Tricyclic oriented graphs with maximal skew energy*

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Abstract

Let G^σ be an oriented graph obtained by assigning an orientation σ to the edge set of a simple undirected graph G. Let $S(G^\sigma)$ be the skew adjacency matrix of G^σ. The skew energy of G^σ is defined as the sum of the absolute values of all eigenvalues of $S(G^\sigma)$. In this paper, we determine the tricyclic oriented graphs of order $n \geq 13$ with the maximal skew energy.

Keywords: Tricyclic graph; Skew-adjacency matrix; Skew energy

AMS Subject Classification (2010): 05C20, 05C50, 05C90

1 Introduction

An important quantum-chemical characteristic of a conjugated molecule is its total $\pi-$electron energy. The energy of a graph has closed links to chemistry. Let G be a simple undirected graph and $A(G)$ be the adjacency matrix of G. Gutman [7] firstly defined the *energy* $E(G)$ of G as follows:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

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where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$. For more results about graph energy, we refer the readers to the surveys [8, 9] and the book [15].

There are various generalizations of graph energy, such as the Randić energy [5, 16], the distance energy [22], the incidence energy [2] and the energy of a polynomial [17]. In this paper, we focus on the *skew energy* of a graph. Let G^σ be an oriented graph obtained by assigning an orientation σ to the edge set of a simple undirected graph G. The *skew adjacency matrix* $S(G^\sigma) = (s_{ij})$ of G^σ is a real skew symmetric matrix, where $s_{ij} = 1$ and $s_{ji} = -1$ if ij is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. Then the authors [1] defined the *skew energy* $E_S(G^\sigma)$ of an oriented graph G^σ as the sum of the absolute values of all eigenvalues of $S(G^\sigma)$. The *skew characteristic polynomial* of G^σ is defined as

$$P_S(G^\sigma; x) = \det(xI - S(G^\sigma)) = \sum_{i=0}^{n} b_i x^{n-i}.$$

Since $S(G^\sigma)$ is a real skew symmetric matrix, we have $b_{2k}(G^\sigma) \geq 0$ and $b_{2k+1}(G^\sigma) = 0$ for all $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$ (see [6]). Thus we have

$$P_S(G^\sigma; x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} b_{2k}(G^\sigma) x^{n-2k}.$$

By the coefficients of $P_S(G^\sigma; x)$, the skew energy $\mathcal{E}_S(G^\sigma)$ can be expressed by the following integral formula as follows [14]:

$$\mathcal{E}_S(G^\sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{t^2} \ln(1 + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} b_{2k} t^{2k}) \mathrm{d}t.$$

So $\mathcal{E}_S(G^\sigma)$ is a strictly monotonically increasing function of $b_{2k}(G^\sigma), k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Consequently, if G^{σ_1} and H^{σ_2} are oriented graphs with

$$b_{2k}(G^{\sigma_1}) \geq b_{2k}(H^{\sigma_2}) \quad \text{for each } k \ (0 \leq k \leq \lfloor \frac{n}{2} \rfloor), \tag{1}$$

then

$$\mathcal{E}_S(G^{\sigma_1}) \geq \mathcal{E}_S(H^{\sigma_2}). \tag{2}$$

Equality in (2) is attained only if (1) is an equality for all $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. If the inequalities (1) hold for all k, then we write $G \succeq H$ or $H \preceq G$. If $G \succeq H$, but not $H \succeq G$, then we write $G \succ H$. That is exactly the *quasi-order relation* defined by Gutman and Polansky [10] on graph energy, which is generalized to the skew-energy of oriented graph. See [3, 11, 12, 14, 19, 20, 23, 25] for some recent results about the spectrum and energy of the skew-adjacency matrix.

Due to the coefficients $b_{2k} \geq 0$, it makes that the skew energy problem is much easier than the adjacency energy problems. Particularly, it is researched thoroughly for the extremal skew energy of unicyclic and bicyclic graphs. For example, Hou et al. [14, 19] determined the unicyclic and bicyclic oriented graphs having the minimum or maximum skew energies respectively. Wang et al. [24] determined the oriented bicyclic graph with the second largest skew energy. In this paper, we will characterize the tricyclic oriented graph of order $n \geq 13$ with maximal skew energy.

For the sake of completeness, we say something about the orientation of G^σ that already exists [19]. Let G^σ be an orientation of a graph G. If C is an even cycle of G, then we say C is *evenly oriented* relative to G^σ if it has an even number of edges oriented in the direction of the routing; otherwise C is *oddly oriented*. Let W be a subset of $V(G)$ and $\overline{W} = V(G)\backslash W$. The orientation $G^{\sigma'}$ of G obtained from G^σ by reversing the orientations of all arcs between $\overline{W}$ and W is said to be obtained from G^σ by a switching with respect to W. Moreover, two orientations G^σ and $G^{\sigma'}$ of a graph G are said to be switching-equivalent if $G^{\sigma'}$ can be obtained from G^σ by a sequence of switchings. As noted in [1], since the skew adjacency matrices obtained by a switching are similar, their spectra and hence skew energies are equal.

It is easy to verify that up to switching equivalence there are just two orientations of a cycle C: (1) Just one edge on the cycle has the opposite orientation to that of others, we call it orientation $+$. (2) All edges on the cycle C have the same orientation, we denote this orientation $-$. So if a cycle is of even length and oddly oriented, then it is equivalent to the orientation $+$; if a cycle is of even length and evenly oriented, then it is equivalent to the orientation $-$. The skew energy of a directed tree is the same as the energy of its underlying tree ([1]). So by switching equivalence, for an oriented unicyclic graph or an oriented bicyclic graph, we only need to consider the orientations of cycles. Simultaneously, we denote by T the oriented tree and omit the superscript σ since the skew energy of a directed tree is independent of its orientations.

We denote by G^+ (resp., G^-) the unicyclic graph on which the orientation of a cycle is of orientation $+$ (resp., $-$), and denote by G^* the unicyclic graph on which the orientation of a cycle is of arbitrary orientation $*$. Let C_x, C_y be two cycles in bicyclic graph G with t ($t \geq 0$) common vertices. If $t \leq 1$, then G contains exactly two cycles, and we denote by $G^{a,b}$ the bicyclic graph on which cycle C_x is of orientation a and cycle C_y is of orientation b, where $a, b \in \{+, -, *\}$. If $t \geq 2$, then G contains exactly three cycles. The third cycle is denoted by C_z, where $z = x + y - 2t + 2$. Without loss of generality, assume that $x \leq z$ and $y \leq z$. Moreover, Let $G^{a,b,c}$ be the bicyclic graph on which cycle C_x is of orientation a, cycle C_y is of orientation b, C_z

is of orientation c, where $a, b, c \in \{+, -, *\}$. If G is tricyclic with none of two cycles intersecting, we denote by $G^{a,b,c}$, where $a, b, c \in \{+, -, *\}$. The other graphs used in the paper are shown as follows.

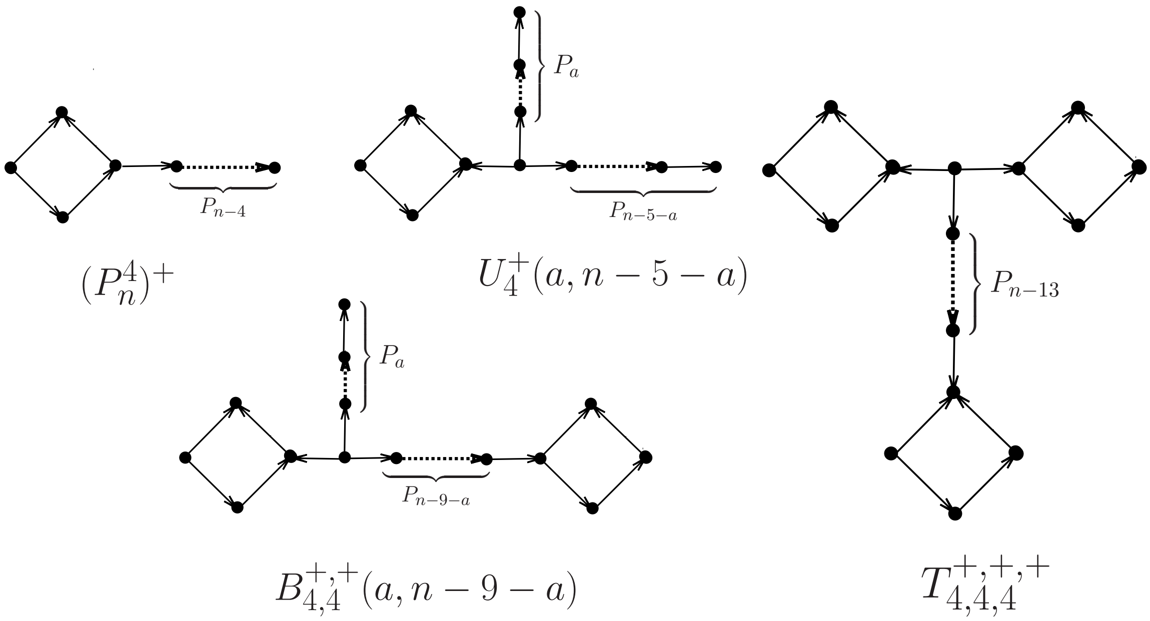

Figure 1: Graphs used in the paper.

The rest of this paper is organized as follows. In section 2, some useful lemmas are stated. In section 3, the tricyclic oriented graph of order $n \geq 13$ with maximal skew energy is determined.

2 Some useful lemmas

Let G be a graph. A *linear subgraph* L of G is a disjoint union of some edges and some cycles in G [4]. We call a linear subgraph L of G *evenly linear* if L contains no cycle with odd length and denote by $\mathcal{EL}_i(G)$ the set of all evenly linear subgraphs of G with i vertices. For a linear subgraph $L \in \mathcal{EL}_i(G)$, denote by $p_e(L)$ (resp., $p_o(L)$) the number of evenly (resp., oddly) oriented cycles in L relative to G^σ.

Lemma 2.1 *[13] Let G^σ be an orientation of a graph G. Then*

$$b_i(G^\sigma) = \sum_{L \in \mathcal{EL}_i} (-2)^{p_e(L)} 2^{p_o(L)}.$$

Lemma 2.1 implies that $b_{2k}(G^\sigma) = m(G^\sigma, k)$ for any orientation of a graph that does not contain any even cycle, particularly for a tree or a unicyclic non-bipartite graph.

Lemma 2.2 *[13] Let $e = uv$ be an edge of G. Then*

$$\begin{aligned} P_S(G^\sigma; x) &= P_S(G^\sigma - e; x) + P_S(G^\sigma - u - v; x) \\ &\quad + 2 \sum_{e \in C \in Od(G^\sigma)} P_S(G^\sigma - C; x) - 2 \sum_{e \in C \in Ev(G^\sigma)} P_S(G^\sigma - C; x). \end{aligned}$$

Corollary 2.1 *[13] Let $e = uv$ be an edge of G that is on no even cycle of G. Then*

$$P_S(G^\sigma; x) = P_S(G^\sigma - e; x) + P_S(G^\sigma - u - v; x). \quad (3)$$

By equating the coefficient of polynomials in Eq.(3), we have

$$b_{2k}(G^\sigma) = b_{2k}(G^\sigma - e) + b_{2k-2}(G^\sigma - u - v). \quad (4)$$

Furthermore, if $e = uv$ is a pendent edge with pendent vertex v, then

$$b_{2k}(G^\sigma) = b_{2k}(G^\sigma - v) + b_{2k-2}(G^\sigma - u - v). \quad (5)$$

A *k-matching* M of a graph G is a disjoint union of k-edges. The number of k-matchings of G is denoted by $m(G, k)$.

Lemma 2.3 *[14] Let $e = uv$ be an edge of G. Then*

(1) $m(G, k) = m(G - e, k) + m(G - u - v, k - 1)$.

(2) if G is a forest, then $m(G, k) \le m(P_n, k)$, $k \ge 1$.

(3) if H is a subgraph of G, then $m(H, k) \le m(G, k)$, $k \ge 1$. Moreover, if H is a proper subgraph of G, then the inequality is strict.

We define $m(G, 0) = 1$ and $m(G, k) = 0$ for $k \ge \frac{n}{2}$.

Lemma 2.4 *[21] Let $a + b = c + d$ with $0 \le a \le b$ and $0 \le c \le d$. Set $a < c$.*

(1) If a is even, then $m(P_a \cup P_b, i) \ge m(P_c \cup P_d, i)$. Furthermore, there exists at least one index i such that the above inequality is strict.

(1) If a is odd, then $m(P_a \cup P_b, i) \le m(P_c \cup P_d, i)$. Furthermore, there exists at least one index i such that the above inequality is strict.

Two immediately results are followed from Lemma 2.3 and 2.4.

Lemma 2.5 *[19] Let F_n be a (oriented) forest of order n. Then $F_n \preceq P_n$. Equality holds if and only if $F_n = P_n$.*

Lemma 2.6 *[19] (1) $P_n \succ P_2 \cup P_{n-2} \succ P_4 \cup P_{n-4} \succ \cdots P_{2k} \cup P_{n-2k} \succ P_{2k+1} \cup P_{n-2k-1} \succ P_{2k-1} \cup P_{n-2k+1} \succ \cdots \succ P_3 \cup P_{n-3} \succ P_1 \cup P_{n-1}$.*
(2) If $a \geq 2$, $P_a \cup (P^4_{n-a})^+ \prec P_2 \cup (P^4_{n-2})^+$.

Let $\mathcal{B}_n^+ = \{U_4^+(a,b) | 0 \leq a \leq b,\ a+b = n-5\}$.

Lemma 2.7 *[25] Let $k = \lfloor \frac{n-5}{2} \rfloor$, $t = \lfloor \frac{k}{2} \rfloor$ and $\ell = \lfloor \frac{k-1}{2} \rfloor$. Then we have the following quasi-order relation in $\mathcal{B}_n^+$, where the graphs are shown in Fig. 1.*

$U_4^+(0, n-5) \succ U_4^+(2, n-7) \succ \cdots \succ U_4^+(2t, n-5-2t) \succ U_4^+(2\ell+1, n-5-2\ell-1) \succ \cdots \succ U_4^+(7, n-12) \succ U_4^+(5, n-10) \succ U_4^+(3, n-8) \succ U_4^+(1, n-6)$.

Let $\mathcal{A}_n^+ = \mathcal{B}_n^+ \backslash \{U_4^+(5, n-10),\ U_4^+(3, n-8),\ U_4^+(1, n-6)\}$. We have the following.

Lemma 2.8 *[25] Let $n \geq 31$. The oriented unicyclic graphs of order n with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies are the oriented unicyclic graphs in $\mathcal{A}_n^+$.*

In [18], the authors determined the bicyclic oriented graphs with the first five largest skew energies, where the graphs are shown in Fig. 1.

Theorem 2.1 *[18] Among all oriented bicyclic graphs with order $n \geq 13$, the graphs $B_{4,4}^{+,+}(0, n-9) \succeq B_{4,4}^{+,+}(n-9, 0) \succeq B_{4,4}^{+,+}(2, n-11) \succeq B_{4,4}^{+,+}(n-11, 2) \succeq B_{4,4}^{+,+}(4, n-13)$ have the first five largest skew energy.*

3 Oriented tricyclic graphs with maximal skew energy

In this section, we determine the tricyclic oriented graph with maximal skew energy. Let $\mathcal{T}(n)$ be the set of all tricyclic graphs of order n. We now divide $\mathcal{T}(n)$ into three subsets, that is, $\mathcal{T}_1(n) = \{G \in \mathcal{T}(n)$: there exists a cycle of G not intersecting other cycles$\}$, $\mathcal{T}_2(n) = \{G \in \mathcal{T}(n) : G \notin \mathcal{T}_1(n)$, there exists a cycle of G intersecting any other cycles at most one vertex$\}$ and $\mathcal{T}_3(n) = \{G \in \mathcal{T}(n) : G \notin \mathcal{T}_1(n)$ and each common vertex of any two cycles is not a cut vertex$\}$ (see Fig. 2-4).

The following lemma is easy but helpful.

Lemma 3.1 *Let G be an arbitrary graph and $v \in G$. We have $m(G,k) \leq m(G \cdot P_{2t}, k+t)$, where $G \cdot P_{2t}$ is the graph obtained by connecting v and an endpoint of P_{2t}.*

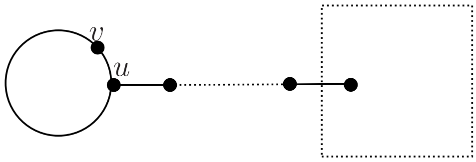

Figure 2: The set $\mathcal{T}_1(n)$, where the right is an arbitrary bicyclic graph.

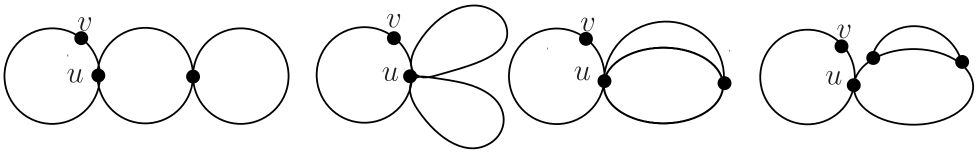

Figure 3: The set $\mathcal{T}_2(n)$.

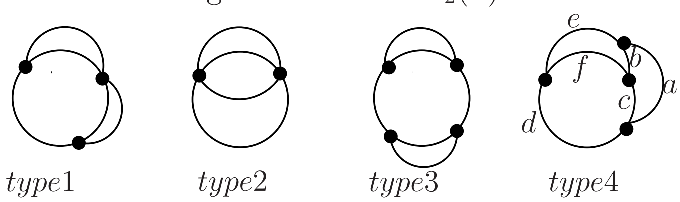

Figure 4: The set $\mathcal{T}_3(n)$.

Proof. Let M be an arbitrary k matching of G and M_t be a t matching of P_{2t}. Then $M' = M \cup M_t$ is a $k+t$ matching of $G \cdot P_{2t}$. Therefore we have $m(G,k) \leq m(G \cdot P_{2t}, k+t)$. $\square$

The following corollary immediately follows from Lemmas 3.1 and 2.1.

Corollary 3.1 *Let* $0 \leq a \leq b \leq \lfloor n/2 \rfloor$. *Then*
(1) $b_{2k-2a}(P_{n-2a}) \geq b_{2k-2b}(P_{n-2b})$;
(2) $b_{2k-2a}((P^4_{n-2a})^+) \geq b_{2k-2b}((P^4_{n-2b})^+)$.

Lemma 3.2 *Let* G^σ *be a tricyclic oriented graph and* $G \in \mathcal{T}_1(n), n \geq 13$. *If* $G^\sigma \neq T^{+,+,+}_{4,4,4}$, *then* $G^\sigma \prec T^{+,+,+}_{4,4,4}$.

Proof. Let C_x be the cycle which does not intersect other cycles and $|C_x| = x$. We can choose the edge $e = uv$ on C_x such that u is a vertex in a path which connects other cycles. Obviously, $G^\sigma - e$ is a bicyclic oriented graph and $G^\sigma - u - v$ is the disjoint union of a forest and a bicyclic oriented graph B_{n_1} with order $n_1 \leq n - x$. Without loss of generality, we can always suppose that $G^\sigma - e \neq B^{+,+}_{4,4}(0, n-9), B^{+,+}_{4,4}(n-9, 0)$ and $B_{n_1} \neq B^{+,+}_{4,4}(0, n_1-9), B^{+,+}_{4,4}(n_1-9, 0)$ (otherwise, one can check that there exists another cycle C_y meeting our requirements).

By Lemmas 2.2 and 2.6 we get

$$\begin{aligned}
&b_{2k}(P_a \cup B_{4,4}^{+,+}(n-a-9,0))\\
=&b_{2k}(C_4^+ \cup P_a \cup (P_{n-a-4}^4)^+) + b_{2k-2}(C_4^+ \cup P_3 \cup P_a \cup P_{n-a-9})\\
<&b_{2k}(C_4^+ \cup P_2 \cup (P_{n-6}^4)^+) + b_{2k-2}(C_4^+ \cup P_3 \cup P_2 \cup P_{n-11})\\
=&b_{2k}(P_2 \cup B_{4,4}^{+,+}(n-11,0)),
\end{aligned}$$

which together with Theorem 2.1 and corollary 3.1 leads to

$$\begin{aligned}
b_{2k}(G^\sigma) \leq &b_{2k}(G^\sigma - e) + b_{2k-2}(G^\sigma - u - v) + 2b_{2k-x}(G^\sigma - C_x^+)\\
<&b_{2k}(B_{4,4}^{+,+}(n-9,0)) + b_{2k-2}(P_{n-n_1-2} \cup B_{4,4}^{+,+}(n_1-9,0))\\
&+ 2b_{2k-x}(B_{4,4}^{+,+}(n_1-9,0) \cup P_{n-n_1-x})\\
\leq &b_{2k}(B_{4,4}^{+,+}(n-9,0)) + b_{2k-2}(P_2 \cup B_{4,4}^{+,+}(n-13,0)) + 2b_{2k-4}(B_{4,4}^{+,+}(n-13,0))\\
=&b_{2k}(T_{4,4,4}^{+,+,+}).
\end{aligned}$$

This completes the proof. $\square$

Lemma 3.3 *Let G^σ be a tricyclic oriented graph and $G \in \mathcal{T}_2(n), n \geq 13$. Then $G^\sigma \prec T_{4,4,4}^{+,+,+}$.*

Proof. By the definition of $\mathcal{T}_2(n)$, let $C_x \cap C_y = \{u\}$ and P be the internal disjoint path on C_y. We divide the proof into two cases.

Case 1. None of the endpoints of P on C_y is u. Suppose $e = uv$ is an edge of C_x. Clearly, $G^\sigma - e$ is a bicyclic oriented graph and $G^\sigma - e \neq B_{4,4}^{+,+}(0, n-9), B_{4,4}^{+,+}(n-9, 0), B_{4,4}^{+,+}(2, n-11), B_{4,4}^{+,+}(n-11, 2)$. $G^\sigma - u - v$ is the disjoint union of a forest and an unicyclic oriented graph U_{n_1} with order $n_1 \leq n - x$. By Lemmas 2.2, 2.6 and Corollary 3.1, we have

$$\begin{aligned}
b_{2k}(G^\sigma) \leq &b_{2k}(G^\sigma - e) + b_{2k-2}(G^\sigma - u - v) + 2b_{2k-x}(G^\sigma - C_x^+)\\
<&b_{2k}(B_{4,4}^{+,+}(4, n-13)) + b_{2k-2}(P_{n-n_1-2} \cup (P_{n_1}^4)^+) + 2b_{2k-x}((P_{n_1}^4)^+ \cup P_{n-n_1-x})\\
\leq &b_{2k}(B_{4,4}^{+,+}(4, n-13)) + b_{2k-2}(P_2 \cup (P_{n-4}^4)^+) + 2b_{2k-4}((P_{n-4}^4)^+)\\
<&b_{2k}(B_{4,4}^{+,+}(4, n-13)) + b_{2k-2}(P_2 \cup B_{4,4}^{+,+}(0, n-13)) + 2b_{2k-4}(B_{4,4}^{+,+}(0, n-13))\\
=&b_{2k}(T_{4,4,4}^{+,+,+}).
\end{aligned}$$

Case 2. At least one endpoint of P on C_y is u. Suppose $e = uv$ is an edge of C_x. Clearly, $G^\sigma - e$ is a bicyclic oriented graph and $G^\sigma - e \notin \{B_{4,4}^{+,+}(0, n-9), B_{4,4}^{+,+}(n -$

$9,0), B_{4,4}^{+,+}(2,n-11), B_{4,4}^{+,+}(n-11,2)\}$ and $G^\sigma - u - v$ is also a forest. Then we have

$$\begin{aligned}
b_{2k}(G^\sigma) \leq & b_{2k}(G^\sigma - e) + b_{2k-2}(G^\sigma - u - v) + 2b_{2k-x}(G^\sigma - C_x^+) \\
< & b_{2k}(B_{4,4}^{+,+}(4,n-13)) + b_{2k-2}(P_2 \cup P_{n-4}) + 2b_{2k-4}(P_{n-4}) \\
\leq & b_{2k}(B_{4,4}^{+,+}(4,n-13)) + b_{2k-2}(P_2 \cup B_{4,4}^{+,+}(0,n-13)) + 2b_{2k-4}(B_{4,4}^{+,+}(0,n-13)) \\
= & b_{2k}(T_{4,4,4}^{+,+,+}).
\end{aligned}$$

This finishes the proof. $\square$

We finally focus on dealing with the set $\mathcal{T}_3(n)$. The following lemma is a fact concerning the orientation of a oriented graph.

Lemma 3.4 *[6] A graph has an orientation under which every cycle of even length is oddly oriented if and only if the graph contains no subgraph which is, after the contraction of an most one cycle of odd length an, even subdivision of $K_{2,3}$, where $K_{2,3}$ denotes the complete bipartite graph with two parts whose order are 2 and 3 respectively.*

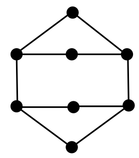

Figure 5: Graph G_1.

Lemma 3.5 *Let G^σ be a tricyclic oriented graph and $G \in \mathcal{T}_3(n), n \geq 13$. If there exists an edge on some cycle in at most three oddly oriented cycles, among which at most one cycle is of length 4, then $G^\sigma \prec T_{4,4,4}^{+,+,+}$.*

Proof. Let $e = uv$ be a such edge and u be an intersection vertex of some two cycles. Assume that e is in three oddly oriented cycles C_x^+, C_y^+ and C_z^+ with $|C_x^+| \geq 4$, $|C_y^+| \geq 6$ and $|C_z^+| \geq 6$(the case e in less than three oddly oriented cycles is easier and obvious from our proof). Without loss of generality, set $|C_z| \geq |C_y| \geq |C_x|$. We first show the following claim.

Claim 3.1 $(P_n^4)^+ \prec B_{4,4}^{+,+}(0,n-11) \cup P_2$.

The proof of Claim 3.1. From Lemma 2.2 it follows that

$$\begin{aligned}
&b_{2k}(B_{4,4}^{+,+}(0,n-11)\cup P_2)-b_{2k}((P_n^4)^+)\\
=&b_{2k-2}((P_{n-6}^4)^+\cup P_2\cup P_2)-b_{2k-2}((P_{n-3}^4)^+\cup P_1)+2b_{2k-4}((P_{n-6}^4)^+\cup P_2)\\
\geq&b_{2k-2}((P_{n-6}^4)^+\cup P_2\cup 2P_1)-b_{2k-2}((P_{n-6}^4)^+\cup P_3\cup P_1)+2b_{2k-4}((P_{n-6}^4)^+\cup P_2)\\
=&-b_{2k-4}((P_{n-6}^4)^+\cup 2P_1)+2b_{2k-4}((P_{n-6}^4)^+\cup P_2)\\
\geq&0.
\end{aligned}$$

It is obvious that there exists at least one index k such that the above inequality is strict and this finishes the proof of Claim 3.1.

Set $G_i^\sigma=G^\sigma-C_i$, where $i\in\{x,y,z\}$. We now discuss the following cases based on the fact that at most one graph in $\{G_x^\sigma,G_y^\sigma,G_z^\sigma\}$ contains at most a cycle.

Case 1. G_x^σ, G_y^σ and G_z^σ are all acyclic. Note that $G^\sigma-uv\neq B_{4,4}^{+,+}(0,n-9),B_{4,4}^{+,+}(n-9,0),B_{4,4}^{+,+}(2,n-11),B_{4,4}^{+,+}(n-11,2)$. Then by Lemma 2.2, Claim 3.1 and Theorem 2.1 we have

$$\begin{aligned}
b_{2k}(G^\sigma)=&b_{2k}(G^\sigma-e)+b_{2k-2}(G^\sigma-u-v)+2b_{2k-x}(G^\sigma-C_x^+)+2b_{2k-y}(G^\sigma-C_y^+)\\
&+2b_{2k-z}(G^\sigma-C_z^+)\\
<&b_{2k}(B_{4,4}^{+,+}(4,n-13))+b_{2k-2}((P_{n-2}^4)^+)+2b_{2k-4}(P_{n-4})+4b_{2k-6}(P_{n-6})\\
\leq&b_{2k}(B_{4,4}^{+,+}(4,n-13))+b_{2k-2}(B_{4,4}^{+,+}(0,n-13)\cup P_2)\\
&+2b_{2k-4}(P_{n-4})+4b_{2k-6}(P_{n-6}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&b_{2k}(T_{4,4,4}^{+,+,+})-b_{2k}(G^\sigma)\\
>&2b_{2k-4}(B_{4,4}^{+,+}(0,n-13))-2b_{2k-4}(P_{n-4})-4b_{2k-6}(P_{n-8}\cup P_2)-4b_{2k-8}(P_{n-9}\cup P_1)\\
\geq&2b_{2k-4}(B_{4,4}^{+,+}(0,n-13))-2b_{2k-4}(P_{n-4})-4b_{2k-6}(P_{n-8}\cup P_2)-4b_{2k-8}(P_{n-8})\\
\geq&0.
\end{aligned}$$

Case 2. One graph in $\{G_x^\sigma,G_y^\sigma,G_z^\sigma\}$ contains a cycle. Recall that $|C_z|\geq|C_y|\geq 6$ and $|C_y|\geq|C_x|\geq 4$. The following subcases are considered.

Subcase 2.1. $|C_x^+|\geq 4$, $|C_y^+|\geq 6$ and $|C_z^+|\geq 8$. Notice that $G^\sigma-e\neq B_{4,4}^{+,+}(0,n-9),B_{4,4}^{+,+}(n-9,0),B_{4,4}^{+,+}(2,n-11),B_{4,4}^{+,+}(n-11,2)$. Then by Lemma 2.2, Claim 3.1 and Corollary 3.1 we have

$$
\begin{aligned}
b_{2k}(G^\sigma) =& b_{2k}(G^\sigma - e) + b_{2k-2}(G^\sigma - u - v) + 2b_{2k-x}(G^\sigma - C_x^+) + 2b_{2k-y}(G^\sigma - C_y^+) \\
& + 2b_{2k-z}(G^\sigma - C_z^+) \\
<& b_{2k}(B_{4,4}^{+,+}(4, n-13)) + b_{2k-2}((P_{n-2}^4)^+) + 2b_{2k-4}((P_{n-4}^4)^+) \\
& + 2b_{2k-6}(P_{n-6}) + 2b_{2k-8}(P_{n-8}) \\
\leq& b_{2k}(B_{4,4}^{+,+}(4, n-13)) + b_{2k-2}(B_{4,4}^{+,+}(0, n-13) \cup P_2) + 2b_{2k-4}((P_{n-4}^4)^+) \\
& + 2b_{2k-6}(P_{n-6}) + 2b_{2k-8}(P_{n-8}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& b_{2k}(T_{4,4,4}^{+,+,+}) - b_{2k}(G^\sigma) \\
>& 2b_{2k-4}(B_{4,4}^{+,+}(0, n-13)) - 2b_{2k-4}((P_{n-4}^4)^+) - 2b_{2k-6}(P_{n-6}) - 2b_{2k-8}(P_{n-8}) \\
\geq& 2b_{2k-4}(B_{4,4}^{+,+}(0, n-13)) - 2b_{2k-4}(P_{n-4}) - 4b_{2k-6}(P_{n-8} \cup P_2) \\
& - 4b_{2k-8}(P_{n-8}) - 2b_{2k-8}(P_{n-9} \cup P_1) - 2b_{2k-8}(P_{n-8}) \\
\geq& 0.
\end{aligned}
$$

Subcase 2.2. $|C_x^+| = 6$, $|C_y^+| = 6$ and $|C_z^+| = 6$. By a similar technique with Subcase 2.1, it is not hard to verify this case whose proof is omitted here.

Subcase 2.3. $|C_x^+| = 4$, $|C_y^+| = 6$ and $|C_z^+| = 6$. Then G^σ must contain G_1 as a subgraph (see Fig. 5). We can observe that there is an even subdivision of $K_{2,3}$ in G_1. Thus by lemma 3.4, there exists at least one evenly oriented cycle. We choose an new edge $u'v'$ on this cycle such that u' is a intersection vetex with other cycles and $u'v'$ is in at most two oddly oriented cycles. With the similar discussion of Subcase 2.1, we can get the result easily. □

Lemma 3.6 *Let G^σ be a tricyclic oriented graph and $G \in \mathcal{T}_3(n), n \geq 13$. If G contains K_4 or $K_{2,4}$ as a subgraph, then we have $G^\sigma \prec T_{4,4,4}^{+,+,+}$.*

Proof. Suppose G contains $K_{2,4}$ as a subgraph. By Lemma 3.4, at least one cycle of $K_{2,3}$ is evenly oriented. If there exists a subgraph $K_{2,3}$ of G^σ with all three cycles being evenly oriented, then each edge of the subgraph $K_{2,3}$ is in at most one oddly oriented cycle of G^σ. By Lemma 3.5, we are done. Now suppose that each subgraph $K_{2,3}$ of G^σ has exactly one cycle with evenly oriented. Then each edge of $K_{2,4}$ is in exactly two oddly oriented cycles and an evenly oriented cycle. Since $n \geq 13$, we can choose the edge $e = uv$ of $K_{2,4}$ such that $G^\sigma - u - v$ is disconnected and acyclic. Clearly, $G^\sigma - e \neq B_{4,4}^{+,+}(0, n-9), B_{4,4}^{+,+}(n-9, 0), B_{4,4}^{+,+}(2, n-11), B_{4,4}^{+,+}(n-11, 2)$.

If G contains K_4 as a subgraph, it is easy to check that each edge is in at most two oddly oriented cycles, we can also choose the edge $e = uv$ satisfying that $G^\sigma - u - v$ is disconnected and acyclic. Clearly, $G^\sigma - e \neq B_{4,4}^{+,+}(0, n-9), B_{4,4}^{+,+}(n-9, 0), B_{4,4}^{+,+}(2, n-11), B_{4,4}^{+,+}(n-11, 2)$. Therefore, for the above both situations, we have

$$
\begin{aligned}
b_{2k}(G^\sigma) =& b_{2k}(G^\sigma - e) + b_{2k-2}(G^\sigma - u - v) + 4b_{2k-4}(G^\sigma - C_4^+) \\
\leq & b_{2k}(G^\sigma - e) + m(P_{n-4} \cup P_2, k-1) + 4m(P_{n-4}, k-2) \\
\leq & b_{2k}(G^\sigma - e) + m(P_{n-4}, k-1) + 5m(P_{n-4}, k-2) \\
< & b_{2k}(B_{4,4}^{+,+}(4, n-13)) + m(P_{n-4}^4, k-1) + m(P_{n-8}^4, k-2) \\
& + 3m(P_{n-4}^4, k-2) + 8m(P_{n-8}^4, k-3) + 4m(P_{n-12}, k-5) \\
& + 15m(P_{n-8}^4, k-4) + 12m(P_{n-12}, k-6) \\
= & b_{2k}(T_{4,4,4}^{+,+,+}).
\end{aligned}
$$

This completes the proof. $\square$

Lemma 3.7 *Let G^σ be a tricyclic oriented graph and $G \in \mathcal{T}_3(n), n \geq 13$. If G contains an odd cycle, then we have $G^\sigma \prec T_{4,4,4}^{+,+,+}$.*

Proof. Let $C_x \in G^\sigma$ be an odd cycle with vertices u and v, where the two paths on the cycle connecting u and u are denoted by P_a and P_b respectively. Since $G \in \mathcal{T}_3(n)$, then there is another path P_c connecting u and v. Since C_x is an odd cycle, we can assume $|P_a| \equiv |P_c|$ $(mod\ 2)$. Notice that the cycle formed by P_c and P_b is also an odd cycle. In order that G is tricyclic, there must be another internal disjoint path P_d connecting two vertices u', v', where u' and v' belong to P_c or C_x. Let $u_1 \in P_b$ be adjacent to u. We finish the proof by dividing it into four cases:

Case 1. $u', v' \in P_a$ (or P_c). It is easy to check that uu_1 is in at most one even cycle. By Lemma 3.5, $G^\sigma \prec T_{4,4,4}^{+,+,+}$.

Case 2. $u', v' \in P_b$. The edge $u'u_1'$, where u_1' is on the path from u' to v' of P_b, is in at most one even cycle. By Lemma 3.5, $G^\sigma \prec T_{4,4,4}^{+,+,+}$.

Case 3. $u' \in P_a$ (or P_c), $v' \in P_b$. For briefly, denote by P_s (resp. P_t) the path from u' through u (resp. v) to v' on P_b and P_c. Then $|P_s| \equiv |P_t|$ $(mod\ 2)$ and one of the cycles, formed by P_d and P_s, P_t respectively, must be of odd length. Without loss of generality, suppose the cycle formed by P_d and P_s is odd, then the edge uu_1 is in at most one even cycle. By Lemma 3.5, $G^\sigma \prec T_{4,4,4}^{+,+,+}$.

Case 4. $u' \in P_a$, $v' \in P_c$. Note that the edge uu_1 is in at most two even cycles. If the new formed cycles containing uu_1 are both of length 4, it is easy to check that

G contains K_4 as subgraph. Then by Lemma 3.6, we are done. If at least one cycle has length more than 4, by Lemma 3.5 we are done. $\square$

Lemma 3.8 *Let G^σ be a tricyclic oriented graph and $G \in \mathcal{T}_3(n), n \geq 13$. Then $G^\sigma \prec T_{4,4,4}^{+,+,+}$.*

Proof. If G contains odd cycles, by Lemma 3.7 we are done. We now consider the case that G contains no odd cycles. We first show the following claim.

Claim 3.2 *There exists an cycle C of G^σ such that each edge of this cycle is in at most three oddly oriented cycles.*

The proof of Claim 3.2. It is obvious for the graphs of type 1,2 and 3 (see Fig. 4). Now let G^σ has the type 4 (see Fig. 4), where a, b, c, d, e, f denote the lengths of the paths respectively. Then each edge of G^σ is in at most 4 cycles. Since G has no odd cycle, then one of $\{a, b, c\}$ must be even. Let $a \equiv 0$ $(mod\ 2)$. Then $a \equiv b + c \equiv d + e \equiv 0$ $(mod\ 2)$. Thus G^σ contains an even subdivision of $K_{2,3}$ as a subgraph. By Lemma 3.4, at least one cycle of G^σ is evenly oriented. Therefore, each edge of this evenly oriented cycle is in at most three oddly oriented cycles. This finishes the proof of Claim 3.2.

Let C be such a cycle in Claim 3.2. If $|C| \geq 6$, it is easy to check there exists one edge of C satisfying the condition of Lemma 3.5. Thus we are done. Now assume that $|C| = 4$, and each edge of C is in at least two oddly oriented 4-cycles. Recall that G^σ has no odd cycle, then G^σ must contain $K_{2,3}$ as a subgraph, set $V(K_{2,3}) = \{u_1, u_2, v_1, v_2, v_3\}$. By Lemma 3.4, suppose that the cycle $C' = u_1v_1u_2v_3u_1$ is evenly oriented. If $C = C'$, then G contains $K_{2,4}$ as a subgraph. By Lemma 3.6, we are done. Hence assume that $C \neq C'$ and $K_{2,4} \nsubseteq G^\sigma$. Without loss of generality, let $\{u_1v_1, u_2v_1\} \subset C$. In order to make sure that u_1v_1, u_2v_1 is in at least two oddly oriented 4-cycles, there is an 2-length internal path $P = v_1wv_2$ connecting v_1 and v_2. Note that the subgraph induced by $\{u_1, u_2, v_1, v_2, w\}$ is also a $K_{2,3}$, thus by Lemma 3.4, one of $\{u_1v_1, u_2v_1\}$ is in at most one oddly oriented 4-cycle only. By Lemma 3.5, we are done. Consequently, we have $G^\sigma \prec T_{4,4,4}^{+,+,+}$. $\square$

Combining Lemmas 3.2,3.3 and 3.8, we get the main result of this paper.

Theorem 3.1 *Among all oriented tricyclic oriented graphs with order $n \geq 13$, the graph $T_{4,4,4}^{+,+,+}$ has the maximal skew energy.*

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