J Comb Optim
DOI 10.1007/s10878-016-0052-6

Rainbow connection numbers of Cayley graphs

Yingbin Ma[1,2] · **Zaiping Lu**[2]

Abstract An edge colored graph is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number, rc-number for short, of a graph Γ, is the smallest number of colors that are needed in order to make Γ rainbow connected. In this paper, we give a method to bound the rc-numbers of graphs with certain structural properties. Using this method, we investigate the rc-numbers of Cayley graphs, especially, those defined on abelian groups and on dihedral groups.

Keywords Edge-coloring · Rainbow connection number · Cayley graph · Dihedral group · Interconnection networks

Mathematics Subject Classification 05C15 · 05C40

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of Bondy and Murty (2008) for those not described here.

✉ Yingbin Ma
mayingbincw@htu.cn

Zaiping Lu
lu@nankai.edu.cn

1 College of Mathematics and Information Science, Henan Normal University, XinXiang 453007, People's Republic of China

2 Center for Combinatorics LPMC-TJKLC, Nankai University, Tianjin 300071, People's Republic of China

Published online: 04 July 2016

For a graph Γ, we denote by $V\Gamma$ and $E\Gamma$ the vertex set and edge set of Γ, respectively. An *edge-coloring* of a graph Γ is a mapping from $E\Gamma$ to some finite set of colors. A path in an edge colored graph is said to be a *rainbow path* if no two edges on this path share the same color. An edge colored graph Γ is *rainbow connected* if each pair of distinct vertices of Γ are joined by some rainbow path, while the coloring is called a *rainbow coloring*. The *rainbow connection number* of a connected graph, *rc-number* for short, is the smallest number of colors that are needed in order to make the graph rainbow connected. For a connected graph Γ, we denote by $rc(\Gamma)$ its rc-number. Then, for a connected graph Γ with diameter $diam(\Gamma)$, it follows from the definition that

$$diam(\Gamma) \leq rc(\Gamma) \leq |E\Gamma|.$$

Moreover, $rc(\Gamma) \leq rc(\Sigma)$ for each connected spanning subgraph Σ of a graph Γ.

The concept of rc-number was introduced in Chartrand et al. (2008), where the rc-numbers of several graph classes were determined. Since then the study of rc-numbers has received considerable attention in the literature, see Li and Sun (2012) for a survey on this topic. It was shown in Chakraborty et al. (2011) that computing the rc-number of an arbitrary graph is an NP-Hard problem. Subsequently, there have been various investigations towards finding good upper (or lower) bounds for rc-numbers in terms of graph parameters such as connectivity, minimum degree, radius etc., see Basavaraju et al. (2014), Caro et al. (2008), Chandran et al. (2012), Krivelevich and Yuster (2009), Li et al. (2012), and Schiermeyer (2009) for example. In particular, it was shown in Li et al. (2012) that a 2-connected graph of order n has rc-number no more than $\lceil \frac{n}{2} \rceil$. In this paper, we focus our attention on a special class of 2-connected graphs.

Let G be a finite group with identity element 1, and let S be a subset of G such that $1 \notin S = S^{-1} := \{s^{-1} \mid s \in S\}$. The *Cayley graph* $\mathrm{Cay}(G, S)$ (on G with respect to S) is defined on G such that two 'vertices' g and h are adjacent if and only if $g^{-1}h \in S$. Then $\mathrm{Cay}(G, S)$ is a well-defined simple regular graph of valency $|S|$. It is well-known that $\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$, that is, S is a generating set of the underlying group G. A subset X of G is a *minimal generating set* if G is generated by X but not by any proper subset of X.

Cayley graphs have been an active topic in algebraic graph theory for a long time. In fact, interconnection networks are often modeled by highly symmetric Cayley graphs (Akers and Krishnamurthy 1989). The rainbow connection number of a graph can be applied to measure the safety of a network. Thus the object of the rainbow connection numbers of Cayley graphs should be meaningful. Li et al. (2011) established an upper bound for the rc-numbers of Cayley graphs on abelian groups by using minimal generating sets. For an element $x \in G$, denote by $|x|$ the order of x in G.

Theorem 1.1 (Li et al. 2011) *Let G be a finite abelian group, and let S be a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then $rc(\Gamma) \leq \sum_{x \in X} \lceil \frac{|x|}{2} \rceil$, where X is an arbitrary minimal generating set of G contained in S.*

This motivates us to consider the rc-numbers of Cayley graphs, especially, those defined on non-abelian groups. In Sect. 2 we establish a method to bound the rc-numbers of (Cayley) graphs satisfying certain structural properties, which leads to a new proof and an improvement of Theorem 1.1. In Sect. 3, applying the method given in Sect. 2, we investigate the rc-numbers of connected Cayley graphs on dihedral groups.

2 rc-numbers of Cayley graphs on abelian groups

Let Γ be a graph. For $V_1, V_2 \subseteq V\Gamma$, we denote by $\Gamma[V_1, V_2]$ the subgraph on $V_1 \cup V_2$ with edge set $\{\{v_1, v_2\} \in E\Gamma \mid v_1 \in V_1, v_2 \in V_2\}$. For a partition $\mathcal{B} = \{V_0, V_1, \ldots, V_{m-1}\}$ of $V\Gamma$, define a graph $\Gamma_{\mathcal{B}}$ on $\mathcal{B}$ such that distinct 'vertices' V_i and V_j are adjacent if and only if some $u \in V_i$ is adjacent to some $v \in V_j$ in Γ. The graph $\Gamma_{\mathcal{B}}$ is called a *quotient graph* of Γ. We first prove a technical lemma.

Lemma 2.1 *Let Γ be a connected graph. Assume that $V\Gamma$ has a partition $\mathcal{B} = \{V_0, V_1, \ldots, V_{m-1}\}$ such that, for each i, the subgraph $\Gamma[V_i, V_i]$ is connected.*

(i) *Suppose that, for $0 \le i < m$, $\Gamma[V_i, V_{i+1}]$is not empty, and every $u \in V_i$ is adjacent to some $v \in V_{i-1}$ or some $w \in V_{i+1}$ in Γ, reading the subscripts modulo m. Then*

$$rc(\Gamma) \le (\max\{rc(\Gamma[V_i, V_i]) \mid 0 \le i < m\} + 1)\left\lceil \frac{m}{2} \right\rceil.$$

(ii) *Suppose that, for $0 \le i, j < m$, every subgraph $\Gamma[V_i, V_j]$ has no isolated vertices provided that it has at least one edge. Then*

$$rc(\Gamma) \le \max\{rc(\Gamma[V_i, V_i]) \mid 0 \le i < m\} + rc(\Gamma_{\mathcal{B}}).$$

Proof Let $\Gamma_i = \Gamma[V_i, V_i]$ and $c = \max\{rc(\Gamma_i) \mid 0 \le i < m\}$.

(i) Now we are ready to show that Γ has a connected spanning subgraph which has rc-number no more than $(c+1)\lceil \frac{m}{2} \rceil$. Consider the spanning subgraph Σ of Γ with edge set

$$E\Sigma = (\cup_{i=0}^{m-1} E\Gamma_i) \cup (\cup_{i=0}^{m-1} E\Gamma[V_i, V_{i+1}]),$$

where $V_m = V_0$. (Note that $\Sigma_{\mathcal{B}}$ is a cycle of length m.) Let $C_0, C_1, \ldots, C_{m-1}$ be c-sets of colors such that $C_i \cap C_j = \emptyset$ if $0 \le i < j < \lceil \frac{m}{2} \rceil$, and $C_i = C_j$ if $i \equiv j \pmod{\lceil \frac{m}{2} \rceil}$. For each graph Γ_i, since $rc(\Gamma_i) \le c$, we choose a rainbow coloring $\theta_i : E\Gamma_i \to C_i$. Choose $\lceil \frac{m}{2} \rceil$ colors $c_1, c_2, \ldots, c_{\lceil \frac{m}{2} \rceil}$ which are not used above. We define an edge-coloring θ of Σ as follows:

$$\theta(e) = \begin{cases} \theta_i(e) & \text{if } e \in E\Gamma_i \text{ for } 0 \le i < m; \\ c_i & \text{if } e \in E\Gamma[V_{i-1}, V_i] \text{ for } 1 \le i \le \lceil \frac{m}{2} \rceil; \\ c_{j-\lceil \frac{m}{2} \rceil} & \text{if } e \in E\Gamma[V_{j-1}, V_j] \text{ for } \lceil \frac{m}{2} \rceil < j \le m. \end{cases}$$

It is straightforwardly checked that Γ is rainbow connected with the edge-coloring θ. Then the lemma follows from enumerating the number of colors used for θ.

(ii) Let C be a set of c colors and D be a set of $rc(\Gamma_{\mathcal{B}})$ colors with $C \cap D = \emptyset$. For each graph Γ_i, choose a rainbow coloring $\theta_i : E\Gamma_i \to C$. For $\Gamma_{\mathcal{B}}$, we choose a rainbow coloring $\bar{\theta} : E\Gamma_{\mathcal{B}} \to D$. We define an edge-coloring θ of Γ as follows:

$$\theta(e) = \begin{cases} \theta_i(e) & \text{if } e \in E\Gamma_i \text{ for } 0 \le i < m; \\ \bar{\theta}(\{V_i, V_j\}) & \text{if } \{V_i, V_j\} \in E\Gamma_{\mathcal{B}} \text{ and } e \in E\Gamma[V_i, V_j]. \end{cases}$$

Then Γ is rainbow connected with the edge-coloring θ. Thus

$$rc(\Gamma) \le |C \cup D| = c + rc(\Gamma_{\mathcal{B}}) = \max\{rc(\Gamma_i) \mid 0 \le i < m\} + rc(\Gamma_{\mathcal{B}}).$$

□

Let G be a group and N a normal subgroup of G. Then all (left) cosets of N in G form a group under the product

$$(gN)(hN) = ghN,$$

which is denoted by G/N and called *the quotient group* of G with respect to N.

Theorem 2.2 *Let G be a finite group and S a generating set of G such that $1 \notin S = S^{-1}$. Suppose that $X \subseteq S$ such that $N := \langle S \setminus (X \cup X^{-1})\rangle \neq G$. Set $\Gamma = \mathrm{Cay}(G, S)$, $Y = S \setminus (X \cup X^{-1})$ and $\Sigma = \mathrm{Cay}(N, Y)$. Suppose that N is normal in G. Then $rc(\Gamma) \le rc(\Sigma) + rc(\mathrm{Cay}(\bar{G}, \bar{X}))$, where $\bar{G} = G/N$ and $\bar{X} = \{xN \mid x \in S \setminus N\}$.*

Proof Since N is normal in G, we have $G = \langle X, Y\rangle \le \langle X, N\rangle = \langle X\rangle N$, and so $G = \langle X\rangle N$. Let m be the index of N in G. Then $m = \frac{|G|}{|N|}$. Let $g_0N = N, g_1N, \ldots g_iN, \ldots, g_{m-1}N$ be all distinct left cosets of N in G. Set $V_i = g_iN$ for $0 \le i < m$. Then $\mathcal{B} = \{V_i \mid 0 \le i < m\}$ is a partition of $V\Gamma$. It is easily shown that $V_0 \to V_i, g \mapsto g_ig$ is an isomorphism from $\Gamma[V_0, V_0]$ to $\Gamma[V_i, V_i]$. Thus each subgraph $\Gamma[V_i, V_i]$ contains a spanning subgraph isomorphic to the connected Cayley graph $\Sigma = \mathrm{Cay}(N, Y)$, and so $rc(\Gamma[V_i, V_i]) \le rc(\Sigma)$.

Note that $gNh = ghN$ for $\forall g, h \in G$. Assume that $\Gamma[V_i, V_j]$ is not empty, where $i \neq j$. Then there are some $g, h \in N$ and $x \in S \setminus N$ such that $g_igx = g_jh$. Thus

$$g_iNx = g_igNx = g_igx(x^{-1}Nx) = g_igxN = g_jhN = g_jN.$$

It follows that $\Gamma[V_i, V_j]$ contains a perfect matching, and so $\Gamma[V_i, V_j]$ has no isolated vertices. By Lemma 2.1 (ii), $rc(\Gamma) \le rc(\Sigma) + rc(\Gamma_{\mathcal{B}})$. Consider the quotient graph $\Gamma_{\mathcal{B}}$. Then V_i and V_j are adjacent if and only if $g_jN = g_iNx = (g_iN)(xN)$ for some $x \in S \setminus N$. It follows that $\Gamma_{\mathcal{B}} = \mathrm{Cay}(\bar{G}, \bar{X})$, and hence the result follows. □

Recall that a graph is called *vertex transitive* if for any two vertices there is an automorphism of the graph mapping one vertex to the other one. It is well-known that a connected vertex transitive graph of order no less than three must be 2-connected (see Godsil and Royle 2001, Theorem 3.4.2). Thus, by Li et al. (2012, Theorem 2.4), if Γ is a connected vertex transitive graph then $rc(\Gamma) \le \lceil\frac{|V\Gamma|}{2}\rceil$. Note that a Caley graph must be vertex-transitive. Then the next two results follow from Theorem 2.2 directly.

Corollary 2.3 *Let Γ, G and N be as in Theorem 2.2. Then* $rc(\Gamma) \le \lceil \frac{|N|}{2} \rceil + \lceil \frac{|G|}{2|N|} \rceil$.

Corollary 2.4 *Let G be a finite abelian group and S a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then either*

(i) *G is cyclic and S consists of generators of G; or*
(ii) *there are two proper divisors m and n of $|G|$ such that $|G| = mn$ and $rc(\Gamma) \le \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$.*

Proof If $\langle x \rangle = G$ for each $x \in S$, then part (i) follows. Thus we assume that there are $x \in S$ and $Y \subseteq S$ such that $|Y| \ge 1$ and $\langle x, Y \rangle = G$ but $\langle Y \rangle \ne G$. Set $N = \langle Y \rangle$. Then, by Theorem 2.2, $rc(\Gamma) \le rc(\mathrm{Cay}(G, Y \cup Y^{-1} \cup \{x, x^{-1}\})) \le rc(\mathrm{Cay}(N, Y \cup Y^{-1})) + rc(\mathrm{Cay}(G/N, \{xN, x^{-1}N\}))$. Note that the Cayley graph $\mathrm{Cay}(G/N, \{xN, x^{-1}N\})$ is either a cycle or the complete graph on two vertices. Then part (ii) follows by setting $|N| = m$ and $|G/N| = n$. □

Now we give a new proof of a known result by using the above simple lemma.

Theorem 2.5 (Li et al. 2011) *Let G be a finite abelian group and S a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then $rc(\Gamma) \le \sum_{x \in X} \lceil \frac{|x|}{2} \rceil$, where X is an arbitrary minimal generating set of G contained in S.*

Proof We prove the result by induction on the orders of groups. Let X be an arbitrary minimal generating set of G contained in S. Take $x \in X$, set $Y = X \setminus \{x\}$ and $N = \langle Y \rangle$. Then $G = \langle X \rangle = N\langle x \rangle$, and $|G/N| \le |\langle x \rangle| = |x|$. By Theorem 2.2, $rc(\mathrm{Cay}(G, X \cup X^{-1})) \le rc(\mathrm{Cay}(N, Y \cup Y^{-1})) + \lceil \frac{|x|}{2} \rceil$. Since $|N| < |G|$, by induction, we may assume that the result holds for $\mathrm{Cay}(N, Y \cup Y^{-1})$. It is easily shown that Y is also a minimal generating set of N. Thus $rc(\mathrm{Cay}(N, Y \cup Y^{-1})) \le \sum_{y \in Y} \lceil \frac{|y|}{2} \rceil$, and so $rc(\mathrm{Cay}(G, X \cup X^{-1})) \le \sum_{x \in X} \lceil \frac{|x|}{2} \rceil$. Then the theorem follows because $rc(\Gamma) \le rc(\mathrm{Cay}(G, X \cup X^{-1}))$. □

For integers $n \ge 1$ and $m \ge 3$, denote by $\mathbb{Z}_n$ the cyclic group of order n, and by $\mathbf{C}_m$ the cycle of length m. For graphs $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$, the *Cartesian product* $\Sigma_1 \Box \Sigma_2 \Box \cdots \Box \Sigma_r$ is the graph defined on $V\Sigma_1 \times \cdots \times V\Sigma_r$ such that two vertices $(u_1, u_2, \ldots, u_r)$ and $(v_1, v_2, \ldots, v_r)$ are adjacent if and only if there is some $1 \le i \le r$ such that $\{u_i, v_i\} \in E\Sigma_i$ and $u_j = v_j$ for all $j \ne i$.

Theorem 2.6 *Let G be a finite abelian group and S a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then either*

(i) *$rc(\Gamma) < \min\{\sum_{x \in X} \lceil \frac{|x|}{2} \rceil \mid X \subseteq S$ is a minimal generating set of $G\}$; or*
(ii) *$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ and Γ has a connected spanning subgraph $\Sigma_1 \Box \Sigma_2 \Box \cdots \Box \Sigma_r$, where Σ_i is either the n_i-cycle $\mathbf{C}_{n_i}$ if $n_i \ge 3$, or the complete graph K_2 on two vertices if $n_i = 2$.*

Proof By Theorem 2.5, either part (i) follows or there is a minimal generating set $X = \{x_1, x_2, \dots, x_r\}$ of G such that $rc(\Gamma) = \sum_{i=1}^{r} \lceil \frac{|x_i|}{2} \rceil$, where X is a subset of S. Assume that the latter case occurs. Note that the minimal generating set X is a subset of S. Set $Y = X \setminus \{x_r\}$, $N = \langle Y \rangle$ and $\Sigma = \mathrm{Cay}(N, Y \cup Y^{-1})$. Then Y is a minimal generating set of N. By Theorems 2.2 and 2.5,

$$\sum_{i=1}^{r} \left\lceil \frac{|x_i|}{2} \right\rceil = rc(\Gamma) \le rc(\Sigma) + \left\lceil \frac{|G|}{2|N|} \right\rceil \le rc(\Sigma) + \left\lceil \frac{|x_r|}{2} \right\rceil \le \sum_{i=1}^{r} \left\lceil \frac{|x_i|}{2} \right\rceil.$$

It follows that $rc(\Sigma) = \sum_{i=1}^{r-1} \lceil \frac{|x_i|}{2} \rceil$ and $\lceil \frac{|G|}{2|N|} \rceil = \lceil \frac{|x_r|}{2} \rceil$. Since $G = \langle x_r, Y \rangle = \langle x_r \rangle N$, we have $|G| = |\langle x_r \rangle N| = \frac{|x_r||N|}{|\langle x_r \rangle \cap N|}$, and hence $\frac{|G|}{|N|}$ is a divisor of $|x_r|$. Thus $\lceil \frac{|G|}{2|N|} \rceil = \lceil \frac{|x_r|}{2} \rceil$ implies that either $\frac{|G|}{|N|} = |x_r|$ or $\frac{\frac{|G|}{|N|}+1}{2} = \frac{|x_r|}{2}$. The latter case implies that $\frac{|G|}{|N|}$ and $|x_r|$ are coprime. It follows that $\frac{|G|}{|N|} = 1$ as $\frac{|G|}{|N|}$ is a divisor of $|x_r|$, yielding $G = N$, a contradiction. Therefore, $\frac{|G|}{|N|} = |x_r|$. Thus $|\langle x_r \rangle \cap N| = \frac{|x_r||N|}{|G|} = 1$. Then $G = N\langle x_r \rangle = N \times \langle x_r \rangle$. Since Y is a minimal generating set of N, by induction on r, we may assume that $N = \langle x_1 \rangle \times \dots \times \langle x_{r-1} \rangle$. Thus $G = \langle x_1 \rangle \times \dots \times \langle x_{r-1} \rangle \times \langle x_r \rangle \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_r}$, where $n_i = |x_i|$ for $1 \le i \le r$. Notice that each element of G has the form of $x_1^{e_1} \dots x_i^{e_i} \dots x_r^{e_r}$ for integers $e_1, e_2, \dots, e_r$. Consider the connected spanning subgraph $\mathrm{Cay}(G, X \cup X^{-1})$ of Γ. Then two vertices $x_1^{e_1} \dots x_i^{e_i} \dots x_r^{e_r}$ and $x_1^{f_1} \dots x_i^{f_i} \dots x_r^{f_r}$ are adjacent in $\mathrm{Cay}(G, X \cup X^{-1})$ if and only if there is some $1 \le j \le r$ such that $e_j - f_j \equiv \pm 1 \pmod{n_j}$ and $e_i - f_i \equiv 0 \pmod{n_i}$ for $i \ne j$. Set $\Sigma_i = \mathrm{Cay}(\langle x_i \rangle, \{x_i, x_i^{-1}\})$ for $1 \le i \le r$. Then $\mathrm{Cay}(G, X \cup X^{-1}) = \Sigma_1 \Box \Sigma_2 \Box \dots \Box \Sigma_r$, and so part (ii) follows. □

3 rc-numbers of Cayley graphs on dihedral groups

Let $n \ge 1$ be an integer. We use D_{2n} to denote *the dihedral group* generated by two elements, say a and b, such that

$$|a| = n,\ |b| = 2,\ b^{-1}ab = a^{-1}.$$

(Note that $\mathrm{D}_2 = \mathbb{Z}_2$ and $\mathrm{D}_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.) Then

$$\mathrm{D}_{2n} = \langle a \rangle \cup \langle a \rangle b = \{a^i \mid 0 \le i < n\} \cup \{a^i b \mid 0 \le i < n\}.$$

Let $\mathbf{Z}(\mathrm{D}_{2n})$ be the center of D_{2n}. Then, for $n \ge 3$, either $\mathbf{Z}(\mathrm{D}_{2n}) = 1$, or $\mathbf{Z}(\mathrm{D}_{2n}) = \langle a^{\frac{n}{2}} \rangle$ while n is even. For convenience, we collect some basic facts about dihedral groups by considering the involutions, elements of order 2 in D_{2n}.

Lemma 3.1 (i) *For* $0 \le i \le n-1$, *each* $a^i b$ *is an involution.*

(ii) *If* n *is odd, then* D_{2n} *has a unique conjugacy class of involutions, which is* $\{a^i b \mid 0 \le i \le n-1\}$.

(iii) *If n is even, then* D_{2n} *has exactly three conjugacy classes of involutions, which are* $\{a^{\frac{n}{2}}\}$, $\{a^{2i}b \mid 0 \le i < \frac{n}{2}\}$ *and* $\{a^{2i+1}b \mid 0 \le i < \frac{n}{2}\}$.

(iv) *If m is a divisor of n then $\langle a\rangle$ has a unique subgroup of order m, which is* $\langle a^{\frac{n}{m}}\rangle$. *If $N \le \langle a\rangle$, then N is normal in* D_{2n} *and the quotient group* D_{2n}/N *is a dihedral group generated by* $\{aN, bN\}$.

(v) *If X is a (minimal) generating set of* D_{2n}, *then X contains some involution $a^s b$, and* $(X \cap \langle a\rangle) \cup \{xa^s b \mid a^s b \ne x \in X \setminus \langle a\rangle\}$ *is a (minimal) generating set of* $\langle a\rangle$.

(vi) *Set* $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ *for distinct primes p_i. If Y is a minimal generating set of $\langle a\rangle$, then $|Y| \le r$. If X is a minimal generating set of* D_{2n}, *then* $|X| \le r+1$.

Proof Items (i)–(iv) can be found in most text books about elementary group theory. Here we only prove items (v) and (vi).

Let X be an arbitrary generating set of D_{2n}. Then $\langle a\rangle \ne \mathrm{D}_{2n} = \langle X\rangle$, yielding $a^s b \in X$ for some integer s. Set $Y = (X \cap \langle a\rangle) \cup \{xa^s b \mid a^s b \ne x \in X \setminus \langle a\rangle\}$. Then $Y \subseteq \langle a\rangle$ as $xa^s b \in \langle a\rangle$ for $x \in X \setminus \langle a\rangle$, and so $\langle Y\rangle$ is normal in D_{2n}. Thus

$$\mathrm{D}_{2n} = \langle X\rangle = \langle X \cap \langle a\rangle, X \setminus \langle a\rangle\rangle = \langle Y, a^s b\rangle = \langle Y\rangle\langle a^s b\rangle.$$

It follows that $\langle a\rangle = \langle Y\rangle$. Assume that Y is not a minimal generating set of $\langle a\rangle$. Then $\langle Y \setminus \{y\}\rangle = \langle a\rangle$ for some $y \in Y$. If $y \in X \cap \langle a\rangle$ then $\mathrm{D}_{2n} = \langle X \setminus \{y\}\rangle$, and so X is not a minimal generating set of D_{2n}. Suppose that $y = za^s b$ for some $z \in X \setminus \langle a\rangle$ with $z \ne a^s b$. Set $Y_1 = Y \setminus \{za^s b\}$. Then

$$\langle X \setminus \{z\}\rangle = \langle X \cap \langle a\rangle, X \setminus (\langle a\rangle \cup \{z, a^s b\}), a^s b\rangle = \langle Y_1\rangle\langle a^s b\rangle = \langle Y\rangle\langle a^s b\rangle = \mathrm{D}_{2n}.$$

It implies that X is not a minimal generating set of D_{2n}. Thus item (v) holds.

Now set $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ for distinct primes p_i. For $1 \le i \le r$, set $a_i = a^{\prod_{j\ne i} p_j^{e_j}}$. Then $\langle a\rangle = \langle a_i \mid 1 \le i \le r\rangle = \langle a_1\rangle \times \cdots \times \langle a_r\rangle$, and so each element in $\langle a\rangle$ can be written as a product of the form of $\prod_{i=1}^r a_i^{m_i}$. Let $Y = \{y_j \mid 1 \le j \le t\}$ be an arbitrary generating set of $\langle a\rangle$. Write $y_j = \prod_{i=1}^r a_i^{f_i(y_j)}$, where all f_i are integral valued functions on Y. Then

$$\langle a\rangle = \langle Y\rangle = \langle a_i^{f_i(y_j)} \mid 1 \le i \le r, 1 \le j \le t\rangle.$$

For each $1 \le j \le r$, we choose one element $y \in Y$ such that $|a_j^{f_j(y)}| = \max\{|a_j^{f_j(y_i)}| \mid 1 \le i \le t\}$. Thus we obtain r elements contained in Y which are not necessarily distinct, say $y_1', y_2', \dots, y_r'$. Set $Y_1 = \{y_j' \mid 1 \le j \le r\}$. Then

$$\langle Y_1\rangle = \langle a_i^{f_i(y_j')} \mid 1 \le i, j \le r\rangle.$$

By item (iv) and the choice of elements $a_j^{f_j(y_j')}$, we have

$$\langle a_i^{f_i(y_j')} \mid 1 \le i, j \le r\rangle = \langle a_j^{f_j(y_j')} \mid 1 \le j \le r\rangle = \langle a_i^{f_i(y_j)} \mid 1 \le i \le r, 1 \le j \le t\rangle.$$

It follows that $\langle Y_1\rangle = \langle Y\rangle = \langle a\rangle$. Thus each generating set of $\langle a\rangle$ contains a generating set of size no more than r. Then the first part of item (vi) follows, while the second part follows from item (v). $\square$

It is well-known and easily shown that the ladder graph L_n and Möbius ladder graph M_n (of order $2n$) are cubic Cayley graphs on the dihedral group D_{2n}. Cai et al. (to appear) proved that

$$rc(L_n) = \left\lceil \frac{n+1}{2} \right\rceil \text{ and } \quad rc(M_n) = \left\lceil \frac{n}{2} \right\rceil.$$

In this section we investigate the rc-numbers of the whole class of connected Cayley graphs on dihedral groups. By Theorem 2.2, the following lemma holds.

Lemma 3.2 *Let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected Cayley graph. Then*

$$rc(\Gamma) \le rc(\mathrm{Cay}(\langle S_1\rangle, S_1)) + rc(\mathrm{Cay}(\mathrm{D}_{2m}, \bar{S})),$$

where $S_1 = \langle a\rangle \cap S$, $m = \frac{n}{|\langle S_1\rangle|}$, $\bar{S}$ is a set of involutions in D_{2m} and either $m \le 2$ or $\bar{S} \cap \mathbf{Z}(\mathrm{D}_{2m}) = \emptyset$.

Note that, in the above lemma, $\mathrm{Cay}(\langle S_1\rangle, S_1)$ is a Cayley graph on a cyclic group. Thus, in view of Theorem 2.6, we next consider the Cayley graphs $\mathrm{Cay}(\mathrm{D}_{2n}, S)$ under the assumption that $S \cap \langle a\rangle = \emptyset$.

Theorem 3.3 *Let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected Cayley graph. Assume that $S = \{a^{e_i} b \mid 1 \le i \le r\}$ for integers $e_1, e_2, \ldots, e_r$. Then one of the following holds.*

- (i) *Γ is a cycle;*
- (ii) *$r \ge 3$, n is odd and $e_i - e_j$ are coprime to n, where $i \ne j$ and $1 \le i, j \le r$;*
- (iii) *$rc(\Gamma) \le (rc(Cay(\mathrm{D}_{2l}, X))+1)\lceil \frac{m}{2}\rceil$ for some divisors l and m of n with $l, m \ge 2$ and $n = lm$, where X is a set of involutions in D_{2l} such that $|X| \le |S| - 1$ and either $l = 2$ or $X \cap \mathbf{Z}(\mathrm{D}_{2l}) = \emptyset$.*

Proof We assume first that D_{2n} is generated by any 2-subset of S. If $r = 2$ then part (i) follows. Suppose that $r \ge 3$. By the assumption, for distinct i and j,

$$\mathrm{D}_{2n} = \langle a^{e_i} b, a^{e_j} b\rangle = \langle a^{e_i - e_j}, a^{e_j} b\rangle = \langle a^{e_i - e_j}\rangle\langle a^{e_j} b\rangle.$$

It follows that $\langle a\rangle = \langle a^{e_i - e_j}\rangle$, which yields that $e_i - e_j$ is coprime to n. Since $r \ge 3$, take $k \ne i, j$. Then $e_i - e_j$, $e_i - e_k$ and $e_j - e_k$ are coprime to n. Note that one of $e_i - e_j$, $e_i - e_k$ and $e_j - e_k$ must be even. It implies that n is odd.

Let $G = \mathrm{D}_{2n}$. Now we may assume that there are $x \in S$ and $Y \subseteq S$ such that $|Y| \ge 2$ and $\langle x, Y\rangle = G$ but $\langle Y\rangle \ne G$. Without loss of generality, we may set $x = a^{e_r} b$ and $Y = \{a^{e_j} b \mid 1 \le j \le s\}$, where $2 \le s \le r - 1$. Let $H = \langle Y\rangle$. Then $H = \langle Y\rangle = \langle Y a^{e_s} b, a^{e_s} b\rangle$. Observe that $Y a^{e_s} b \setminus \{1\} = \{a^{e_j - e_s} \mid 1 \le j \le s - 1\} \subset \langle a\rangle$, we conclude that $H = \langle Y\rangle = L\langle a^{e_s} b\rangle$, where $L = \langle a^e\rangle$ for some integer e. Clearly, $|H| \ge 4$, $l := |a^e| = |L| = \frac{|H|}{2} \ge 2$ and $m := \frac{|G|}{|H|} = \frac{n}{l} \ge 2$. Moreover,

$a^m \in L = \langle a^e \rangle$ and H has m left cosets in G. Note that $G = \langle a, b \rangle = \langle a, a^{e_s} b \rangle = \langle a \rangle \langle a^{e_s} b \rangle$. Then each left coset of H has the form of $a^j H$ for some $0 \le j < m$. Notice that

$$G = \langle x, Y \rangle = \langle x, H \rangle = \langle a^{e_r} b, a^e, a^{e_s} b \rangle = \langle a^e, a^{e_s - e_r} \rangle \langle x \rangle = \langle a^e, a^{e_r - e_s} \rangle \langle x \rangle.$$

It follows that $\langle a \rangle = \langle a^e, a^{e_r - e_s} \rangle$. Thus for each $0 \le j < m$, there are some integers j_1 and j_2 such that $a^j = (a^{e_r - e_s})^{j_1} (a^e)^{j_2}$, and so $a^j H = a^{j_1(e_r - e_s)} H$. Recalling $a^m \in \langle a^e \rangle \le H$, we have $a^{m(e_r - e_s)} \in H$. It follows that each left coset of H has the form of $a^{i(e_r - e_s)} H$ for some $0 \le i < m$. Set $V_i = a^{i(e_r - e_s)} H$ for $0 \le i < m$. Then each subgraph $\Gamma[V_i, V_i]$ contains a spanning subgraph isomorphic to the connected Cayley graph $\mathrm{Cay}(H, Y)$.

Note that

$$H = \langle a^e \rangle \cup \langle a^e \rangle a^{e_s} b = \langle a^e \rangle \cup \langle a^e \rangle a^{e_s - e_r} a^{e_r} b = \langle a^e \rangle \cup a^{e_s - e_r} \langle a^e \rangle x.$$

Then

$$V_i = a^{i(e_r - e_s)} H = a^{i(e_r - e_s)} \langle a^e \rangle \cup a^{(i-1)(e_r - e_s)} \langle a^e \rangle x.$$

Set $V_i^1 = a^{i(e_r - e_s)} \langle a^e \rangle$ and $V_i^2 = a^{(i-1)(e_r - e_s)} \langle a^e \rangle x$. Then $\Gamma[V_i^1, V_{i+1}^2]$ contains a perfect matching (on $V_i^1 \cup V_{i+1}^2$), and $\Gamma[V_{i-1}^1, V_i^2]$ contains a perfect matching (on $V_{i-1}^1 \cup V_i^2$). Thus, noting that $H \cong \mathrm{D}_{2l}$, the result follows from Lemma 2.1 (i). □

Recall that a cycle of length $2m$ has rc-number m. Then we get the following consequence of Theorem 3.3 by induction on the sizes of minimal generating sets of D_{2n}.

Corollary 3.4 *Let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected Cayley graph with n even. Assume that $S = \{a^{e_i} b \mid 1 \le i \le r\}$ is a minimal generating set of D_{2n}. Then either Γ is a cycle of length $2n$, or*

$$rc(\Gamma) \le m_1 \prod_{j=2}^{r-1} \left\lceil \frac{m_j}{2} \right\rceil + \sum_{i=2}^{r-1} \prod_{j=i}^{r-1} \left\lceil \frac{m_j}{2} \right\rceil$$

for some divisors m_j of n with $n = \prod_{j=1}^{r-1} m_j$ and $m_j \ge 2$ for $1 \le j \le r-1$.

In the next result we investigate the rc-numbers of cubic Cayley graphs on dihedral groups.

Theorem 3.5 *Let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected cubic Cayley graph. Then one of the following cases occurs.*

(i) $rc(\Gamma) = \lceil \frac{n+1}{2} \rceil$, *and Γ is the ladder graph of order $2n$.*
(ii) $rc(\Gamma) = \lceil \frac{n}{2} \rceil$, *and Γ is the Möbius ladder of order $2n$.*
(iii) $\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, \{b, a^s b, a^t b\})$ *for some integers s and t, and either*

(iii.1) $rc(\Gamma) \le (l+1)\lceil \frac{m}{2} \rceil$, *where* $l \in \{|a^s|, |a^t|, |a^{s-t}|\}$ *and* $m = \frac{n}{l} \ge 2$*; or*
(iii.2) n *is odd, and* s, t *and* $s-t$ *are coprime to* n.

Proof Since $|S| = 3$ and $S^{-1} = S$, we conclude that S contains an involution z of D_{2n}. Set $S = \{x, y, z\}$. Then either $x = y^{-1}$ has order no less than 3, or x, y and z are distinct involutions. Thus one of the following cases occurs:

(1) $x = y^{-1}$ has order no less than 3;
(2) n is even, x, y and z are involutions, and $a^{\frac{n}{2}} \in S$.
(3) $x, y, z \in \{a^i b \mid 0 \le i \le n-1\}$.

We next prove this theorem by dealing with these three cases separately.

Assume that (1) holds. Since Γ is connected, $\langle S \rangle = \mathrm{D}_{2n}$. It follows from the assumption that $x = a^i$ and $z = a^j b$ for some integers i and j. Then $zxz = x^{-1}$ and $\mathrm{D}_{2n} = \langle x, z \rangle = \langle x \rangle \langle z \rangle$. Thus $2n = |D_{2n}| = |\langle x \rangle \langle z \rangle| = |x||z| = 2|x|$, yielding $|x| = n$. Then Γ has two cycles of length n, say $(1, x, x^2, \ldots, x^k, \ldots, x^{n-1}, 1)$ and $(z, zx, \ldots, zx^k, \ldots, zx^{n-1}, z)$. Moreover, Γ is constructed from these two cycles by adding a perfect matching $\{\{x^i, zx^{n-i}\} \mid 0 \le i < n\}$. This says that Γ is (isomorphic to) the ladder graph of order $2n$. By Cai et al. (to appear), $rc(\Gamma) = \lceil \frac{n+1}{2} \rceil$, and so part (i) of this theorem occurs.

Assume that (2) holds. Without loss of generality, we may set $z = a^{\frac{n}{2}}$ and write $x = a^s b$ and $y = a^t b$ for some integers s and t. Thus $xy = a^{s-t}$ and $\langle x, y \rangle = \langle xy \rangle \langle x \rangle$.

Suppose that $\langle x, y \rangle$ is a proper subgroup of D_{2n}. Since Γ is connected, $\mathrm{D}_{2n} = \langle x, y, z \rangle$. Note that $z = a^{\frac{n}{2}}$ lies in the center of D_{2n}. Thus $\mathrm{D}_{2n} = \langle x, y, z \rangle = \langle z \rangle \langle x, y \rangle \ne \langle x, y \rangle$. It implies that $|\langle x, y \rangle| = n$. For $0 \le i < \frac{n}{2}$, set $u_{2i} = (xy)^i$, $u_{2i+1} = (xy)^i x$, $w_{2i} = (xy)^i z$ and $w_{2i+1} = (xy)^i xz$. Then Γ is constructed from two cycles $(u_0, u_1, \ldots, u_{n-1}, u_0)$ and $(w_0, w_1, \ldots, w_{n-1}, w_0)$ by adding a perfect matching $\{\{u_i, w_i\} \mid 0 \le i < n\}$. Thus Γ is the the ladder graph of order $2n$ and, by Cai et al. (to appear), part (i) of this theorem occurs.

Suppose that $\mathrm{D}_{2n} = \langle x, y \rangle$. Then $|xy| = n$, and so $s-t$ is coprime to n; in particular, $s-t$ is odd. It follows that $z = a^{\frac{n}{2}} = (a^{s-t})^{\frac{n}{2}} = (xy)^{\frac{n}{2}}$. Set $v_{2i} = (xy)^i$ and $v_{2i+1} = (xy)^i x$ for $0 \le i < n$. Then Γ has a hamiltonian cycle $(v_0, v_1, v_2, \ldots, v_{2n-1}, v_0)$ and a perfect matching $\{\{v_i, v_{i+n}\} \mid 0 \le i < n\}$. Hence Γ is (isomorphic to) the Möbius ladder of order $2n$. By Cai et al. (to appear), part (ii) of this theorem occurs.

Now we deal with case (3). By Lemma 3.1(ii) and (iii), one of x, y and z is conjugate to b. Without loss of generality, we assume that $b = g^{-1}zg$ for some $g \in \mathrm{D}_{2n}$. Write $g^{-1}xg = a^s b$ and $g^{-1}yg = a^t b$. Set $T = \{b, a^s b, a^t b\}$ and $\Sigma = \mathrm{Cay}(\mathrm{D}_{2n}, T)$. It is easily shown that $V\Gamma \to V\Sigma, h \mapsto g^{-1}hg$ is an isomorphism from Γ to Σ. Note that two isomorphic graphs have the same rc-number. Thus $rc(\Gamma) = rc(\Sigma)$, and part (iii) of this theorem follows from Theorem 3.3 and Li et al. (2012, Theorem 2.4). □

We end this section by considering the rc-numbers of Cayley graphs on $G = \mathrm{D}_{2p^k}$ or D_{2pq}, where $k \ge 1$ is an integer, p and q are distinct primes.

Let X be a minimal generating set of G, $S = X \cup X^{-1}$ and $\Gamma = \mathrm{Cay}(G, S)$. Then $2 \le |X| \le 3$ by Lemma 3.1 (vi). Suppose that $|X| = 2$. Then either $X = \{a^i b, a^j b\}$ or $X = \{a^i, a^j b\}$ for some integers i and j. Thus $S = X$ or $S = \{a^i, a^{-i}, a^j b\}$. It follows that Γ is either a cycle or a ladder graph.

Therefore, we assume next that $|X| = 3$. By Lemma 3.1 (v), $\langle a\rangle$ has a minimal generating set of size 2. It follows that $|a| = |\langle a\rangle|$ is not a power of some prime. Thus $|a| = pq$ and $G = \mathrm{D}_{2pq}$. By Lemma 3.1 (v), we have $a^s b \in X$ for some integer s. Suppose that $\langle a\rangle \cap X = \emptyset$. Then $X = \{a^r b, a^t b, a^s b\}$ with $\{|a^{r-s}|, |a^{t-s}|\} = \{p, q\}$, where r and t are integers. It follows that $a^{r-t} = a^{r-s}a^{-(t-s)}$ has order pq, and hence $\langle a^r b, a^t b\rangle = \mathrm{D}_{2pq}$, a contradiction. Thus $\langle a\rangle \cap X \neq \emptyset$, and hence one of the following cases occurs:

I. $X = \{a^r, a^t, a^s b\}$ with $\{|a^r|, |a^t|\} = \{p, q\}$.
II. $X = \{a^r, a^t b, a^s b\}$ with $\{|a^r|, |a^{t-s}|\} = \{p, q\}$.

For case I, by Theorems 2.2 and 2.5, we get $rc(\Gamma) \leq \lceil \frac{p}{2}\rceil + \lceil \frac{q}{2}\rceil + 1$. In the following we discuss case II.

Assume case II occurs. Set $|a^r| = l, |a^{t-s}| = m$ and $H = \langle a^t b, a^s b\rangle$. Then $H \cong \mathrm{D}_{2m}, \mathrm{D}_{2pq} = \langle a^r\rangle H$ and $\langle a^r\rangle \cap H = 1$. In particular, every element of D_{2pq} has the form of $a^{ir}h$, where $0 \leq i < l$ and $h \in H$. Thus it is easily shown that $\Gamma = \Sigma_1 \square \Sigma_2$, where $\Sigma_1 = \mathrm{Cay}(\langle a^r\rangle, \{a^r, a^{-r}\}) \cong \mathsf{K}_2$ or $\mathbf{C}_l$, and $\Sigma_2 = \mathrm{Cay}(H, \{a^s b, a^t b\}) \cong \mathbf{C}_{2m}$. Suppose $\Sigma_1 \cong \mathsf{K}_2$. Then Γ is isomorphic to the ladder graph L_{2m}. Suppose $\Sigma_1 \cong \mathbf{C}_l$. By Liang (2012), $rc(\mathbf{C}_l \square \mathbf{C}_{2m}) = \frac{l-1}{2} + m$. Therefore, $rc(\Gamma) = \frac{l-1}{2} + m$ with $\{l, m\} = \{p, q\}$.

By the foregoing argument we obtain the following result.

Theorem 3.6 *Let $G = \mathrm{D}_{2p^k}$ or D_{2pq}, where $k \geq 1$ is an integer, p and q are distinct primes. Let X be a minimal generating set of G. Set $S = X \cup X^{-1}$ and $\Gamma = \mathrm{Cay}(G, S)$. Then one of the following statements holds.*

(i) *Γ is either a cycle or a ladder graph.*
(ii) *$G = \mathrm{D}_{2pq}$, $|X| = 3$ and either*
(ii.1) *$|\langle a\rangle \cap X| = 2$ and $rc(\Gamma) \leq \lceil \frac{p}{2}\rceil + \lceil \frac{q}{2}\rceil + 1$; or*
(ii.2) *$|\langle a\rangle \cap X| = 1$ and $rc(\Gamma) = \lfloor \frac{l}{2}\rfloor + m$ with $\{l, m\} = \{p, q\}$.*

Acknowledgments The authors are very grateful to the referees for helpful comments and suggestions. This work partially supported by the NSFC (Nos. 11271267, 11371204, 11526082, 11526081 and 11501181), and the Scientific Research Foundation for Ph.D. of Henan Normal University (No. qd14143).

References

Akers SB, Krishnamurthy B (1989) A group-theoretic model for symmetric interconnection networks. IEEE Trans Comput 38:555–566

Basavaraju M, Chandran LS, Rajendraprasad D, Ramaswamy A (2014) Rainbow connection number and radius. Graphs Combin 30(2):275–285

Bondy JA, Murty USR (2008) Graph theory. Springer, Berlin

Cai QQ, Ma YB, Song JL. Rainbow connection numbers of ladders and Möbius ladders. Ars Combin (to appear)

Caro Y, Lev A, Roditty Y, Tuza Z, Yuster R (2008) On rainbow connection. Electron J Combin 15:R57

Chakraborty S, Fischer E, Matsliah A, Yuster R (2011) Hardness and algorithms for rainbow connection. J Comb Optim 21:330–347

Chandran LS, Das A, Rajendraprasad D, Varma NM (2012) Rainbow connection number and connected dominating sets. J Graph Theory 71:206–218

Chartrand G, Johns GL, McKeon KA, Zhang P (2008) Rainbow connection in graphs. Math Bohem 133:85–98

Godsil C, Royle G (2001) Algebraic graph theory. Springer, New York
Krivelevich M, Yuster R (2009) The rainbow connection of a graph is (at most) reciprocal to its minimum degree. J Graph Theory 63:185–191
Li HZ, Li XL, Liu SJ (2011) The (strong) rainbow connection numbers of Cayley graphs on Abelian groups. Comput Math Appl 62:4082–4088
Li XL, Liu SJ, Chandran LS, Mathew R, Rajendraprasad D (2012) Rainbow connection number and connectivity. Electron J Combin 19:R20
Li XL, Sun YF (2012) Rainbow connections of graphs. Springer, New York
Liang YJ (2012) Rainbow connection numbers of Cartesian product of graphs. 2012 Workshop on Graph Theory and Combinatorics and 2012 Symposium for Young Combiantorialists, August, pp. 10–12
Schiermeyer I (2009) Rainbow connection in graphs with minimum degree three, IWOCA 2009. LNCS, vol 5874. Springer, Berlin, pp 432–437