The Laplacian energy and Laplacian Estrada index of random multipartite graphs*

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Abstract

Let G be a simple connected graph on n vertices and m edges and $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Laplacian matrix of G. The Laplacian energy of G is defined as $\mathcal{E}_L(G) = \sum_{i=1}^{n} |\mu_i - 2m/n|$ and the Laplacian Estrada index of G is defined as $LEE(G) = \sum_{i=1}^{n} e^{\mu_i - 2m/n}$. In this paper, we establish asymptotic lower and upper bounds to the Laplacian energy and Laplacian Estrada index, respectively, for random multipartite graphs.

Keywords: Random multipartite graph; Laplacian energy ; Laplacian Estrada index

Mathematics Subject Classification: 05C50, 15A18

1 Introduction

Let G be a simple undirected graph with vertex set $V_G = \{v_1, v_2, \ldots, v_n\}$ and edge set E_G. The *adjacency matrix* $A(G)$ of G is the symmetric matrix $[A_{ij}]$, where $A_{ij} = A_{ji} = 1$ if vertices v_i and v_j are adjacent, otherwise $A_{ij} = A_{ji} = 0$. The number of edges incident to the vertex $v_i \in V_G$ is the *degree* of v_i, denoted by $d_G(v_i)$. Denote by $d_G = \Sigma_{v_i \in V_G} d_G(v_i)$ the *degree sum* of G. The *Laplacian matrix* of G is the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the *degree matrix*, which is a diagonal matrix with the diagonal entries the degrees of G.

The *eigenvalues* of a graph G are the eigenvalues of its adjacency matrix $A(G)$. As usual, we denote them by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, or simply $\lambda_1, \lambda_2, \ldots, \lambda_n$. The *energy*

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of G was first defined by Gutman [12] in 1978 as

$$\mathcal{E}(G)=\sum_{i=1}^{n}|\lambda_i|,$$

which is derived from the total π-electron energy [23] from chemistry. Since then, graph energy has been studied extensively by lots of mathematicians and chemists. For results on the study of the energy of graphs, we refer the reader to the book [16] and new book [13].

In 2006, Gutman *et al.* [14] introduced a new matrix $\overline{L}(G)$ for a graph G, *i.e.*,

$$\overline{L}(G):=L(G)-\sum_{i=1}^{n}\frac{d_G(v_i)}{n}I_n=L(G)-2\sum_{i=1}^{n}\sum_{i>j}\frac{A_{ij}}{n}I_n,$$

where I_n is the identity matrix of order n. Based on $\overline{L}(G)$, they defined the *Laplacian energy* of G as

$$\mathcal{E}_L(G)=\sum_{i=1}^{n}|\mu_i-2m/n|=\sum_{i=1}^{n}|\xi_i|, \tag{1.1}$$

where m is the number of edges of G, $\mu_1,\mu_2,\ldots,\mu_n$ are the eigenvalues of $L(G)$ and $\xi_1,\xi_2,\ldots,\xi_n$ are the eigenvalues of $\overline{L}(G)$. Obviously, Laplacian energy can be regarded as a variant of graph energy. Up until now, a lot of results have been obtained on Laplacian energy. The reader can be referred to [3, 4, 5, 11, 18, 19, 21, 26, 27, 28, 29].

In 2009, Fath-Tabar *et al.* [10] first proposed the *Laplacian Estrada index* of graphs. For a graph G, its Laplacian Estrada index is defined as

$$LEE_1(G)=\sum_{i=1}^{n}e^{\mu_i}.$$

Independently, also in 2009, Li *et al.* [17] defined the *Laplacian Estrada index* as

$$LEE_2(G)=\sum_{i=1}^{n}e^{\mu_i-2m/n}=\sum_{i=1}^{n}e^{\xi_i}. \tag{1.2}$$

Clearly, $LEE_1(G)=e^{2m/n}LEE_2(G)$. Thus, these two definitions of the Laplacian Estrada index are essentially equivalent. In this paper, we adopt the Definition (1.2) and denote $LEE_2(G)$ simply by $LEE(G)$ for convenience. For more properties of this index, we refer the reader to [1, 6, 10, 15, 17, 24, 25].

In 1950s, Erdős and Rényi [8] founded the theory of random graphs. The Erdős-Rényi random graph $\mathcal{G}_n(p)$ consists of all graphs on n vertices in which the edges are chosen independently with probability p, where $0<p<1$. In [7], Du *et al.* have considered the Laplacian energy of the Erdős-Rényi model $\mathcal{G}_n(p)$. They obtained a lower bound and a upper bound of the Laplacian energy of $\mathcal{G}_n(p)$, and showed that for almost all $G_n(p)\in\mathcal{G}_n(p)$, $\mathcal{E}(G_n(p))$ is no more than $\mathcal{E}_L(G_n(p))$.

The purpose of this paper is to study the Laplacian energy and Laplacian Estrada index of random multipartite graphs. We use $K_{n;\beta_1,\ldots,\beta_k}$ to denote the complete k-partite graph with vertex set V ($|V| = n$), whose parts are $V_1, \ldots, V_k$ ($2 \le k = k(n) \le n$) satisfying $|V_i| = n\beta_i = n\beta_i(n)$, $i = 1, 2, \ldots, k$. The random k-partite graphs $\mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$ consist of all random k-partite graphs in which the edges are chosen independently with probability p from the set of edges of $K_{n;\beta_1,\ldots,\beta_k}$. We denote by $A_{n,k} := A(G_{n;\beta_1,\ldots,\beta_k}(p)) = (x_{ij})_{n\times n}$ the adjacency matrix of random k-partite graphs $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$, where x_{ij} is a random indicator variable for $\{v_i, v_j\}$ being an edge with probability p, for $i \in V_l$ and $j \in V\backslash V_l$, $i \neq j$, $1 \le l \le k$. Then $A_{n,k}$ satisfies the following properties:

- x_{ij}'s, $1 \le i < j \le n$, are independent random variables with $x_{ij} = x_{ji}$;
- $Pr(x_{ij} = 1) = 1 - Pr(x_{ij} = 0) = p$ if $i \in V_l$ and $j \in V\backslash V_l$, while $Pr(x_{ij} = 0) = 1$ if $i \in V_l$ and $j \in V_l$, $1 \le l \le k$.

Note that when $k = n$, $\mathcal{G}_{n;\beta_1,\ldots,\beta_k} = \mathcal{G}_n(p)$, that is, the random multipartite graphs can be viewed as a generalization to the Erdős-Rényi model.

The paper is structured as follows. In Section 2, we consider the Laplacian energy of the random k-partite graph model $\mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$, and establish a lower bound and an upper bound to $\mathcal{E}_L(G_{n;\beta_1,\ldots,\beta_k}(p))$ for almost all $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$. In Section 3, we establish a lower bound and an upper bound to $LEE(G_{n;\beta_1,\ldots,\beta_k}(p))$ for almost all $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$. As a corollary, we obtain the Laplacian Estrada index for almost all $G_n(p) \in \mathcal{G}_n(p)$.

2 Laplacian energy of random multipartite graphs

In this section, we shall formulate a lower bound and an upper bound to the Laplacian energy for random multipartite graphs $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$. Before proceeding, we give some definitions and lemmas.

Let M be a real symmetric matrix. Denote by $\mathcal{E}(M)$ the sum of the absolute values of the eigenvalues of M. Sometimes, $\mathcal{E}(M)$ is called the *energy* of M.

Lemma 1 (Fan [9]). *Let X, Y, Z be real symmetric matrices of order n such that $X+Y = Z$. Then*

$$\mathcal{E}(X) + \mathcal{E}(Y) \ge \mathcal{E}(Z).$$

We say that an event in a probability space holds asymptotically *almost surely* (*a.s.* for short) if its probability goes to one as n tends to infinity.

Lemma 2 (Shiryaev [20]). *Let $X_1, X_2, \ldots$ be an infinite sequence of independent identically distributed (i.i.d.) random variables with expected value $E(X_1) = E(X_2) = \cdots = \mu$, and $E|X_j| < \infty$. Then*

$$\overline{X}_n := \frac{1}{n}(X_1 + X_2 + \cdots + X_n) \to \mu \quad a.s.$$

Let $f(n), g(n)$ be two functions of n. Then $f(n) = o(g(n))$ means that $f(n)/g(n) \to 0$, as $n \to \infty$; $f(n) = O(g(n))$ means that there exists a constant C such that $|f(n)| \leq Cg(n)$, as $n \to \infty$.

Lemma 3 (Du et al.[7]). *Almost every random graph $G_n(p)$ satisfies*

$$\left(\frac{2\sqrt{2}}{3}\sqrt{p(1-p)} + o(1)\right) n^{3/2} \leq \mathcal{E}_L(G_n(p)) \leq \left(\sqrt{2p - p^2} + o(1)\right) n^{3/2}.$$

Theorem 1. *Let $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$ with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k$ and r $(1 \leq r \leq k-1)$ be an integer such that $\beta_{r+1} \leq \sum_{l=1}^{k} \beta_l^2 \leq \beta_r$. Then almost surely*

$$\begin{aligned}
& 2(p+o(1))n^2 \left(\sum_{l=1}^{r} \beta_l^2 - \beta_r \sum_{l=1}^{r} \beta_l\right) - \left(\sqrt{2p-p^2} + \frac{2\sqrt{2p(1-p)}}{3}\sum_{i=1}^{k} \beta_i^{3/2} + o(1)\right) n^{3/2} \\
\leq\ & \mathcal{E}_L(G_{n;\beta_1,\ldots,\beta_k}(p)) \\
\leq\ & 2(p+o(1))n^2 \left(\sum_{l=1}^{r} \beta_l^2 - \beta_{r+1} \sum_{l=1}^{r} \beta_l\right) + \left(\sqrt{2p-p^2} + \frac{2\sqrt{2p(1-p)}}{3}\sum_{i=1}^{k} \beta_i^{3/2} + o(1)\right) n^{3/2}.
\end{aligned}$$

Proof. Note that the parts $V_1, \ldots, V_k$ of random k-partite graph $G_{n;\beta_1,\ldots,\beta_k}(p)$ satisfy $|V_i| = n\beta_i, \ i = 1, 2, \ldots, k$. Then the adjacency matrix $A_{n,k}$ of $G_{n;\beta_1,\ldots,\beta_k}(p)$ satisfies

$$A_{n,k} + A'_{n,k} = A_n,$$

where

$$A'_{n,k} = \begin{pmatrix} A_{n\beta_1} & & & \\ & A_{n\beta_2} & & \\ & & \ddots & \\ & & & A_{n\beta_k} \end{pmatrix}_{n\times n,}$$

and $A_n := A(G_n(p)), \ A_{n\beta_i} := A(G_{n\beta_i}(p)), \ i = 1, 2, \ldots, k$.

The degree matrix $D_{n,k} := D(G_{n;\beta_1,\ldots,\beta_k}(p))$ of $G_{n;\beta_1,\ldots,\beta_k}(p)$ satisfies

$$D_{n,k} + D'_{n,k} = D_n,$$

where

$$D'_{n,k} = \begin{pmatrix} D_{n\beta_1} & & & \\ & D_{n\beta_2} & & \\ & & \ddots & \\ & & & D_{n\beta_k} \end{pmatrix}_{n\times n,}$$

and $D_n := D(G_n(p)),\ D_{n\beta_i} := D(G_{n\beta_i}(p)),\ i = 1, 2, \ldots, k.$

The Laplacian matrix $L_{n,k} := L(G_{n;\beta_1,\ldots,\beta_k}(p))$ of $G_{n;\beta_1,\ldots,\beta_k}(p)$ satisfies

$$L_{n,k} + L'_{n,k} = L_n,$$

where

$$L'_{n,k} = \begin{pmatrix} L_{n\beta_1} & & & \\ & L_{n\beta_2} & & \\ & & \ddots & \\ & & & L_{n\beta_k} \end{pmatrix}_{n\times n,}$$

and $L_n := L(G_n(p)),\ L_{n\beta_i} := L(G_{n\beta_i}(p)),\ i = 1, 2, \ldots, k.$

Note that $L_{n,k} = L_n - L'_{n,k}$, $A_{n,k} = A_n - A'_{n,k}$, and

$$\overline{L_n} = L_n - \sum_{i=1}^{n} \frac{d_{G_n(p)}(v_i)}{n} I_n = L_n - 2\sum_{i=1}^{n}\sum_{i>j} \frac{(A_n)_{ij}}{n} I_n.$$

Then

$$\begin{aligned} \overline{L_{n,k}} &= L_{n,k} - 2\sum_{i=1}^{n}\sum_{i>j} \frac{(A_{n,k})_{ij}}{n} I_n \\ &= L_n - L'_{n,k} - 2\sum_{i=1}^{n}\sum_{i>j} \frac{(A_n - A'_{n,k})_{ij}}{n} I_n \\ &= L_n - 2\sum_{i=1}^{n}\sum_{i>j} \frac{(A_n)_{ij}}{n} I_n - L'_{n,k} + \frac{2}{n}\sum_{l=1}^{k}\sum_{i=1}^{n\beta_l}\sum_{i>j} (A_{n\beta_l})_{ij} I_n \\ &= \overline{L_n} - B_n - C_n, \end{aligned} \tag{2.1}$$

where

$$B_n = \begin{pmatrix} \overline{L_{n\beta_1}} & & \\ & \ddots & \\ & & \overline{L_{n\beta_k}} \end{pmatrix}_{n\times n}$$

with

$$\overline{L_{n\beta_l}} = L_{n\beta_l} - 2\frac{\sum_{i=1}^{n\beta_l}\sum_{i>j}(A_{n\beta_l})_{ij}}{n\beta_l} I_{n\beta_l},\ \text{for } 1 \le l \le k,$$

and

$$C_n = \begin{pmatrix} C_{n\beta_1} & & \\ & \ddots & \\ & & C_{n\beta_k} \end{pmatrix}_{n\times n}$$

with

$$C_{n\beta_l} = \left(2\frac{\sum_{i=1}^{n\beta_l}\sum_{i>j}(A_{n\beta_l})_{ij}}{n\beta_l} - \frac{2}{n}\sum_{l=1}^{k}\sum_{i=1}^{n\beta_l}\sum_{i>j}(A_{n\beta_l})_{ij} \right) I_{n\beta_l},\ \text{for } 1 \le l \le k.$$

By (2.1) and Lemma 1, we have

$$|\mathcal{E}(\overline{L_n} - B_n) - \mathcal{E}(C_n)| \leq \mathcal{E}(\overline{L_{n,k}}) \leq \mathcal{E}(\overline{L_n}) + \mathcal{E}(B_n) + \mathcal{E}(C_n). \tag{2.2}$$

Note that

$$\mathcal{E}_L(G_n(p)) = \sum_{i=1}^{n} |\mu(L_n) - \frac{Tr(D_n)}{n}| = \sum_{i=1}^{n} |\xi_i(\overline{L_n})| = \mathcal{E}(\overline{L_n}),$$

and

$$\mathcal{E}_L(G_{n,k}(p)) = \sum_{i=1}^{n} |\mu_i(L_{n,k}) - \frac{Tr(D_{n,k})}{n}| = \sum_{i=1}^{n} |\xi_i(\overline{L_{n,k}})| = \mathcal{E}(\overline{L_{n,k}}).$$

Then

$$\mathcal{E}(B_n) = \mathcal{E}(\overline{L_{n\beta_1}}) + \cdots + \mathcal{E}(\overline{L_{n\beta_k}}) = \mathcal{E}_L(G_{n\beta_1}(p)) + \cdots + \mathcal{E}_L(G_{n\beta_k}(p)).$$

Thus, Lemma 3 implies that

$$\begin{aligned}
\mathcal{E}(\overline{L_n}) - \mathcal{E}(B_n) &= \mathcal{E}_L(G_n(p)) - [\mathcal{E}_L(G_{n\beta_1}(p)) + \cdots + \mathcal{E}_L(G_{n\beta_k}(p))] \\
&\geq \left(\frac{2\sqrt{2}}{3}\sqrt{p(1-p)} + o(1)\right) n^{3/2} - \left(\sqrt{2p-p^2} + o(1)\right) n^{3/2} \sum_{i=1}^{k} \beta_i^{3/2} \\
&= \left(\frac{2\sqrt{2}}{3}\sqrt{p(1-p)} - \sqrt{2p-p^2}\sum_{i=1}^{k} \beta_i^{3/2} + o(1)\right) n^{3/2} \quad a.s.,
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
\mathcal{E}(\overline{L_n}) + \mathcal{E}(B_n) &= \mathcal{E}_L(G_n(p)) + [\mathcal{E}_L(G_{n\beta_1}(p)) + \cdots + \mathcal{E}_L(G_{n\beta_k}(p))] \\
&\leq \left(\sqrt{2p-p^2} + o(1)\right) n^{3/2} + \left(\frac{2\sqrt{2}}{3}\sqrt{p(1-p)} + o(1)\right) n^{3/2} \sum_{i=1}^{k} \beta_i^{3/2} \\
&= \left(\sqrt{2p-p^2} + \frac{2\sqrt{2}}{3}\sqrt{p(1-p)}\sum_{i=1}^{k} \beta_i^{3/2} + o(1)\right) n^{3/2} \quad a.s.
\end{aligned} \tag{2.4}$$

By Lemma 1, we have

$$\mathcal{E}(\overline{L_n}) - \mathcal{E}(B_n) \leq \mathcal{E}(\overline{L_n} - B_n) \leq \mathcal{E}(\overline{L_n}) + \mathcal{E}(B_n). \tag{2.5}$$

Next, by estimating $\mathcal{E}(C_n)$, we compare $\mathcal{E}(\overline{L_n} - B_n)$ and $\mathcal{E}(C_n)$. Since $(A_n)_{ij} (i > j)$ are $i.i.d.$ with mean p and variance $\sqrt{p(1-p)}$. It follows from Lemma 2 that, with probability 1,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}\sum_{i>j}(A_n)_{ij}}{\frac{n(n-1)}{2}} = p.$$

Thus, we have

$$\sum_{i=1}^{n}\sum_{i>j}(A_n)_{ij} = (p/2 + o(1))n^2 \quad a.s.$$

Similarly, for $l = 1, 2, \dots, k$,

$$\sum_{i=1}^{n\beta_l} \sum_{i>j} (A_{n\beta_l})_{ij} = (p/2 + o(1)) n^2 \beta_l^2 \quad a.s.$$

Since $\beta_1 \geq \cdots \geq \beta_k$ and $\beta_{r+1} \leq \sum_{l=1}^{k} \beta_l^2 \leq \beta_r$, we have

$$\begin{aligned}
\mathcal{E}(C_n) &= \sum_{l=1}^{k} \left| 2\frac{\sum_{i=1}^{n\beta_l} \sum_{i>j} (A_{n\beta_l})_{ij}}{n\beta_l} - \frac{2}{n} \sum_{l=1}^{k} \sum_{i=1}^{n\beta_l} \sum_{i>j} (A_{n\beta_l})_{ij} \right| \cdot n\beta_l \\
&= \sum_{l=1}^{k} \left| (p + o(1)) n\beta_l - (p + o(1)) n \sum_{i=1}^{k} \beta_i^2 \right| \cdot n\beta_l \\
&= (p + o(1)) n^2 \sum_{l=1}^{k} \left| \beta_l - \sum_{i=1}^{k} \beta_i^2 \right| \cdot \beta_l \\
&= 2(p + o(1)) n^2 \left(\sum_{l=1}^{r} \beta_l^2 - \sum_{l=1}^{k} \beta_l^2 \cdot \sum_{l=1}^{r} \beta_l \right) \quad a.s.
\end{aligned}$$

Note that

$$\sum_{l=1}^{r} \beta_l^2 - \sum_{l=1}^{k} \beta_l^2 \cdot \sum_{l=1}^{r} \beta_l \geq \sum_{l=1}^{r} \beta_l^2 - \beta_r \cdot \sum_{l=1}^{r} \beta_l \geq 0.$$

Hence

$$\mathcal{E}(C_n) \geq \mathcal{E}(\overline{L_n} - B_n). \tag{2.6}$$

Since $\beta_{r+1} \leq \sum_{l=1}^{k} \beta_l^2 \leq \beta_r$, we have

$$\begin{aligned}
2(p + o(1)) n^2 \left(\sum_{l=1}^{r} \beta_l^2 - \beta_r \sum_{l=1}^{r} \beta_l \right) &\leq \mathcal{E}(C_n) \\
&\leq 2(p + o(1)) n^2 \left(\sum_{l=1}^{r} \beta_l^2 - \beta_{r+1} \sum_{l=1}^{r} \beta_l \right). \qquad (2.7)
\end{aligned}$$

By (2.2), (2.5) and (2.6), we have

$$\begin{aligned}
\mathcal{E}(C_n) - (\mathcal{E}(\overline{L_n}) + \mathcal{E}(B_n)) &\leq \mathcal{E}(C_n) - \mathcal{E}(\overline{L_n} - B_n) \\
&\leq \mathcal{E}(\overline{L_{n,k}}) \\
&\leq \mathcal{E}(C_n) + \mathcal{E}(\overline{L_n}) + \mathcal{E}(B_n).
\end{aligned}$$

Then by (2.4) and (2.7), we have

$$\begin{aligned}
&2(p + o(1)) n^2 \left(\sum_{l=1}^{r} \beta_l^2 - \beta_r \sum_{l=1}^{r} \beta_l \right) - \left(\sqrt{2p - p^2} + \frac{2\sqrt{2p(1-p)}}{3} \sum_{i=1}^{k} \beta_i^{3/2} + o(1) \right) n^{3/2} \\
\leq\ & \mathcal{E}(\overline{L_{n,k}}) \\
\leq\ & 2(p + o(1)) n^2 \left(\sum_{l=1}^{r} \beta_l^2 - \beta_{r+1} \sum_{l=1}^{r} \beta_l \right) \\
&+ \left(\sqrt{2p - p^2} + \frac{2\sqrt{2p(1-p)}}{3} \sum_{i=1}^{k} \beta_i^{3/2} + o(1) \right) n^{3/2} \quad a.s.
\end{aligned}$$

This completes the proof. □

Next, we consider the case when each part of $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$ has the same size as n tends to infinity.

Theorem 2. *Let $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$ satisfying $\lim_{n\to\infty} \frac{\beta_i}{\beta_j} = 1,\ 1 \le i,j \le k$. Then almost surely*

$$\begin{aligned}
&\left(\frac{2\sqrt{2p(1-p)}}{3} - \sqrt{\frac{2p-p^2}{k}} + o(1)\right) n^{3/2}\\
\le\ & \mathcal{E}_L(G_{n;\beta_1,\ldots,\beta_k}(p))\\
\le\ & \left(\sqrt{2p-p^2} + \frac{2}{3}\sqrt{\frac{2p(1-p)}{k}} + o(1)\right) n^{3/2}.
\end{aligned}$$

Proof. Note that $\lim_{n\to\infty} \frac{\beta_i}{\beta_j} = 1,$ for $1 \le i,j \le k$. Then for $l,t = 1,\cdots,k$, we have

$$\frac{\sum_{i=1}^{n\beta_l}\sum_{i>j}(A_{n\beta_l})_{ij}}{n\beta_l} = \frac{\sum_{i=1}^{n\beta_t}\sum_{i>j}(A_{n\beta_t})_{ij}}{n\beta_t} = \frac{\sum_{l=1}^{k}\sum_{i=1}^{n\beta_l}\sum_{i>j}(A_{n\beta_l})_{ij}}{n} \quad a.s.$$

Then

$$C_n = 0 \quad a.s.$$

So, by (2.1), we have

$$\overline{L_{n,k}} = \overline{L_n} - B_n \quad a.s.$$

According to Lemma 1, we have

$$\mathcal{E}(\overline{L_n}) - \mathcal{E}(B_n) \le \mathcal{E}(\overline{L_{n,k}}) \le \mathcal{E}(\overline{L_n}) + \mathcal{E}(B_n). \tag{2.8}$$

Note that $\lim_{n\to\infty} \frac{\beta_i}{\beta_j} = 1$ implies that $\lim_{n\to\infty} \beta_i = \frac{1}{k}$, for $1 \le i \le k$. From (2.3) and (2.4), we have

$$\begin{aligned}
\mathcal{E}(\overline{L_n}) - \mathcal{E}(B_n) &\ge \left(\frac{2\sqrt{2}}{3}\sqrt{p(1-p)} - \sqrt{2p-p^2}\sum_{i=1}^{k}\beta_i^{3/2} + o(1)\right) n^{3/2}\\
&= \left(\frac{2\sqrt{2}}{3}\sqrt{p(1-p)} - \sqrt{\frac{2p-p^2}{k}} + o(1)\right) n^{3/2} \quad a.s.,
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\mathcal{E}(\overline{L_n}) + \mathcal{E}(B_n) &\le \left(\sqrt{2p-p^2} + \frac{2\sqrt{2}}{3}\sqrt{p(1-p)}\sum_{i=1}^{k}\beta_i^{3/2} + o(1)\right) n^{3/2}\\
&= \left(\sqrt{2p-p^2} + \frac{2\sqrt{2}}{3}\sqrt{\frac{p(1-p)}{k}} + o(1)\right) n^{3/2} \quad a.s.
\end{aligned} \tag{2.10}$$

Then (2.8), (2.9) and (2.10) imply that

$$\begin{aligned}
&\left(\frac{2\sqrt{2p(1-p)}}{3} - \sqrt{\frac{2p-p^2}{k}} + o(1)\right) n^{3/2}\\
\le\ & \mathcal{E}_L(G_{n;\beta_1,\ldots,\beta_k}(p))\\
\le\ & \left(\sqrt{2p-p^2} + \frac{2}{3}\sqrt{\frac{2p(1-p)}{k}} + o(1)\right) n^{3/2}.
\end{aligned}$$

This completes the proof. $\square$

3 Laplacian Estrada index of random multipartite graphs

In this section, we will establish a lower bound and an upper bound to $LEE(G_{n;\beta_1,\ldots,\beta_k}(p))$ for almost all $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$. Recall that $A_{n,k}$, $L_{n,k}$ and $\overline{L_{n,k}}$ denote $A(G_{n;\beta_1,\ldots,\beta_k}(p))$, $L(G_{n;\beta_1,\ldots,\beta_k}(p))$ and $\overline{L}(G_{n;\beta_1,\ldots,\mu_k}(p))$, respectively.

Lemma 4 (Bryc *et al.* [2]). *Let X be a symmetric random matrix satisfying that the entries X_{ij}, $1 \le i < j$, are a collection of i.i.d. random variables with $E(X_{12}) = 0, Var(X_{12}) = 1$ and $E(X_{12}^4) < \infty$. Define $S := diag(\sum_{i\neq j} X_{ij})_{1\le i\le n}$ and let $M = S - X$, where $diag\{\cdot\}$ denotes diagonal matrix. Denote by $\parallel M \parallel$ the spectral radius of M. Then*

$$\lim_{n\to\infty} \frac{\parallel M \parallel}{\sqrt{2n\log n}} = 1 \quad a.s.,$$

i.e., with probability 1, $\frac{\|M\|}{\sqrt{2n\log n}}$ converges weakly to 1 as n tends to infinity.

Lemma 5 (Weyl [22]). *Let X, Y and Z be $n\times n$ Hermitian matrices such that $X = Y + Z$. Suppose that X, Y, Z have eigenvalues, respectively, $\lambda_1(X) \ge \cdots \ge \lambda_n(X)$, $\lambda_1(Y) \ge \cdots \ge \lambda_n(Y)$, $\lambda_1(Z) \ge \cdots \ge \lambda_n(Z)$. Then for $i = 1, 2, \ldots, n$ the following inequalities hold:*

$$\lambda_i(Y) + \lambda_n(Z) \le \lambda_i(X) \le \lambda_i(X) + \lambda_1(Z).$$

Theorem 3. *Let $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$. Then almost surely*

$$\begin{aligned}
& (n-1+e^{-np})e^{np(\sum_{i=1}^k \beta_i^2 - \max_{1\le i\le k}\{\beta_i\}) + o(1)n} \\
\le\ & LEE(G_{n;\beta_1,\ldots,\beta_k}(p)) \\
\le\ & (n-1+e^{-np})e^{np\sum_{i=1}^k \beta_i^2 + o(1)n}.
\end{aligned}$$

Proof. Define an auxiliary matrix

$$\widetilde{L_n} := L_n - p(n-1)I_n + p(J_n - I_n) = (D_n - p(n-1)I_n) - (A_n - p(J_n - I_n)),$$

where J_n is the all-ones matrix. Let

$$S = \frac{1}{\sqrt{p(1-p)}}[D_n - p(n-1)I_n]$$

and

$$X = \frac{1}{\sqrt{p(1-p)}}[A_n - p(J_n - I_n)].$$

Then $E(X_{12}) = 0, Var(X_{12}) = 1$, and

$$E(X_{12}^4) = \frac{1}{p^2(1-p)^2}(p - 4p^2 + 6p^3 - 3p^4) < \infty.$$

By Lemma 4, we have

$$\lim_{n\to\infty} \frac{\| \widetilde{L_n} \|}{\sqrt{2p(1-p)n\log n}} = 1 \quad a.s.$$

Then

$$\lim_{n\to\infty} \frac{\| \widetilde{L_n} \|}{n} = 0 \quad a.s.,$$

i.e.,

$$\| \widetilde{L_n} \| = o(1)n \quad a.s.$$

Let $R_n := p(n-1)I_n - p(J_n - I_n)$. Then $\widetilde{L_n} + R_n = L_n$. Suppose that $L_n, \widetilde{L_n}, R_n$ have eigenvalues, respectively, $\mu_1(L_n) \geq \cdots \geq \mu_n(L_n)$, $\lambda_1(\widetilde{L_n}) \geq \cdots \geq \lambda_n(\widetilde{L_n})$, $\lambda_1(R_n) \geq \cdots \geq \lambda_n(R_n)$. It follows from Lemma 5 that

$$\lambda_i(R_n) + \lambda_n(\widetilde{L_n}) \leq \mu_i(L_n) \leq \lambda_i(R_n) + \lambda_1(\widetilde{L_n}), \text{ for } i = 1, 2, \ldots, n.$$

Notice that $\lambda_i(R_n) = pn$ for $i = 1, 2, \ldots, n-1$ and $\lambda_n(R_n) = 0$. We have

$$\mu_i(L_n) = (p + o(1))n \;\; a.s., \text{ for } 1 \leq i \leq n-1 \tag{3.1}$$

and

$$\mu_n(L_n) = o(1)n \;\; a.s. \tag{3.2}$$

In the following, we first evaluate the eigenvalues of $L_{n,k}$ according to the spectral distribution of L_n and $L'_{n,k}$.

Since $L_{n,k} = L_n - L'_{n,k}$, Lemma 5 implies that for $1 \leq i \leq n$,

$$\mu_i(L_n) + \mu_n(-L'_{n,k}) \leq \mu_i(L_{n,k}) \leq \mu_i(L_n) + \mu_1(-L'_{n,k}), \tag{3.3}$$

where $\mu_n(-L'_{n,k})$ and $\mu_1(-L'_{n,k})$ are the minimum and maximum eigenvalues of $-L'_{n,k}$ respectively. By (3.1), (3.2) and (3.3), we have

$$np(1 - \max_{1\leq i\leq k}\{\beta_i\}) + o(1)n \leq \mu_i(L_{n,k}) \leq np + o(1)n \quad a.s., \tag{3.4}$$

and

$$-np \max_{1\leq i\leq k}\{\beta_i\} + o(1)n \leq \mu_n(L_{n,k}) \leq o(1)n \quad a.s. \tag{3.5}$$

Now we consider the trace $Tr(D_{n,k})$ of $D_{n,k}$. Note that $Tr(D_{n,k}) = 2\sum_{i>j}(A_{n,k})_{ij}$. Since that $(A_n)_{ij}(i > j)$ are *i.i.d.* with mean p and variance $\sqrt{p(1-p)}$. According to Lemma 2, we obtain that with probability 1,

$$\lim_{n\to\infty} \frac{\sum_{i>j}(A_n)_{ij}}{\frac{n(n-1)}{2}} = p,$$

i.e.,

$$\sum_{i>j}(A_n)_{ij} = (p/2 + o(1))n^2 \quad a.s.$$

Then

$$Tr(D_n) = (p + o(1))n^2 \quad a.s. \tag{3.6}$$

Similarly, for $i = 1, \ldots, k$,

$$Tr(D_{n\beta_i}) = (p + o(1))n^2\beta_i^2 \quad a.s.$$

Thus,

$$\begin{aligned}
Tr(D_{n,k}) &= 2\sum_{i>j}(A_{n,k})_{ij} = 2\sum_{i>j}(A_n - A'_{n,k})_{ij} \\
&= 2\sum_{i>j}(A_n)_{ij} - 2\sum_{i>j}(A'_{n,k})_{ij} \\
&= 2\sum_{n\geq i>j\geq 1}(A_n)_{ij} - 2\left[\sum_{n\beta_1\geq i>j\geq 1}(A_{n\beta_1})_{ij} + \cdots + \sum_{n\beta_k\geq i>j\geq 1}(A_{n\beta_k})_{ij}\right] \qquad (3.7)\\
&= (p + o(1))n^2 - [(p + o(1))(n\beta_1)^2 + \cdots + (p + o(1))(n\beta_k)^2] \\
&= p(1 - \sum_{i=1}^{k}\beta_i^2)n^2 + o(1)n^2 \quad a.s.
\end{aligned}$$

Note that $L_{n,k} - \frac{Tr(D_{n,k})}{n} = \overline{L_{n,k}}$. Then $\mu_i(L_{n,k}) - \frac{Tr(D_{n,k})}{n} = \xi_i(\overline{L_{n,k}})$, for $i = 1, \ldots, n$, where $\mu_i(L_{n,k})$, $\xi_i(\overline{L_{n,k}})$ are eigenvalues of $L_{n,k}$ and $\overline{L_{n,k}}$ respectively. By (3.4), (3.5) and (3.7), we have

$$np(\sum_{i=1}^{k}\beta_i^2 - \max_{1\leq i\leq k}\{\beta_i\}) + o(1)n \leq \xi_i(\overline{L_{n,k}}) \leq np\sum_{i=1}^{k}\beta_i^2 + o(1)n \quad a.s., \text{ for } 1 \leq i \leq n-1, \tag{3.8}$$

and

$$np(\sum_{i=1}^{k}\beta_i^2 - \max_{1\leq i\leq k}\{\beta_i\} - 1) + o(1)n \leq \xi_n(\overline{L_{n,k}}) \leq np(\sum_{i=1}^{k}\beta_i^2 - 1) + o(1)n \quad a.s. \tag{3.9}$$

Hence we have

$$(n-1)e^{np(\sum_{i=1}^{k}\beta_i^2 - \max_{1\leq i\leq k}\{\beta_i\}) + o(1)n} \leq \sum_{i=1}^{n-1}e^{\xi_i(\overline{L_{n,k}})} \leq (n-1)e^{np\sum_{i=1}^{k}\beta_i^2 + o(1)n} \quad a.s. \tag{3.10}$$

and

$$e^{np(\sum_{i=1}^{k}\beta_i^2 - \max_{1\leq i\leq k}\{\beta_i\} - 1) + o(1)n} \leq e^{\xi_n(\overline{L_{n,k}})} \leq e^{np(\sum_{i=1}^{k}\beta_i^2 - 1) + o(1)n} \quad a.s. \tag{3.11}$$

Then (3.10) and (3.11) imply that

$$\begin{aligned}
LEE(G_{n;\mu_1,\ldots,\mu_k}(p)) &\geq (n-1)e^{np(\sum_{i=1}^{k}\beta_i^2 - \max_{1\leq i\leq k}\{\beta_i\}) + o(1)n} \\
&\quad + e^{np(\sum_{i=1}^{k}\beta_i^2 - \max_{1\leq i\leq k}\{\beta_i\} - 1) + o(1)n} \qquad (3.12)\\
&= (n - 1 + e^{-np})e^{np(\sum_{i=1}^{k}\beta_i^2 - \max_{1\leq i\leq k}\{\beta_i\}) + o(1)n} \quad a.s.,
\end{aligned}$$

and

$$\begin{aligned} LEE(G_{n;\beta_1,\ldots,\beta_k}(p)) &\le (n-1)e^{np\sum_{i=1}^k \beta_i^2 + o(1)n} + e^{np(\sum_{i=1}^k \beta_i^2 - 1) + o(1)n} \\ &= (n-1+e^{-np})e^{np\sum_{i=1}^k \beta_i^2 + o(1)n} \quad a.s. \end{aligned} \tag{3.13}$$

This completes the proof. □

Corollary 1. *Let $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$. Then*

$$LEE(G_{n;\beta_1,\ldots,\beta_k}(p)) = (n-1+e^{-np})e^{o(1)n} \quad a.s. \tag{3.14}$$

if and only if $\max\{n\beta_1, \ldots, n\beta_k\} = o(1)n$.

Proof. By (3.8), (3.9), (3.10) and (3.11), we have that (3.14) holds if and only if

$$\xi_i(\overline{L_{n,k}}) = o(1)n \quad a.s., \text{for } 1 \le i \le n-1 \tag{3.15}$$

and

$$\xi_n(\overline{L_{n,k}}) = -np + o(1)n \quad a.s. \tag{3.16}$$

By (3.4) and (3.5), (3.15) and (3.16) hold if and only if $\max\{n\beta_1, \ldots, n\beta_k\} = o(1)n$. □

Note that if $k = n$, then $G_{n;\beta_1,\ldots,\beta_k}(p) = G_n(p)$, that is, $\beta_i = \frac{1}{n}$, $1 \le i \le k$. By Corollary 1, we have the following result immediately.

Corollary 2. *Let $G_n(p) \in \mathcal{G}_n(p)$ be a random graph. Then almost surely $LEE(G_n(p)) = (n-1+e^{-np})e^{o(1)n}$.*

Corollary 3. *Let $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$ satisfying $\lim\limits_{n\to\infty} \max\{\beta_1, \beta_2, \cdots, \beta_k\} > 0$ and $\lim\limits_{n\to\infty} \frac{\beta_i}{\beta_j} = 1$. Then*

$$(n-1+e^{-np})e^{o(1)n} \le LEE(G_{n;\beta_1,\ldots,\beta_k}(p)) \le (n-1+e^{-np})e^{(p/k+o(1))n} \quad a.s.$$

Corollary 4. *Let $G_{n;\beta_1,\ldots,\beta_k}(p) \in \mathcal{G}_{n;\beta_1,\ldots,\beta_k}(p)$ satisfying $\lim\limits_{n\to\infty} \max\{\beta_1, \beta_2, \cdots, \beta_k\} > 0$, and there exist β_i and β_j such that $\lim\limits_{n\to\infty} \frac{\beta_i}{\beta_j} < 1$, that is, there exists an integer $r \ge 1$ such that $|V_1|, \ldots, |V_r|$ are of order $O(n)$ and $|V_{r+1}|, \ldots, |V_k|$ are of order $o(n)$. Then*

$$\begin{aligned} & (n-1+e^{-np})e^{np(\sum_{i=1}^r \beta_i^2 - \max_{1\le i\le r}\{\beta_i\}) + o(1)n} \\ \le\ & LEE(G_{n;\beta_1,\ldots,\beta_k}(p)) \\ \le\ & (n-1+e^{-np})e^{np(\sum_{i=1}^r \beta_i^2) + o(1)n} \quad a.s. \end{aligned}$$

References

[1] H. Bamdad, F. Ashraf, I. Gutman, Lower bounds for Estrada index and Laplacian Estrada index, *Appl. Math. Lett.,* **23** (2010) 739–742.

[2] W. Bryc, A. Dembo, T. Jiang, Spectral measure of large random Hankel, Markov and Toeplitz matrices, *Ann. Probab.,* **34** (2006) 1–38.

[3] A. Chang, B. Deng, On the Laplacian energy of trees with perfect matchings, *MATCH Commun. Math. Comput. Chem.,* **68** (2012) 767–776.

[4] K.C. Das, S.A. Mojallal, On Laplacian energy of graphs. *Discrete Math.,* **325** (2014) 52–64.

[5] K.C. Das, S.A. Mojallala, I. Gutman, On Laplacian energy in terms of graph invariants, *Appl. Math. Comput.,* **268** (2015) 83–92.

[6] H. Deng, J. Zhang, A note on the Laplacian Estrada index of trees, *MATCH Commun. Math. Comput. Chem.,* **63** (2010) 777–782.

[7] W. Du, X. Li, Y. Li, The Laplacian energy of random graphs, *J. Math. Anal. Appl.,* **368** (2010) 311–319.

[8] P. Erdős, A. Rényi, On random graphs I, *Publ. Math. Debrecen.,* **6** (1959) 290–297.

[9] K. Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Natl. Acad. Sci. USA,* **37** (1951) 760–766.

[10] G.H. Fath-Tabar, A.R. Ashrafi, I. Gutman, Note on Estrada and L-Estrada indices of graphs, *Bull. Cl. Sci. Math. Nat. Sci. Math.,* **139** (2009) 1–16.

[11] E. Fritscher, C. Hoppen, I. Rocha, V. Trevisan, On the sum of the Laplacian eigenvalues of a tree, *Linear. Algebra. Appl.,* **435 (2)** (2011) 371–399.

[12] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. Forschungsz. Graz.,* **103** (1978) 1–22.

[13] I. Gutman, X. Li, Energies of Graphs–Theory and Applications, Mathematical Chemistry Monographs No.17, Kragujevac, 2016, pp.III+290. ISBN: 978-86-6009-033-3.

[14] I. Gutman, B. Zhou, Laplacian energy of a graph, *Linear. Algebra. Appl.,* **414** (2006) 29–37.

[15] A. Ilić, B. Zhou, Laplacian Estrada index of trees, *MATCH Commun. Math. Comput. Chem.,* **63** (2010) 769–776.

[16] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[17] J. Li, W.C. Shiu, A. Chang, On the Laplacian Estrada index of a graph, *Appl. Anal. Discrete Math.,* **3** (2009) 147–156.

[18] M. Robbiano, R. Jimnez, Applications of a theorem by Ky Fan in the theory of Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.,* **62** (2009) 537–552.

[19] M. Robbiano, E. A. Martins, R. Jimnez, B. S. Martin, Upper bounds on the Laplacian energy of some graphs, *MATCH Commun. Math. Comput. Chem.,* **64** (2010) 97–110.

[20] A.N. Shiryaev, *Probability, 2nd edition,* New York: Springer-Verlag,1996.

[21] C.T.M. Vinagre, R.R.Del-Vecchio, D.A.R. Justo, V. Trevisan, Maximum Laplacian energy among threshold graphs, *Linear. Algebra. Appl.,* **439** (2013). 1479–1495.

[22] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, *Math. Ann.,* **71** (2010) 441–479.

[23] K. Yates, *Hückel Molecular Orbital Theory,* Academic Press, New York, 1978.

[24] B. Zhou, I. Gutman, More on the Laplacian Estrada index, *Appl. Anal. Discrete Math.,* **3** (2009) 371–378.

[25] B. Zhou, On sum of powers of Laplacian eigenvalues and Laplacian Estrada index of graphs, *MATCH Commun. Math. Comput. Chem.,* **62** (2009) 611–619.

[26] B. Zhou, On Laplacian eigenvalues of a graph, *Z. Naturforsch.,* **59a** (2004) 181–184.

[27] B. Zhou, New upper bounds for Laplacian energy, *MATCH Commun. Math. Comput. Chem.,* **62** (2009) 553–560.

[28] B. Zhou, I. Gutman, On Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.,* **57** (2007) 211–220.

[29] B. Zhou, I. Gutman, T. Aleksić, A note on Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.,* **60** (2008) 441–446.