Upper bounds of proper connection number of graphs*

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Abstract

A path in an edge-colored graph is called a proper path if no two adjacent edges of the path are colored with one same color. An edge-colored graph is proper connected if any two vertices of the graph are connected by a proper path in the graph. The smallest number of colors that are needed in order to make G proper connected is called the proper connection number of G, denoted by $pc(G)$. In this paper, we present an upper bound for the proper connection number of a graph G in terms of the bridge-block tree of G. We also use this bound as an efficient tool to investigate the Erdös-Gallai-type problems for proper connection number of a graph G.

Keywords: proper connection number, bridge-block tree, Erdös-Gallai-type problem

AMS subject classification 2010: 05C15, 05C35, 05C38, 05C40.

1 Introduction

In this paper we are concerned with simple connected finite graphs. We follow the terminology and the notation of Bondy and Murty [2]. For a graph $G = (V, E)$ and two disjoint subsets X and Y of V, denote by $B_G[X, Y]$ the bipartite subgraph of G with vertex set $X \cup Y$ and edge set $E(X, Y)$, where $E(X, Y)$ is the set of edges of G that have one end in X and the other in Y. An induced subgraph denoted by $G[X]$ is the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which

*Supported by NSFC No.11371205 and 11531011, and PCSIRT.

have both ends in X. For a vertex set D, let $N(D) = \{u \in V(G) : dist(u, D) = 1\}$ and $D^* = G[D \cup N(D)]$. A vertex is called a simplical vertex if $G[N(v)]$ is a clique.

An edge-coloring of a graph G is an assignment c of colors to the edges of G, one color to each edge of G. If adjacent edges of G are assigned different colors by c, then c is a *proper (edge-)coloring*. For a graph G, the minimum number of colors needed in a proper coloring of G is referred to as the *chromatic index* of *edge-chromatic number* of G and denoted by $\chi'(G)$. A path in an edge-colored graph with no two edges sharing the same color is called a *rainbow path*. An edge-colored graph G is said to be *rainbow connected* if every pair of distinct vertices of G is connected by at least one rainbow path in G. Such a coloring is called a *rainbow coloring* of the graph. For a connected graph G, the minimum number of colors needed in a rainbow coloring of G is referred to as the *rainbow connection number* of G and denoted by $rc(G)$. The concept of rainbow coloring was first introduced by Chartrand et al. in [5]. In recent years, the rainbow coloring has been extensively studied and a variety of nice results have been obtained, see [4, 6, 10, 12, 15] for examples. For more details we refer to a survey paper [16] and a book [17].

Inspired by the rainbow coloring and proper coloring of graphs, Andrews et al. [1] introduced the concept of proper-path coloring. Let G be an edge-colored graph, where adjacent edges may be colored with the same color. A path in G is called a *proper path* if no two adjacent edges of the path are colored with a same color. Similarly, we call a cycle *proper cycle* in G if no two adjacent edges of the cycle are colored with a same color. An edge-coloring c is a *proper-path coloring* of a connected graph G if every pair of distinct vertices u, v of G is connected by a proper $u - v$ path in G. A graph with a proper-path coloring is said to be *proper connected*. If k colors are used, then c is referred to as a *proper-path k-coloring*. An edge-colored graph G is *k-proper connected* if any two vertices are connected by k internally pairwise vertex-disjoint proper paths. For a k-connected graph G, the *k-proper connection number* of G, denoted by $pc_k(G)$, is defined as the smallest number of colors that are needed in order to make G k-proper connected. Clearly, if a graph is k-proper connected, then it is also k-connected. Conversely, any k-connected graph has an edge-coloring that makes it k-proper connected; the number of colors is easily bounded by the edge-chromatic number which is well known to be at most $\Delta(G)$ or $\Delta(G) + 1$ by Vizings Theorem [19] (where $\Delta(G)$, or simply Δ, is the maximum degree of G). Thus $pc_k(G) \leq \Delta(G) + 1$ for any k-connected graph G. For $k = 1$, we write $pc(G)$ as opposed to $pc_1(G)$, and call it the *proper connection number* of G.

Let G be a nontrivial connected graph of order n and size m. Then the proper connection number of G has the following apparent bounds:

$$1 \leq pc(G) \leq \min\{\chi'(G), rc(G)\} \leq m.$$

Furthermore, $pc(G) = 1$ if and only if $G = K_n$ and $pc(G) = m$ if and only if $G = K_{1,m}$ is a star of size m.

The Erdös-Gallai-type problem is an important and interesting problem in extremal graph theory, which is to determine the maximum or minimum value of a graph parameter with some given properties. The Erdös-Gallai-type questions for rainbow connection number $rc(G)$ were considered in [9, 11, 13, 14, 18].

The paper is organized as follows: In Section 2, we give the basic definitions and some useful lemmas. In Section 3, we present an upper bound $\max\{3, \Delta^*(G)\}$ of the proper connection number of a graph G, where $\Delta^*(G)$ is the maximum degree of the bridge-block tree of G. In Section 4, we study two kinds of Erdös-Gallai-type problems for $pc(G)$ by using the upper bound we give in Section 3.

2 Preliminaries

At the beginning of this section, we list some fundamental results on proper-path coloring.

Lemma 2.1. *[1] If G is a connected graph and H is a connected spanning subgraph of G, then $pc(G) \leq pc(H)$. In particular, $pc(G) \leq pc(T)$ for every spanning tree T of G.*

Lemma 2.2. *[1] If T is a tree, then $pc(T) = \chi'(T) = \Delta(T)$.*

Given a colored path $P = v_1v_2 \ldots v_{s-1}v_s$ between any two vertices v_1 and v_s, we denote by $start(P)$ the color of the first edge in the path, i.e., $c(v_1v_2)$, and by $end(P)$ the last color, i.e., $c(v_{s-1}v_s)$. If P is just the edge v_1v_s, then $start(P) = end(P) = c(v_1v_s)$.

Definition 2.1. *Let c be an edge-coloring of G that makes G proper connected. We say that G has the strong property under c if for any pair of vertices $u, v \in V(G)$, there exist two proper paths P_1, P_2 between them (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

In [3], the authors studied the proper-connection numbers in 2-connected graphs. Also, they presented a result which improves the upper bound $\Delta(G) + 1$ of $pc(G)$ to the best possible whenever the graph G is 2-connected.

Lemma 2.3. *[3] Let G be a graph. If G is bipartite and 2(-edge)-connected, then $pc(G) = 2$ and there exists a 2-edge-coloring c of G such that G has the strong property under c.*

As a result of Lemma 2.3, the authors in [3] obtained a corollary.

Corollary 2.4. *[3] Let G be a graph. If G is 3-connected and noncomplete, then $pc(G) = 2$ and there exists a 2-edge-coloring c of G such that G has the strong property under c.*

Lemma 2.5. *[1] Let G be a connected graph and v a vertex not in G. If $pc(G) = 2$, then $pc(G \cup v) = 2$ as long as $d(v) \geq 2$, that is, we connect v to G by using at least two edges.*

Lemma 2.6. *[3] Let G be a graph. If G is 2(-edge)-connected, then $pc(G) \leq 3$ and there exists a 3-edge-coloring c of G such that G has the strong property under c.*

Lemma 2.7. *[8] Let $H = G \cup \{v_1\} \cup \{v_2\}$. If there is a proper-path k-coloring c of G such that G has the strong property under c. Then $pc(H) \leq k$ as long as H is connected.*

As a result of Lemma 2.7, we obtain the following corollary.

Corollary 2.8. *Let H be the graph that is obtained by identifying u_i of G to v_i of a path P^i for $i = 1, 2$, where v_i is an end vertex of P_i. If there is a proper-path k-coloring c of G such that G has the strong property under c, then $pc(H) \leq k$.*

3 Upper bounds of proper connection number

Let $B \subseteq E$ be the set of cut-edges of a graph G. Let $\mathcal{C}$ denote the set of connected components of $G' = (V; E \setminus B)$. There are two types of elements in $\mathcal{C}$, singletons and connected bridgeless subgraphs of G. Let $\mathcal{S} \subseteq \mathcal{C}$ denote the singletons and let $\mathcal{D} = \mathcal{C} \setminus \mathcal{S}$. Each element of $\mathcal{S}$ is, therefore, a vertex, and each element of $\mathcal{D}$ is a connected bridgeless subgraph of G.

Contracting each element of $\mathcal{D}$ to a vertex, we obtain a new graph G^*. It is easy to see that G^* is the well-known *bridge-block tree* of G, and the edge set of G^* is B.

Lemma 3.1. *Let G be a graph and $H = G - PV(G)$, where $PV(G)$ is the set of the pendent vertices of G. If H is bridgeless, then $pc(G) \leq \max\{3, |PV(G)|\}$.*

Proof. Since H is bridgeless, one has that $pc(H) \leq 3$ and there is a proper-path 3-coloring c of H such that H has the strong property under c by Lemma 2.6. Assume that $PV(G) = \{v_1, v_2, \ldots, v_k\}$. If $k \leq 2$, we have that $pc(G) \leq 3$ by Lemma 2.7. So we consider the case that $k \geq 3$. Let u_i be the neighbor of v_i in G for $i = 1, 2, \ldots, k$, and let

$\{1,2,3\}$ be the color-set of c. We first assign color j to u_jv_j for $j = 4,\ldots,k$. Then we color the remaining edges u_1v_1, u_2v_2, u_3v_3 by colors $1,2,3$ by the following strategy.

If $u_1 = u_2 = u_3$, we assign color i to u_iv_i for $i = 1,2,3$. If $u_1 = u_2 \neq u_3$, let P be a proper path of G connecting u_1 and u_3. Then there are two different colors in $\{1,2,3\} \setminus \{start(P)\}$. We assign these two colors to u_1v_1 and u_2v_2, respectively, and choose a color that is distinct from $end(P)$ in $\{1,2,3\}$ for u_3v_3. If $u_i \neq u_j$ for $1 \leq i \neq j \leq 3$, suppose that P_{ij} is a proper path of G between u_i and u_j. We choose a color that is distinct from $start(P_{12})$ and $start(P_{13})$ in $\{1,2,3\}$ for u_1v_1. Similarly, we color u_2v_2 by a color in $\{1,2,3\} \setminus \{end(P_{12}), start(P_{23})\}$, and color u_3v_3 by a color in $\{1,2,3\} \setminus \{end(P_{13}), end(P_{23})\}$.

One can see that in all these cases, v_i and v_j are proper connected for $1 \leq i \neq j \leq k$. Moreover, as H has the strong property under edge-coloring c, it is obvious that v_i and u are proper connected for $1 \leq i \leq k$ and $u \in V(H)$. Therefore, we have that $pc(H) \leq k = |PV(G)|$. Hence, we obtain that $pc(G) \leq \max\{3, |PV(G)|\}$. □

Lemma 3.2. *Let G be a graph with a cut-edge v_1v_2, and G_i be the connected graph obtained from G by contracting the connected component containing v_i of $G - v_1v_2$ to a vertex v_i, where $i = 1,2$. Then $pc(G) = \max\{pc(G_1), pc(G_2)\}$*

Proof. First, it is obvious that $pc(G) \geq \max\{pc(G_1), pc(G_2)\}$. Let $pc(G_1) = k_1$ and $pc(G_2) = k_2$. Without loss of generality, suppose $k_1 \geq k_2$. Let c_1 be a k_1-proper coloring of G_1 and c_2 be a k_2-proper coloring of G_2 such that $c_1(v_1v_2) = c_2(v_1v_2)$ and $\{c_2(e) : e \in E(G_2)\} \subseteq \{c_1(e) : e \in E(G_1)\}$. Let c be the edge-coloring of G such that $c(e) = c_1(e)$ for any $e \in E(G_1)$ and $c(e) = c_2(e)$ otherwise. Then c is an edge-coloring of G using k_1 colors. We will show that c is a proper-path coloring of G. For any pair of vertices $u, v \in V(G)$, we can easily find a proper path between them if $u, v \in V(G_1)$ or $u, v \in V(G_2)$. Hence we only need to consider that $u \in V(G_1) \setminus \{v_1, v_2\}$ and $v \in V(G_2) \setminus \{v_1, v_2\}$. Since c_1 is a k_1-proper coloring of G_1, there is a proper path P_1 in G_1 connecting u and v_1. Since c_2 is a k_2-proper coloring of G_2, there is a proper path P_2 in G_2 connecting v and v_2. As $c_1(v_1v_2) = c_2(v_1v_2)$, then we know that $P = uP_1v_2v_1P_2v$ is a proper path connecting u and v in G. Therefore, we have that $pc(G) \leq k_1$, and the proof is thus complete. □

Theorem 3.3. *If G is a connected graph, then $pc(G) \leq \max\{3, \Delta(G^*)\}$.*

Proof. If G is bridgeless, we have that $pc(G) \leq 3$ by Lemma 2.6. Otherwise, let $B \subseteq E$ be the set of cut-edges of graph G. Let $\mathcal{C}$ denote the set of connected components of $G' = (V; E \setminus B)$. We claim that $pc(D^*) \leq \max\{3, \Delta(G^*)\}$ for any $D \in \mathcal{C}$. Note that if D is a singleton, it is obvious that $D^* \cong K_{1,|N(D)|}$ and $pc(D^*) = |N(D)| \leq \max\{3, \Delta(G^*)\}$. If

D is bridgeless, by Lemma 3.1, we have that $pc(D^*) \leq \max\{3, |N(D)|\} \leq \max\{3, \Delta(G^*)\}$. Hence by Lemma 3.2, we have that $pc(G) = \max_{D \in \mathcal{C}} pc(D^*) \leq 3$. $\square$

Let rK_t be the disjoint union of r copies of the complete graph K_t, We use S_r^t to denote the graph obtained from rK_t by adding an extra vertex v and joining v to one vertex of each K_t.

Corollary 3.4. *If G is a connected graph with n vertices and minimum degree $\delta \geq 2$, then $pc(G) \leq \max\{3, \frac{n-1}{\delta+1}\}$. Moreover, if $\frac{n-1}{\delta+1} > 3$, and $n \geq \delta(\delta+1)+1$, we have that $pc(G) = \frac{n-1}{\delta+1}$ if and only if $G \cong S_r^t$, where $t-1=\delta$ and $rt+1=n$.*

Proof. Since the minimum degree of G is $\delta \geq 2$, we know that each leaf of G^* is obtained by contracting an element with at least $\delta+1$ vertices of $\mathcal{D}$. Therefore, $\mathcal{D}$ has at most $\frac{n-1}{\delta+1}$ such elements, and so, one can see that $\Delta(G^*) \leq \frac{n-1}{\delta+1}$. From Theorem 3.3, we know that $pc(G) \leq \max\{3, \frac{n-1}{\delta+1}\}$.

If $\frac{n-1}{\delta+1} > 3$ and $pc(G) = \frac{n-1}{\delta+1}$, one can see that G^* is a star with $\Delta(G^*) = \frac{n-1}{\delta+1}$, and each leaf of G^* is obtained by contracting an element with $\delta+1$ vertices of $\mathcal{D}$, that is, $G \cong S_r^t$, where $t=\delta$ and $rt+1=n$. On the other hand, if $G \cong S_r^t$, where $t=\delta$ and $rt+1=n$, we can easily check that $pc(G) = r = \frac{n-1}{\delta+1}$. $\square$

4 Erdös-Gallai-type results for proper connection numbers of graphs

In this section, we study two kinds of Erdös-Gallai-type problems for $pc(G)$. We consider the following two problems:

Problem A. For every k with $2 \leq k \leq n-1$, compute and minimize the function $f(n,k)$ with the following property: for any connected graph G with n vertices, if $|E(G)| \geq f(n,k)$, then $pc(G) \leq k$.

Problem B. For every k with $2 \leq k \leq n-1$, compute and maximize the function $g(n,k)$ with the following property: for any connected graph G with n vertices, if $|E(G)| \leq g(n,k)$, then $pc(G) \geq k$.

It is worth mentioning that the two parameters $f(n,k)$ and $g(n,k)$ are equivalent to another two parameters. For $2 \leq k \leq n-1$, let

$$s(n,k) = \max\{|E(G)| : |V(G) = n, pc(G) \geq k\}$$

and

$$t(n,k) = \min\{|E(G)| : |V(G)| = n, pc(G) \leq k\}.$$

It is easy to see that $g(n,k) = t(n,k-1) - 1$ and $f(n,k) = s(n,k+1) + 1$.

Since the graphs considered here are connected, we have that $t(n,k) \geq n-1$. On the other hand, we can get that $t(n,k) \leq n-1$ since $pc(P_n) = 2 \leq k$. Hence, $t(n,k) = n-1$ for any $2 \leq k \leq n-1$. This implies that $g(n,k) = n-2$ for $3 \leq k \leq n-1$. Hence we know that $g(n,k)$ is meaningless for $3 \leq k \leq n-1$. For $k = 2$, we can get that $g(n,2) = \binom{n}{2} - 1$ from the definition of $g(n,k)$.

We now show a lower bound for $f(n,k)$.

Proposition 4.1. $f(n,k) \geq \binom{n-k-1}{2} + k + 2$.

Proof. We construct a graph G_k as follows: Take a K_{n-k-1} and a star S_{k+2}. Identify the center-vertex of S_{k+2} with an arbitrary vertex of K_{n-k-1}. The resulting graph G_k has order n and size $E(G_k) = \binom{n-k-1}{2} + k + 1$. It can be easily checked that $pc(G_k) = k+1$. Hence, $f(n,k) \geq \binom{n-k-1}{2} + k + 2$. □

By using Theorem 3.3, the value of $f(n,k)$ for $k \geq 3$ can be completely determined.

Theorem 4.2. *For $k \geq 3$, one has that $f(n,k) = \binom{n-k-1}{2} + k + 2$.*

Proof. By the definition of $f(n,k)$, we need to prove that $pc(G) \leq k$ when $E(G) \geq \binom{n-k-1}{2} + k + 2$. Suppose to the contrary that $pc(G) \geq k+1$. From Theorem 3.3, we know that $\Delta(G^*) \geq k+1$, where G^* is the bridge-block tree of G. By some simple computations, we know that $|E(G)| \leq \binom{n-k-1}{2} + k + 1$, which contradicts the assumption. Hence, $pc(G) \leq k$. □

To compute the value of $f(n,2)$, we need the following Lemmas.

Lemma 4.3. *Let G be a graph with n ($n \geq 6$) vertices and at least $\binom{n-1}{2} + 3$ edges. Then for any $u, v \in V(G)$, there is a 2-connected bipartite spanning subgraph of G with u, v in the same part.*

Proof. Let $\overline{G}$ be the complement of G. Then we have that $|E(\overline{G})| \leq n-4$. Let $S = N(u) \cap N(v)$, we have that $|S| \geq 2$. Otherwise, $|S| \leq 1$, then for any $w \in V(G) \setminus (S \cup \{u,v\})$, either $uw \in E(\overline{G})$ or $vw \in E(\overline{G})$, and thus $|E(\overline{G})| \geq n-3$, which contradicts the fact that $|E(\overline{G})| \leq n-4$. Therefore, we know that $B_G[S, \{u,v\}]$ is a 2-connected bipartite subgraph of G with u, v in the same part.

Suppose that $H = B_G[X,Y]$ is a 2-connected bipartite subgraph of G with u, v in the same part and H has as many vertices as possible. Then, if $V(G) \setminus V(H) \neq \emptyset$, one has that there exists a vertex $w \in V(G) \setminus V(H)$, such that $|N(w) \cap X| \geq 2$ or $|N(w) \cap Y| \geq 2$. Since otherwise,

$$|E(\overline{G})| \geq (n - |V(H_1)|)(|V(H_1)| - 2) \geq n-3,$$

which contradicts the fact that $|E(\overline{G})| \leq n-4$. Then w can be added to X if $|N(w) \cap X| \geq 2$ or added to Y otherwise, which contradicts the maximality of H. So, we know that H is a 2-connected bipartite spanning subgraph of G with u, v in the same part, which completes the proof. □

Lemma 4.4. *Every 2-connected graph on n ($n \geq 12$) vertices with at least $\binom{n-1}{2} - 5$ edges contains a 2-connected bipartite spanning subgraph.*

Proof. The result is trivial if G is complete. We will prove our result by induction on n for noncomplete graphs. First, if $|V(G)| = 12$ and $|E(G)| \geq 50$, one can find a 2-connected bipartite spanning subgraph of G. So we suppose that the result holds for all 2-connected graphs on n_0 ($13 < n_0 < n$) vertices with at least $\binom{n_0-1}{2} - 5$ edges. For a 2-connected graph G on n vertices with $|E(G)| \geq \binom{n-1}{2} - 5$, let v be a vertex with minimum degree of G, and let $H = G - v$. If $d(v) = 2$, then $|E(H)| \geq \binom{n-1}{2} - 7$. Let $N_G(v) = \{v_1, v_2\}$. We know that H contains a 2-connected bipartite spanning subgraph with v_1, v_2 in the same part by Lemma 4.3. Clearly, G contains a 2-connected bipartite spanning subgraph. Otherwise, $3 \leq d(v) \leq n-2$, then $|E(H)| \geq \binom{n-1}{2} - 5 - (n-2) = \binom{(n-1)-1}{2} - 5$ and $\delta(H) \geq 2$. If H has a cut vertex u, then each connected component of $H - u$ contains at least 2 vertices. We have that $|E(H)| \leq \binom{n-3}{2} + 3 < \binom{n-2}{2} - 5$, a contradiction. Hence, H is 2-connected. By the induction hypothesis, we know that H contains a 2-connected bipartite spanning subgraph $B_H[X, Y]$. Since $d(v) \geq 3$, at least one of X and Y contains at least 2 neighbors of v. Hence, G contains a 2-connected bipartite spanning subgraph. □

Theorem 4.5. *Let G be a connected graph of order $n \geq 14$. If $\binom{n-3}{2} + 4 \leq |E(G)| \leq \binom{n}{2} - 1$, then $pc(G) = 2$.*

Proof. The result clearly holds if G is 3-connected by Corollary 2.4. We only consider of the graphs with connectivity at most 2. So we can partition $V(G)$ into three parts V_1, V_2, S such that $1 \leq |S| \leq 2$, $|V_1| \leq |V_2|$, and there is no edge between V_1 and V_2 in G.

If $|V_1| \geq 4$, then we must have $|V_1| = 4$, $|S| = 2$ and both $G[V_1 \cup S]$ and $G[V_2 \cup S]$ must be complete graphs since $n \geq 14$ and $|E(G)| \geq \binom{n-3}{2} + 4$. In this case, we can easily check that $pc(G) = 2$ from the structure of G. Thus we may assume that $|V_1| \leq 3$. It follows that $\delta(G) \leq 4$.

Let v be a vertex with the minimum degree in G, and let $H = G - v$. Then $|V(H)| = n - 1$ and $|E(H)| \geq \binom{n-3}{2} + 4 - 4 = \binom{n-3}{2}$. Note that if H is 3-connected, one can get that $pc(H) \leq 2$ by Corollary 2.4. Then from Lemma 2.7, one has that $pc(G) \leq 2$. So we may assume that the connectivity of H is at most 2. By the similar analysis, we can get that $\delta(H) \leq 3$.

Let u be a vertex with the minimum degree in H, and let $F = H-u = G-v-u$. Then $|V(F)| = n-2$ and $|E(F)| \geq \binom{n-3}{2}-3 = \binom{(n-2)-1}{2}-3$. If F is 2-connected, we know that F contains a bipartite 2-connected spanning subgraph by Lemma 4.4, and hence $pc(H) \leq 2$. By Lemma 2.7, we have that $pc(G) \leq 2$. Now we assume that the connectivity of F is at most 1. Since $|E(F)| \geq \binom{n-3}{2} - 3 = \binom{(n-2)-1}{2} - 3$, we know that F has a vertex w with $d_F(w) \leq 1$. Let $F' = F - w = G - u - v - w$, then $|E(F')| \geq \binom{n-3}{2} - 4$. From Lemma 4.3, we know that F' contains a 2-connected bipartite spanning subgraph, and so $pc(F') \leq 2$. If $d_G(w) = 1$, then u and v are also pendent vertices in G. We have that $|E(G)| \leq \binom{n-3}{2}+3$, which contradicts the fact that $|E(G)| \geq \binom{n-3}{2}+4$. Thus, $d_G(w) \geq 2$. If $uv \in E(G)$, one can see that $pc(G) = 2$ by Corollary 2.8. If $uv \notin E(G)$, we have that u has a neighbor in F'. Since otherwise, $d_G(u) = 1$ and $d_G(v) = 1$, $|E(G)| \leq \binom{n-3}{2}+3$, a contradiction. So we know that either v has a neighbor in F' or $wv \in E(G)$. By Corollary 2.8, we have that $pc(G) = 2$. The proof is thus complete.

□

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