Rainbow connection number of graphs with diameter 3^*

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Abstract

A path in an edge-colored graph G, where adjacent edges may have the same color, is called a rainbow path if no two edges of the path are colored the same. The rainbow connection number $rc(G)$ of G is the minimum integer i for which there exists an i-edge-coloring of G such that every two distinct vertices of G are connected by a rainbow path. It is known that almost all graphs have small diameters, and for any fixed integer $k \geq 2$, deciding if $rc(G) = k$ is NP-Complete. In foregoing papers, we showed that a bridgeless graph with diameter 2 has rainbow connection number at most 5. In this paper, we prove that a bridgeless graph with diameter 3 has rainbow connection number at most 8.

Keywords: edge-coloring, rainbow path, rainbow connection number, diameter

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to book [3] for graph theoretical notation and terminology not described here. A path in an

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edge-colored graph G, where adjacent edges may have the same color, is called a *rainbow path* if no two edges of the path are colored the same. An edge-coloring of graph G is a *rainbow edge-coloring* if every two distinct vertices of graph G are connected by a rainbow path. The *rainbow connection number* $rc(G)$ of G is the minimum integer i for which there exists an i-edge-coloring of G such that every two distinct vertices of G are connected by a rainbow path. It is easy to see that $diam(G) \leq rc(G)$ for any connected graph G, where $diam(G)$ is the diameter of G.

The rainbow connection number was introduced by Chartrand et al. in [11]. It is of great use in transferring information of high security in multicomputer networks. We refer the readers to [9, 12] for details.

Chakraborty et al. investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph G, deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard. Later, Ananth and Nasre [1] showed that for any fixed integer $k \geq 3$, deciding if $rc(G) = k$ is NP-Complete. Bounds for the rainbow connection number of a graph have also been obtained in terms of other graph parameters, for example, radius, diameter, dominating number, minimum degree, connectivity [2, 10, 11, 12, 14, 16, 18], etc.

It is known that almost all graphs have small diameters. In foregoing papers [13, 16], we showed that a bridgeless graph with diameter 2 has rainbow connection number at most 5. In this paper, we will prove that a bridgeless graph with diameter 3 has rainbow connection number at most 8.

This paper is organized as follows. In Section 2, we introduce a class of extremal graphs which are spanning subgraphs of bridgeless graphs with diameter 3. In Section 3, we present a vertex set partition of this graph class, and 8-edge-coloring under this partition. In Section 4, we show the above 8-edge-coloring is rainbow.

2 Extremal graphs

We introduce a class of well-known extremal graphs. Denote by $G(n, k, \lambda, s)$ the class of graphs with n vertices and diameter at most k having the property that by deleting any s or fewer edges the resulting subgraphs have diameters at most $\lambda > k$. Furthermore, denote by $\mathrm{Min}G(n, k, \lambda, s)$ the subclass (of $G(n, k, \lambda, s)$) of graphs with minimum number of edges, and denote by $M(n,k,\lambda,s)$ the minimum possible number of edges.

Clearly, each bridgeless graph with diameter 3 has a spanning subgraph which belongs to $\mathrm{Min}G(n,3,\lambda,1)$, where $\lambda \geq 4$. Moreover, if a spanning subgraph of a graph G admits an orientation with diameter d, then G also admits an orientation with diameter at most d. So, we only need to study the oriented graphs of $\mathrm{Min}G(n,3,\lambda,1)$.

In [8], Caccetta gave the following observation and lemma.

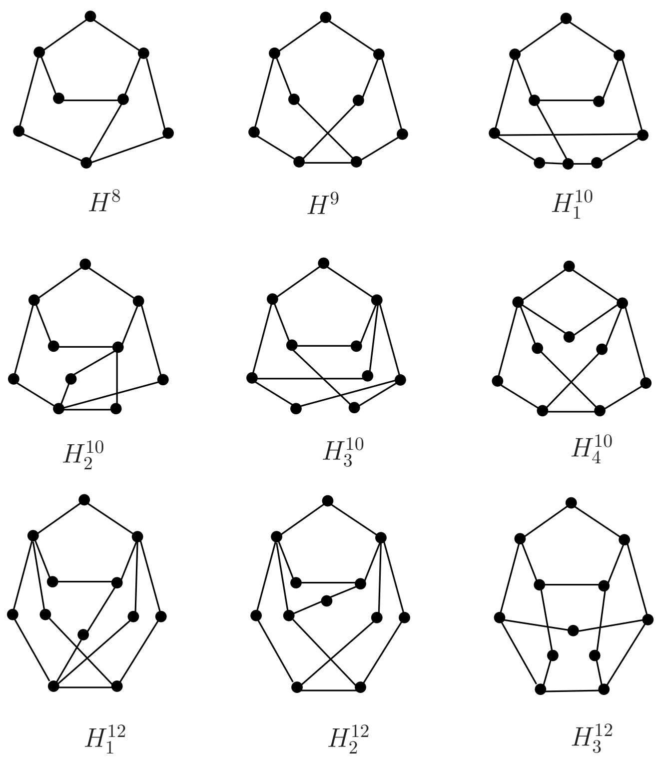

Figure 1. The graphs $H^8, H^9, H_j^{10} (j = 1,2,3,4)$ and $H_j^{12} (j = 1,2,3)$.

Observation 1. *[8]*

(1) *A $G(n,k,\lambda,s)$ graph is also a $G(n,k',\lambda',s')$ graph, whenever $k' \geq k, \lambda' \geq \lambda$ and $s' \leq s$. Consequently the functions $M(n,k,\lambda,s)$ are monotonic non-decreasing in s, and monotonic non-increasing in k and λ.*

(2) *In a $G(n,k,\lambda,s)$ there will be at least $s+1$ edge disjoint paths of lengths $\leq \lambda$ between any two vertices, at least one of which has length $\leq k$.*

(3) *The degree of every vertex of G is at least $s+1$, that is, $\delta(G) \geq s+1$.*

(4) *If $\delta(G) = s+1$, then every vertex of G which is not adjacent to x with degree $\delta(G)$ must be connected to each of the $s+1$ vertices adjacent to x by a path of length $\leq \lambda - 1$ (from (2)).*

Lemma 1. *[8] Let $G \in \mathrm{Min}G(n, 3, \lambda, 1)$, where $\lambda \geq 4$. Then G possesses two adjacent vertices of degree 2 for every $n \geq 5$ except possibly $n = 8, 9, 10$ and 12. Furthermore, if G does not possess two adjacent vertices of degree 2, then the only possible structures are the graphs $H^8, H^9, H_j^{10}(j = 1, 2, 3, 4)$ and $H_j^{12}(j = 1, 2, 3)$ as shown Figure 2.*

There are many interesting results on $\mathrm{Min}G(n, k, \lambda, s)$. We refer the readers to [4, 5, 6, 7, 8] for more results and details.

3 Vertex set partition and edge-coloring

In this section, let $G \in \mathrm{Min}G(n, 3, \lambda, 1)$, where $n \geq 5$ and $\lambda \geq 4$, and let u and v be two adjacent vertices of degree 2 in G. Suppose $u(v)$ is adjacent to $x(y)$. Let X, Y and Z denote the sets $N(x) \setminus N(y)$, $N(y) \setminus N(x)$ and $N(x) \cap N(y)$, respectively.

Let $A = X \cup Y \cup Z \cup \{u, v, x, y\}$. For $s \in V(G) \setminus A$, clearly $d_G(s, u) = d_G(s, v) = 3$, since $G \in G(n, 3, \lambda, 1)$, that is, $N(s) \cap N(x) \neq \emptyset$ and $N(s) \cap N(y) \neq \emptyset$. We partition this set based on the distribution of the neighbors of s.

$$
\begin{aligned}
W &= (N(X) \cap N(Y)) \setminus A, \\
I &= (N(X) \cap N(Z)) \setminus A, \\
K &= (N(Y) \cap N(Z)) \setminus A, \\
J &= V(G) \setminus (W \cup I \cup K \cup A),
\end{aligned}
$$

see Figure 2 for details.

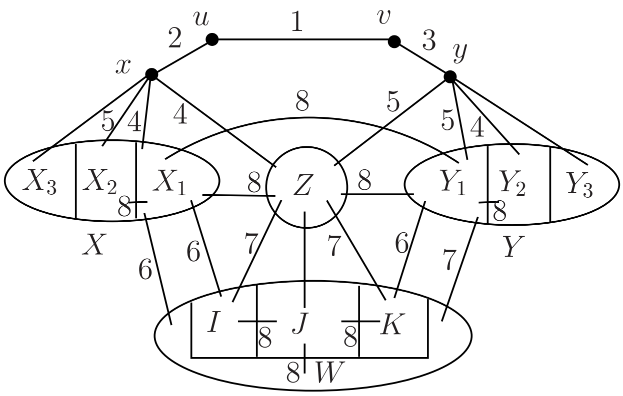

Figure 2. A partial edge-coloring of G.

At this point, we partition X and Y as follows:

$$
\begin{aligned}
&X_1 = \{x \in X \mid x \text{ has neighbors in } Y \cup Z \cup I \cup W\}.\\
&X_2 = \{x \in X \setminus X_1 \mid x \text{ is an isolated vertex in } G[X \setminus X_1]\},\\
&X_3 = X \setminus (X_1 \cup X_2),\\
&Y_1 = \{y \in Y \mid y \text{ has neighbors in } X \cup Z \cup K \cup W\},\\
&Y_2 = \{y \in Y \setminus Y_1 \mid y \text{ is an isolated vertex in } G[Y \setminus Y_1]\},\\
&Y_3 = Y \setminus (Y_1 \cup Y_2).
\end{aligned}
$$

Note that, in Figure 2, if s and t lie in distinct ellipses and there exists no edge joining the two ellipses, then s and t are nonadjacent in G. In general, the edges drawn in Figure 2 do not represent that the subgraph is complete bipartite. The following observation holds for the above vertex set partition.

Observation 2. (1) *For every $x \in X_2$, $N(x) \cap X_1 \neq \emptyset$.* (2) *For every $y \in Y_2$, $N(y) \cap Y_1 \neq \emptyset$.*

We give a partial 8-edge-coloring of G as follows:

$$c(e)=\begin{cases}1, & if\ e=uv;\\ 2, & if\ e=xu;\\ 3, & if\ e=vy;\\ 4, & if\ e\in E[x,X_1\cup Z]\cup E[y,Y_2];\\ 5, & if\ e\in E[x,X_2]\cup E[y,Y_1\cup Z];\\ 6, & if\ e\in E[X_1,I\cup W]\cup E[Y_1,K];\\ 7, & if\ e\in E[Z,I\cup K]\cup E[Y_1,W];\\ 8, & if\ e\in E[X_1,X_2]\cup E[Y_1,Y_2]\cup E[X_1,Y_1]\cup E[J,I\cup K\cup W],\end{cases}$$

see Figure 2 for details.

To complete our edge-coloring, we further partition J as follows:

$$\begin{aligned}&J_1=J\cap N(K),\ J_2=(J\cap N(I))\setminus J_1,\ J_3=(J\cap N(W))\setminus(J_1\cup J_2),\\ &J_4=J\setminus(J_1\cup J_2\cup J_3)=\{s\in J\mid s\text{ has no neighbor in }I\cup K\cup W\},\\ &J_{4,1}=\{s\in J_4\mid N(s)\in Z\},\\ &J_{4,2}=J_4\setminus J_{4,1}=\{\ s\in J_4\mid s\in N(Z)\cup N(J)\}.\end{aligned}$$

Now we further color the edges of G as follows: color the edges in $E[Z,J_1\cup J_2\cup J_3]$ by color 7; for any $s\in J_{4,1}$, color one of $E[s,Z]$ by 7, color the others of the edges in $E[s,Z]$ by 6 (there exists at least one such edge since $G\in \mathrm{Min}G(n,3,\lambda,1)$).

To color the remaining edge, we need the following lemma.

Let S and S' be two disjoint vertex sets. We use $E[S,S']$ to denote the set of edges having one endpoint in each one of S and S'.

Lemma 2. *In a graph G, let S and T be disjoint vertex sets such that $S\subseteq N(T)$. If the induced subgraph $G[S]$ has no trivial components, then there is an $\{\alpha,\beta,\gamma\}$-edge-coloring of $G[S]\cup E[S,T]$, where the edges in $E[S,T]$ are colored by colors α and β, and the edges in $G[S]$ are colored by color γ, such that there exist two rainbow paths P_1 and P_2 between s and T for every $s\in S$. Furthermore, if P_1 admits color $\{\alpha\}$, then P_2 admits colors $\{\beta,\gamma\}$; if P_1 admits color $\{\beta\}$, then P_2 admits colors $\{\alpha,\gamma\}$.*

Proof. Let F be a spanning forest of $G[S]$, and let X and Y be any one of the bipartition defined by this forest F. We give a 3-edge-coloring $c:E(G[S])\cup$

$E[S,T] \to \{\alpha, \beta, \gamma\}$ of G by defining

$$c(e) = \begin{cases} \alpha, & if\ e \in E[T,X]; \\ \beta, & if\ e \in E[T,Y]; \\ \gamma, & otherwise. \end{cases}$$

Clearly, for the above edge-coloring, there exist two rainbow paths P_1 and P_2 between s and T for every $s \in S$. Furthermore, if P_1 admits color $\{\alpha\}$, then P_2 admits colors $\{\beta, \gamma\}$; if P_2 admits color $\{\beta\}$, then P_2 admits colors $\{\alpha, \gamma\}$. □

We can complete our edge-coloring by Lemma 2. The edge-coloring in Lemma 2 is called a $c^{\gamma}_{\alpha,\beta}$-edge-coloring of $G[S] \cup E[S,T]$. By the above definition, we know that $G[J_{4,2}]$ has not trivial components. Thus, we can give $G[J_{4,2}] \cup E[J_{4,2}, Z]$ a $c^{8}_{6,7}$-edge-coloring. Similarly, we can give $G[X_3] \cup E[X_3, \{x\}]$ and $G[Y_3] \cup E[Y_3, \{y\}]$ a $c^{8}_{4,5}$-edge-coloring. The remaining edges can be colored arbitrarily.

4 Rainbow connection number of bridgeless graphs with diameter 3

In this section, we show that every bridgeless graph with diameter 3 has a rainbow coloring with at most 8 colors.

At first, we need some notation and terminology. Let c be a rainbow edge-coloring of G. If an edge e is colored by i, we say that e is an *i-color edge*. Let P be a rainbow path. If $c(e) \in \{i_1, i_2, \ldots, i_r\}$ for any $e \in E(P)$, then P is called an *$\{i_1, i_2, \ldots, i_r\}$-rainbow path*. Let $X_1, X_2, \ldots X_k$ be disjoint vertex subsets of G. The notation $X_1, X_2, \cdots, X_k$ means that there exists some desired rainbow path $P = (x_1, x_2, \ldots, x_k)$, where $x_i \in X_i,\ i = 1, 2, \ldots, k$.

Observation 3. (1) *For every $s \in X_1$, there exists a $\{5, 6, 7, 8\}$-rainbow path between y and s under the above edge-coloring.* (2) *For every $s \in Y_1$, there exists a $\{4, 6, 7, 8\}$-rainbow path between x and s under the above edge-coloring.*

Proof. Since the proof methods are similar, we only show (1). For any $s \in X_1$, by the definition of set X_1, we know that s has neighbors in $Y \cup Z \cup I \cup W$. Let t be a neighbor of s in $Y \cup Z \cup I \cup W$. If $t \in Y$, then $t \in Y_1$ by the definition of set Y_1. Thus y, t, s is a $\{5, 6, 7, 8\}$-rainbow path between y and s under the above edge-coloring. If $t \in Z$, then y, t, s is a $\{5, 6, 7, 8\}$-rainbow path between y and s under the above edge-coloring. If $t \in I$, then y, z, t, s is a $\{5, 6, 7, 8\}$-rainbow path between y and s under the above edge-coloring, where z is a neighbor of t in Z (Note that such a vertex z must exist by the definition of set I. Otherwise, $t \in W$, then y, y', t, s is a $\{5, 6, 7, 8\}$-rainbow path between y and s under the above edge-coloring, where y' is a neighbor of t in Y_1. □

Lemma 3. *Let $G \in \mathrm{Min}G(n, 3, \lambda, 1)$, where $n \geq 5$ and $\lambda \geq 4$. If G possesses two adjacent vertices of degree 2, then $rc(G) \leq 8$.*

Proof. It suffices to show that the edge-coloring of G given in Section 3 is a rainbow coloring, that is, for any pair of vertices $(s, t) \in (V(G), V(G))$ there exists a rainbow path between s and t. Consider the following three cases.

Case 1. $(s, t) \in (\{x, u, v, y\}, V(G))$.

It is easy to check that the conclusion holds from Figure 2 and the definition of the vertex set partition.

Case 2. $(s, t) \in (X \cup Y \cup Z, V(G))$.

By Case 1, there exist rainbow paths between $X \cup Y \cup Z$ and $\{x, y, u, v\}$. Thus, suppose $(s, t) \notin (X \cup Y \cup Z \cup \{x, y, u, v\})$. Moreover, we omit some rainbow paths, which are denoted by "$\cdots$", by symmetricity. Table 1 presents a rainbow path between s and t.

Note that, in Figure 2, some sets may be empty. Though we do not indicate all kinds of cases, in fact, we can get useful information from these vertex pairs. For example, if $(s, t) \in (J_3, X_3)$, then J_3 and X_3 are nonempty. By the definition of set J_3, we know that W, X_1 and Y_1 are also nonempty. In Table 1, $(s, t) \in (X_3, Z)$ corresponds to "Lem 2 and x, u, v, y, Z". By Figure 2 and the definition of the vertex set partition, there exists a $\{1, 2, 3, 5\}$-rainbow path between x and every vertex t. Moreover, "Lem 2" means that there exists a $\{4, 8\}$-rainbow path between s and x by the definition of the above edge-coloring and Lem 2. Thus s and t is connected by a $\{1, 2, 3, 4, 5, 8\}$-rainbow path.

Case 3. $(s, t) \in (I \cup J \cup K \cup W, V(G))$.

By Cases 1 and 2, there exists a rainbow path between every pair of vertices in $(I \cup J \cup K \cup W, X \cup Y \cup Z \cup \{x, y, u, v\})$. Thus, suppose $(s, t) \notin (I \cup J \cup K \cup W, X \cup Y \cup Z \cup \{x, y, u, v\})$, that is, $(s, t) \in (I \cup J \cup K \cup W, I \cup J \cup K \cup W)$. Table 2 presents a rainbow path between s and t. □

Lemma 4. *Let $G \in \mathrm{Min}G(n, 3, \lambda, 1)$, where $n \geq 5$ and $\lambda \geq 4$. If G does not*

	X_1	X_2	X_3	Y_1	Y_2	Y_3	Z
X_1	X_1, x, u, v, y, and Obs 3	X_1, x, X_2	X_1, x, X_2, and Lem 2	X_1, x, u, v, y, Y_1	$X_1, x, u, v, y, Y_1, Y_2$	X_1, x, u, v, y, and Lem 2	X_1, x, u, v, y, Z
X_2	$\cdots$	X_2, X_1, x, X_2	X_2, x, and Lem 2	$X_2, X_1, x, u, v, y, Y_1$	X_2, x, u, v, y, Y_2	X_2, x, u, v, y and Lem 2	X_2, x, u, v, y, Z
X_3	$\cdots$	$\cdots$	Lem 2 and x, X_3	Lem 2 and x, u, v, y, Y_1	Lem 2 and x, u, v, y, Y_2	Lem 2 and x, u, v, y, Y_3	Lem 2 and x, u, v, y, Z
Y_1	$\cdots$	$\cdots$	$\cdots$	Y_1, y, v, u, x and Obs 3	Y_1, y, Y_2	Y_1, y, and Lem 2	Y_1, y, v, u, x Z
Y_2	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Y_2, Y_1, y, Y_2	Y_2, Y_1, y, and Lem 2	Lem 2, and y, v, u, x, Z
Y_3	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Y_3, y, and Lem 2	Lem 2, and y, Z
Z	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Z, x, u, v, y, Z
I	I, Z, y, v, u, x, X_1	$I, Z, y, v, u, x, X_1, X_2$	I, Z, y, v, u, x, and Lem 2	I, Z, x, u, v, y, Y_1	$I, Z, x, u, v, y, Y_1, Y_2$	I, Z, x, u, v, y, and Lem 2	I, Z, x, u, v, y, Z
K	K, Z, y, v, u, x, X_1	$k, Z, y, v, u, x, X_1, X_2$	K, Z, y, v, u, x, and Lem 2	K, Z, x, u, v, y, Y_1	$K, Z, x, u, v, y, Y_1, Y_2$	K, Z, x, u, v, y, and Lem 2	K, Z, x, u, v, y, Z
W	W, Y_1, y, v, u, x, X_1	$W, Y_1, y, v, u, x, X_1, X_2$	W, Y_1, y, v, u, x, and Lem 2	W, X_1, x, u, v, y, Y_1	$W, X_1, x, u, v, y, Y_1, Y_2$	W, X_1, x, u, v, y, and Lem 2	W, Y_1, y, v, u, Z
J_1	J_1, Z, y, v, u, x, X_1	$J_1, Z, y, v, u, x, X_1, X_2$	J_1, Z, y, v, u, x, and Lem 2	J_1, Z, x, u, v, y, Y_1	$J_1, Z, x, u, v, y, Y_1, Y_2$	J_1, Z, x, u, v, y, and Lem 2	J_1, Z, x, u, v, y, Z
J_2	J_2, Z, y, v, u, x, X_1	$J_2, Z, y, v, u, x, X_1, X_2$	J_2, Z, y, v, u, x, and Lem 2	J_2, Z, x, u, v, y, Y_1	$J_2, Z, x, u, v, y, Y_1, Y_2$	J_2, Z, x, u, v, y, and Lem 2	J_2, Z, x, u, v, y, Z
J_3	J_3, Z, y, v, u, x, X_1	$J_3, Z, y, v, u, x, X_1, X_2$	J_3, Z, y, v, u, x, and Lem 2	J_3, Z, x, u, v, y, Y_1	$J_3, Z, x, u, v, y, Y_1, Y_2$	J_3, Z, x, u, v, y, and Lem 2	J_3, Z, x, u, v, y, Z
$J_{4,1}$	$J_{4,1}, Z, y, v, u, x, X_1$	$J_{4,1}, Z, y, v, u, x, X_1, X_2$	$J_{4,1}, Z, y, v, u, x$, and Lem 2	$J_{4,1}, Z, x, u, v, y, Y_1$	$J_{4,1}, Z, x, u, v, y, Y_1, Y_2$	$J_{4,1}, Z, x, u, v, y$, and Lem 2	$J_{4,1}, Z, x, u, v, y, Z$
$J_{4,2}$	$J_{4,2}, Z, y, v, u, x, X_1$	$J_{4,2}, Z, y, v, u, x, X_1, X_2$	$J_{4,2}, Z, y, v, u, x$, and Lem 2	$J_{4,2}, Z, x, u, v, y, Y_1$	$J_{4,2}, Z, x, u, v, y, Y_1, Y_2$	$J_{4,2}, Z, x, u, v, y$, and Lem 2	$J_{4,2}, Z, x, u, v, y, Z$

Table 1. The rainbow paths in G for Case 2.

	I	K	W	J_1	J_2	J_3	$J_{4,1}$	$J_{4,2}$
I	I, Z, y, v, u, x, X_1, I	I, Z, x, u, v, y, Y_1, K	$I, X_1, x, u, v, y, Y_1, W$	$I, X_1, x, u, v, y, Z, J_1$	$I, X_1, x, u, v, y, Z, J_2$	$I, X_1, x, u, v, y, Z, J_3$	$I, X_1, x, u, v, y, Z, J_4$	I, X_1, x, u, v, y, Z, and Lem 2
K	$\cdots$	K, Z, x, u, v, y, Y_1, K	K, Z, x, u, v, y, Y_1, W	K, Z, x, u, v, y, Z, J_1	K, Z, x, u, v, y, Z, J_2	K, Z, x, u, v, y, Z, J_3	K, Z, x, u, v, y, Z, J_4	K, Z, x, u, v, y, Z, and Lem 2
W	$\cdots$	$\cdots$	$W, X_1, x, u, v, y, Y_1, W$	$W, X_1, x, u, v, y, Z, J_1$	$W, X_1, x, u, v, y, Z, J_2$	$W, X_1, x, u, v, y, Z, J_3$	$W, X_1, x, u, v, y, Z, J_4$	W, X_1, x, u, v, y, Z, and Lem 2
J_1	$\cdots$	$\cdots$	$\cdots$	$J_1, Z, x, u, v, y, Z, J_1$	$J_1, K, Y_1, y, v, u, x, Z, J_2$	$W, X_1, x, u, v, y, Z, J_3$	$W, X_1, x, u, v, y, Z, J_4$	W, X_1, x, u, v, y, Z, and Lem 2
J_2	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$J_2, I, X_1, x, u, v, y, Z, J_2$	$J_2, I, X_1, x, u, v, y, Z, J_3$	$J_2, I, X_1, x, u, v, y, Z, J_{4,1}$	J_2, Z, x, u, v, y, Z, and Lem 2
J_3	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$J_3, W, X_1, x, u, v, y, Z, J_3$	$J_3, W, X_1, x, u, v, y, Z, J_{4,1}$	J_3, Z, x, u, v, y, Z, and Lem 2
$J_{4,1}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$J_{4,1}, Z, x, u, v, y, Z, J_{4,1}$	$J_{4,1}, Z, x, u, v, y, Z$, and Lem 2
$J_{4,2}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$J_{4,2}, Z, x, u, v, y, Z$, and Lem 2

Table 2. The rainbow paths in G for Case 3.

possess two adjacent vertices of degree 2*, then* G *has a rainbow coloring with at most* 8 *colors.*

Proof. Let $G \in \text{Min}G(n, 3, \lambda, 1)$ which does not possess two adjacent vertices of degree 2, where $n \geq 5$ and $\lambda \geq 4$. By Lemma 2, G must be one of the graphs H^8, H^9, H_j^{10} $(j = 1, 2, 3, 4)$ and H_j^{12} $(j = 1, 2, 3)$, as shown Figure 4. It suffices to show that these graphs have a rainbow coloring with at most 8 colors. In fact, we can color these graphs by 8 colors as Figure 3. It is easy to check that these edge-colorings are rainbow. Thus we are done. $\square$

Combining Lemmas 3 and 4, the following result is immediate.

Theorem 1. *Let* $G \in \text{Min}G(n, 3, \lambda, 1)$*, where* $n \geq 5$ *and* $\lambda \geq 4$*. Then* $rc(G) \leq 8$*.*

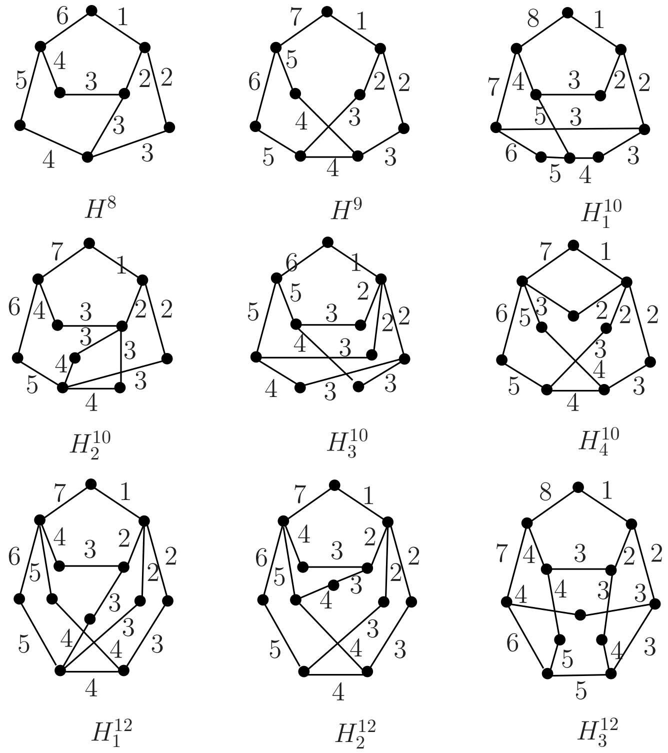

Figure 3. The rainbow edge-colorings of graphs H^8, H^9, H^{10}_j $(j = 1, 2, 3, 4)$ and H^{12}_j $(j = 1, 2, 3)$.

Corollary 1. *The rainbow connection number of a bridgeless graph with diameter* 3 *is at most* 8.

Proof. Let G be a bridgeless graph with diameter 3. Since each bridgeless graph with diameter 3 has a spanning subgraph which belongs to $\mathrm{Min}G(n, 3, \lambda, 1)$, where $\lambda \geq 4$. Moreover, if a spanning subgraph of a graph G has rainbow connection number c, then G has rainbow connection number at most c. Thus, the rainbow connection number of G is at most 8 by Theorem 1. $\square$

We must say that at the moment we have not found examples showing that the upper bound 8 is not possible. However, by a similar method in [13] we can give the following example of graphs with diameter 3 for which the rainbow connection number reaches 7.

Example 1. Let K_n be a complete graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, where

$n \geq 217$. For every v_i, we hang a path $P_i = (v_i, v_{i,1}, v_{i,2}, v_{i,3})$, and then we identify the vertex $v_{i,3}$ with a vertex v. The resulting graph is denoted by G. Clearly, $diam(G) = 3$. Let c be any 6-edge-coloring of G with colors $\{1, 2, \ldots, 6\}$. Since $6^3 = 216$ and there exist 217 hanging paths P_i, at least two of them have the same color. Without loss generality, say P_1 and P_2, that is, $c(v_1v_{1,1}) = c(v_2v_{2,1}), c(v_{1,1}v_{1,2}) = c(v_{2,1}v_{2,2})$ and $c(v_{1,2}v_{1,3}) = c(v_{2,2}v_{2,3})$. By the structure of G, it is easy to see that there exists no rainbow path between $v_{1,1}$ and $v_{2,1}$ in G under c. Thus $rc(G) \geq 7$.

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