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Arc-transitive graphs of square-free order and small valency

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ABSTRACT

This paper is one of a series of papers devoted to characterizing edge-transitive graphs of square-free order. It presents a complete list of locally-primitive arc-transitive graphs of square-free order and valency $d \in \{5, 6, 7\}$.

1. Introduction

All graphs and groups considered in this paper are assumed to be finite.

Let $\Gamma = (V, E)$ be a simple connected graph with vertex set V and edge set E. The number of vertices $|V|$ is called the *order* of Γ. Let $\mathrm{Aut}\Gamma$ be the automorphism group of Γ and let G be a subgroup of $\mathrm{Aut}\Gamma$, written as $G \le \mathrm{Aut}\Gamma$. Then the graph Γ is said to be *G-vertex-transitive* or *G-edge-transitive* if G acts transitively on V and E, respectively. Recall that an *arc* in Γ is an ordered pair of adjacent vertices. The graph Γ is said to be *G-arc-transitive* if G acts transitively on the set of all arcs in Γ. For $\alpha \in V$, we denote by G_α and $\Gamma(\alpha)$ respectively the stabilizer of α in G and the set of neighbors of α in Γ, that is,

$$G_\alpha = \{g \in G \mid \alpha^g = \alpha\} \quad \text{and} \quad \Gamma(\alpha) = \{\beta \in V \mid \{\alpha, \beta\} \in E\}.$$

The graph Γ is called *G-locally-primitive* if for every $\alpha \in V$ the stabilizer G_α acts primitively on $\Gamma(\alpha)$. It is easy to see that Γ is G-edge-transitive if it is G-locally-primitive. Moreover, if Γ is both G-vertex-transitive and G-locally-primitive, then Γ must be G-arc-transitive; in this case, Γ is said to be *G-locally-primitive arc-transitive*.

The study of graphs with square-free order has a long history, see for example [1,16,17,19] for those graphs of order being a product of two primes. This paper is devoted to classifying arc-transitive graphs of square-free order and small valency.

In recent work [14], the authors gave a reduction for connected locally-primitive arc-transitive of square-free order. We proved that, for a connected locally-primitive arc-transitive graph Γ of square-free order and valency d, if it is not a complete bipartite graph then either $\mathrm{Aut}\Gamma$ is soluble, or Γ is a cover of one of the 'basic' graphs associated with $\mathrm{PSL}(2, p)$, $\mathrm{PGL}(2, p)$ and a finite number (depending only on the valency d) of other almost simple groups. Then for some small values of d we may determine most 'basic' graphs, which makes it possible to give a classification of such graphs of small valencies.

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Thus a natural question is to find a classification of locally-primitive arc-transitive graphs of square-free order and small valency d. This question was solved for $d = 3$ and 4 in [13] and [15], respectively. In this paper we deal with the case where $d \in \{5, 6, 7\}$. Our main result is stated as follows.

Theorem 1.1. *Let Γ be a connected G-locally-primitive arc-transitive graph of square-free order and valency $d = 5, 6$ or 7. Then one of the following statements holds.*

(i) $G = \mathrm{D}_{2n}{:}\mathbb{Z}_d$ *with* $d \in \{5, 7\}$, *and Γ is a graph given by Construction 4.1.*
(ii) *Γ is isomorphic to one of the following graphs:*
  K_6, K_7, $\mathrm{K}_{5,5}$, $\mathrm{K}_{7,7}$ *and* $\mathrm{K}_{7,7} - 7\mathrm{K}_2$*;*
  the incidence graphs of $\mathrm{PG}(3, 2)$, $\mathrm{PG}(2, 4)$, $\mathrm{PG}(2, 5)$ *and* $\mathrm{GQ}(4)$*;*
  the graphs given in Examples 4.2–4.5.
(iii) $G = \mathrm{PSL}(2, p)$ *or* $\mathrm{PGL}(2, p)$ *for odd prime p, and for an edge $\{\alpha, \beta\}$ of Γ the pair $(G_\alpha, G_{\alpha\beta})$ is listed in Table 3.*

For groups, we follow the notation used in the Atlas [6] while we sometimes use $\mathbb{Z}_l$ and $\mathbb{Z}_p^k$ to denote respectively the cyclic group of order l and the elementary abelian group of order p^k.

2. Preliminaries

Let $\Gamma = (V, E)$ be a graph of valency d, let $\{\alpha, \beta\} \in E$ and $G \le \mathrm{Aut}\Gamma$. Set $G_{\alpha\beta} = G_\alpha \cap G_\beta$, call the *arc-stabilizer* of (α, β) (and (β, α)). Assume that Γ is G-arc-transitive. Then G_α is transitive on $\Gamma(\alpha)$, and $d = |\Gamma(\alpha)| = |G_\alpha : G_{\alpha\beta}|$. Take $x \in G$ with $(\alpha, \beta)^x = (\beta, \alpha)$. Then

$$x \in \mathbf{N}_G(G_{\alpha\beta}) \setminus G_{\alpha\beta}, \quad x^2 \in G_{\alpha\beta}.$$

(In particular, the index $|\mathbf{N}_G(G_{\alpha\beta}) : G_{\alpha\beta}|$ is even.) Obviously, this x may be chosen as a 2-element in the normalizer $\mathbf{N}_G(G_{\alpha\beta})$. Moreover, Γ is connected if and only if $\langle x, G_\alpha\rangle = G$. Since G is transitive on V, the map $\alpha^g \mapsto G_\alpha g$ is a bijection between V and $[G : G_\alpha]$, the set of right cosets of G_α in G. It is easy to show that this map is an isomorphism from the graph Γ to a coset graph defined as follows.

Let G be a finite group and H be a core-free subgroup of G, where core-free means that $\cap_{g\in G} H^g = 1$. For $x \in G \setminus H$, the *coset graph* $\mathrm{Cos}(G, H, H\{x, x^{-1}\}H)$ is defined on $[G : H]$ such that Hg_1 and Hg_2 are adjacent whenever $g_2g_1^{-1} \in HxH \cup Hx^{-1}H$. Note that G may be viewed as a subgroup of $\mathrm{AutCos}(G, H, H\{x, x^{-1}\}H)$, where G acts on $[G : H]$ by right multiplication. The following statements for coset graphs are well-known.

Lemma 2.1. *Let G be a finite group and H a core-free subgroup of G. Set $\Gamma = \mathrm{Cos}(G, H, H\{x, x^{-1}\}H)$, where $x \in G \setminus H$. Then Γ is both G-vertex-transitive and G-edge-transitive, and*

(i) *Γ is G-arc-transitive if and only if $HxH = HyH$ for some 2-element $y \in \mathbf{N}_G(H \cap H^x) \setminus H$ with $y^2 \in H \cap H^x$; in this case, Γ has valency $|H : (H \cap H^y)|$;*
(ii) *Γ is connected if and only if $\langle H, x\rangle = G$.*

Let $\Gamma = (V, E)$ be a connected graph and $G \le \mathrm{Aut}\Gamma$. For $\alpha \in V$, the stabilizer G_α induces a permutation group $G_\alpha^{\Gamma(\alpha)}$. Let $G_\alpha^{[1]}$ be the kernel of this action. Then $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha/G_\alpha^{[1]}$. Consider the actions of Sylow subgroups of $G_\alpha^{[1]}$ on V. It is easily shown that the next lemma holds, see [5] for example.

Lemma 2.2. *Let $\Gamma = (V, E)$ be a connected regular graph, $G \le \mathrm{Aut}\Gamma$ and $\alpha \in V$. Assume that $G_\alpha \ne 1$. Let p be a prime divisor of $|G_\alpha|$. Then $p \le |\Gamma(\alpha)|$. If further Γ is G-vertex-transitive, then p divides $|G_\alpha^{\Gamma(\alpha)}|$ and, for $\beta \in \Gamma(\alpha)$, each prime divisor of $|G_{\alpha\beta}|$ is less than $|\Gamma(\alpha)|$.*

Lemma 2.3. *Assume that $\Gamma = (V, E)$ is a connected G-vertex-transitive graph. Let $N \lhd G$ be a normal subgroup of G such that $N_\alpha^{\Gamma(\alpha)}$ is semiregular for some $\alpha \in V$. Then $N_\alpha^{[1]} = 1$, that is, N_α is faithful on $\Gamma(\alpha)$.*

Proof. Let $\beta \in \Gamma(\alpha)$. Then $\beta = \alpha^x$ for some $x \in G$, and hence $N_\beta = N \cap G_{\alpha^x} = (N_\alpha)^x$. It follows that $N_\beta^{\Gamma(\beta)}$ and $N_\alpha^{\Gamma(\alpha)}$ are permutation isomorphic; in particular, $N_\beta^{\Gamma(\beta)}$ is semiregular on $\Gamma(\beta)$. Thus $N_\alpha^{[1]}$ acts trivially on $\Gamma(\beta)$, and so $N_\alpha^{[1]} = N_\beta^{[1]}$. Since Γ is connected, $N_\alpha^{[1]}$ fixes each vertex of Γ, and hence $N_\alpha^{[1]} = 1$. □

Lemma 2.4. *Let $\Gamma = (V, E)$ be a connected graph, $N \lhd G \le \mathrm{Aut}\Gamma$ and $\alpha \in V$. Assume that either N is regular on V, or Γ is a bipartite graph such that N is regular on both the bipartition subsets of Γ. Then $G_\alpha^{[1]} = 1$.*

Proof. Set $X = NG_\alpha^{[1]}$. Then $X_\alpha = G_\alpha^{[1]}$ and $X_\alpha^{[1]} = G_\alpha^{[1]}$, and hence $X_\alpha^{\Gamma(\alpha)} = 1$.

Assume first that N is regular on V. Then $G = NG_\alpha$. It follows that X is normal in G. Thus our result follows from Lemma 2.3.

Now assume that Γ is a bipartite graph with bipartition subsets U and W, and that N is regular on both U and W. For each $\delta \in U \cup W$, we have $NX_\alpha = X = NX_\delta$, and $|X_\delta| = |X_\alpha|$. Since $X_\alpha = G_\alpha^{[1]}$ acts trivially on $\Gamma(\alpha)$, we have $X_\alpha \le X_\beta$ for each $\beta \in \Gamma(\alpha)$, and so $X_\alpha = X_\beta$ as $|X_\beta| = |X_\alpha|$. For $\alpha' \in U$, there exists some $x \in N$ such that $\alpha' = \alpha^x$. Then $X_{\alpha'} = X_{\alpha^x} = X_\alpha^x$ and $\Gamma(\alpha') = \Gamma(\alpha)^x$. It follows that $X_{\beta'} = X_{\alpha'}$ for every $\beta' \in \Gamma(\alpha')$. This implies that $X_\delta = X\gamma$ for an arbitrary edge $\{\delta, \gamma\}$ of Γ. By the connectedness of Γ, we conclude that $G_\alpha^{[1]}$ fixes each vertex of Γ. Thus $G_\alpha^{[1]} = 1$. □

Let $\Gamma = (V, E)$ be a connected G-locally-primitive graph, where $G \le \mathrm{Aut}\Gamma$. Then Γ is G-edge-transitive, and G has at most two orbits on V. Let N be a normal subgroup of G. Note that $G_\alpha^{\Gamma(\alpha)}$ is a primitive permutation group for each $\alpha \in V$. If Γ is G-vertex-transitive then, by Lemma 2.3, either N is semiregular on V, or N_α is transitive on $\Gamma(\alpha)$; the latter case implies that Γ is N-edge-transitive. Then we have

Lemma 2.5. *Let $\Gamma = (V, E)$ be a connected G-locally-primitive arc-transitive graph, where $G \le \mathrm{Aut}\Gamma$. Let N be a normal subgroup of G. If N is not semiregular on V then for $\alpha \in V$ the stabilizer N_α is transitive on $\Gamma(\alpha)$; in particular, N is transitive on E and has at most two orbits on V.*

Suppose that N is intransitive on every G-orbit on V. For $\alpha \in V$, we use $\bar{\alpha}$ to denote the N-orbit containing α. The *normal quotient* Γ_N is defined as the graph with vertex set $\overline{V} = \{\bar{\alpha} \mid \alpha \in V\}$ and edge set $\{\{\bar{\alpha}, \bar{\beta}\} \mid \{\alpha, \beta\} \in E\}$. The graph Γ is called a (normal) *cover* of Γ_N if, for every edge of $\{\bar{\alpha}, \bar{\beta}\}$ of Γ_N, the subgraph of Γ induced by $\bar{\alpha} \cup \bar{\beta}$ is a matching. If Γ is a cover of Γ_N then, noting that Γ is connected and G-vertex-transitive, it is easily shown that N is semiregular on V and N itself is the kernel of G acting on $\overline{V}$. Moreover, the following lemma holds.

Lemma 2.6. *Let $\Gamma = (V, E)$ be a connected G-locally-primitive graph, where $G \le \mathrm{Aut}\Gamma$. Let N be a normal subgroup of G. Assume that N is intransitive on every G-orbit on V. Then one of the following statements holds.*

(i) *Γ is a cover of Γ_N, N is semiregular on V and N itself is the kernel of G acting on $\overline{V}$, and Γ_N is (G/N)-locally-primitive.*
(ii) *N has two orbits on V, Γ is a G-arc-transitive bipartite graph, and either Γ is N-edge-transitive or $G_\alpha^{[1]} = 1$ for every $\alpha \in V$.*

Proof. Assume that N has two orbits on V. Then, by the choice of N, we know that G is transitive on V, and so Γ is bipartite and G-arc-transitive. Thus part (ii) of this lemma follows from Lemmas 2.4 and 2.5.

Assume that N has at least three orbits on V. If G has two orbits on V then part (i) of this lemma occurs by [9, Lemma 5.1].

Assume further that G is transitive on V. Take an arbitrary vertex $\alpha \in V$, and set $\Delta = \{\Gamma(\alpha) \cap \bar{\beta} \mid \beta \in \Gamma(\alpha)\}$. Then Δ is a G_α-invariant partition of $\Gamma(\alpha)$. Since G_α acts primitively on $\Gamma(\alpha)$, either $|\Delta| = 1$ or $|\Gamma(\alpha) \cap \bar{\beta}| = 1$ for each $\beta \in \Gamma(\alpha)$. On other hand, Γ_N is connected and of order no less 3, we have $|\Delta| \ge 2$. Thus $|\Gamma(\alpha) \cap \bar{\beta}| = 1$ for each $\beta \in \Gamma(\alpha)$. This yields that, for every edge of $\{\bar{\alpha}, \bar{\beta}\}$ of Γ_N, the subgraph of Γ induced by $\bar{\alpha} \cup \bar{\beta}$ is a matching. Then part (i) follows. □

3. The structure of stabilizers

Let $\Gamma = (V, E)$ be a group and $G \le \mathrm{Aut}\Gamma$. For an edge $\{\alpha, \beta\} \in E$, let $G_{\alpha\beta}^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]}$, the kernel of the edge stabilizer $G_{\{\alpha,\beta\}}$ acting on $\Gamma(\alpha) \cup \Gamma(\beta)$. Then

$$G_\alpha^{[1]}/G_{\alpha\beta}^{[1]} \cong (G_\alpha^{[1]}G_\beta^{[1]})/G_\beta^{[1]} \lhd G_{\alpha\beta}/G_\beta^{[1]} \cong G_{\alpha\beta}^{\Gamma(\beta)} = (G_\beta^{\Gamma(\beta)})_\alpha.$$

Moreover, the following result is well-known, see [8].

Theorem 3.1. *Let $\Gamma = (V, E)$ be a connected G-locally-primitive arc-transitive graph. If $\{\alpha, \beta\} \in E$ then $G_{\alpha\beta}^{[1]}$ is a p-group for some prime p.*

Since $G_\alpha/G_\alpha^{[1]} \cong G_\alpha^{\Gamma(\alpha)}$ and $G_\alpha^{[1]}/G_{\alpha\beta}^{[1]}$ is isomorphic to a normal subgroup of $(G_\beta^{\Gamma(\beta)})_\alpha$, if $G_\alpha^{\Gamma(\alpha)}$, $(G_\beta^{\Gamma(\beta)})_\alpha$ and $G_{\alpha\beta}^{[1]}$ are soluble then G_α is soluble. Note that $((G_\beta)^{\Gamma(\beta)})_\alpha \cong ((G_\alpha)^{\Gamma(\alpha)})_\beta$ if Γ is G-arc-transitive. Then Theorem 3.1 implies the next result.

Lemma 3.2. *Let Γ be a connected G-locally-primitive arc-transitive graph. Then G_α is soluble if and only if $G_\alpha^{\Gamma(\alpha)}$ is soluble.*

For a positive integer s, an *s-arc* in Γ is an $(s+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices such that $\alpha_{i-1} \in \Gamma(\alpha_i)$ for $1 \le i \le s$ and $\alpha_{i-1} \ne \alpha_{i+1}$ for $1 \le i \le s-1$. The graph Γ is said to be (G, s)*-arc-transitive* if it contains at least one s-arc and G acts transitively on both V and the set of s-arcs, and said to be (G, s)*-transitive* if it is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive. (Note that s-arc-transitivity yields $(s-1)$-arc-transitivity and locally-primitivity for all $s > 1$.) For the stabilizers of s-transitive graphs, we formulate the following theorem from [20,22,23].

Theorem 3.3. *Let $\Gamma = (V, E)$ be a connected (G, s)-transitive graph with $s \ge 2$, and let $\{\alpha, \beta\} \in E$. Then one of the following holds.*

(1) *$G_{\alpha\beta}^{[1]} = 1$ and $s \le 3$;*
(2) *$G_{\alpha\beta}^{[1]}$ is a non-trivial p-group, $G_\alpha^{\Gamma(\alpha)} \rhd \mathrm{PSL}(n, p^f)$, $|\Gamma(\alpha)| = \frac{p^{fn}-1}{p^f-1}$, and either*
  (2.1) *$n \ge 3$ and $s \in \{2, 3\}$; or*
  (2.2) *$n = 2$, $s \ge 4$ and one of the following holds:*

(i) $s = 4$ *and* $G_\alpha = [p^{2f}]{:}(a.\mathrm{PGL}(2, p^f)).R$, *where* $a = \frac{p^f-1}{(3,p^f-1)}$ *and* $|R|$ *is a divisor of* $(3, p^f - 1)f$;
(ii) $s = 5$, $p = 2$ *and* $G_\alpha = [2^{3f}]{:}\mathrm{GL}(2, 2^f).b$, *where* b *is a divisor of* f;
(iii) $s = 7$, $p = 3$ *and* $G_\alpha = [3^{5f}]{:}\mathrm{GL}(2, 3^f).b$, *where* b *is a divisor of* f.

For the case (2.1) of Theorem 3.3, the structure of G_α is determined by Trofimov in a series of papers, see [18]. Theorems 3.1 and 3.3 and Trofimov's results are important tools in the study of locally-primitive arc-transitive graphs. For convenience, we produce here an explicit list for the stabilizers of locally-primitive graphs of valency $d \in \{5, 6, 7\}$, which is of course a reproduction of the above results.

Theorem 3.4. *Let* $\Gamma = (V, E)$ *be a connected* G*-locally-primitive arc-transitive graph of valency* $d \in \{5, 6, 7\}$. *Let* $\alpha \in V$. *Then one of the following holds.*

(i) Γ *is not* $(G, 2)$*-arc-transitive, and* G_α *is (isomorphic to) one of the groups:*

$$\mathbb{Z}_5,\ \mathrm{D}_{10},\ \mathrm{D}_{20};\ \mathbb{Z}_7,\ \mathrm{D}_{14},\ \mathrm{D}_{28},\ 7{:}3,\ 3\times(7{:}3).$$

(ii) Γ *is* (G, s)*-transitive with* $s \geq 2$, *and* G_α *lies in the following list:*

$d = 5$:

s	2	3	4	5
G_α	$5{:}4,\ 2\times(5{:}4)$ $\mathrm{A}_5,\ \mathrm{S}_5$	$4\times(5{:}4),\ \mathrm{A}_4\times\mathrm{A}_5,$ $(\mathrm{A}_4\times\mathrm{A}_5).2,\ \mathrm{S}_4\times\mathrm{S}_5$	$[4^2]{:}\mathrm{SL}(2,4),$ $[4^2]{:}\mathrm{GL}(2,4)$ $[4^2]{:}\Gamma\mathrm{L}(2,4)$	$[4^3]{:}\mathrm{GL}(2,4)$ $[4^3]{:}\Gamma\mathrm{L}(2,4)$

$d = 6$:

s	2	3	4
G_α	$\mathrm{A}_6,\ \mathrm{S}_6$ $\mathrm{A}_5,\ \mathrm{S}_5$	$\mathrm{A}_5\times\mathrm{A}_6,\ (\mathrm{A}_5\times\mathrm{A}_6).2,\ \mathrm{S}_5\times\mathrm{S}_6$ $\mathrm{D}_{10}\times\mathrm{PSL}(2,5),\ (5\times\mathrm{PSL}(2,5)).2$ $\mathrm{D}_{10}\times\mathrm{PGL}(2,5),\ (5{:}4)\times\mathrm{PGL}(2,5)$	$5^2{:}\mathrm{GL}(2,5)$

$d = 7$:

s	2	2, 3	3
G_α	$7{:}6,\ 2\times(7{:}6),\ 3\times(7{:}6)$ $\mathrm{SL}(3,2)$ $2^3.\mathrm{SL}(3,2)$ $[2^4]{:}\mathrm{SL}(3,2)$	A_7 S_7	$6\times(7{:}6),\ \mathrm{A}_6\times\mathrm{A}_7,\ (\mathrm{A}_6\times\mathrm{A}_7).2$ $\mathrm{S}_6\times\mathrm{S}_7,\ \mathrm{A}_4\times\mathrm{SL}(3,2),\ \mathrm{S}_4\times\mathrm{SL}(3,2)$ $[2^6].(\mathrm{SL}(2,2)\times\mathrm{SL}(3,2))$ $[2^{20}].(\mathrm{SL}(2,2)\times\mathrm{SL}(3,2))$

Proof. Assume that Γ is (G, s)-transitive. Note that $G_\alpha^{\Gamma(\alpha)}$ is a primitive permutation group of degree d. Then either
(a) $G_\alpha^{\Gamma(\alpha)}$ 2-transitive and $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_7$ or $\mathrm{PSL}(3, 2)$; or
(b) $G_\alpha^{\Gamma(\alpha)} \cong \mathbb{Z}_d{:}\mathbb{Z}_l$ with $d \in \{5, 7\}$ and l a divisor of $d - 1$.

If $G_\alpha^{[1]} = 1$ then $G_\alpha \cong G_\alpha^{\Gamma(\alpha)}$ is known and, by [12, Proposition 2.6], either $s \leq 2$ or $(d, G_\alpha) = (7, \mathrm{A}_7)$ or $(7, \mathrm{S}_7)$. Thus we next suppose that $G_\alpha^{[1]} \neq 1$. Let $\beta \in \Gamma(\alpha)$.

Assume first that $G_{\alpha\beta}^{[1]}$ is a non-trivial p-group. Then by Theorem 3.3 and [21], $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{PSL}(2, 4)$ or $\mathrm{PSL}(3, 2)$. Thus, by Theorem 3.3 and [18], the triple (d, s, G_α) lies in the following table:

d	s	G_α
5	4	$[4^2]{:}\mathrm{GL}(2,4),\ [4^2]{:}\Gamma\mathrm{L}(2,4),\ [4^2]{:}\mathrm{SL}(2,4)$
	5	$[4^3]{:}\mathrm{GL}(2,4),\ [4^3]{:}\Gamma\mathrm{L}(2,4)$
6	4	$5^2{:}\mathrm{GL}(2,5)$
7	2	$2^3.\mathrm{SL}(3,2),\ [2^4]{:}\mathrm{SL}(3,2)$
	3	$[2^6].(\mathrm{SL}(2,2)\times\mathrm{SL}(3,2)),\ [2^{20}].(\mathrm{SL}(2,2)\times\mathrm{SL}(3,2))$

Now let $G_{\alpha\beta}^{[1]} = 1$. Then $G_\alpha^{[1]}$ acts faithfully on $\Gamma(\beta)$, and $G_\alpha^{[1]}$ is isomorphic to a normal subgroup of $(G_\beta^{\Gamma(\beta)})_\alpha$. Since $G_{\alpha\beta}^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]}$, we have

$$G_{\alpha\beta} \cong G_{\alpha\beta}/(G_\alpha^{[1]} \cap G_\beta^{[1]}) \lesssim G_{\alpha\beta}/G_\alpha^{[1]} \times G_{\alpha\beta}/G_\beta^{[1]} \cong G_{\alpha\beta}^{\Gamma(\alpha)} \times G_{\alpha\beta}^{\Gamma(\beta)}.$$

Note that $G_\beta^{\Gamma(\beta)} \cong G_\alpha^{\Gamma(\alpha)}$ is explicitly known, and so is the stabilizer $(G_\beta^{\Gamma(\beta)})_\alpha$. This gives us a strategy to determine the stabilizer $G_\alpha = G_\alpha^{[1]}.G_\alpha^{\Gamma(\alpha)}$, a group extension of $G_\alpha^{[1]}$ by $G_\alpha^{\Gamma(\alpha)}$. Moreover, we have the following useful observation. Recall that Γ is connected and G-arc-transitive. Then $\mathrm{Aut}\Gamma \geq G = \langle x, G_\alpha\rangle$ for some $x \in \mathbf{N}_G(G_{\alpha\beta})$. It follows that G_α contains no non-trivial normal subgroups which are characteristic in $G_{\alpha\beta}$. In particular, $G_\alpha^{[1]}$ is not a characteristic subgroup of $G_{\alpha\beta}$.

(1) Let $d = 5$. Then $G_\alpha^{\Gamma(\alpha)}$ is not regular on $\Gamma(\alpha)$ by Lemma 2.3, and so $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{D}_{10}, 5{:}4, A_5$ or S_5.

Assume that $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{D}_{10}$ or $5{:}4$. Then $(G_\beta^{\Gamma(\beta)})_\alpha \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$, and hence $G_\alpha^{[1]} \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$, respectively. Thus $G_\alpha = G_\alpha^{[1]}.G_\alpha^{\Gamma(\alpha)} = (G_\alpha^{[1]} \times 5).(G_\alpha^{\Gamma(\alpha)})_\beta = 5{:}G_{\alpha\beta}$. Noting that $G_{\alpha\beta} \lesssim \mathbb{Z}_4 \times \mathbb{Z}_4$ and $G_\alpha^{[1]}$ is faithful on $\Gamma(\alpha) \setminus \{\alpha\}$, it follows that either G_α is one of D_{20} and $2 \times (5{:}4)$, or Γ is $(G, 3)$-transitive and $G_\alpha = 4 \times (5{:}4)$.

Assume $G_\alpha^{\Gamma(\alpha)} \cong A_5$. Then $(G_\beta^{\Gamma(\beta)})_\alpha \cong A_4$, and so $G_\alpha^{[1]} \cong \mathbb{Z}_2^2$ or A_4. If $G_\alpha^{[1]} \cong \mathbb{Z}_2^2$ then $G_\alpha = \mathbb{Z}_2^2 \times A_5$, and so both G_α and $G_{\alpha\beta}$ contain a characteristic subgroup isomorphic to $\mathbb{Z}_2^2$, which is a contradiction. Thus $G_\alpha^{[1]} \cong (G_\beta^{\Gamma(\beta)})_\alpha \cong A_4$, and so $G_\alpha = A_4 \times A_5$ and Γ is $(G, 3)$-transitive.

Assume $G_\alpha^{\Gamma(\alpha)} \cong S_5$. Then $(G_\beta^{\Gamma(\beta)})_\alpha \cong S_4$, and so $G_\alpha^{[1]} \cong \mathbb{Z}_2^2$, A_4 or S_4. Suppose that $G_\alpha^{[1]} \cong \mathbb{Z}_2^2$. Then $G_\alpha = G_\alpha^{[1]}.S_5 = (G_\alpha^{[1]} \times A_5).2$ and $G_{\alpha\beta} = G_\alpha^{[1]}.S_4 = (G_\alpha^{[1]} \times A_4).2$. This implies that both G_α and $G_{\alpha\beta}$ have the same center isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$, a contradiction. Thus $G_\alpha^{[1]} \cong A_4$ or S_4, and so Γ is $(G, 3)$-transitive and $G_\alpha = (A_4 \times A_5).2$, or $S_4 \times S_5$.

(2) Let $d = 6$. Then $G_\alpha^{\Gamma(\alpha)} \cong A_6,\ S_6,\ \mathrm{PSL}(2,5)$ or $\mathrm{PGL}(2,5)$, and $(G_\beta^{\Gamma(\beta)})_\alpha \cong A_5, S_5, D_{10}$ or 5:4, respectively. If $G_\alpha^{[1]} \cong A_5$ or S_5, then $G_\alpha^{[1]} \cong A_5$ or S_5, and so $G_\alpha = A_5 \times A_6$, $(A_5 \times A_6).2$ or $S_5 \times S_6$.

Assume that $(G_\beta^{\Gamma(\beta)})_\alpha \cong D_{10}$. Then $G_\alpha^{[1]} \cong \mathbb{Z}_5$ or D_{10}, and $G_\alpha = \mathrm{PSL}(2,5) \times G_\alpha^{[1]}$. If $G_\alpha^{[1]} \cong \mathbb{Z}_5$ then both G_α and $G_{\alpha\beta}$ have the same center $G_\alpha^{[1]}$, a contradiction. Thus $G_\alpha^{[1]} \cong D_{10}$ and $G_\alpha = D_{10} \times \mathrm{PSL}(2,5)$.

Finally, if $(G_\beta^{\Gamma(\beta)})_\alpha \cong$ 5:4 then $G_\alpha^{[1]} = \mathbb{Z}_5, D_{10}$ or 5:4, this yields that $G_\alpha = (5 \times \mathrm{PSL}(2,5)).2$, $D_{10} \times \mathrm{PGL}(2,5)$, or $(5{:}4) \times \mathrm{PGL}(2,5)$.

(3) Let $d = 7$. Then $G_\alpha^{\Gamma(\alpha)}$ is not regular on $\Gamma(\alpha)$ by Lemma 2.3, and so $G_\alpha^{\Gamma(\alpha)} \cong D_{14}$, 7:3, 7:6, $\mathrm{SL}(3,2)$, A_7 or S_7. For $G_\alpha^{\Gamma(\alpha)} \cong D_{14}$, 7:3 or 7:6, we have $G_\alpha = D_{28}$, $3 \times (7{:}3)$, $2 \times (7{:}6)$, $3 \times (7{:}6)$ or $6 \times (7{:}6)$. For $G_\alpha^{\Gamma(\alpha)} \cong A_7$ or S_7, we have $G_\alpha = A_6 \times A_7$, $(A_6 \times A_7).2$ or $S_6 \times S_7$. Assume that $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{SL}(3,2)$. Then $(G_\beta^{\Gamma(\beta)})_\alpha = S_4$, and so $G_\alpha^{[1]} = \mathbb{Z}_2^2$, A_4 or S_4. The group $\mathbb{Z}_2^2$ is excluded by considering the centers of G_α and $G_{\alpha\beta}$. Thus $G_\alpha^{[1]} \cong A_4$ or S_4, and so $G_\alpha = A_4 \times \mathrm{SL}(3,2)$ or $S_4 \times \mathrm{SL}(3,2)$. □

Consider the orders of the groups G_α listed in Theorem 3.4. We have

Corollary 3.5. *Let $\Gamma = (V, E)$ be a connected G-locally-primitive arc-transitive graph of valency $d \in \{5, 6, 7\}$. For $\alpha \in V$, the following statements hold.*

(1) *None of 2^{25}, 3^5, 5^4 and 7^2 is a divisor of $|G_\alpha|$.*
(2) *If $|G_\alpha|$ is divisible by 2^{10} then $|G_\alpha| = 2^{10} \cdot 3^2 \cdot 7$ or $2^{24} \cdot 3^2 \cdot 7$.*
(3) *If $|G_\alpha|$ is not divisible by 3 then 2^5 is not a divisor of $|G_\alpha|$.*
(4) *If $d = 7$ then one of 2^9 and 3^3 is not a divisor of $|G_\alpha|$.*

4. Examples

We describe in this section some arc-transitive graphs of square-free order. For a square-free number n, the complete graph K_n is such a graph, and so is the complete bipartite graph $K_{n,n}$ if in addition n is odd. Also for an odd square-free number n, the standard double cover of K_n is such an example, which is isomorphic to $K_{n,n} - nK_2$. Note that $K_6, K_7, K_{5,5}, K_{7,7}$ and $K_{7,7} - 7K_2$ are involved in Theorem 1.1.

The *odd graph* $\mathbf{O}_d$ is defined on the set consisting of $(d-1)$-subsets of a set of size $2d-1$ such that two vertices are adjacent whenever they disjoint. Then $\mathrm{Aut}\mathbf{O}_d = S_{2d-1}$ which acts 3-arc-transitively on $\mathbf{O}_d$ with stabilizer $S_d \times S_{d-1}$. The graph $\mathbf{O}_d$ has valency d and order $\binom{2d-1}{d-1}$. The graph $\mathbf{O}_6$ is involved in Theorem 1.1.

Let $\mathrm{PG}(2,q)$ be the *projective plane* over the finite field of order q. Then $\mathrm{PG}(2,q)$ has q^2+q+1 points and q^2+q+1 lines, and the group $\mathrm{PGL}(3,q)$ acts transitively on the flags of $\mathrm{PG}(2,q)$. The incidence graph of $\mathrm{PG}(2,q)$ is a $(G,4)$-arc-transitive graph of valency $q+1$ and order $2(q^2+q+1)$, where $G = \mathrm{PGL}(3,q).\langle\tau\rangle$ with τ being transpose-inverse automorphism of $\mathrm{PGL}(3,q)$. For $q = 4$ and 5, the resulting graphs are involved in Theorem 1.1.

Let $\mathrm{PG}(3,2)$ be the 3-dimensional projective geometry over the field of order 2. Then $\mathrm{PG}(3,2)$ have 15 points and 15 hyperplanes. The point–hyperplane incidence graph of $\mathrm{PG}(3,2)$ appears in Theorem 1.1, which is a $(G,2)$-arc-transitive graph of valency 7 and order 30, where $G = S_7$ or $\mathrm{PSL}(4,2).2$.

Let $\mathrm{GQ}(q)$ be the *generalized quadrangle* of order $q = 2^f$, which has $(q^2+1)(q+1)$ points and lines. The symplectic group $\mathrm{PSp}(4,q)$ acts on the geometry $\mathrm{GQ}(q)$ flag-transitively. For convenience, denote by $\mathrm{GQ}(q)$ the incidence graph of itself. Then the graph $\mathrm{GQ}(q)$ is $(G,5)$-arc-transitive of valency $q+1$, where $G = \mathrm{PSp}(4,q).2$. The graph $\mathrm{GQ}(4)$ appears in Theorem 1.1, which has valency 5 and order 170.

Let R be a group, and S a inverse-closed subset of R which does not contain the identity of R. Then the Cayley graph $\Gamma = \mathrm{Cay}(R,S)$ is the graph with vertex set R, where two vertices $x, y \in R$ are adjacent if and only if $yx^{-1} \in S$. It easily follows that $\mathrm{Aut}\Gamma$ has a subgroup $\hat{R}$ which is isomorphic to R and regular on the vertex set of Γ.

Construction 4.1. Let $R = \langle a\rangle{:}\langle b\rangle = D_{2n}$, where $n > 1$ is odd square-free. Let d be a prime. Assume that there is some integer r such that $\sum_{i=0}^{d-1} r^i \equiv 0 \pmod{n}$. Let s be an integer coprime to n, and let $\sigma \in \mathrm{Aut}(R)$ such that $a^\sigma = a^r$ and $b^\sigma = a^s b$. Then σ has order d and $R = \langle S\rangle$, where $S = \{b^{\sigma^i} \mid 0 \le i \le d-1\}$. Hence $G := R{:}\langle\sigma\rangle \cong D_{2n}{:}\mathbb{Z}_d$, and $\mathrm{Cay}(R,S)$ is a connected bipartite G-arc-regular graph of valency d. For example, taking $n = 155$ and $r = 2$, we get a graph of order 310 and valency 5.

Next we give several examples by using coset graphs.

Example 4.2. We identify $H = \mathrm{PSL}(2,5)$ with a transitive subgroup of A_6 containing $K = \langle \sigma, \tau \rangle$, where $\sigma = (1\,2\,3\,4\,5)$ and $\tau = (1\,5)(2\,4)$. Then $\mathbf{N}_{\mathrm{A}_7}(K) = \langle \sigma, \pi \rangle \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\langle \pi, H \rangle = \mathrm{A}_7$ and $\pi^2 \in K$, where $\pi = (1\,4\,5\,2)(6\,7)$. Thus $\mathsf{Cos}(\mathrm{A}_7, H, H\pi H)$ is a connected 2-arc-transitive graph of valency 6 and order 42.

Example 4.3. Checking by GAP, we know that the first Janko group J_1 has exactly two conjugation classes of subgroups isomorphic to A_5. Let H_1 and H_2 be two subgroups isomorphic to A_5 such that they are not conjugate in J_1. Then one of them is self-normalized and the other one has normalizer isomorphic to $2 \times \mathrm{A}_5$. Assume that $\mathbf{N}_{\mathrm{J}_1}(H_1) = H_1$ and $\mathbf{N}_{\mathrm{J}_1}(H_2) \cong 2 \times \mathrm{A}_5$.

(1) Take $\mathrm{A}_4 \cong K_1 \le H_1$. Then $\mathbf{N}_{\mathrm{J}_1}(K_1) = \langle x \rangle \times K_1 \cong \mathbb{Z}_2 \times \mathrm{A}_4$. Checking the maximal subgroups of J_1, we conclude that $\langle x, H_1 \rangle = \mathrm{J}_1$. Thus $\mathsf{Cos}(\mathrm{J}_1, H_1, H_1 x H_1)$ is a $(\mathrm{J}_1, 2)$-arc-transitive graph of valency 5 and order $2 \cdot 7 \cdot 11 \cdot 19$.

(2) Checking by GAP, if a subgroup $K \cong D_{10}$ is contained in H_1 or H_2 then $N_{J_1}(K) \cong D_{20}$. Take $\mathrm{D}_{10} \cong K_2 \le H_1$. Then $\mathbf{N}_{\mathrm{J}_1}(K_2) = \langle y \rangle \times K_2 \cong \mathrm{D}_{20}$. Checking the maximal subgroups of J_1, we conclude that $\langle y, H_1 \rangle = \mathrm{J}_1$. Thus $\mathsf{Cos}(\mathrm{J}_1, H_1, H_1 y H_1)$ is a $(\mathrm{J}_1, 2)$-arc-transitive graph of valency 6 and order $2 \cdot 7 \cdot 11 \cdot 19$.

Example 4.4. Let H be a maximal subgroup of M_{22} with $H \cong 2^3{:}\mathrm{SL}(3,2)$. By the Atlas [6], $\mathrm{SL}(3,2)$ has two conjugate classes of subgroups isomorphic to S_4. Then H has two conjugate classes of subgroups isomorphic to $2^3{:}\mathrm{S}_4$. Checking by GAP, we know that the subgroups in one of these classes are self-normalizing in M_{22}, and the subgroups in the other class have normalizers isomorphic to $2^4{:}\mathrm{S}_4$. Take $K < H$ with $K \cong 2^3{:}\mathrm{S}_4$ and $\mathbf{N}_{\mathrm{M}_{22}}(K) \cong 2^4{:}\mathrm{S}_4$. Let $g \in \mathbf{N}_{\mathrm{M}_{22}}(K) \setminus H$. Then $\langle H, g \rangle = \mathrm{M}_{22}$, $H^g \cap H = K$, and so $\Gamma = \mathsf{Cos}(\mathrm{M}_{22}, H, HgH)$ is a connected $(\mathrm{M}_{22}, 2)$-arc-transitive graph of valency 7. Note that this graph is a distance-transitive graph with automorphism group $\mathrm{M}_{22}.2$, see [3, Section 6.10].

Example 4.5. By the Atlas [6], $T = \mathrm{PSL}(2, 25)$ contains exactly two conjugation classes of elements of order 5, which appear respectively in two distinct conjugation classes of maximal subgroups isomorphic to S_5 in T. It follows that T has exactly two conjugation classes of subgroups isomorphic to 5:4. Computation of the number of the pairs with type $(\mathrm{S}_5, 5{:}4)$ of subgroups of T, we conclude that each subgroup 5:4 is contained in exactly one subgroup S_5.

Let $\mathbb{Z}_5{:}\mathbb{Z}_4 \cong H \le M \le T$, $M \cong \mathrm{S}_5$ and $\mathbb{Z}_4 \cong K \le H$. Then $\mathbf{N}_M(K) \cong \mathrm{D}_8$ and $\mathbf{N}_T(K) \cong \mathrm{D}_{24}$. Set $\mathbf{N}_M(K) = K{:}\langle z \rangle$ and $\mathbf{N}_T(K) = K{:}(\langle y \rangle{:}\langle z \rangle)$ with $\langle y \rangle{:}\langle z \rangle \cong \mathrm{D}_6$. By the above argument, we have $\langle y^i z, H \rangle = T$ for $i = 1$ and 2. Then $\mathsf{Cos}(T, H, HyzH)$ and $\mathsf{Cos}(T, H, Hy^2 zH)$ are two $(T, 2)$-arc-transitive graphs of valency 5 and order 390.

5. The automorphism groups

Let $\Gamma = (V, E)$ be a connected G-locally-primitive arc-transitive graph of square-free order and valency d, where $G \le \mathsf{Aut}\Gamma$ and $d \in \{5, 6, 7\}$. Let $\alpha \in V$.

5.1. Assume that G is soluble. Then $G_\alpha^{\Gamma(\alpha)}$ is a soluble primitive group of degree d. This implies that $d = 5$ or 7. Moreover, the next result holds.

Lemma 5.1. *Assume that G is soluble. Then $d \in \{5, 7\}$ and either $\Gamma \cong \mathsf{K}_{d,d}$ and $\mathrm{soc}(G) \cong \mathbb{Z}_d^2$, or Γ is isomorphic to a graph constructed in* Construction 4.1.

Proof. Let F be the Fitting subgroup of G. Then $\mathbf{C}_G(F) \le F \ne 1$, and every Sylow subgroup of F is normal in G. Take an arbitrary prime divisor p of $|F|$, and let P be the Sylow p-subgroup of F. Then $P \lhd G$. If $|P| > p$ then, by Lemma 2.5, it is easily shown that $\Gamma \cong \mathsf{K}_{p,p}$; in this case, $d = p \in \{5, 7\}$ and $\mathrm{soc}(G) = P \cong \mathbb{Z}_d^2$. Thus we assume next that $|F|$ is square-free. Then F is cyclic, and so $\mathbf{C}_G(F) = F$ and $\mathsf{Aut}(F)$ is abelian. It is easily shown that F is semiregular on V.

Note that $G/F = \mathbf{N}_G(F)/\mathbf{C}_G(F) \lesssim \mathsf{Aut}(F)$. If F has at least three orbits on V then the quotient graph Γ_F has valency d and admits an abelian group acting transitively on its arcs, which is impossible. Thus F has at most two orbits on V. Suppose that F is transitive on V. Then F is a normal regular subgroup of G, and so $\Gamma \cong \mathsf{Cay}(F, S)$, where $S = S^{-1} = \{x^\sigma \mid \sigma \in A\}$ for some $x \in F$ and $A \le \mathsf{Aut}(F)$. Since Γ has odd valency, S contains an involution, and so S consists of involutions. Since Γ is connected and F is cyclic, $F = \langle S \rangle \cong \mathbb{Z}_2$. Then $|V| = |F| = 2$, which is impossible. Therefore, F has exactly two orbits on V, and so $|G : (FG_\alpha)| = 2$, where $\alpha \in V$. Since $G_\alpha \cong G_\alpha F/F \le G/F \lesssim \mathsf{Aut}(F)$, we know that G_α is abelian. By Lemma 2.3, $G_\alpha \cong \mathbb{Z}_d$, and so $G = F.\mathbb{Z}_{2d}$. Thus G has a normal regular subgroup $F{:}\mathbb{Z}_2$. Then $\Gamma \cong \mathsf{Cay}(F{:}\mathbb{Z}_2, S)$, where $S = \{s^{\sigma^i} \mid 0 \le i \le d-1\}$ for an involution $s \in F{:}\mathbb{Z}_2$ and $\sigma \in \mathsf{Aut}(F{:}\mathbb{Z}_2)$ of order d such that $\langle S \rangle = F{:}\mathbb{Z}_2$. Noting that $|F{:}\mathbb{Z}_2|$ is square-free, we conclude that $F{:}\mathbb{Z}_2$ is a dihedral group. Then the lemma follows. □

5.2. In this part we analyze the structure of G while G is insoluble.

Lemma 5.2. *Assume that G is insoluble. Let M be a soluble normal subgroup of G. Then M is semiregular and has at least three orbits on V, Γ is a cover of Γ_M and $G = M{:}X$ for some $X \le G$.*

Proof. Suppose that $M_\alpha \ne 1$ for $\alpha \in V$. Then M_α is transitive on $\Gamma(\alpha)$, and so $G_\alpha^{\Gamma(\alpha)}$ has a soluble transitive normal subgroup isomorphic to $M_\alpha G_\alpha^{[1]}/G_\alpha^{[1]} \cong M_\alpha/M_\alpha^{[1]}$. Noting that $G_\alpha^{\Gamma(\alpha)}$ is a primitive group of degree $d \in \{5, 6, 7\}$, it follows that $G_\alpha^{\Gamma(\alpha)}$ is soluble. Then G_α is soluble by Lemma 3.2, and so MG_α is soluble. By Lemma 2.5, M has at most two orbits on V, it follows that $|G : MG_\alpha| \le 2$. This implies that G is soluble, a contradiction. Thus M is semiregular on V.

Suppose that M has at most two orbits on V. Then $|G : MG_\alpha| \le 2$, and $G_\alpha \cong G_\alpha^{\Gamma(\alpha)}$ by Lemma 2.4. Since $|G : MG_\alpha| \le 2$ and G is insoluble, G_α is insoluble, and hence $G_\alpha^{\Gamma(\alpha)}$ is an almost simple 2-transitive permutation group of degree $d \in \{5, 6, 7\}$. Thus we have $\mathrm{soc}(G_\alpha) \cong \mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{A}_5, \mathrm{A}_6, \mathrm{PSL}(3, 2)$ or A_7. Since M is semiregular on V, we know that M has square-free order, and so $\mathrm{Aut}(M)$ is soluble. Note that

$$M/\mathbf{C}_{M\mathrm{soc}(G_\alpha)}(M) = \mathbf{N}_{M\mathrm{soc}(G_\alpha)}(M)/\mathbf{C}_{M\mathrm{soc}(G_\alpha)}(M) \lesssim \mathrm{Aut}(M).$$

It follows that $\mathrm{soc}(G_\alpha) \le \mathbf{C}_{M\mathrm{soc}(G_\alpha)}(M)$, and hence $M\mathrm{soc}(G_\alpha) = M{:}\mathrm{soc}(G_\alpha) = M \times \mathrm{soc}(G_\alpha)$. It is easily shown that $\mathrm{soc}(G_\alpha)$ is a characteristic subgroup group of $M\mathrm{soc}(G_\alpha)$, and so $\mathrm{soc}(G_\alpha) \lhd MG_\alpha$.

Take $\beta \in \Gamma(\alpha)$. Since Γ is G-vertex-transitive, G_α and G_β are conjugate, and hence $\mathrm{soc}(G_\alpha) \cong \mathrm{soc}(G_\beta) \lhd MG_\beta$. Let U and W be the M-orbits containing α and β, respectively. (Note that $U = W = V$ if M is transitive on V.) Then $\mathrm{soc}(G_\alpha)$ and $\mathrm{soc}(G_\beta)$ act trivially on U and W, respectively. Note that $MG_\alpha = G_U = G_W = MG_\beta$. Then both $\mathrm{soc}(G_\alpha)$ and $\mathrm{soc}(G_\beta)$ are normal in MG_α, and so $\mathrm{soc}(G_\alpha) \cap \mathrm{soc}(G_\beta)$ is normal in MG_α. Since $\mathrm{soc}(G_\alpha)$ and $\mathrm{soc}(G_\beta)$ are nonabelian simple groups, either $\mathrm{soc}(G_\alpha) = \mathrm{soc}(G_\beta)$ or $\mathrm{soc}(G_\alpha) \cap \mathrm{soc}(G_\beta) = 1$. If $\mathrm{soc}(G_\alpha) \cap \mathrm{soc}(G_\beta) = 1$ then $\mathrm{soc}(G_\beta) \cong \mathrm{soc}(G_\alpha)\mathrm{soc}(G_\beta)/\mathrm{soc}(G_\alpha) \le MG_\alpha/\mathrm{soc}(G_\alpha)$; however, $MG_\alpha/\mathrm{soc}(G_\alpha)$ is soluble, a contradiction. Thus $\mathrm{soc}(G_\alpha) = \mathrm{soc}(G_\beta)$. This implies that $\mathrm{soc}(G_\alpha)$ fixes $V = U \cup W$ point-wise, which contradicts $1 \ne \mathrm{soc}(G_\alpha) \le \mathrm{Aut}\Gamma$. Then M has at least three orbits on V, and Γ is a cover of Γ_M by Lemma 2.6.

Now we show that $G = M{:}X$ for some $X \le G$ by induction on $|M|$. This is trivial for $M = 1$. Thus we assume that $|M| > 1$ in the following.

Let p be the largest prime divisor of $|M|$. Then, since M has square-free order, M has a unique Sylow p-subgroup, say P. Thus P is a characteristic subgroup of M, and so $P \lhd G$. Clearly P has at least three orbits on V. By Lemma 2.6, Γ is a normal cover of Γ_P and Γ_P is G/P-locally-primitive arc-transitive. Note that each M-orbit on V is the union of some P-orbits. Then M/P has at least three orbits on the vertex set of Γ_P. Then, by induction, we may assume that $G/P = (M/P){:}(Y/P)$ for a subgroup $Y \le G$ with $Y \cap M = P$. (Note that $Y = G$ if $P = M$.) Clearly, Y acts transitively on the vertex set of Γ_P, and so Y is transitive on V. Note that Γ_P has order $\frac{|V|}{p}$. Then $\frac{|V|}{p} = |Y : Y_B|$ for a P-orbit B on V. Since $|V|$ is square-free, $|Y : Y_B|$ is coprime to p, and then Y_B contains a Sylow p-subgroup of Y. Since $P \le Y_B$ is transitive on B, we have $Y_B = PY_\alpha = P{:}Y_\alpha$ for $\alpha \in B$. It follows that Y_B and hence Y has a Sylow p-subgroup $P{:}Q$, where Q is a Sylow p-subgroup of Y_α. Then, by Gaschütz' Theorem (see [2, 10.4]), the extension $Y = P.(Y/P)$ splits over P. Thus $Y = P{:}X$ for $X < Y$ with $X \cap P = 1$. Then $G = MY = MX$ and $X \cap M = X \cap (Y \cap M) = X \cap P = 1$, and our result follows. □

Lemma 5.3. *Assume that $T^l \cong N \lhd G$, where $l \ge 2$ and T is a non-abelian simple group. Then $l = 2$, $T \cong \mathrm{A}_5, \mathrm{A}_7$ or $\mathrm{PSL}(3, 2)$, and $\Gamma \cong \mathrm{K}_{d,d}$ with $d \in \{5, 7\}$.*

Proof. Since $|V|$ is square-free, N is not semiregular on V, and so N has at most two orbits on V by Lemma 2.5. Let $\alpha \in V$ and U be the N-orbit containing α. Then $U = V$ or $|U| = \frac{|V|}{2}$. Note that $|T|^l = |N| = |U||N_\alpha|$ and $|U|$ is square-free. Then $|N_\alpha|$ is divisible by $|T|^{l-1}$, and so $|G_\alpha|$ is divisible by $|T|^{l-1}$. Suppose that $G_\alpha^{\Gamma(\alpha)}$ is soluble. By Lemma 3.2, G_α is soluble, and so G_α is explicitly known by Theorem 3.4. This implies that $|G_\alpha|$ is not divisible by the order of some non-abelian simple group, a contradiction. Thus $G_\alpha^{\Gamma(\alpha)}$ is insoluble, and then $G_\alpha^{\Gamma(\alpha)}$ is an almost 2-transitive permutation group of degree $d \in \{5, 6, 7\}$; in particular, $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{A}_5, \mathrm{A}_6, \mathrm{PSL}(3, 2)$ or A_7. Since N is not semiregular on V, by Lemma 2.5, N_α induces a normal transitive subgroup of $G_\alpha^{\Gamma(\alpha)}$. It follows that N_α acts 2-transitively on $\Gamma(\alpha)$.

Set $N = T_1 \times T_2 \times \cdots \times T_l$, where $T_1 \cong T_2 \cong \cdots \cong T_l \cong T$. Suppose that $U = V$, that is, N is transitive on V. Then Γ is $(N, 2)$-arc-transitive and every T_i acts non-trivially on V. In particular, by Lemma 2.5, T_i has at most two orbits on V. Since T_j has no subgroups of index 2, each T_j fixes every T_i-orbit setwise, and so does N. It follows that every T_i is transitive on V. Then T_i is regular on V (see [7, Theorem 4.2A]), a contradiction. Thus N has two orbits on V, say U and W.

If some T_i is intransitive on both U and W then, by Lemma 2.6, T_i semiregular on U, and so $|T_i|$ is square-free, a contradiction. Thus every T_i is transitive on at least one of U and W. Without loss of generality, we assume that T_1 acts transitively on U. Then, by [7, Theorem 4.2A], T_2 induces a semiregular permutation group on U, and hence T_2 acts trivially on U. Thus T_2 is transitive on W. This implies that Γ is a complete bipartite graph. Since $|V|$ is square-free, $\Gamma \cong \mathrm{K}_{5,5}$ or $\mathrm{K}_{7,7}$, and $T_1 \cong T_2 \cong \mathrm{A}_5, \mathrm{A}_7$ or $\mathrm{PSL}(3, 2)$. If $l \ge 3$, then a similar argument as above implies that T_3 is trivial on both U and W, a contradiction. Thus the lemma follows. □

Lemma 5.4. *Assume that G has no soluble minimal normal subgroups. Then $\mathrm{soc}(G)$ is a minimal normal subgroup of G, and either G is almost simple, or $\mathrm{soc}(G) \cong T^2$ and $\Gamma \cong \mathrm{K}_{d,d}$ with $d \in \{5, 7\}$, where $T \cong \mathrm{A}_5, \mathrm{A}_7$ or $\mathrm{PSL}(3, 2)$.*

Proof. Note that every minimal normal subgroup of G is a directed product of isomorphic non-abelian simple groups. Suppose that G has two distinct minimal normal subgroups N and M. Then $NM = N \times M$. Since $|V|$ is square-free, N is not semiregular on V, and so N has at most two orbits on V by Lemma 2.5. Let U be an N-orbit on V. Then $U = V$ or $|U| = \frac{|V|}{2}$. Noting that M has no subgroups of index 2, we conclude that M fixes U setwise, and then U is also an M-orbit. Then N and M induce two regular permutation groups on U (see [7, Theorem 4.2A]), which is impossible. Thus G has a unique minimal normal subgroup, that is, $\mathrm{soc}(G)$ is a minimal normal subgroup of G. Finally, the lemma follows from Lemma 5.3. □

Lemma 5.5. *Assume that $\mathrm{soc}(G) = T$ is a non-abelian simple group. Then, up to isomorphism, T is one of the following simple groups:*

(i) A_c *for* $c \in \{5, 6, 7, 8, 10, 11, 12, 13, 14\}$;
(ii) $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1$;
(iii) $PSL(2, 2^f)$ *for* $4 \le f \le 10$, $PSL(3, 4)$, $PSL(3, 8)$, $PSL(5, 2)$, $PSU(3, 4)$, $PSU(5, 2)$, $PSp(4, 4)$, $Sz(8)$;
(iv) $PSL(3, 3)$, $PSL(3, 5)$, $PSL(2, 3^4)$, $PSL(2, 25)$, $PSL(2, 5^4)$;
(v) $PSL(2, p)$ *for prime* $p \ge 7$.

Proof. Let $\alpha \in V$. Since T is normal in G, every T-orbit on V has length $|T : T_\alpha|$, which is a divisor of $|V| = |G : G_\alpha|$. Thus $|T : T_\alpha|$ is square-free, and so T has a maximal subgroup (containing T_α) of square-free index.

Assume that T is an alternating simple group. By Corollary 3.5, 3^5 is not a divisor of $|G_\alpha|$, and hence $|G|$ is not divisible by 3^6 as $|G : G_\alpha|$ is square-free. In particular, $|T|$ is not divisible by 3^6. It follows that $T \cong A_c$ with $5 \le c \le 14$. Checking the subgroups of A_9 in the Atlas [6], A_9 has no maximal subgroup of square-free index. Thus $c \ne 9$.

Assume that T is one of sporadic simple groups. Note that, by Corollary 3.5, $|G|$ and hence $|T|$ is not divisible by $2^{11} \cdot 5^2 \cdot 7$. Checking the order of T (see [11, Table 5.1.C] for example), we know that T is isomorphic to one of M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3 and HS. The groups J_2, J_3 and HS are excluded as they have no maximal subgroup of square-free index (see the Atlas [6]).

Now let T be one of simple groups of Lie type with characteristic p. Check the order $|T|$ of T and consider the maximal power of p dividing $|T|$, see [11, pp. 170]. Then, noting the isomorphisms among simple groups (see [11, Proposition 2.9.1 and Theorem 5.1.1]), we may get a finite list of candidates for T. For odd prime p, we conclude that either $T \cong PSL(2, p)$ with $p > 7$, or T is isomorphic to one of the following simple groups:

$PSL(2, 3^f)$ with $2 \le f \le 5$, $PSL(3, 3)$, $PSU(3, 3)$, $PSp(4, 3)(\cong PSU(4, 2))$;
$PSL(2, 5^f)$ with $1 \le f \le 4$, $PSL(3, 5)$, $PSU(3, 5)$, $PSp(4, 5)$;
$PSL(2, 7)$, $PSL(2, 49)$.

The groups $PSL(2, 3^3)$, $PSL(2, 3^5)$, $PSL(2, 5^3)$, $PSL(2, 49)$, $PSU(3, 3)$, $PSU(3, 5)$ and $PSp(4, 5)$ are easily excluded as they have no maximal subgroup of square-free index (see [10, II. 8.27] and the Atlas [6]).

Assume that T is one of exceptional groups of Lie type with characteristic 2. By Corollary 3.5, $|G_\alpha|$ is not divisible by 2^{25}, and hence $|G|$ is not divisible by 2^{26}. Then $|T|$ is not divisible by 2^{26}. It follows from [11, Table 5.1.B] that T is isomorphic to one of $G_2(2^f)$ (with $2 \le f \le 4$), ${}^2B_2(2^{2m+1})$ (with $1 \le m \le 5$), ${}^3D_4(2)$ and ${}^3D_4(4)$. If $T \cong {}^2B_2(2^{2m+1})$ for $m = 2, 3, 5$, then $|G|$ is not divisible by 3, which contradicts (3) of Corollary 3.5. If $T \cong {}^2B_2(2^9)$ then $|G|$ is divisible by 2^{18} but not by 2^{19}; however, by Corollary 3.5, we know that $|G|$ is either not divisible by 2^{12} or divisible by 2^{24}, a contradiction. By Corollary 3.5 (2), we conclude that none of 5^2, 3^4 and 17^2 is a divisor of $|G|$. This observation excludes the groups $G_2(2^f)$, where $2 \le f \le 4$. Similarly, ${}^3D_4(2)$ and ${}^3D_4(4)$ are easily excluded as they have orders divisible by $2^{12} \cdot 3^4$. Thus $T \cong {}^2B_2(2^3) = Sz(8)$.

Let T be one of classical groups of Lie type with characteristic 2. If $|T|$ is divisible by 2^{11}, then a similar argument as above yields that $T \cong PSL(2, 2^f)$ with $11 \le f \le 25$. If $|T|$ is not divisible by 2^{11} then, checking the order of T, we know that T is isomorphic to one of the following simple groups:

$PSL(2, 2^f)$ with $2 \le f \le 10$, $PSL(3, 2)$, $PSL(3, 4)$, $PSL(3, 8)$, $PSL(4, 2)$, $PSL(5, 2)$, $PSU(3, 4)$, $PSU(3, 8)$, $PSU(4, 2)$, $PSU(5, 2)$, $PSp(4, 4)$ and $PSp(6, 2)$.

Checking the Atlas [6], the groups $PSL(2, 8)$, $PSU(3, 8)$, $PSU(4, 2)$ and $PSp(6, 2)$ are excluded as they have no maximal subgroup of square-free index. Thus the lemma follows by noting that $PSL(3, 2) \cong PSL(2, 7)$, $PSL(2, 4) \cong PSL(2, 5) \cong A_5$, $PSL(2, 9) \cong A_6$ and $PSL(4, 2) \cong A_8$.

6. The graphs associated with almost simple groups

Assume that $\Gamma = (V, E)$ is a connected G-locally-primitive arc-transitive graph of square-free order and valency d, where $G \le \mathrm{Aut}\Gamma$ and $d \in \{5, 6, 7\}$. Assume further that $\mathrm{soc}(G) = T$ is a non-abelian simple group. Then T is not semiregular on V. Let $\alpha \in V$. By Lemma 2.5, T_α induces a transitive normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. Thus

(∗) $|T : T_\alpha|$ is square-free, either $d = |\Gamma(\alpha)| \in \{5, 7\}$ and $|T_\alpha|$ is divisible by d, or $d = 6$ and T_α has a composition factor isomorphic to A_5 or A_6.

This simple observation is helpful to the further argument.

6.1. In this part we assume that $T = \mathrm{soc}(G) = A_c$ with $c \ge 5$. By Lemma 5.5, $c \in \{5, 6, 7, 8, 10, 11, 12, 13, 14\}$. If $c = 14$ then $7^2 \cdot 5^2 \cdot 3^5 \cdot 2^{10}$ is a divisor $|T|$, so $|T_\alpha|$ is divisible by $7 \cdot 5 \cdot 3^4 \cdot 2^9$, which contradicts Corollary 3.5.

Suppose that $c = 13$. If $G = S_{13}$ then $|G|$ is divisible by $2^{10} \cdot 3^5 \cdot 5^2$ and hence $|G_\alpha|$ is divisible by $2^9 \cdot 3^4 \cdot 5$; but such a G_α does not satisfy Theorem 3.4, a contradiction. Assume that $G = A_{13}$. Then $|G|$ is divisible by $2^9 \cdot 3^5 \cdot 5^2$, and hence $|G_\alpha|$ is divisible by $2^8 \cdot 3^4 \cdot 5$. By the Atlas [6], the stabilizer $G_\alpha \cong A_{12}$ or S_{11}. Then Γ has valency at least 11 by Lemma 2.2, a contradiction.

Suppose that $c = 12$. If $G = S_{12}$ then $|G_\alpha|$ is divisible by $2^9 \cdot 3^4 \cdot 5$, but such a G_α does not satisfy Theorem 3.4, a contradiction. Assume that $G = A_{12}$. Then $|G_\alpha|$ is divisible by $2^8 \cdot 3^4 \cdot 5$. By Theorem 3.4, we conclude that $G_\alpha \cong S_6 \times S_7$. However, $S_6 \times S_7$ is not isomorphic to a subgroup of A_{12}, a contradiction.

Suppose that $c = 10$. Then $5^2 \cdot 3^4 \cdot 2^7$ divides $|G|$, so $|G_\alpha|$ is divisible by $2^6 \cdot 3^3 \cdot 5$. By Theorem 3.4, we know that $A_5 \times A_6$ or $A_6 \times A_7$ is isomorphic to a subgroup of G_α. But S_{10} cannot contains such a subgroup, a contradiction.

Therefore, $T = A_5, A_6, A_7, A_8$ or A_{11}, and the next lemma holds.

Table 1
Graphs associated with alternating groups.

G	G_α	d	Graph
A_5, S_5	D_{10}, $5{:}4$	5	K_6
S_5	$5{:}4$	5	K_6
A_6, S_6	A_5, S_5	5	K_6
A_7, S_7	A_6, S_6	6	K_7
S_7	A_6	6	$K_{7,7}-7K_2$
A_7, S_7	A_5, S_5	6	Example 4.2
S_7	$SL(3,2)$	7	$PG(3,2)$
S_8	$2^3{:}SL(3,2)$	7	$PG(3,2)$
A_{11}, S_{11}	$(A_5\times A_6).2$, $S_5\times S_6$	6	$\mathbf{O}_6$

Lemma 6.1. *If T is one of the alternating groups, then one line of Table 1 occurs.*

Proof. (1) If $T = A_5$ then, by the observation $(*)$ ahead this subsection, either $G\cong A_5$ and $G_\alpha\cong D_{10}$, or $G\cong S_5$ and $G_\alpha\cong \mathbb{Z}_5{:}\mathbb{Z}_4$, yielding $\Gamma\cong K_6$.

(2) Assume that $T = A_6$. Then $G\cong A_6$, S_6, $PGL(2,9)$, M_{10} or $P\Gamma L(2,9)$. Checking the subgroups of G satisfying $(*)$, either $G\cong A_6$ and $G_\alpha\cong A_5$, or $G\cong S_6$ and $G_\alpha\cong S_5$. It follows that $\Gamma\cong K_6$.

(3) Assume that $T = A_8$. Then $|T_\alpha|$ is divisible by $2^5\cdot 3$. Recall that $|T_\alpha|$ is divisible by 5 or 7. By the Atlas [6], we conclude that $T_\alpha = 2^3{:}SL(3,2)$ and Γ has valency 7. Then, noting $A_8\cong PSL(4,2)$, the graph Γ is the incidence graph of the projective geometry $PG(3,2)$.

(4) Assume that $T = A_{11}$. Then $|T|$ is divisible by $2^7\cdot 3^4\cdot 5^2$, and hence $|T_\alpha|$ is divisible by $2^6\cdot 3^3\cdot 5$. By the Atlas [6] and Theorem 3.4, we conclude that $T_\alpha\cong (A_5\times A_6).2$ and Γ is of valency 6. This graph is actually the odd graph $\mathbf{O}_6$. Moreover, $G = \mathrm{Aut}\Gamma = S_{11}$, $G_\alpha = S_5\times S_6$, and Γ is 3-arc-transitive.

(5) Assume that $T = A_7$. Then $|T_\alpha|$ is divisible by 12. Checking the subgroups of T satisfying $(*)$, we conclude from Theorem 3.4 that $T_\alpha\cong S_5$, A_6, A_5 or $PSL(3,2)$.

Suppose that $T_\alpha\cong S_5$. Then the vertices in each T-orbit on V may be viewed as the 2-subsets of $\{1,2,3,4,5,6,7\}$. Then $|\Gamma(\alpha)| = |\{\beta \mid \alpha\cap\beta=\emptyset\}|$ or $|\{\beta\neq\alpha \mid \alpha\cap\beta\neq\emptyset\}|$, which is 10 and not in the case.

If $T_\alpha\cong A_6$, then $G\cong A_7$ or S_7, and then $\Gamma\cong K_7$ or $K_{7,7}-7K_2$, respectively.

Assume that $T_\alpha\cong A_5$. Then Γ has valency 5 or 6. Further, $|T:T_\alpha| = 42$ is even, and so T is transitive on V; in particular, Γ is T-arc-transitive. Consider the action of T_α corresponding to the natural action of A_7 on $\Pi := \{1,2,3,4,5,6,7\}$. Suppose that a T_α-orbit on Π has size 5. Then T_α fixes two points in Π. Let $\beta\in\Gamma(\alpha)$. It is easily shown that $T_{\alpha\beta}$ has an orbit on Π of size at least 4. Then we get $\mathbf{N}_T(T_{\alpha\beta})\leq \mathrm{Sym}(\Pi\setminus\Pi_0)\times\mathrm{Sym}(\Pi_0)$, where Π_0 is the set of points fixed by $T_{\alpha\beta}$. Then there is no 2-element $x\in\mathbf{N}_T(T_{\alpha\beta})$ such that $\langle T_\alpha, x\rangle = T$, a contradiction. Thus T_α fixes exactly one point, say 7, and acts transitively on $\Pi_1 = \{1,2,3,4,5,6\}$. If Γ is of valency 5, then $T_{\alpha\beta}\cong A_4$ is transitive on Π_1, and so $\mathbf{N}_T(T_{\alpha\beta})\leq\mathrm{Sym}(\Pi_1)$, which yields a similar contradiction as above. Thus Γ is of valency 6. Then $T_{\alpha\beta}\cong\mathbb{Z}_5\rtimes\mathbb{Z}_2$, and $T_{\alpha\beta}$ fixes only one point in Π_1, say 6. We may set $T_{\alpha\beta} = \langle\sigma,\tau\rangle$, where $\sigma = (1\,2\,3\,4\,5)$ and $\tau = (1\,5)(2\,4)$. Then $\mathbf{N}_T(T_{\alpha\beta}) = \langle\sigma,\pi\rangle\cong\mathbb{Z}_5\rtimes\mathbb{Z}_4$, where $\pi = (1\,4\,5\,2)(6\,7)$. It is easily shown that Γ is isomorphic to the graph given in Example 4.2.

Assume finally that $T_\alpha\cong PSL(3,2)$. If $G = A_7$, then $|V| = |T:T_\alpha| = 15$; in particular, Γ is of even valency, which yields $|\Gamma(\alpha)| = 8$. We do not consider this case here. Then $G = S_7$ and $G_\alpha\cong PSL(3,2)$. Hence Γ is a bipartite graph with two bipartition subsets, say U and W, having size 15 respectively. Further, A_7 is primitive on both U and W and transitive on E, the edge set of Γ. Suppose that the actions of A_7 on U and on W are permutation equivalent. Then A_7 is a primitive permutation group with degree 15 and a suborbit of size $|\Gamma(\alpha)|$. It is easy to see that such a primitive permutation group is 2-transitive. Thus $|\Gamma(\alpha)| = 14$, and $\Gamma\cong K_{15,15}-15K_2$. This is not the case we considered. Therefore, we may assume that U is the point set while W the hyperplane set of the projective geometry $PG(3,2)$, respectively. (Note that A_7 is viewed as a transitive subgroup of $PSL(4,2)\cong A_8$ on projective points or on hyperplanes.) Then Γ is the incidence graph of the projective geometry $PG(3,2)$.

6.2. In this part we assume that $T = \mathrm{soc}(G)$ is a sporadic simple group. By Lemma 5.5, $T = M_{11}$, M_{12}, M_{22}, M_{23}, M_{24} or J_1. Then either $G = T$ or $G = M_{12}.2$.

Lemma 6.2. *T is not one of M_{11}, M_{12}, M_{23} and M_{24}.*

Proof. We shall exclude one by one the simple groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24}.

(1) Suppose that $T = M_{11}$. Then $G = T$ and the order $|T|$ is divisible by $2^4\cdot 3^2$. Since $|T:T_\alpha|$ is square-free, $|T_\alpha|$ is divisible by $2^3\cdot 3$ and not divisible by 2^5, 3^3 or 5^2. Check the groups which appear in Theorem 3.4 and satisfy $(*)$. We conclude that $T_\alpha\cong S_5$, $A_4\times A_5$, A_6 or S_6. By the Atlas [6], only one of A_6 and S_5 may be isomorphic to a subgroup of M_{11}. Thus $T_\alpha\cong S_5$ or A_6.

Suppose that $T_\alpha\cong S_5$. Then Γ is $(T,2)$-transitive and of valency 5 or 6. Thus $T_{\alpha\beta} = 5{:}4$ or S_4, where $\beta\in\Gamma(\alpha)$. Checking the subgroups of M_{11}, we have $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}$. Therefore, there exists no element $x\in N_T(T_{\alpha\beta})$ with $\langle T_\alpha, x\rangle = T$, a contradiction.

Suppose that $T_\alpha = A_6$. Then Γ is $(T, 2)$-transitive and of valency 6. For $\beta \in \Gamma(\alpha)$, the arc-stabilizer $T_{\alpha\beta} \cong A_5$ is contained in a maximal subgroup of T isomorphic to M_{10}. Note that M_{11} has two conjugation classes of subgroups isomorphic to A_5 (confirmed by GAP). Then, checking the subgroups of M_{11} in the Atlas [6], we conclude that $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}$, a contradiction.

(2) Suppose that $T = M_{12}$. Then the order $|T|$ is divisible by $2^6 \cdot 3^3$, and hence $|T_\alpha|$ is divisible by $2^5 \cdot 3^2$. By the Atlas [6], we conclude that $T_\alpha \cong M_{10}.2$; however, by Theorem 3.4, such a group cannot be the stabilizer of any graph of valency 5, 6 or 7.

(3) Suppose that $T = M_{23}$. Then $|T|$ is divisible by $2^7 \cdot 3^2$. Since $|T : T_\alpha|$ is square-free, $|T_\alpha|$ is divisible by $2^6 \cdot 3$. Further $|T_\alpha|$ is not divisible by 2^8 or 3^3. By Theorem 3.4 and checking the subgroups of M_{23}, we know that T_α is isomorphic to $[4^2].\mathrm{SL}(2,4)$, $[4^2].\mathrm{GL}(2,4)$ or $[4^2].\Gamma L(2,4)$. In particular, Γ has valency 5 and $|V|$ is even, and so $T_\alpha \ncong [4^2].\Gamma L(2,4)$. Then $T_\alpha \cong [4^2].\mathrm{SL}(2,4)$ or $[4^2].\mathrm{GL}(2,4)$; in this case, both $\mathbf{N}_T(T_{\alpha\beta})$ and T_α are contained in a maximal subgroup of T isomorphic to $[4^2].\Gamma L(2,4)$ (confirmed by GAP), a contradiction.

(4) Suppose that $T = M_{24}$. Then $|T|$ is divisible by $2^{10} \cdot 3^3$, and hence $|T_\alpha|$ is divisible by $2^9 \cdot 3^2$. By Theorem 3.4, $T_\alpha = [4^3].\Gamma L(2,4) \cong 2^6{:}((3 \times A_5).2)$, and Γ is of valency 5. In this case, both $\mathbf{N}_T(T_{\alpha\beta})$ and T_α are contained in a maximal subgroup of T isomorphic to $2^6{:}3^{\cdot}S_6$ (confirmed by GAP), a contradiction. □

Lemma 6.3. *Assume that $T = \mathrm{soc}(G)$ is a sporadic simple group. Then either $G = J_1$ and Γ is isomorphic to one of the graphs given in Example 4.3; or $T = M_{22}$ and Γ is isomorphic to the graph given in Example 4.4.*

Proof. By Lemmas 5.5 and 6.2, $T = J_1$ or M_{22}.

Assume first that $T = M_{22}$. Then $G = M_{22}$ or $M_{22}.2$. Note that $|G|$ is divisible by $2^7 \cdot 3^2|G : T|$ but not by $2^8|G : T|$ or 3^3. Then $|G_\alpha|$ is divisible by $2^6 \cdot 3|G : T|$ but not by $2^8|G : T|$ or 3^3.

Let $G = M_{22}$. Then $|G_\alpha|$ is divisible by $2^6 \cdot 3$ but not by 2^8 or 3^3. By Theorem 3.4, G_α is isomorphic to one of $S_4 \times S_5$, $[4^2]{:}\mathrm{SL}(2,4)$, $[4^2]{:}\mathrm{GL}(2,4)$, $[4^2]{:}\Gamma L(2,4)$, $S_4 \times \mathrm{SL}(3,2)$, $2^4{:}\mathrm{SL}(3,2)$ and $2^3{:}\mathrm{SL}(3,2)$. Checking the subgroups of M_{22}, we have $G_\alpha \cong 2^3{:}\mathrm{SL}(3,2)$. Then Γ has valency 7 and Γ is isomorphic to the graph given in Example 4.4.

Let $G = M_{22}.2$. Then $|G_\alpha|$ is divisible by $2^7 \cdot 3$ but not by 2^9 or 3^3. By Theorem 3.4, G_α is isomorphic to one of $[4^2]{:}\mathrm{GL}(2,4)$, $[4^2]{:}\Gamma L(2,4)$ and $2^4{:}\mathrm{SL}(3,2)$. Checking the subgroups of $M_{22}.2$, we conclude that $G_\alpha \cong 2^4{:}\mathrm{SL}(3,2)$, and so Γ has valency 7 and order 330. Since $T = M_{22}$ is not semiregular on $V\Gamma$, by Lemma 2.5, T has at most two orbits on $V\Gamma$. If T has two orbits on $V\Gamma$, then $T_\alpha = G_\alpha$; however, M_{22} has no subgroup isomorphic to $2^4{:}\mathrm{SL}(3,2)$, a contradiction. Thus T is transitive on $V\Gamma$, and hence Γ is T-arc-transitive. Then Γ is isomorphic to the graph given in Example 4.4.

Assume that $T = J_1$. Then $G = T$ and the order of T is divisible by $2^3 \cdot 3 \cdot 5$. Since $|T : T_\alpha|$ is square-free, $|T : T_\alpha|$ is divisible by 2^2 but not divisible by 2^4, 5^2 or 3^2. By Theorem 3.4 and the observation $(*)$, $T_\alpha \cong D_{20}$, $5{:}4$, $2 \times (5{:}4)$, A_5, S_5 or $2 \times (7{:}6)$. However, by the Atlas [6], J_1 has no subgroups isomorphic to one of S_4, S_5, $5{:}4$, $2 \times (5{:}4)$ and $2 \times (7{:}6)$. Thus $G_\alpha \cong D_{20}$ or A_5.

Suppose that $T_\alpha = D_{20}$. Then $T_{\alpha\beta} = \mathbb{Z}_2^2$ and Γ is of valency 5, where $\beta \in \Gamma(\alpha)$. Note that T_α is contained in the normalizer $N = D_6 \times D_{10}$ of a Sylow 5-subgroup of T, and that T_α is a Hall subgroup of N. We conclude that all subgroups isomorphic to D_{20} are conjugate in T. Thus we may assume that T_α is contained in a maximal subgroup $M \cong 2 \times A_5$ of T. Let x be a 2-element in $\mathbf{N}_T(T_{\alpha\beta})$ with $\langle x, T_\alpha\rangle = T$. Then $x \notin M$ and $P = \langle x, T_{\alpha\beta}\rangle$ is a Sylow 2-subgroup of T. Let $X \cong 2^3{:}7{:}3$ be a maximal subgroup of T with $P \le X$. Let Q be a Sylow 2-subgroup of M which contains $T_{\alpha\beta}$. Then $1 \ne T_{\alpha\beta} \lhd \langle P, Q\rangle$. Hence $\langle P, Q\rangle \ne T$, and it follows that $\langle P, Q\rangle \le X$. Thus $P = Q$, and so $x \in Q \le M$, a contradiction.

Now let $T_\alpha \cong A_5$. Suppose that $\mathbf{N}_T(T_\alpha) \cong 2 \times A_5$ and Γ has valency 5. Then $T_{\alpha\beta} = A_4$ and $N_G(T_{\alpha\beta}) = 2 \times A_4$ for $\beta \in \Gamma(\alpha)$. However, $\langle g, T_\alpha\rangle \le \mathbf{N}_T(T_\alpha) \ne T$ for any $g \in N_G(T_{\alpha\beta})$, a contradiction. Thus either $\mathbf{N}_T(T_\alpha) = T_\alpha$ or Γ has valency 6. Then Γ is isomorphic to one of the graphs given in Example 4.3. □

6.3. In this part we assume that $T = \mathrm{soc}(G)$ is one of the simple groups listed in parts (iii)–(v) of Lemma 5.5. We first exclude most candidates for T.

Lemma 6.4. $T = \mathrm{PSL}(3,4)$, $\mathrm{PSp}(4,4)$, $\mathrm{PSL}(3,5)$, $\mathrm{PSL}(2,25)$ *or* $\mathrm{PSL}(2,p)$.

Proof. Suppose that $T = \mathrm{PSL}(2,2^f)$ for $4 \le f \le 25$. Note that $|T : T_\alpha|$ is square-free. Checking the subgroups of T (see [10, II. 8.27]), we conclude that $\mathbb{Z}_2^{f-1} \lesssim T_\alpha \lesssim \mathbb{Z}_2^f{:}\mathbb{Z}_{2^f-1}$. In particular, T_α is soluble and, by Lemma 2.5, T_α induces a soluble transitive normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. This yields that $G_\alpha^{\Gamma(\alpha)}$ is soluble, and so G_α is soluble by Lemma 3.2. By Theorem 3.4, $|G_\alpha|$ is not divisible by 2^5. This implies that $f = 4$ or 5. Again by Theorem 3.4, $G_\alpha \cong 4 \times (5{:}4)$; however, such a G_α has no subgroups isomorphic to $\mathbb{Z}_2^{f-1}$, a contradiction.

Suppose that $T = \mathrm{PSL}(2,3^4)$. Then $|T_\alpha|$, and hence $|G_\alpha|$, is divisible by 3^3. By Theorem 3.4, G_α has a subgroup isomorphic to $A_5 \times A_6$. In particular, $|G|$ is divisible by 5^2, which is impossible.

Suppose that $T = \mathrm{PSL}(2,5^4)$. Then $|T_\alpha|$, and hence $|G_\alpha|$, is divisible by 5^3. By Theorem 3.4, $G_\alpha \cong 5^2.\mathrm{GL}(2,5)$ and Γ has valency 6. In particular, $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{PSL}(2,5)$. By Lemma 2.5, T_α induces a transitive normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. It follows that T_α has a composition factor isomorphic to $\mathrm{PSL}(2,5)$. However, by [10, II.8.27], $\mathrm{PSL}(2,5^4)$ has no such a subgroup T_α of square-free index, a contradiction.

Note that the rest candidates for T lie in the Atlas [6]. By the information given in the Atlas, we have the following arguments.

Suppose that $T = \mathrm{PSU}(3,4)$, $\mathrm{PSU}(5,2)$ or $\mathrm{Sz}(8)$. Check the subgroups of T of square-free index. We conclude that T_α is soluble, and so G_α is soluble. By Theorem 3.4, $|G_\alpha|$ is not divisible by 2^5, and so $|V| = |G : G_\alpha|$ is divisible by 4, a contradiction.

Table 2
Incidence graphs.

G	G_α	d	Graph
PSL(3, 4).2	2^4:A_5	5	PG(2, 4)
PSp(4, 4).2	$[4^3]$:GL(2, 4)	6	GQ(4)
PSL(3, 5).2	5^2:GL(2, 5)	6	PG(2, 5)

Table 3
PSL(2, p)-graphs.

G	G_α	d	$G_{\alpha\beta}$	$\mathbf{N}_G(G_{\alpha\beta})$	Remark
PSL(2, p)	A_5	5	A_4	S_4	$p^2 \equiv 1 \pmod 5$, $p \equiv \pm 1 \pmod 8$
PGL(2, p)	A_5	5	A_4	S_4	$p^2 \equiv 1 \pmod 5$, $p \equiv \pm 3 \pmod 8$
PSL(2, p)	A_5	6	D_{10}	D_{20}	$p^2 \equiv 1 \pmod 5$, $p \equiv \pm 1 \pmod 8$, $4 \mid p+\epsilon$
PGL(2, p)	A_5	6	D_{10}	D_{20}	$p^2 \equiv 1 \pmod 5$, $p \equiv \pm 3 \pmod 8$, $4 \nmid p+\epsilon$
PSL(2, p)	D_{2r}	r	$\mathbb{Z}_2$	$D_{p\pm 1}$	$p^2 \equiv 1 \pmod r$, $p \equiv \pm 3 \pmod 8$, $r \in \{5, 7\}$
PGL(2, p)	D_{4r}	r	$\mathbb{Z}_2^2$	S_4	$p^2 \equiv 1 \pmod r$, $p \equiv \pm 3 \pmod 8$, $r \in \{5, 7\}$
PSL(2, p)	D_{4r}	r	$\mathbb{Z}_2^2$	S_4	$p^2 \equiv 1 \pmod r$, $p \equiv \pm 1 \pmod 8$, $r \in \{5, 7\}$
PGL(2, p)	D_{4r}	r	$\mathbb{Z}_2^2$	S_4	$p^2 \equiv 1 \pmod r$, $p \equiv \pm 3 \pmod 8$, $r \in \{5, 7\}$

Suppose that $T = \mathrm{PSL}(5, 2)$. Then $G = \mathrm{PSL}(5, 2)$ or $\mathrm{PSL}(5, 2).2$. Note that $|G|$ is divisible by 2^{10}, and so $|G_\alpha|$ is divisible by 2^9. Then $G_\alpha = [4^3]{:}\Gamma L(2, 4)$ by Theorem 3.4; however, G has no such a subgroup.

Suppose that $T = \mathrm{PSL}(3, 8)$. Then $|T_\alpha|$ is divisible by $2^8 \cdot 3 \cdot 7$, and hence $G_\alpha \cong S_6 \times S_7$ or $[2^6]{:}\mathrm{SL}(3, 2)$ by Theorem 3.4; however, G has no such a subgroup.

Finally, this lemma follows from Lemma 5.5. □

Lemma 6.5. *Let $\{\alpha, \beta\}$ be an edge of Γ. Then either Γ is isomorphic to one of the graphs given in Example 4.5, or one line of Tables 2 and 3 occurs, where $\epsilon = \pm 1$ with $p + \epsilon$ divisible by 5.*

Proof. By Lemma 6.4, $T = \mathrm{PSL}(3, 4)$, $\mathrm{PSp}(4, 4)$, $\mathrm{PSL}(3, 5)$, $\mathrm{PSL}(2, 25)$ or $\mathrm{PSL}(2, p)$.

Let $T = \mathrm{PSL}(3, 4)$. Then $|T_\alpha|$ is divided by $2^5 \cdot 3$. By Theorem 3.4 and checking the subgroups of T in the Atlas [6], we conclude that $T_\alpha \cong 2^4{:}A_5$ and Γ has valency 5. This implies that Γ is the incidence graph of the projective plane PG(2, 4).

Let $T = \mathrm{PSp}(4, 4)$. Then $|T_\alpha|$ is divided by $2^7 \cdot 3 \cdot 5$. By Theorem 3.4 and checking the subgroups of T in the Atlas, we conclude that $G_\alpha = T_\alpha \cong [4^3]{:}\mathrm{GL}(2, 4)$ and Γ has valency 5. Then Γ is the $(T.2, 5)$-arc-transitive graph GQ(4) of order 170.

Let $T = \mathrm{PSL}(3, 5)$. Then $|T_\alpha|$, and hence $|G_\alpha|$, is divisible by $2^4 \cdot 5^2$ but not by 7. By Theorem 3.4, G_α is insoluble and Γ has valency 6. Checking the subgroups of G, we conclude that $T_\alpha = G_\alpha \cong 5^2{:}\mathrm{GL}(2, 5)$. This implies that Γ is the incidence graph of the projective plane PG(2, 5), and $G = \mathrm{Aut}(\mathrm{PSL}(3, 5)) = \mathrm{PSL}(3, 5).2$.

Let $T = \mathrm{PSL}(2, 25)$. Then $G = T.\mathbb{Z}_2^l$ for $l \in \{0, 1, 2\}$, and $|G_\alpha|$ is divisible by $2^2 \cdot 5$ but not by 3^2, 7 or 2^6. By Theorem 3.4 and checking the subgroups of G of square-free index, we conclude that either $d = 5$ and $G_\alpha \cong 5{:}4$, or $d = 6$ and $G_\alpha \cong S_5$ or A_5. Suppose that $G_\alpha \cong S_5$. Then $G = T$ or $T.2$, and $G_{\alpha\beta} \cong 5{:}4$ for $\beta \in \Gamma(\alpha)$. Checking the subgroups of G in the Atlas [6], we conclude that both $\mathbf{N}_G(G_{\alpha\beta})$ and G_α are contained in a maximal subgroup of G, a contradiction. If $G_\alpha \cong A_5$ then $G = T$ and $G_{\alpha\beta} \cong \mathbb{Z}_5{:}\mathbb{Z}_2$, which yields a similar contradiction as above. Thus $G_\alpha \cong 5{:}4$. Then $G = T$ and Γ is isomorphic to a graph given in Example 4.5.

Finally, let $T = \mathrm{PSL}(2, p)$ for prime $p \geq 7$. Check the subgroups of T, see [10, II.8.27]. If $p^2 \not\equiv 1 \pmod 5$ and $p^2 \not\equiv 1 \pmod 7$, then T has no subgroups satisfying $(*)$. Moreover, either $p^2 \equiv 1 \pmod 5$ and $T_\alpha \cong A_5$, or $T_\alpha \cong D_{2r}$ or D_{4r} for $r = d \in \{5, 7\}$ with $p^2 \equiv 1 \pmod r$. Let $\beta \in \Gamma(\alpha)$.

(1) Assume that $T_\alpha \cong A_5$. Note that $G = T$ or $\mathrm{PGL}(2, p)$. Check the subgroups of $\mathrm{PGL}(2, p)$, see [4, Theorem 2]. We have $G_\alpha = T_\alpha$.

Assume that Γ has valency $d = 5$. Then $G_{\alpha\beta} \cong A_4$. This implies that $\mathbf{N}_G(G_{\alpha\beta}) \cong S_4$, and either $G = \mathrm{PSL}(2, p)$ with $p \equiv \pm 1 \pmod 8$, or $G = \mathrm{PGL}(2, p)$ with $p \equiv \pm 3 \pmod 8$; otherwise, $\mathbf{N}_G(G_{\alpha\beta}) = G_{\alpha\beta}$, a contradiction.

Assume that Γ has valency $d = 6$. Then $G_{\alpha\beta} \cong D_{10}$. Let $\epsilon = \pm 1$ such that $p + \epsilon$ is divisible by 5. Then $\mathbf{N}_G(G_{\alpha\beta}) \cong D_{20}$, and either $G = T$ and $p + \epsilon$ is divisible by 4, or $G = \mathrm{PGL}(2, p)$ with $p \equiv \pm 3 \pmod 8$ and $p + \epsilon$ not divisible by 4.

(2) Assume that $T_\alpha \cong D_{2r}$. Then $p \equiv \pm 3 \pmod 8$, and either $G = T$, or $G = \mathrm{PGL}(2, p)$ and $G_\alpha \cong D_{4r}$. For the latter case, $\mathbf{N}_G(G_{\alpha\beta}) \cong S_4$.

(3) Assume that $T_\alpha \cong D_{4r}$. Then $G_\alpha = T_\alpha$, $G_{\alpha\beta} \cong \mathbb{Z}_2^2$ and $\mathbf{N}_G(G_{\alpha\beta}) \cong S_4$. Moreover, either $G = T$ and $p \equiv \pm 1 \pmod 8$, or $p \equiv \pm 3 \pmod 8$ and $G = \mathrm{PGL}(2, p)$. □

7. The proof of Theorem 1.1

Let $\Gamma = (V, E)$ be a connected G-locally-primitive arc-transitive graph of valency $d = 5$, 6 or 7. If G is soluble then Γ and G are known by Lemma 5.1. Thus we assume further that G is insoluble.

Table 4
Candidates for $(X, X_{\bar{\alpha}})$.

X	$X_{\bar{\alpha}}$	d	t	$\|M\|$
A_5, S_5	D_{10}, 5:4	5	1	Odd
S_5	5:4	5	1	Odd
A_6, S_6	A_5, S_5	5	1	Odd
A_7, S_7	A_6, S_6	6	1	
S_7	$A_6 \leq T$	6	2	Odd
A_7, S_7	A_5, S_5	6	1	Odd
S_7	$SL(3, 2) \leq T$	7	2	Odd
S_8	2^3:SL(3, 2) $\leq T$	7	2	Odd
A_{11}, S_{11}	$(A_5 \times A_6).2$, $S_5 \times S_6$	6	1	Odd

Let M be the maximal soluble normal subgroup of G. By Lemma 5.2, $G = M{:}X$ for $X < G$, M is semiregular on V and Γ is a normal cover of $\Sigma := \Gamma_M$. We identify X with a subgroup of $\mathrm{Aut}\Sigma$. Then Σ is X-locally-primitive arc-transitive. Denote by $\overline{V}$ the vertex set of Σ, that is, the set of M-orbits on V. Then $|V| = |M||\overline{V}|$. Thus if $|\overline{V}|$ is even then $|M|$ is odd. If $M = 1$ then G and Γ are known by Lemmas 5.4, 5.5, 6.1, 6.3 and 6.5. We next assume that $M \neq 1$.

By the choice of M, we know that X has no soluble minimal normal subgroups. By Lemma 5.4, $\mathrm{soc}(X)$ is the unique minimal normal subgroup of X. Set $N = M\mathrm{soc}(X)$. Then $N \lhd G$, and so $\mathbf{C}_N(M) \lhd G$ and $M\mathbf{C}_N(M) \lhd G$. Since $|M|$ is square-free, $\mathrm{Aut}(M)$ is soluble. Note that $N/\mathbf{C}_N(M) = \mathbf{N}_N(M)/\mathbf{C}_N(M) \lesssim \mathrm{Aut}(M)$. It follows that $\mathrm{soc}(X) \leq \mathbf{C}_N(M)$, and so $M\mathbf{C}_N(M) = M \times \mathrm{soc}(X)$. This implies that $\mathrm{soc}(X)$ is a characteristic subgroup of $M\mathbf{C}_N(M)$, yielding $\mathrm{soc}(X) \lhd G$. Suppose that X is not almost simple. By Lemma 5.4, $\Sigma \cong \mathsf{K}_{d,d}$ with $d \in \{5, 7\}$. Since $\mathrm{soc}(X) \lhd G$, by Lemma 5.3, $\Gamma \cong \mathsf{K}_{d,d}$. Then $M = 1$ as $2d = |V| = |M||\overline{V}| = 2d|M|$, a contradiction. Thus $T := \mathrm{soc}(X)$ is a non-abelian simple group. Then $MT = M \times T$, $T \lhd G$ and the pair (X, Σ) is known by Lemmas 6.1, 6.3 and 6.5. Let $\alpha \in V$ and $\bar{\alpha} \in \overline{V}$ with $\alpha \in \bar{\alpha}$.

(1) Assume first (X, Σ) satisfies Lemma 6.1. Then one line of Table 4 occurs, where t is the number of T-orbits on $\overline{V}$.

Suppose that $|M|$ is odd. Recall that T has at most two orbits on V, see Lemma 2.5. Then M fixes each T-orbit on V. Let U be a T-orbit on V. Choose $\alpha \in U$. Then $\bar{\alpha} \subseteq U$, $MT_{\bar{\alpha}}$ fixes $\bar{\alpha}$ setwise, and both M and $T_{\bar{\alpha}}$ are transitive on $\bar{\alpha}$. Thus, since $MT_{\bar{\alpha}} = M \times T_{\bar{\alpha}}$, both M and $T_{\bar{\alpha}}$ induce two regular permutation groups on $\bar{\alpha}$. In particular, $T_{\bar{\alpha}}$ has a normal subgroup of odd index $|\bar{\alpha}| = |M| \neq 1$, which is impossible by checking one by one the possible $T_{\bar{\alpha}}$ in Table 4. Therefore, $|M|$ is even, $T = A_7$ and $\Sigma \cong \mathsf{K}_7$. If T is transitive on V then, noting that $T_{\bar{\alpha}} \cong A_6$ is simple, a similar argument implies a contradiction. Thus $\Sigma \cong \mathsf{K}_7$ and $T = A_7$ has two orbits on V.

Since $G_\alpha^{\Gamma(\alpha)}$ is a primitive group of degree $d = 6$, we have $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) \cong A_6$. By Lemma 2.5, T_α induces a transitive normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. It follows that $T_\alpha \cong A_6$. Thus $7|M| = |M||\overline{V}| = |V| = 2|T : T_\alpha| = 14$, and so $M \cong \mathbb{Z}_2$. Then $G = M{:}X = M \times X$, and Γ is isomorphic to the standard double cover of K_7, that is, $\Gamma \cong \mathsf{K}_{7,7} - 7\mathsf{K}_2$.

(2) Suppose that (X, Σ) is known by Lemmas 6.3 and 6.5. Then Σ has even order $|\overline{V}|$, and so $|M|$ is odd. Then we conclude that $M = 1$ by a similar argument as in the case (1), a contradiction. This completes the proof of Theorem 1.1.

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