Some upper bounds for the 3-proper index of graphs*

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Abstract

A tree T in an edge-colored graph is a *proper tree* if no two adjacent edges of T receive the same color. Let G be a connected graph of order n and k be a fixed integer with $2 \leq k \leq n$. For a vertex subset $S \subseteq V(G)$ with $|S| \geq 2$, a tree containing all the vertices of S in G is called an S-tree. An edge-coloring of G is called a *k-proper coloring* if for every k-subset S of $V(G)$, there exists a proper S-tree in G. For a connected graph G, the *k-proper index* of G, denoted by $px_k(G)$, is the smallest number of colors that are needed in a k-proper coloring of G. In this paper, we show that for every connected graph G of order n and minimum degree $\delta \geq 3$, $px_3(G) \leq n\frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1)) + 2$. We also prove that for every connected graph G with minimum degree at least 3, $px_3(G) \leq px_3(G[D]) + 3$ when D is a connected 3-way dominating set of G and $px_3(G) \leq px_3(G[D]) + 1$ when D is a connected 3-dominating set of G. In addition, we obtain sharp upper bounds of the 3-proper index for two special graph classes: threshold graphs and chain graphs. Finally, we prove that $px_3(G) \leq \lfloor \frac{n}{2} \rfloor$ for any 2-connected graph with at least four vertices.

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1 Introduction

We follow [2] for graph theoretical notation and terminology not described here. Let G be a graph, we use $V(G), E(G), \Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and minimum degree of G, respectively. For $D \subseteq V(G)$, let $\overline{D} = V(G) \backslash D$, and let $G[D]$ denote the subgraph of G induced from D. For $v \in V(G)$, let $N(v)$ denote the set of neighbors of v in G. For two disjoint subsets X and Y of $V(G)$, $E[X, Y]$ denotes the set of edges of G between X and Y. The *join* of two graphs G and H, denoted by $G \vee H$, is the graph obtained from a disjoint union of G and H by adding edges joining every vertex of G to every vertex of H.

Let G be a nontrivial connected graph with an associated *edge-coloring* $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. If adjacent edges of G are assigned different colors by c, then c is a *proper (edge-)coloring*. For a graph G, the minimum number of colors needed in a proper coloring of G is referred to as the *chromatic number* of G and denoted by $\chi'(G)$. A path in an edge-colored graph G is said to be a *rainbow path* if no two edges on the path have the same color. The graph G is called *rainbow connected* if for any two vertices there is a rainbow path of G connecting them. An edge-coloring of a connected graph is a *rainbow connecting coloring* if it makes the graph rainbow connected. For a connected graph G, the *rainbow connection number* $rc(G)$ of G is the smallest number of colors that are needed in order to make G rainbow connected. These concepts of rainbow connection of graphs were introduced by Chartrand et al. [7] in 2008. The readers who are interested in this topic can see [14, 15] for a survey.

In [8], Chartrand et al. generalized the concept of rainbow connection to rainbow index. A tree T in an edge-colored graph is a *rainbow tree* if no two edges of T receive the same color. Let G be a connected graph of order n and k be a fixed integer with $2 \leq k \leq n$. For a vertex subset $S \subseteq V(G)$ with $|S| \geq 2$, a tree containing all the vertices of S in G is called an S-tree. An edge-coloring of G is called a *k-rainbow coloring* if for every k-subset S of $V(G)$, there exists a rainbow S-tree in G. For a connected graph G, the *k-rainbow index* of G, denoted by $rx_k(G)$, is the minimum number of colors that are needed in a k-rainbow coloring of G. We refer to [4, 5, 10, 17] for more details.

Motivated by rainbow coloring and proper coloring in graphs, Andrews et al. [1] and Borozan et al. [3] introduced the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called a *proper path* if no two adjacent edges of the path are colored with the same color. An edge-coloring of a connected graph G is a *proper-path coloring* if every pair of distinct vertices of G are connected by a proper path in G. An edge-colored graph G is *proper connected* if any two vertices of G are connected by a proper path. For a connected graph G, the *proper connection number* of G, denoted by $pc(G)$, is defined as the smallest number of colors that are needed in order to make G proper connected. For more details, we refer to [11, 12, 16] and a dynamic survey [13].

Inspired by the k-rainbow index and the proper-path coloring, Chen et al. [9] introduced the concept of k-proper index of a connected graph G. A tree T in an edge-colored graph is a *proper tree* if no two adjacent edges of T receive the same color. Let G be a connected graph of order n and k be a fixed integer with $2 \leq k \leq n$. An edge-coloring of G is called a *k-proper coloring* if for every k-subset S of $V(G)$, there exists a proper S-tree in G. In this case, G is called *k-proper connected.* For a connected graph, the *k-proper index* of G, denoted by $px_k(G)$, is defined as the minimum number of colors that are needed in a k-proper coloring of G. Clearly, when $k = 2$, $px_2(G)$ is exactly the proper connection number $pc(G)$ of G. Hence, we will study $px_k(G)$ only for k with $3 \leq k \leq n$ here. Let G be a nontrivial connected graph of order n and size m, it is easy to see that $pc(G) \leq px_3(G) \leq \cdots \leq px_n(G) \leq m$.

The rest of this paper is organised as follows. In Section 2, we list some basic definitions and fundamental results on the k-proper index of graphs. In Section 3, we study the 3-proper index by using connected 3-way dominating sets and 3-dominating sets. We first show that for every connected graph G with minimum degree at least 3, $px_3(G) \leq px_3(G[D]) + 3$ when D is a connected 3-way dominating set of G. Then, we can easily get that for every connected graph G on n vertices with minimum degree $\delta \geq 3$, $px_3(G) \leq n\frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1)) + 2$. At last, we show that $px_3(G) \leq px_3(G[D]) + 1$ when D is a connected 3-dominating set of G. In addition, we obtain the sharp upper bounds of the 3-proper index for two special graph classes: threshold graphs and chain graphs. In Section 4, we prove that $px_3(G) \leq \lfloor \frac{n}{2} \rfloor$ for any 2-connected graph with at least four vertices.

2 Preliminaries

To begin with this section, we present the following basic concepts.

Definition 2.1 *Given a graph G, a set $D \subseteq V(G)$ is called a dominating set if every vertex of $\overline{D}$ is adjacent to at least one vertex of D. Furthermore, if the subgraph $G[D]$ is connected, it is called a connected dominating set of G. The domination number $\gamma(G)$ is the number of vertices in a minimum dominating set of G. Similarly, the connected dominating number $\gamma_c(G)$ is the number of vertices in a minimum connected dominating set of G.*

Definition 2.2 *Let s be a positive integer. A dominating set D in a graph G is called an s-way dominating set if every vertex of $\overline{D}$ has at least s neighbours in G. In addition, if $G[D]$ is connected, we call D a connected s-way dominating set.*

Definition 2.3 *A set $D \subseteq G$ is called an s-dominating set of G if every vertex of $\overline{D}$ is adjacent to at least s distinct vertices of D. Furthermore, if $G[D]$ is connected, then D is called a connected s-dominating set. Obviously, a (connected) s-dominating set is also a (connected) s-way dominating set.*

Definition 2.4 *BFS (breadth-first search) is an algorithm for traversing or searching tree or graph data structures. It starts at the tree root and explores the neighbor vertices first, before moving to the next level neighbors. A BFS-tree (breadth-first search tree) is a spanning rooted tree returned by BFS. Let T be a BFS-tree with root r. For a vertex v, the level of v is the length of the unique $\{v, r\}$-path in T, the ancestors of v are the vertices on the unique $\{v, r\}$-path in T, the parent of v is its neighbor on the unique $\{v, r\}$-path in T. Its other neighbors are called the children of v. The siblings of v are the vertices in the same level as v. The left (resp. right) siblings of v are the siblings of v visited before (resp. after) v in BFS.*

Remark: BFS-trees have a nice property: every edge of the graph joins vertices on the same level or consecutive levels. It is not possible for an edge to skip a level. Thus, a neighbor of a vertex v has three possibilities: (1) a sibling of v; (2) the parent of v or a right sibling of the parent of v; (3) a child of v or a left sibling of the children of v.

Next, we state some fundamental results on the k-proper index of graphs which will be used in the sequel.

Proposition 2.5 *[9] If G is a nontrivial connected graph of order $n \geq 3$ and H is a connected spanning subgraph of G, then $px_k(G) \leq px_k(H)$ for each integer k with $3 \leq k \leq n$. In particular, $px_k(G) \leq px_k(T)$ for every spanning tree T of G.*

Proposition 2.6 *[9] If T is a nontrivial tree of order $n \geq 3$, then $px_k(T) = \chi'(G) = \Delta(G)$ for each integer k with $3 \leq k \leq n$.*

Propositions 2.5 and 2.6 provide an upper bound of the k-proper index for a graph.

Proposition 2.7 *[9] Let G be a nontrivial connected graph of order $n \geq 3$. Then, $2 \leq px_3(G) \leq \ldots \leq px_n(G) \leq \min\{\Delta(T) : T$ is a spanning tree of $G\}$.*

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is a *traceable graph.* The following is an immediate consequence of Proposition 2.7.

Corollary 2.8 *[9] If G is a traceable graph of order n, then for each integer k with $3 \leq k \leq n$, $px_k(G) = 2$.*

Obviously, for any integer k with $3 \leq k \leq n$, $px_k(P_n) = px_k(C_n) = px_k(W_n) = px_k(K_n) = px_k(K_{n,n}) = 2$.

Lemma 2.9 *If G is a connected graph with order n_G and H is a connected subgraph of G with order n_H, then for each integer k with $3 \leq k \leq n_H$, we have $px_k(G) \leq px_k(H) + n_G - n_H$; for each integer k with $n_H \leq k \leq n_G$, we have $px_k(G) \leq px_{n_H}(H) + n_G - n_H$.*

Proof. Let G' be a graph obtained from G by contracting H to a single vertex. Then, G' is a connected graph of order $n_G - n_H + 1$. Thus, by Proposition 2.7, $px_{k'}(G') \leq n_G - n_H$ for each integer k' with $3 \leq k' \leq n_G - n_H + 1$. Given an edge-coloring of G' with $n_G - n_H$ colors such that G' is k'-proper connected ($3 \leq k' \leq n_G - n_H + 1$). Now, go back to G, and color each edge outside H with the color of the corresponding edge in G'. For H, if $3 \leq k \leq n_H$, then we assign $px_k(H)$ new colors to the edges of H such that H is k-proper connected; if $n_H \leq k \leq n_G$, then we assign $px_{n_H}(H)$ new colors to the edges of H such that H is n_H-proper connected. The resulting edge-coloring makes G k-proper connected. Therefore, for each integer k with $3 \leq k \leq n_H$, we have $px_k(G) \leq px_k(H) + n_G - n_H$; for each integer k with $n_H \leq k \leq n_G$, we have $px_k(G) \leq px_{n_H}(H) + n_G - n_H$. This completes the proof. □

3 The 3-proper index and connected dominating sets

In this section, we give some upper bounds of the 3-proper index for a graph G by using connected 3-way dominating sets and 3-dominating sets.

Let G be a graph, $D \subseteq V(G)$, and $v \in \overline{D}$. We call a path $P = v_0 v_1 \cdots v_t$ a $v - D$ *path* if $v_0 = v$ and $V(P) \cap D = \{v_t\}$. Two or more paths are called *internally disjoint* if none of them contains an inner vertex of another. If P is edge-colored, then we denote by $end(P)$ the color of the last edge $v_{t-1}v_t$. Now we give our main results.

Theorem 3.1 *If D is a connected 3-way dominating set of a connected graph G, then $px_3(G) \leq px_3(G[D]) + 3$. Moreover, this bound is sharp.*

Proof. Let D be a connected 3-way dominating set of a connected graph G. For $v \in \overline{D}$, its neighbors in D are called the *feet* of v, and the corresponding edges are called the *legs* of v. A set of proper $v - D$ paths $\{P_1, P_2, \ldots, P_\ell\}$ are called *strong-proper* if $end(P_i) \neq end(P_j)$ $(1 \leq i < j \leq \ell)$. For a vertex v in $\overline{D}$, we call it *good* if there are three internally disjoint strong-proper $v - D$ paths. Otherwise, we call v *bad*. Denote by $c(e)$ the color of an edge e. Let T be a rooted BFS-tree. Pick a vertex v in T, and let $\ell(v)$ be the level of v, $p(v)$ the parent of v, $ch(v)$ the child of v, $\alpha(v)$ the ancestor of v in the first level.

We now review the ideas in the proof. At first, we color the edges in $E[D, \overline{D}]$ and $E(G[\overline{D}])$ with three colors from $\{1, 2, 3\}$ such that every vertex v of $\overline{D}$ is good. Then, we extend the coloring to the whole graph by coloring the edges in $G[D]$ with $px_3(G[D])$ fresh colors. Finally, we prove this edge-coloring is a 3-proper coloring of G.

Assume that $A_1, \ldots, A_s$, $B_1, \ldots, B_t$, $C_1, \ldots, C_q$ are the connected components of the subgraph $G - D$ such that $|V(A_i)| = 1$ $(1 \leq i \leq s)$, $|V(B_j)| = 2$ $(1 \leq j \leq t)$ and $|V(C_k)| \geq 3$ $(1 \leq k \leq q)$, where s, t and q are nonnegative integers, and $s = 0$ or $t = 0$ or $q = 0$ means that there is no A_i-component or B_j-component or C_k-component.

For each A_i $(1 \leq i \leq s)$, let v be an isolated vertex of A_i. Then, v has at least three legs, we color one of them with 1, one of them with 2, and all the others with 3. Thus, v is good.

For each B_j $(1 \leq j \leq t)$, let uv be the edge of B_j. Then, u has at least two legs, we color one of them with 1, and all the others with 2. Also, v has at least two legs. We color one of them with 2, and all the others with 3. Finally, we color uv with 2. Thus, both u and v are good.

For each C_k $(1 \leq k \leq q)$, since $|V(C_k)| \geq 3$, it follows that there exists a vertex v_0 in C_k having at least two neighbors in C_k. Starting from v_0, we construct a BFS-tree T of C_k. Suppose that the neighbors of v_0 in C_k are $\{v_1, v_2, \ldots, v_p\}$ $(p \geq 2)$, which form the first level of T. We call the subtree of T rooted at v_i $(1 \leq i \leq p-1)$ of *type I* and the subtree of T rooted at v_p of *type II*. There may be many subtrees of type I, but only one subtree of type II. For each vertex v in C_k, we denote one leg of v by e_v, the corresponding foot of v by $t(v)$, the unique edge joining v and its parent $p(v)$ in T by f_v. Now, we color the edges e_v and f_v as follows: $c(e_{v_0}) = 3$; $c(f_{v_i}) = 2$ and $c(e_{v_i}) = 1$ for $(1 \leq i \leq p-1)$; $c(f_{v_p}) = 1$ and $c(e_{v_p}) = 2$; for each vertex v in $V(C_k) \setminus \{v_1, v_2, \ldots, v_p\}$, if $\alpha(v) = v_p$, then set $c(f_v) = 2$ and $c(e_v) = 3$ when $\ell(v) \equiv 0$ $(mod\ 3)$, $c(f_v) = 1$ and $c(e_v) = 2$ when $\ell(v) \equiv 1$ $(mod\ 3)$, $c(f_v) = 3$ and $c(e_v) = 1$ when $\ell(v) \equiv 2$ $(mod\ 3)$; if $\alpha(v) = v_i$ $(1 \leq i \leq p-1)$, then set $c(f_v) = 1$ and $c(e_v) = 3$ when $\ell(v) \equiv 0$ $(mod\ 3)$, $c(f_v) = 2$ and $c(e_v) = 1$ when $\ell(v) \equiv 1$ $(mod\ 3)$, $c(f_v) = 3$ and $c(e_v) = 2$ when $\ell(v) \equiv 2$ $(mod\ 3)$. Note that the subtrees of the same type are colored in the same way.

Now, for any non-leaf vertex v in T, there exist three internally disjoint strong-proper $v-D$ paths. As for the root v_0, $P_1^{v_0} = v_0t(v_0)$; $P_2^{v_0} = v_0v_1t(v_1)$; $P_3^{v_0} = v_0v_pt(v_p)$. As for any other non-leaf vertex v in T, $P_1^v = vt(v)$; $P_2^v = vp(v)t(p(v))$; $P_3^v = vch(v)t(ch(v))$. Hence, all the non-leaf vertices of T are good.

It remains to deal with the leaves of T. Pick a leaf w in T. Since w has no children, it has exactly two internally disjoint strong-proper $w-D$ paths: $P_1^w = wt(w)$; $P_2^w = wp(w)t(p(w))$. In order to make w good, we need to provide the third path P_3^w which is internally disjoint with P_1^w and P_2^w. Since $w \in \overline{D}$, we have $d(w) \geq 3$. It follows that w has another neighbor which is not $t(w)$, $p(w)$. Let $W = \{w = w_1, w_2, \ldots, w_a\}$ be the children of $p(w)$ such that w_i $(1 \leq i \leq a)$ is a leaf of T and in the subtrees of the same type. Then, $G[W]$ contains a spanning subgraph H which consists of the components of the following two types: (1) a star, (2) an isolated vertex, where the isolated vertices of H are just the isolated vertices of $G[W]$. For each component of type (1), let S be the star and $V(S) = \{w_{i_1}, w_{i_2}, \ldots, w_{i_r}\}$ $(r \geq 2)$, where w_{i_1} is the central vertex of S. Now we recolor the edge $e_{w_{i_1}}$ and color all edges of S. If w_{i_1} is in the subtree of type I, then recolor $e_{w_{i_1}}$ with 1 and color all edges of S with 2 when $\ell(w_{i_1}) \equiv 0$ $(mod\ 3)$; recolor $e_{w_{i_1}}$ with 2 and color all edges of S with 3 when $\ell(w_{i_1}) \equiv 1$ $(mod\ 3)$; recolor $e_{w_{i_1}}$ with 3 and color all edges of S with 1 when $\ell(w_{i_1}) \equiv 2$ $(mod\ 3)$. If w_{i_1} is in the subtree of type II, then recolor $e_{w_{i_1}}$ with 2 and color all edges of S with 1 when $\ell(w_{i_1}) \equiv 0$ $(mod\ 3)$; recolor $e_{w_{i_1}}$ with 1 and color all edges of S with 3 when $\ell(w_{i_1}) \equiv 1$ $(mod\ 3)$; recolor $e_{w_{i_1}}$ with 3 and color all edges of S with 2 when $\ell(w_{i_1}) \equiv 2$ $(mod\ 3)$. Note that the recoloring of the edge $e_{w_{i_1}}$ has no influence on $p(w)$ since $p(w)$ has at least two children in this case. It is easy to check that for the center w_{i_1} of S, there exists a required path $P_3^{w_{i_1}} = w_{i_1}w_{i_2}t(w_{i_2})$, and for every vertex $w_{i_t} \in S$ $(2 \leq t \leq r)$, there exists a required path $P_3^{w_{i_t}} = w_{i_t}w_{i_1}t(w_{i_1})$. Thus, every leaf in the components of type (1) is good.

For each component of type (2), let w be the isolated vertex and w' be another neighbor of w. Note that $w' \notin W$. If $w' \in D$, then we color the edge ww' as follows: if w is in the subtree of type I, then we color $c(ww') = 1$ when $\ell(w) \equiv 0$ $(mod\ 3)$, $c(ww') = 2$ when $\ell(w) \equiv 1$ $(mod\ 3)$, $c(ww') = 3$ when $\ell(w) \equiv 2$ $(mod\ 3)$; if w is in the subtree of type II, then we color $c(ww') = 2$ when $\ell(w) \equiv 0$ $(mod\ 3)$, $c(ww') = 1$ when $\ell(w) \equiv 1$ $(mod\ 3)$, $c(ww') = 3$ when $\ell(w) \equiv 2$ $(mod\ 3)$. Note that for any vertex w in the component of type (2) satisfying $w' \in D$, we have $P_3^w = ww'$. Thus, w is good.

Now we suppose $w' \in T$. Then, w' is either a non-leaf vertex of T or a leaf vertex of T with $p(w') \neq p(w)$. Notice that if $e_{w'}$ is recolored, then w' is a good leaf, and w' has a neighbor w'' such that w'' is a sibling of w'. We distinguish the following four cases:

Case 1: w and w' are in the subtree of type I.

Since T is a BFS-tree, we have that $\ell(w') = \ell(w) - 1$ or $\ell(w') = \ell(w)$ or $\ell(w') = \ell(w) + 1$. Then, we consider the following three subcases.

Subcase 1.1: $\ell(w) \equiv 0\ (mod\ 3)$.

If $\ell(w') = \ell(w) - 1$, then color ww' with 1. Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wt(w)$.

If $\ell(w') = \ell(w)$, then color ww' with 3. Thus, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

If $\ell(w') = \ell(w) + 1$, then color ww' with 2. If $e_{w'}$ is recolored, then w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w'). If $e_{w'}$ is not recolored, then $P_3^w = ww't(w')$. In this situation, if w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

Subcase 1.2: $\ell(w) \equiv 1\ (mod\ 3)$.

If $\ell(w') = \ell(w) - 1$, then color ww' with 2. Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, $P_3^{w'} = w'wt(w)$.

If $\ell(w') = \ell(w)$, then color ww' with 1. If w and w' are in the first level, then w' has at least one child since $p(w') = p(w)$ and is already good. Thus, $P_3^w = ww'ch(w')t(ch(w'))$. Now suppose that w and w' are not in the first level. Then, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

If $\ell(w') = \ell(w) + 1$, then color ww' with 3. If $e_{w'}$ is recolored, then w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w'). If $e_{w'}$ is not recolored, then $P_3^w = ww't(w')$. In this case, if w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

Subcase 1.3: $\ell(w) \equiv 2\ (mod\ 3)$.

If $\ell(w') = \ell(w) - 1$, then color ww' with 3. Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wt(w)$.

If $\ell(w') = \ell(w)$, then color ww' with 2. Thus, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

If $\ell(w') = \ell(w) + 1$, then color ww' with 1. If $e_{w'}$ is recolored, then w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w'). If $e_{w'}$ is not recolored, then $P_3^w = ww't(w')$. In this case, if w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

Thus, both w and w' are good.

Case 2: w is in the subtrees of type I and w' is in the subtree of type II.

Since T is a BFS-tree, it follows that $\ell(w') = \ell(w) - 1$ or $\ell(w') = \ell(w)$. Then, we consider the following three subcases.

Subcase 2.1: $\ell(w) \equiv 0\ (mod\ 3)$.

If $\ell(w') = \ell(w) - 1$, then we distinguish two situations. If $e_{w'}$ is not recolored, then color ww' with 2. Thus, $P_3^w = ww't(w')$. In this situation, if w' is bad, then

$P_3^{w'} = w'wt(w)$. If $e_{w'}$ is recolored, then color ww' with 3 and w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w').

If $\ell(w') = \ell(w)$, then color ww' with 3. Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

Subcase 2.2: $\ell(w) \equiv 1$ $(mod\ 3)$.

If $\ell(w') = \ell(w)-1$, then color ww' with 3. Thus, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

If $\ell(w') = \ell(w)$, then we distinguish two situations. If $e_{w'}$ is not recolored, then color ww' with 3. Thus, $P_3^w = ww't(w')$. In this situation, if w' is bad, then $P_3^{w'} = w'wt(w)$. If $e_{w'}$ is recolored, then color ww' with 2 and w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w').

Subcase 2.3: $\ell(w) \equiv 2$ $(mod\ 3)$.

If $\ell(w') = \ell(w) - 1$, then color ww' with 2, Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

If $\ell(w') = \ell(w)$, then color ww' with 1. Thus, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

Thus, both w and w' are good.

Case 3: If w is in the subtrees of type II and w' is the subtree of type I.

Since T is a BFS-tree, we have that $\ell(w') = \ell(w)$ or $\ell(w') = \ell(w) + 1$. Then, we consider the following three subcases.

Subcase 3.1: $\ell(w) \equiv 0$ $(mod\ 3)$.

If $\ell(w') = \ell(w)$, then color ww' with 3. Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

If $\ell(w') = \ell(w)+1$, then color ww' with 3. Thus, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

Subcase 3.2: $\ell(w) \equiv 1$ $(mod\ 3)$.

If $\ell(w') = \ell(w)$, then we distinguish two situations. If $e_{w'}$ is not recolored, then color ww' with 3. Thus, $P_3^w = ww't(w')$. In this situation, if w' is bad, then $P_3^{w'} = w'wt(w)$. If $e_{w'}$ is recolored, then color ww' with 2 and w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w').

If $\ell(w') = \ell(w) + 1$, then color ww' with 2, Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

Subcase 3.3: $\ell(w) \equiv 2$ $(mod\ 3)$.

If $\ell(w') = \ell(w)$, then color ww' with 1. Thus, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

If $\ell(w') = \ell(w) + 1$, then we distinguish two situations. If $e_{w'}$ is not recolored, then color ww' with 2. Thus, $P_3^w = ww't(w')$. In this situation, if w' is bad, then $P_3^{w'} = w'wt(w)$. If $e_{w'}$ is recolored, then color ww' with 3 and w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w').

Thus, both w and w' are good.

Case 4: If w, w' are in the subtree of type II.

Since T is a BFS-tree, it follows that $\ell(w') = \ell(w) - 1$ or $\ell(w') = \ell(w)$ or $\ell(w') = \ell(w) + 1$. Then, we consider the following three subcases.

Subcase 4.1: $\ell(w) \equiv 0$ $(mod\ 3)$.

If $\ell(w') = \ell(w) - 1$, then color ww' with 2. Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wt(w)$.

If $\ell(w') = \ell(w)$, then color ww' with 3. Thus, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

If $\ell(w') = \ell(w) + 1$, then color ww' with 1. If $e_{w'}$ is recolored, then w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w'). If $e_{w'}$ is not recolored, then $P_3^w = ww't(w')$. In this case, if w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

Subcase 4.2: $\ell(w) \equiv 1$ $(mod\ 3)$.

If $\ell(w') = \ell(w) - 1$, then color ww' with 1. Thus, $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wt(w)$.

If $\ell(w') = \ell(w)$, then color ww' with 2. If w and w' are in the first level, then w' has at least one child since $p(w') = p(w)$ and is already good. Thus, $P_3^w = ww'ch(w')t(ch(w'))$. Now suppose that w and w' are not in the first level. Then, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

If $\ell(w') = \ell(w) + 1$, then color ww' with 3. If $e_{w'}$ is recolored, then w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w'). If $e_{w'}$ is not recolored, then $P_3^w = ww't(w')$. In this case, if w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

Subcase 4.3: $\ell(w) \equiv 2$ $(mod\ 3)$.

If $\ell(w') = \ell(w) - 1$, then color ww' with 3. Thus $P_3^w = ww'p(w')t(p(w'))$. If w' is bad, then $P_3^{w'} = w'wt(w)$.

If $\ell(w') = \ell(w)$, then color ww' with 1. Thus, $P_3^w = ww'p(w')p(p(w'))t(p(p(w')))$. If w' is bad, then $P_3^{w'} = w'wp(w)p(p(w))t(p(p(w)))$.

If $\ell(w') = \ell(w) + 1$, then color ww' with 2. If $e_{w'}$ is recolored, then w' is already good. Thus, $P_3^w = ww'w''t(w'')$ (where w'' is a sibling of w'). If $e_{w'}$ is not recolored, then $P_3^w = ww't(w')$. In this case, if w' is bad, then $P_3^{w'} = w'wp(w)t(p(w))$.

Thus, both w and w' are good.

After the above process, w becomes good, and so does w' if w' is bad. Note that all the good vertices are still good since we just color the edge ww'. As a result, every vertex in T is good.

If there still remain uncolored edges in $E[D, \overline{D}]$ and $E(G[\overline{D}])$, then color them with 1. Now we have a coloring of all the edges in $E[D, \overline{D}]$ and $E(G[\overline{D}])$ using three colors from $\{1, 2, 3\}$ such that all the vertices in $\overline{D}$ are good. Next, we color the edges in $G[D]$ with $px_3(G[D])$ fresh colors such that for each triple of vertices in D, there is a proper tree in $G[D]$ connecting them. Thus, we provide an edge-coloring c of G using $px_3(G[D]) + 3$ colors.

Now we show that this edge-coloring c is a 3-proper coloring of G, which implies $px_3(G) \leq px_3(G[D]) + 3$. We first claim that for any three vertices u, v, w in $\overline{D}$, there exists a proper $u - D$ path P^u, a proper $v - D$ path P^v and a proper $w - D$ path P^w such that $P^u \cup P^v \cup P^w$ is also proper. Since this edge-coloring makes every vertex of $\overline{D}$ good, we only need to consider the situation that u, v, w are in the same component of $G - D$. So, $u, v, w \in C_k$ $(1 \leq k \leq q)$. Note that for any vertex $x \neq v_0 \in C_k$, there are three internally disjoint strong-proper $x - D$ paths P_1^x, P_2^x, P_3^x such that $P_1^x = xt(x)$ and $P_2^x = xp(x)t(p(x))$. For $v_0 \in C_k$, the three internally disjoint strong-proper $v_0 - D$ paths are $P_1^{v_0} = v_0t(v_0)$, $P_2^{v_0} = v_0v_1t(v_1)$ and $P_3^{v_0} = v_0v_pt(v_p)$. If $\{c(e_u), c(e_v), c(e_w)\}$ contains three distinct colors, then $P_1^u \cup P_1^v \cup P_1^w$ is also proper. If $\{c(e_u), c(e_v), c(e_w)\}$ contains two distinct colors, without loss of generality, assume $c(e_u) \neq c(e_v)$, then it is easy to check that either $P_1^u \cup P_1^v \cup P_2^w$ or $P_1^u \cup P_1^v \cup P_3^w$ is proper. Now we assume that $c(e_u) = c(e_v) = c(e_w)$. If u, v, w are in the subtrees of the same type, then we distinguish the following situations. If one of $\{e_u, e_v, e_w\}$ is recolored, without loss of generality, assume that e_u is recolored, then $P_2^u \cup P_1^v \cup P_2^w$ is proper. If two of $\{e_u, e_v, e_w\}$ are recolored, without loss of generality, assume e_w is not recolored, then $P_2^u \cup P_1^v \cup P_2^w$ is proper. If e_u, e_v and e_w are simultaneously recolored or not recolored, without loss of generality, assume v is visited before w in T, then $P_1^u \cup P_2^v \cup P_3^w$ is proper. Now suppose that u, v, w are in the subtrees of different types. Without loss of generality, assume u, v are in the subtree of the same type, and w is in the subtree of the other type. If e_u, e_v and e_w are simultaneously recolored or not recolored, then $P_1^u \cup P_2^v \cup P_2^w$ is proper. If e_u and e_v are recolored, e_w is not recolored, then $P_1^u \cup P_3^v \cup P_2^w$ is proper. If one of $\{e_u, e_v\}$ is recolored, e_w is recolored, without loss of generality, assume e_u is recolored, then $P_2^u \cup P_1^v \cup P_2^w$ is proper. If one of $\{e_u, e_v\}$ is recolored, e_w is not recolored, without loss of generality, assume e_u is recolored, then $P_1^u \cup P_2^v \cup P_2^w$ is proper. If e_u and e_v are not recolored, e_w is recolored, then $P_1^u \cup P_2^v \cup P_3^w$ is proper. Thus, the claim holds.

Next, it is sufficient to show that for any three vertices u, v, w of G, there exists a proper tree connecting them. If all of them are in D, then there is already a proper tree connecting them in $G[D]$. If one of them is in $\overline{D}$, without loss of generality, say $u \in \overline{D}$, then any leg of u (colored by 1, 2 or 3) together with the proper tree connecting v, w, and the corresponding foot of u in $G[D]$ forms a proper $\{u, v, w\}$-tree. If two of them are in $\overline{D}$, without loss of generality, say $u, v \in \overline{D}$, then there exists a proper $u - D$ path P^u, a proper $v - D$ path P^v such that $P^u \cup P^v$ is also proper. Assume that the endvertices of P^u, P^v in D are u', v', respectively. Then, the proper tree connecting u', v' and w together with the paths P^u and P^v forms a proper $\{u, v, w\}$-tree. If all of them are in $\overline{D}$, then there exists a proper $u - D$ path P^u, a proper $v - D$ path P^v and a proper $w - D$ path P^w such that $P^u \cup P^v \cup P^w$ is also proper. Assume that the endvertices of P^u, P^v and P^w in D are u', v', w', respectively. Then, the proper tree in $G[D]$ connecting u', v', w' together with the paths P^u, P^v and P^w forms a proper $\{u, v, w\}$-tree.

To complete the proof of Theorem 3.1, we show the sharpness of the bound with the graph class $\mathcal{G}$. Let p be an integer with $p \geq 3$, $\mathcal{G} = \{G$: G is a graph obtained by taking p complete graphs $K_{i_1}, K_{i_2}, \ldots, K_{i_p}$ with just a vertex in common, say v_0 for $i_j \geq 4$ when $1 \leq j \leq p\}$. For any graph G in $\mathcal{G}$, it is obvious that $D = \{v_0\}$ is a connected 3-way dominating set. By Theorem 3.1, we have $px_3(G) \leq px_3(G[D]) + 3 = 3$. On the other hand, it is easy to show that $px_3(G) = 3$. Thus, the bound is sharp. $\square$

Corollary 3.2 *Let G be a connected graph with minimum degree $\delta(G) \geq 3$. Then, $px_3(G) \leq \gamma_c(G) + 2$.*

Proof. Since $\delta(G) \geq 3$, every connected dominating set of G is a connected 3-way dominating set. Consider a minimum connected dominating set D with size $\gamma_c(G)$. Then, $px_3(G[D]) \leq |D| - 1 = \gamma_c(G) - 1$. We have that $px_3(G) \leq px_3(G[D]) + 3 \leq \gamma_c(G) + 2$ by Theorem 3.1. $\square$

Caro et al. [6] showed that for every connected graph G of order n and minimum degree δ, $\gamma_c(G) = n\frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1))$. With the help of Corollary 3.2, we obtain the following result.

Corollary 3.3 *Let G be a connected graph with minimum degree $\delta(G) \geq 3$. Then, $px_3(G) \leq n\frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1)) + 2$.*

Next, we will give another upper bound for the 3-proper index of graphs with respect to the connected 3-dominating set.

Theorem 3.4 *If D is a connected 3-dominating set of a connected graph G with minimum degree $\delta(G) \geq 3$, then $px_3(G) \leq px_3(G[D]) + 1$. Moreover, the bound is sharp.*

Proof. Since D is a connected 3-dominating set, every vertex in $\overline{D}$ has at least three neighbors in D. Let $t = px_3(G[D])$. We first color the edges in $G[D]$ with t different colors from $\{2, 3, \ldots, t+1\}$ such that for every triple of vertices in D, there exists a proper tree in $G[D]$ connecting them. Then, we color the remaining edges with color 1.

Next, we will show that this edge-coloring makes G 3-proper connected. For any triple $\{u, v, w\}$ of vertices in G, if all of them are in D, then there is already a proper tree connecting them in $G[D]$. If one of them is in $\overline{D}$, without loss of generality, say $u \in \overline{D}$, then let u_1 be the neighbor of u in D. Thus, the proper tree connecting u_1, v, w in $G[D]$ together with the edge uu_1 forms a proper $\{u, v, w\}$-tree in G. If two of them are in $\overline{D}$, without loss of generality, say $u, v \in \overline{D}$, then let u_1, v_1 be the two distinct neighbors of u, v in D, respectively. Thus, the proper tree connecting u_1, v_1, w in $G[D]$ together with two edges uu_1, vv_1 forms a proper $\{u, v, w\}$-tree in G. If all of them are in $\overline{D}$, then let u_1, v_1, w_1 be the three distinct neighbors of u, v, w in D, respectively. Thus, the proper tree connecting u_1, v_1, w_1 in $G[D]$ together with three edges uu_1, vv_1, ww_1 forms a proper $\{u, v, w\}$-tree in G.

The sharpness of the bound can be seen from the following corollaries. □

Next, we give some sharp upper bounds for the 3-proper index of two special graph classes: threshold graphs and chain graphs, which implies the sharpness of the bound in Theorem 3.4. A graph G is called a *threshold graph*, if there exists a weight function w: $V(G) \to \mathbb{R}$ and a real constant t such that two vertices u, $v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$. We call t the threshold of G. A bipartite graph $G(U, V)$ is called a *chain graph*, if the vertices of U can be ordered as $U = \{u_1, u_2, \ldots, u_s\}$ such that $N(u_1) \subseteq N(u_2) \subseteq \cdots \subseteq N(u_s)$.

Corollary 3.5 *Let G be a connected threshold graph with $\delta(G) \geq 3$. Then, $px_3(G) \leq 3$, and the bound is sharp.*

Proof. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$. Since $\delta(G) \geq 3$, v_1, v_2, v_3 are adjacent to all the other vertices in G. Thus, $D = \{v_1, v_2, v_3\}$ is a connected 3-dominating set of G. Since $G[D] = K_3$, we have $px_3(G[D]) = 2$. It follows that $px_3(G) \leq px_3(G[D]) + 1 = 3$ by Theorem 3.4.

Next, we give a class of threshold graphs which have $px_3(G) = 3$. Consider the graph $G = rK_1 \vee K_3$, where $r \geq 2 \times 2^3 + 1$. Let $V(rK_1) = \{v_1, v_2, \ldots, v_r\}$ and $V(K_3) = \{u_1, u_2, u_3\}$. Obviously, it is a threshold graph (u_1, u_2, u_3 can be given a weight 1, others a weight 0 and the threshold 1). We will show that $px_3(G) \geq 3$. By contradiction, we assume that G has a 3-proper coloring with 2 colors. For each vertex $v_i \in rK_1$, there exists a 3-tuple $C_i = (c_1, c_2, c_3)$ so that $c(v_iu_j) = c_j$ for $1 \leq j \leq 3$.

Therefore, each vertex $v_i \in rK_1$ has 2^3 different ways of coloring its incident edges using 2 colors. Since $r \geq 2 \times 2^3 + 1$, there exist at least three vertices v_i, $v_j, v_k \in V$ such that $C_i = C_j = C_k$. It is easy to check that there is no proper tree connecting v_i, v_j, v_k in G, a contradiction. $\square$

Corollary 3.6 *Let G be a connected chain graph with $\delta(G) \geq 3$. Then, $px_3(G) \leq 3$, and the bound is sharp.*

Proof. Let $G = G(U, V)$ be a connected chain graph, where $U = \{u_1, u_2, \ldots, u_s\}$, $V = \{v_1, v_2, \ldots, v_t\}$ such that $N(u_1) \subseteq N(u_2) \subseteq \cdots \subseteq N(u_s)$. Since the minimum degree of G is at least three, $u_i(s-2 \leq i \leq s)$ is adjacent to all the vertices in V, and $N(u_1)$ has at least three vertices, say $\{v_1, v_2, v_3\}$. Clearly, v_1, v_2, v_3 are adjacent to all the vertices in U. Therefore, $D = \{v_1, v_2, v_3, u_{s-2}, u_{s-1}, u_s\}$ is a connected 3-dominating set of G. Moreover, $G[D] = K_{3,3}$ is a traceable graph, we have $px_3(K_{3,3}) = 2$. By Theorem 3.4 we have that $px_3(G) \leq px_3(K_{3,3}) + 1 \leq 3$.

Now, we give a class of chain graphs which have $px_3(G) = 3$. Consider the chain graph $G = G[U, V]$, where $U = \{u_1, u_2, \ldots, u_s\}$, $V = \{v_1, v_2, \ldots v_t\}$ such that $N(u_1) = N(u_2) = \cdots = N(u_{s-3}) = \{v_1, v_2, v_3\}$, $N(u_{s-2}) = N(u_{s-1}) = N(u_s) = \{v_1, v_2, \ldots, v_t\}$ and $t \geq 2 \times 2^3 + 4$. Next, we show that $px_3(G) \geq 3$. Suppose not, we assume that G has a 3-proper coloring with 2 colors. For each vertex $v_i \in V$ for $4 \leq i \leq t$, there exists a 3-tuple $C_i = (c_1, c_2, c_3)$ such that $c(u_j v_i) = c_j$ for $s-2 \leq j \leq s$. Therefore, each vertex $v_i \in V$ $(4 \leq i \leq t)$ has 2^3 different ways of coloring its incident edges using 2 colors. Since $t - 3 \geq 2 \times 2^3 + 1$, there exist at least three vertices v_i, $v_j, v_k \in V \setminus \{v_1, v_2, v_3\}$ such that $C_i = C_j = C_k$. It is easy to check that there is no proper tree connecting v_i, v_j, v_k in G, a contradiction. $\square$

4 The 3-proper index of 2-connected graphs

In this section, we give an upper bound for the 3-proper index of 2-connected graphs. The following notation and terminology are needed in the sequel.

Definition 4.1 *Let F be a subgraph of a graph G. An ear of F in G is a nontrivial path in G whose endvertices are in F but whose internal vertices are not. A nested sequence of graphs is a sequence $(G_0, G_1, \ldots, G_k)$ of graphs such that $G_i \subset G_{i+1}$, $0 \leq i < k$. An ear-decomposition of a 2-connected graph G is a nested sequence $(G_0, G_1, \ldots, G_k)$ of 2-connected subgraphs of G such that: (1) G_0 is a cycle; (2) $G_i = G_{i-1} \cup P_i$, where P_i is an ear of G_{i-1} in G, $1 \leq i \leq k$; (3) $G_k = G$.*

From Corollary 2.8, we have that if G is a 2-connected Hamiltonian graph of order n ($n \geq 3$), then $px_3(G) = 2$. Thus, we only need to consider the non-Hamiltonian graphs.

Let G be a 2-connected non-Hamiltonian graph of order n ($n \geq 4$). Then, G must have an even cycle. In fact, since G is 2-connected, G must have a cycle C. If C is an even cycle, we are done. Otherwise, C is an odd cycle, we then choose an ear P of C such that $V(C) \cap V(P) = \{a, b\}$. Since the lengths of the two segments between a, b on C have different parities, P joining one of the two segments forms an even cycle. Then, starting from an even cycle G_0, there exists a nonincreasing ear-decomposition $(G_0, G_1, \dots, G_t, G_{t+1}, \dots, G_k)$ of G, such that $G_i = G_{i-1} \cup P_i$ $(1 \leq i \leq k)$ and P_i is a longest ear of G_{i-1}, i.e., $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$, where $\ell(P_i)$ denotes the length of P_i. Suppose that $V(P_i) \cap V(G_{i-1}) = \{a_i, b_i\}$ $(1 \leq i \leq k)$. We call the distinct vertices a_i, b_i $(1 \leq i \leq k)$ the *endpoints* of the ear P_i, the edges incident to the endpoints in P_i the end-edges of P_i, the other edges the internal edges of P_i. Without loss of generality, suppose that $\ell(P_t) \geq 2$ and $\ell(P_{t+1}) = \cdots = \ell(P_k) = 1$. So, G_t is a 2-connected spanning subgraph of G. Since G is non-Hamiltonian graph, we have $t \geq 2$. Denote the order of G_i $(0 \leq i \leq k)$ by n_i.

Theorem 4.2 *Let G be a 2-connected non-Hamiltonian graph of order n ($n \geq 4$). Then, $px_3(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Since G_t ($t \geq 2$) in the nonincreasing ear-decomposition is a 2-connected spanning subgraph of G, it only needs to show that G_t has a 3-proper coloring with at most $\lfloor \frac{n}{2} \rfloor$ colors by Proposition 2.5.

Next, we will give an edge-coloring c of G_t using at most $\lfloor \frac{n}{2} \rfloor$ colors. Since G_1 is Hamiltonian, It follows from Corollary 2.8 that we can color the edges of G_1 with two different colors from $\{1, 2\}$ such that for every triple of vertices in G_1, there exists a proper tree in G_1 connecting them. Then, we color the end-edges of P_{2j-4} and P_{2j-3} with fresh color j for $3 \leq j \leq \lceil \frac{t+3}{2} \rceil$. Finally, we color the internal edges of P_i $(2 \leq i \leq t)$ with two colors from $\{1, 2\}$ such that P_i is a proper path if $\ell(P_i) \geq 3$. One can see that we color all the edges of G_t with $\lceil \frac{t+3}{2} \rceil$ colors. Since $n_0 + \sum_{i=1}^{t}(\ell(P_i) - 1) = n$ and $n_0 \geq 4$, we have that $\lceil \frac{t+3}{2} \rceil \leq \lfloor \frac{n}{2} \rfloor$, the equality holds if and only if $n_0 = 4$ and $\ell(P_i) = 2$.

Now we show that this edge-coloring is a 3-proper coloring of G_t. We apply induction on t ($t \geq 2$). If $t = 2$, then let u, v, w be any three vertices of G_2. If all of $\{u, v, w\}$ are in G_1, then there is already a proper tree connecting them in G_1. If two of $\{u, v, w\}$ are in G_1, without loss of generality, assume that $u \in V(P_2) \setminus \{a_2, b_2\}$, then the proper tree connecting a_2, v, w in G_1 together with the proper path uP_2a_2 forms a proper $\{u, v, w\}$-tree in G_2. If one of $\{u, v, w\}$ is in G_1, without loss of generality, assume that $u, v \in V(P_2) \setminus \{a_2, b_2\}$ and v is on the proper path uP_2a_2, then the proper tree connecting a_2, w in G_1 together with the proper path uP_2a_2 forms a proper $\{u, v, w\}$-tree in G_2. If none of $\{u, v, w\}$ is in G_1, then $\{u, v, w\} \subset V(P_2) \setminus \{a_2, b_2\}$. Thus, there is already a proper path connecting them in P_2. Now we assume that this edge-coloring makes G_i $(1 \leq i \leq t-1)$ 3-proper connected. It is sufficient to show that this edge-coloring makes G_t 3-proper connected. For any three vertices $\{u, v, w\}$ of G_t, if all of them are in G_{t-1}, then there is already a proper tree in G_{t-1} connecting them. If two of $\{u, v, w\}$ are in G_{t-1}, without loss of generality, say $u \in V(P_t) \setminus \{a_t, b_t\}$. If t is even, then the color of the end-edges of P_t does not appear in G_{t-1}. Thus, the proper tree connecting a_t, v, w in G_{t-1} together with the proper path uP_ta_t forms a proper $\{u, v, w\}$-tree in G_t. If t is odd, then the end-edges of P_{t-1} and P_t have the same color which does not appear in G_{t-2}. We consider the following two cases.

Case 1. $|[V(P_t) \cap V(P_{t-1})] \setminus V(G_{t-2})| \leq 1$.

Without loss of generality, assume that $a_t \in V(G_{t-2})$ and $a_t \neq b_{t-1}$. If both of v and w are in G_{t-2}, then the proper tree connecting a_t, v, w in G_{t-2} together with the proper path uP_ta_t forms a proper $\{u, v, w\}$-tree in G_t. If $v \in V(G_{t-2})$ and $w \in V(P_{t-1}) \setminus \{a_{t-1}, b_{t-1}\}$, then the proper tree connecting a_t, v, b_{t-1} in G_{t-2} together with the proper paths uP_ta_t and $wP_{t-1}b_{t-1}$ forms a proper $\{u, v, w\}$-tree in G_t. If $v, w \in V(P_{t-1}) \setminus \{a_{t-1}, b_{t-1}\}$, without loss of generality, assume that v is on the proper path $wP_{t-1}b_{t-1}$. Thus, the proper tree connecting a_t, b_{t-1} in G_{t-2} together with the proper paths uP_ta_t and $wP_{t-1}b_{t-1}$ forms a proper $\{u, v, w\}$-tree in G_t.

Case 2. $|[V(P_t) \cap V(P_{t-1})] \setminus V(G_{t-2})| = 2$.

One can see that $\ell(P_{t-1}) \geq 3$. Without loss of generality, assume that a_t is on the proper path of $b_tP_{t-1}a_{t-1}$ and b_t is on the proper path of $a_tP_{t-1}b_{t-1}$. If both of v and w are in G_{t-2}, then the proper tree connecting b_{t-1}, v, w in G_{t-2} together with the proper path $uP_ta_tP_{t-1}b_{t-1}$ forms a proper $\{u, v, w\}$-tree in G_t. If $v \in V(G_{t-2})$ and $w \in V(P_{t-1}) \setminus \{a_{t-1}, b_{t-1}\}$, without loss of generality, assume that w is on the proper path $a_tP_{t-1}b_{t-1}$, then the proper tree connecting v, b_{t-1} in G_{t-2} together with the proper path $uP_ta_tP_{t-1}b_{t-1}$ forms a proper $\{u, v, w\}$-tree in G_t. If $v, w \in V(P_{t-1}) \setminus \{a_{t-1}, b_{t-1}\}$, without loss of generality, assume that v is on the proper path $a_tP_{t-1}b_{t-1}$. If w is on the proper path $a_tP_{t-1}b_{t-1}$, then the path $uP_ta_tP_{t-1}b_{t-1}$ is a proper path connecting u, v, w in G_t. If w is on the proper path $a_tP_{t-1}a_{t-1}$, then the proper tree connecting a_{t-1}, b_{t-1} in G_{t-2} together with the proper paths $uP_ta_tP_{t-1}b_{t-1}$ and $wP_{t-1}a_{t-1}$ forms a proper $\{u, v, w\}$-tree in G_t.

If one of $\{u, v, w\}$ is in G_{t-1}, then we can easily get a proper $\{u, v, w\}$-tree in G_t in a way similar to the situation that two of $\{u, v, w\}$ are in G_{t-1}. If none of $\{u, v, w\}$ is in G_{t-1}, then $\{u, v, w\} \subset V(P_t) \setminus \{a_t, b_t\}$. Thus, there is also already a proper path in P_t connecting them. Hence, we complete the proof. $\Box$

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