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RAINBOW VERTEX-CONNECTION AND FORBIDDEN SUBGRAPHS

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Abstract

A path in a vertex-colored graph is called *vertex-rainbow* if its internal vertices have pairwise distinct colors. A vertex-colored graph G is *rainbow vertex-connected* if for any two distinct vertices of G, there is a vertex-rainbow path connecting them. For a connected graph G, the *rainbow vertex-connection number* of G, denoted by $rvc(G)$, is defined as the minimum number of colors that are required to make G rainbow vertex-connected. In this paper, we find all the families $\mathcal{F}$ of connected graphs with $|\mathcal{F}| \in \{1, 2\}$, for which there is a constant $k_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph G, $rvc(G) \leq diam(G) + k_{\mathcal{F}}$, where $diam(G)$ is the diameter of G.

Keywords: vertex-rainbow path; rainbow vertex-connection; forbidden subgraphs.

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1. Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow the terminology and notation of Bondy and Murty in [2] for those not defined here.

Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. A path in G is called *a rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G, there is a rainbow path connecting them. For a connected edge-colored graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make G rainbow connected. Observe that if G has n vertices, then $diam(G) \leq rc(G) \leq n-1$. It is easy to verify that $rc(G) = 1$ if and only if G is a complete graph, and $rc(G) = n - 1$ if and only if G is a tree. The concept of rainbow connection of graphs was first introduced by Chartrand et al. in [3], and has been well-studied since then. For further details, we refer the reader to a survey paper [10] and a book [11].

Let G be a nontrivial connected graph with a *vertex-coloring* $c : V(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent vertices may be colored with the same color. A path of G is called *vertex-rainbow* if any two internal vertices of the path have distinct colors. The vertex-colored graph G is *rainbow vertex-connected* if any two vertices of G are connected by a vertex-rainbow path. For a connected graph G, the *rainbow vertex-connection number* of G, denoted by $rvc(G)$, is the minimum number of colors used in a vertex-coloring of G to make G rainbow vertex-connected. The concept of rainbow vertex-connection of graphs was proposed by Krivelevich and Yuster in [6]. They showed that if G is a connected graph with n vertices and minimum degree δ, then $rvc(G) \leq 11n/\delta$. In [9], Li and Shi improved this bound. In [4], it was shown that computing the rainbow vertex-connection number of a graph is NP-hard. Recently, Li et al. in [7] proved that it is NP-complete to decide whether a given vertex-colored graph is rainbow vertex-connected even when the graph is bipartite.

For the rainbow vertex-connection number of graphs, the following observations are immediate.

Proposition 1. Let G be a connected graph with n vertices. Then

(*i*) $diam(G) - 1 \leq rvc(G) \leq n - 2$;

(*ii*) $rvc(G) = diam(G) - 1$ if $diam(G) = 1$ or 2, with the assumption that complete graphs have rainbow vertex-connection number 0.

Note that the difference $rvc(G) - diam(G)$ can be arbitrarily large. In fact, if G is a subdivision of a star $K_{1,n}$, then we have $rvc(G) - diam(G) = (n+1) - 4 =$

$n-3$, since in a rainbow vertex-connected coloring of G, the internal vertices must have distinct colors.

In [8], Li and Liu studied the rainbow vertex-connection number for any 2-connected graph, and determined the precise value of the rainbow vertex-connection number of the cycle C_n ($n \geq 3$).

Theorem 1. [8] Let C_n be a cycle of order n ($n \geq 3$). Then,

$$rvc(C_n) = \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n = 4, 5; \\ 3 & \text{if } n = 9; \\ \lceil \frac{n}{2} \rceil - 1 & \text{if } n = 6, 7, 8, 10, 11, 12, 13, \text{or } 15; \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 16 \text{ or } n = 14. \end{cases}$$

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is *$\mathcal{F}$-free* if G does not contain any induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that G is *X-free*, and for $\mathcal{F} = \{X, Y\}$ we say that G is *(X, Y)-free*. The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs*, and for $|\mathcal{F}| = 2$ we also say that $\mathcal{F}$ is a *forbidden pair*.

In [5], Holub et al. considered the question: For which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph G satisfies $rc(G) \leq diam(G) + k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$)? They gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$ in the following two results (where N denotes the *net*, a graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem 2. [5] Let X be a connected graph. Then there is a constant $k_{\mathcal{F}}$ such that every connected X-free graph G satisfies $rc(G) \leq diam(G) + k_X$, if and only if $X = P_3$.

Theorem 3. [5] Let X, Y be connected graphs such that $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rc(G) \leq diam(G) + k_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$ ($r \geq 4$) and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.

Naturally, we may consider an analogous question concerning the rainbow vertex-connection number of graphs. In this paper, we will consider the following question.

For which families $\mathcal{F}$ of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $rvc(G) \leq diam(G) + k_{\mathcal{F}}$?

We give a complete answer for $|\mathcal{F}| = 1$ in Section 3, and for $|\mathcal{F}| = 2$ in Section 4.

2. Preliminaries

In this section, we introduce some further notations and facts that will be needed for the proofs of our main results.

If G is a graph and $A \subset V(G)$, then $G[A]$ denotes the subgraph of G induced by the vertex set A, and $G - A$ the graph $G[V(G) \setminus A]$. An edge is called a *pendant edge* if one of its end vertices has degree one. The *subdivision* of a graph G is the graph obtained from G by adding a vertex of degree 2 to each edge of G. For $x, y \in V(G)$, a path in G from x to y will be referred to as an (x, y)-*path*, and, whenever necessary, it will be considered as oriented from x to y. For a subpath of a path P with origin u and terminus v (also referred to as a (u, v)-*arc* of P), we will use the notation uPv. If w is a vertex of a path with a fixed orientation, then w^- and w^+ denote the predecessor and successor of w, respectively.

For graphs X and G, we write $X \subset G$ if X is a subgraph of G, $X \overset{\text{IND}}{\subset} G$ if X is an induced subgraph of G, and $X \simeq G$ if X is isomorphic to G. For two vertices $x, y \in V(G)$, we use $dist_G(x, y)$ to denote the distance between x and y in G. The diameter of G is defined as the maximum of $dist_G(x, y)$ among all pairs of vertices x, y of G, and will be denoted by $diam(G)$. A shortest path joining two vertices at distance $diam(G)$ will be referred to as a *diameter path*. The *distance between a vertex $u \in V(G)$ and a set $S \subset V(G)$* is defined as $dist_G(u, S) := min_{v \in S} dist_G(u, v)$. A set $D \subset V(G)$ is called *dominating* if every vertex in $V(G) \setminus D$ has a neighbor in D. In addition, if $G[D]$ is connected, then we call D a *connected dominating set*. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers.

For a set $S \subset V(G)$ and $k \in \mathbb{N}$, the *kth-neighborhood* of S is the set $N_G^k(S)$ of all vertices of G at distance k from S. In the special case $k = 1$, we simply write $N_G(S)$ for $N_G^1(S)$, and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subset V(G)$, we denote $N_M^k(S) = N_G^k(S) \cap M$ and $N_M^k(x) = N_G^k(x) \cap M$, and as above, we simply use $N_M(S)$ for $N_M^1(S)$ and $N_M(x)$ for $N_M^1(x)$. For a subgraph $P \subset G$, we write $N_P(x)$ for $N_{V(P)}(x)$. Finally, we will use P_k to denote the path on k vertices.

We end up this section with an important result that will be used in our proofs.

Theorem 4. [1] Let G be a connected P_5-free graph. Then G has a dominating clique or a dominating P_3.

3. Families with one forbidden subgraph

In this section, we characterize all connected graphs X such that every connected X-free graph G satisfies $rvc(G) \leq diam(G) + k_X$, where k_X is a constant.

Theorem 5. Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $rvc(G) \leq diam(G) + k_X$, if and only if $X = P_3$ or P_4.

Proof. We have $diam(G) \leq 2$ since G is P_4-free. Then it follows from Proposition 1 that $rvc(G) = diam(G) - 1 \leq 1$.

Conversely, let $t \geq k_X + 5$, and G_1^t be the subdivision of $K_{1,t}$, and let G_2^t denote the graph obtained by attaching a pendant edge to each vertex of the complete graph K_t (see Fig.1). Since $rvc(G_1^t) = t + 1$ but $diam(G_1^t) = 4$, X is an induced subgraph of G_1^t. Clearly, $rvc(G_2^t) = t$ but $diam(G_2^t) = 3$, and G_2^t is $K_{1,3}$-free and P_5-free. Hence, X is an induced subgraph of P_4.

The proof is thus complete. ■

Figure 1: The graphs G_1^t and G_2^t.

4. Families with a pair of forbidden subgraphs

For $i, j, k \in \mathbb{N}$, let $S_{i,j,k}$ denote the graph obtained by identifying one endvertex from each of three vertex-disjoint paths of lengths i, j, k, and $N_{i,j,k}$ denote the graph obtained by identifying each vertex of a triangle with an endvertex of one of three vertex-disjoint paths of lengths i, j, k (see Fig.2). In this context, we will also write K_t^h for the graph G_2^t introduced in the proof of Theorem 5.

The following statement, which is the main result of this section, characterizes all forbidden pairs X, Y for which there is a constant k_{XY} such that G being (X, Y)-free implies $rvc(G) \leq diam(G) + k_{XY}$. By virtue of Theorem 5, we exclude the case that one of X, Y is an induced subgraph of P_4. Recall that the *net* is the graph $N = N_{1,1,1}$.

Theorem 6. Let $X, Y \neq P_3$ or P_4 be a pair of connected graphs. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rvc(G) \leq diam(G) + k_{XY}$, if and only if (up to symmetry) $X = P_5$ and $Y \overset{\text{IND}}{\subset} K_r^h$ $(r \geq 4)$, or $X \overset{\text{IND}}{\subset} S_{1,2,2}$ and $Y \overset{\text{IND}}{\subset} N$.

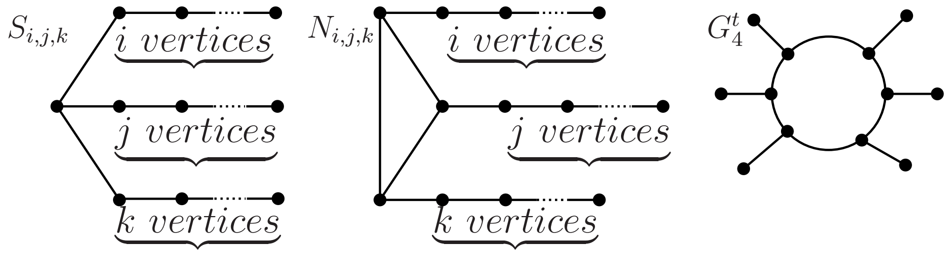

Figure 2: The graphs $S_{i,j,k}$, $N_{i,j,k}$ and G_4^t.

The proof of Theorem 6 will be divided into three separate results: we prove the necessity in Proposition 2, and Theorems 7 and 8 will establish the sufficiency of the forbidden pairs given in Theorem 6.

Proposition 2. Let $X, Y \neq P_3$ or P_4 be a pair of connected graphs for which there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rvc(G) \leq diam(G)+k_{XY}$. Then, (up to symmetry) $X = P_5$ and $Y \overset{\text{IND}}{\subset} K_r^h$ $(r \geq 4)$, or $X \overset{\text{IND}}{\subset} S_{1,2,2}$ and $Y \overset{\text{IND}}{\subset} N$.

Proof. Let $t \geq 2k_{XY} + 5$, and let (see Fig.2):

• $G_3^t = N_{t-1,t-1,t-1}$;

• G_4^t be the graph obtained by attaching a pendant edge to each vertex of a cycle C_t.

We will also use the graphs G_1^t and $G_2^t (= K_t^h)$ shown in Fig.1.

For the graphs G_1^t and G_2^t, we have $diam(G_1^t) = 4$ but $rvc(G_1^t) = t + 1$, and $diam(G_2^t) = 3$ but $rvc(G_2^t) = t$, respectively. For the graph G_3^t, we observe that $diam(G_3^t) = 2t - 1$ while $rvc(G_3^t) = 3(t-1) = \frac{3}{2}(diam(G_3^t) - 1)$, since all internal vertices must have mutually distinct colors. Analogously, for the graph G_4^t, we have $diam(G_4^t) = \lfloor \frac{t}{2} \rfloor + 2$, but $rvc(G_4^t) = t \geq 2(diam(G_4^t) - 2)$. Thus, each of the graphs G_1^t, G_2^t, G_3^t and G_4^t must contain an induced subgraph isomorphic to one of the graphs X, Y.

Consider the graph G_1^t. Up to symmetry, we have that X is an induced subgraph of G_1^t excluding P_3 and P_4. Now we consider the graph G_2^t. Obviously, G_2^t is X-free since G_2^t is $K_{1,3}$-free. Hence, G_2^t contains Y, implying $Y \overset{\text{IND}}{\subset} K_r^h$ for some $r \geq 3$ (for $r \leq 2$ we get $Y \overset{\text{IND}}{\subset} P_4$, which is excluded by the assumptions).

Now we consider the graph G_3^t. There are two possibilities:

(i) $Y \overset{\text{IND}}{\subset} G_3^t$. Then $Y \overset{\text{IND}}{\subset} N$. Now we consider the graph G_4^t. G_4^t is N-free, so we get $X \overset{\text{IND}}{\subset} S_{1,2,2}$.

(ii) $X \overset{\text{IND}}{\subset} G_3^t$. Then $X = P_5$. As the case $X = P_5$ and $Y = N$ is already covered by case (i), we have that $X = P_5$ and $Y \overset{\text{IND}}{\subset} K_r^h$, $r \geq 4$.

This completes the proof. ■

It is easy to observe that if $X \overset{\text{IND}}{\subset} X'$, then every (X, Y)-free graph is also (X', Y)-free. Thus, when proving the sufficiency of Theorem 6, we will be always interested in *maximal pairs* of forbidden subgraphs, i.e., pairs X, Y such that, if replacing one of X, Y, say X, with a graph $X' \neq X$ such that $X \overset{\text{IND}}{\subset} X'$, then the statement under consideration is not true for (X', Y)-free graphs.

Theorem 7. Let G be a connected (P_5, K_r^h)-free graph for some $r \geq 4$. Then, $rvc(G) \leq diam(G) + r$.

Proof. From Theorem 4, we have that G has a dominating clique or a dominating P_3.

Case 1. G has a dominating P_3.

We color the vertices of P_3 with colors $1, 2, 3$ and color the remaining vertices arbitrarily (e.g., all of them have color 1). One can easily check that this vertex-coloring can make G rainbow vertex-connected. So, in this case, $rvc(G) \leq 3 \leq diam(G) + r$.

Case 2. G has a dominating clique, denoted by K_p.

Set $W = V(G)\backslash V(K_p)$, $H = G\backslash E(K_p)$. Let A be an independent set in $G[W]$ and $B \subset V(K_p)$ such that $H[A \cup B] = \ell K_2$ (that is, a matching of order ℓ) and ℓ is maximal. Then $\ell < r$, for otherwise, $G[A \cup B]$ contains an induced K_r^h. Moreover, for $x \in W\backslash A$, $N_{A\cup B}(x) \neq \emptyset$, since ℓ is maximal. Now we define the following vertex-coloring of G. Use colors $1, 2, \ldots, \ell$ to color each vertex in B, color the vertices of A with color $\ell + 1$, the vertices of $V(K_p)\backslash B$ with color $\ell + 2$, and color the remaining vertices arbitrarily (e.g., all of them have color 1). Thus, pairs of vertices in $(A \cup V(K_p)) \times V(G)$ are rainbow vertex-connected. As for $x_1, x_2 \in W\backslash A$, let $y_1 \in N_{A\cup B}(x_1)$, $y_2 \in N_{K_p}(x_2)$. Then, there is a vertex-rainbow (x_1, x_2)-path containing y_1 and y_2. So, $rvc(G) \leq \ell + 2 \leq r + 1 \leq diam(G) + r$.

The proof is complete. ■

Now let G be an $(S_{1,2,2}, N)$-free graph, let $x, y \in V(G)$, and let $P : x = v_0, v_1, \ldots, v_k = y$ $(k \geq 3)$ be a shortest (x, y)-path in G. Let $z \in V(G)\backslash V(P)$. If $|N_P(z)| \geq 2$ and $\{v_i, v_j\} \subset N_P(z)$, then $|i - j| \leq 2$ and $|N_P(z)| \leq 3$, since P is a shortest path. Moreover, the following facts are easily observed.

- If $|N_P(z)| = 1$, then, since G is $S_{1,2,2}$-free, z is adjacent to x, v_1, v_{k-1} or y.
- If $|N_P(z)| = 3$, then the vertices of $N_P(z)$ must be consecutive on P, since P is a shortest path.

This motivates the following notations:

- $A_i := \{z \in V(G)\backslash V(P) | N_P(z) = \{v_i\}\}$ for $i = 0, 1, k-1, k$;

- $L_i := \{z \in V(G)\backslash V(P) | N_P(z) = \{v_{i-1}, v_{i+1}\}\}$ for $1 \leq i \leq k-1$;
- $M_i := \{z \in V(G)\backslash V(P) | N_P(z) = \{v_{i-1}, v_i\}\}$ for $1 \leq i \leq k$;
- $N_i := \{z \in V(G)\backslash V(P) | N_P(z) = \{v_{i-1}, v_i, v_{i+1}\}\}$ for $1 \leq i \leq k-1$.

We further set $S = V(P) \cup N_G(P)$ and $R = V(G)\backslash S$.

Lemma 1. Let G be an $(S_{1,2,2}, N)$-free graph, let $x, y \in V(G)$ be such that $dist_G(x, y) \geq 4$ and let $P : x = v_0, v_1, \ldots, v_k = y$, be a shortest (x, y)-path in G. Then

(i) $N_G(M_i) \subset S,\ i = 2, \ldots, k-1$;
(ii) $N_G(N_i) \subset S,\ i = 2, \ldots, k-2$;
(iii) $N_G(L_i) \subset S,\ i = 1, \ldots, k-1$;
(iv) $N_P(R) = \emptyset$;
(v) $N_S(R) \subset A_0 \cup M_1 \cup N_1 \cup N_{k-1} \cup M_k \cup A_k$.

Proof. If $zv \in E(G)$ for some $z \in R$ and $v \in M_i$, $2 \leq i \leq k-1$, then we have $G[\{v_{i-2}, v_{i-1}, v_i,\ v_{i+1}, v, z\}] \simeq N$, a contradiction. Hence, (i) follows. To show (ii), we observe that if $zv \in E(G)$ for some $z \in R$ and $v \in N_i$, $2 \leq i \leq k-2$, then we have $G[\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v, z\}] \simeq S_{1,2,2}$, a contradiction. Similarly, for (iii), if $zv \in E(G)$ for some $z \in R$ and $v \in L_i$, $1 \leq i \leq k-1$, then, for $i = 1$ we have $G[\{v_1, v_2, v_3, v_4, v, z\}] \simeq S_{1,2,2}$, for $2 \leq i \leq k-2$ we have $G[\{z, v, v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2}\}] \simeq S_{1,2,2}$, and for $i = k-1$, $G[\{v_{k-1}, v_{k-2}, v_{k-3}, v_{k-4}, v, z\}] \simeq S_{1,2,2}$, a contradiction. Part (iv) follows immediately from the definition of R, and by (i) through (iii), we have $N_S(R) \subset A_0 \cup A_1 \cup M_1 \cup N_1 \cup N_{k-1} \cup M_k \cup A_{k-1} \cup A_k$. But if $zv \in E(G)$ for some $z \in R$ and $v \in A_1$, then $G[\{v_0, v_1, v_2, v_3, v, z\}] \simeq S_{1,2,2}$, a contradiction. Similarly, we have $N_{A_{k-1}}(R) = \emptyset$, implying (v).

The proof is complete. ■

Theorem 8. Let G be a connected $(S_{1,2,2}, N)$-free graph. Then, $rvc(G) \leq diam(G) + 11$.

Proof. Let G be a connected $(S_{1,2,2}, N)$-free graph. If $diam(G) \leq 2$, then $rvc(G) = diam(G) - 1$. Thus, for the rest of the proof we suppose that $diam(G) = d \geq 3$. Let $v_0, v_d \in V(G)$ be such that $dist_G(v_0, v_d) = d$, let $P : v_0 v_1 v_2 \ldots v_d$ be a diameter path in G, and let A_i, L_i, M_i, N_i, S, R be defined as above.

We distinguish three cases according to the value of d.

Case 1. $d = 3$.

First, we partition $V(G)$ into four parts $P, N_G(P), N_G^2(P)$ and $N_G^3(P)$ according to the distance from P. Then, for the vertices in $N_G(P)$, we can partition them into three parts $X_1 = A_0 \cup M_1 \cup L_1 \cup N_1$, $X_2 = A_3 \cup M_3 \cup L_2 \cup N_2$ and $X_3 = A_1 \cup M_2 \cup A_2$. We must point out that $X_1 \cap X_2 = \emptyset$ and $N_R(X_3) = \emptyset$, whose proof is similar to that of Lemma 1. Then, we denote Y_i the set of vertices in $N_G^2(P)$ such that for each $v \in Y_i$, $N_{N(P)}(v) \subset X_i, i = 1, 2$, and $Y_3 = N_G^2(P) \backslash (Y_1 \cup Y_2)$. With a similar reason as above, $N_{N_G^3(P)}(Y_3) = \emptyset$. So, analogously we can partition $N_G^3(P)$ into three parts Z_1, Z_2 and Z_3. It should be noticed that $Z_1 = \emptyset$; otherwise there exists a vertex $z \in Z_1$ such that $dist_G(z, v_3) \geq 4$, a contradiction. Symmetrically, we have $Z_2 = \emptyset$.

Now, we define a vertex-coloring of G that uses at most 14 colors. Color the vertices of P with colors $0, 1, 2, 3$ and color the vertices in $A_0, M_1, L_1, N_1, N_2, L_2, M_3, A_3, Y_1$ and Y_2 with colors $4, 5, \ldots, 13$, respectively. Then, color the remaining vertices arbitrarily (e.g., all of them have color 0). We can show that this vertex-coloring can make G rainbow vertex-connected. We only need to verify that for a pair of vertices $x, y \in (Y_1 \times Y_1) \cup (Y_2 \times Y_2)$, there exists a vertex-rainbow path connecting them. Without loss of generality, we suppose $(x, y) \in Y_1 \times Y_1$. If $dist_G(x, y) \leq 2$, then there is nothing left to do. Next we consider $dist_G(x, y) \geq 3$. Let x' be an arbitrary neighbor of x in X_1, and y' an arbitrary neighbor of y in X_1. We claim that x' and y' cannot have the same color. Otherwise, we suppose that x' and y' are colored with the same color, i.e., they are in the same vertex-class of X_1, and let $i = max\{j : v_j \in N_P(x') \cap N_P(y')\}$. Then, we have $G[\{v_i, v_{i+1}, x', y', x, y\}] \simeq S_{1,2,2}$ if $x'y' \notin E(G)$, or $G[\{v_i, v_{i+1}, x', y', x, y\}] \simeq N$ if $x'y' \in E(G)$, respectively. So, the colors of x' and y' must be different. Then, the (x, y)-path $P_1 : xx'v_0y'y$ is vertex-rainbow. Hence, we have $rvc(G) \leq diam(G) + 11$.

Case 2. $d = 4$.

Similarly, with the partition and the vertex-coloring of Case 1, we can get that $rvc(G) \leq 15 = diam(G) + 11$.

Case 3. $d \geq 5$.

Set $B_c = (\cup_{i=2}^{d-2} N_i) \cup (\cup_{i=2}^{d-1} M_i) \cup (\cup_{i=1}^{d-1} L_i) \cup A_1 \cup A_{d-1} \cup \{v_1, v_2, \ldots, v_{d-1}\}$, $X = A_0 \cup M_1 \cup N_1 \cup N_{d-1} \cup M_d \cup A_d$, $X_1 = A_0 \cup M_1 \cup N_1$, and $X_2 = N_{d-1} \cup M_d \cup A_d$. By virtue of Lemma 1, we have $N_G(B_c) \subset S$.

Subcase 3.1. B_c is a cut-set of G.

We claim that $S \cup N_G(S) = V(G)$. Suppose, to the contrary, that $z \in R$ is at distance 2 from S. Then, by Lemma 1 and the assumption of Case 1, as well as the symmetry, we can assume that $N_S^2(z) \subset X_1$. Let Q be a shortest (z, v_d)-path, let w be the first vertex of Q in B_c (it exists by the assumption of Subcase 3.1), and let w^- be the predecessor of w on Q. By Lemma 1, $dist(w^-, P) = 1$, implying $w^- \in X_1$. Then, $dist_G(w^-, v_d) \geq d-1$; otherwise, the path $v_0w^-Qv_d$ is a (v_0, v_d)-path shorter than P. Since $dist_G(z, w^-) \geq 2$, we have $dist_G(z, v_d) \geq d+1$, contradicting $diam(G) = d$. Hence, we have $S \cup N_G(S) = V(G)$. Moreover, with a similar argument to that of Case 1, we have that for $x, y \in R$ with distance at least 3, their neighbors x' and y' cannot be in the same vertex-class of X.

Now we define a vertex-coloring of G that uses at most $d + 7$ colors. Color the vertices of P with colors $0, 1, \ldots, d$ and color the vertices in A_0, M_1, N_1, N_{d-1}, M_d and A_d with colors $d+1, d+2, \ldots, d+6$, respectively. Then, color the remaining vertices arbitrarily (e.g., all of them have color 0). We can show that this vertex-coloring can make G rainbow vertex-connected. For any pair of vertices in $S \times (S \cup R)$, we can easily find a vertex-rainbow path connecting them. For a pair $(x, y) \in R \times R$, if $dist_G(x, y) \leq 2$, then there is nothing left to do. Next we consider $dist_G(x, y) \geq 3$. From above, we know that their neighbors x' and y' in X are colored differently. So, the (x, y)-path containing x' and y' is vertex-rainbow.

Consequently, we have $rvc(G) \leq diam(G) + 7$.

Subcase 3.2. B_c is not a cut-set of G.

Set $H = G - B_c$. Let $P' : v_d v_{d+1} \ldots v_{d+\ell-1} v_{d+\ell} = v_0$ be a shortest (v_d, v_0)-path in H. Since P is a diameter path, $\ell \geq d \geq 5$. If v_{d+1} is adjacent to v_{d-2}, then $G[\{v_d, v_{d+1}, v_{d-2}, v_{d-3}, v_{d+2}, v_{d+3}\}] \simeq S_{1,2,2}$, a contradiction. So, $v_{d+1} \in A_d \cup M_d$. Similarly, we have $v_{d+\ell-1} \in A_0 \cup M_1$.

Set $P^d : v_{d-1} v_d v_{d+1}$ if $v_{d-1} v_{d+1} \notin E(G)$, or $P^d : v_{d-1} v_{d+1}$ if $v_{d-1} v_{d+1} \in E(G)$, respectively. Similarly, set $P^0 : v_{d+\ell-1} v_0 v_1$ if $v_{d+\ell-1} v_1 \notin E(G)$, or $P^d : v_{d+\ell-1} v_1$ if $v_{d+\ell-1} v_1 \in E(G)$, respectively. Finally, set $C : v_1 P v_{d-1} P^d v_{d+1} P' v_{d+\ell-1} P^0 v_1$. Then, C is a cycle of length at least $2d - 2$.

Claim 1. The cycle C is chordless.

Proof. This proof can be found in [5]. But for the sake of completeness, we provide the proof here. Suppose, to the contrary, that $v_i v_j \in E(G)$ is a chord in C. Since both P and P' are chordless, we can choose the notation such that $1 \leq i \leq d-1$ and $d+1 \leq j \leq d+\ell-1$. Since $v_j \in V(P')$, we have $v_j \notin B_c$ by the definition of P', implying $i = d-1$ and $v_j \in M_d$, or, symmetrically, $i = 1$ and $v_j \in M_1$. This implies that in the first case, $v_j = v_{d+1}$; in the second case, $v_j = v_{d+\ell-1}$; and in both cases, $v_i v_j \in E(C)$ by the definition of C. Thus, C is chordless. □

Claim 2. $\ell \leq d + 2$.

Proof. Assume that $\ell \geq d+3$, and let Q be a shortest (v_0, v_{d+2})-path in G. Then, $|E(Q)| \leq d$ (since $diam(G) = d$). Since $\ell \geq d+3$ and P' is shortest in $H = G - B_c$, we have $dist_H(v_0, v_{d+2}) \geq d+1$. So, Q must contain a vertex from B_c. Let w be the last vertex of Q in B_c, and let w^- and w^+ be its predecessor and successor on Q, respectively (they exist since $v_{d+2} \notin B_c$ by the definition of P'). By Lemma 1, w^+ is at distance at most 1 from P. Since clearly $w^+ \notin \{v_0, v_d\}$, either $w^+ v_0 \in E(G)$ or $w^+ v_d \in E(G)$. If $w^+ v_0 \in E(G)$, then $v_0 w^+ Q v_{d+2}$ is a (v_0, v_{d+2})-path shorter than Q, a contradiction. Thus, $w^+ v_d \in E(G)$. Now, $w^+ \neq v_{d+2}$ since P' is chordless, implying $dist_G(v_0, w^+) \leq d-1$. On the other hand, $dist_G(v_0, w^+) \geq d-1$; otherwise, $v_0 Q w^+ v_d$ is a (v_0, v_d)-path of length at most $d-1$, contradicting the fact that P is a diameter path. Hence, $dist_G(v_0, w^+) = d-1$, implying that $dist_G(v_0, w) = d-2$ and $w^+v_{d+2} \in E(Q)$. Since $v_{d+2}, v_{d+3} \in R$, we have $G[\{v_{d+3}, v_{d+2}, v_d, w^+, w, w^-\}] \simeq S_{1,2,2}$, a contradiction. Hence, $\ell \leq d+2$. □

Claim 3. $C \cup N_G(C) = V(G)$, and every vertex in $V(G)\backslash V(C)$ has at least 2 neighbors in C.

Proof. Suppose that a vertex $x \in V(G)\backslash V(C)$ at distance 1 from C has exactly one neighbor in C, and set $N_C(x) = \{y\}$. Let $z_1, z_2 \in N_C^2(x)$, and let $z_1', z_2' \in N_C^3(x)$. Then, we have $G[\{x, y, z_1, z_2, z_1', z_2'\}] \simeq S_{1,2,2}$, a contradiction.

Secondly, suppose, to the contrary, that $z \in V(G)$ is at distance 2 from C, and y is a neighbor of z at distance 1 from C. Then, $dist_G(z, P) \geq 2$; otherwise, $y = v_0$ or v_d, without loss of generality, we assume $y = v_0$. Then, v_1 must be adjacent to $v_{d+\ell-1}$, and thus, $G[\{z, y, v_1, v_2, v_{d+\ell-1}, v_{d+\ell-2}\}] \simeq N$, a contradiction. Hence, $z \in R$. If $y \in R$, then y is not adjacent to any of v_1, v_2 and v_3. If $y \notin R$, then we have $y \in X$. Without loss of generality, we assume $y \in X_2$. Then, y is not adjacent to any of v_1, v_2 and v_3. Moreover, from above we know that y has at least 2 neighbors in C. Let $x_1, x_2 \in N_C(y)$ be the vertices closest to v_1 and v_3, respectively. Let x_1' and x_2' be their neighbors that are closer to v_1 and v_3 in C, respectively. Then, $G[\{y, z, x_1, x_2, x_1', x_2'\}] \simeq S_{1,2,2}$ if $x_1x_2 \notin E(G)$, or $G[\{y, z, x_1, x_2, x_1', x_2'\}] \simeq N$ if $x_1x_2 \in E(G)$, respectively. Thus, C is a dominating set of G. □

By Claims 1 and 2, we know that C is a chordless cycle of length at most $d+\ell \leq 2d+2$. Now, we define a vertex-coloring of G that uses at most $d+1$ colors. Relabel $C = x_1x_2 \dots x_kx_{k+1}(= x_1)$, $8 \leq 2d-2 \leq k \leq 2d+2$. Then, we assign color i to the vertex x_i if $1 \leq i \leq \lceil \frac{k}{2} \rceil$ and assign color $i - \lceil \frac{k}{2} \rceil$ to x_i if $\lceil \frac{k}{2} \rceil < i \leq k$. We color the remaining vertices arbitrarily. We can show that this vertex-coloring can make G rainbow vertex-connected. From Theorem 1 and Claim 3, we know that under this vertex-coloring, pairs in $C \times V(G)$ are rainbow vertex-connected. For each vertex $z \in N_G(C)$, we may strengthen the result of Claim 3 that z has at least two neighbors colored differently in C. Otherwise, we suppose that z_1 and z_2 are the only two neighbors of z having the same color in C. From the vertex-coloring, we know that $dist_C(z_1, z_2) = \lfloor \frac{k}{2} \rfloor \geq 4$. Then, we can easily find an induced $S_{1,2,2}$, a contradiction. So, for a pair $(x, y) \in N_G(C) \times N_G(C)$, we can find a vertex $x' \in N_C(x)$ and a vertex $y' \in N_C(y)$ such that x' and y' are colored differently. Since there exists a vertex-rainbow path P connecting x' and y' and the internal vertices of P are colored differently from x' and y', the path $xx'Py'y$ is vertex-rainbow and connects x and y. Hence, $rvc(G) \leq d+1$.

The proof of Theorem 8 is complete. ■

Combining Proposition 2 with Theorems 7 and 8, we have proved Theorem 6.

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