{5}(5k' + 2) \rceil$;

When $t = 4$, $n(H') = 5k' + 1$ and $e(H') = 6k' + 2 = \lceil \frac{6}{5}(5k' + 1) \rceil$. 

12
Note that $k' \geq 2$. Therefore, for all $n \geq 4$ but $n = 9, 10$, we can always find a graph of order $n$ attaining the lower bound.

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