s-Inversion Sequences and P-Partitions of Type B

William Y.C. Chen[1], Alan J.X. Guo[2], Peter L. Guo[3]
Harry H.Y. Huang[4], Thomas Y.H. Liu[5]

[1]Center for Applied Mathematics
Tianjin University
Tianjin 300072, P.R. China

[2,3,4]Center for Combinatorics, LPMC
Nankai University
Tianjin 300071, P.R. China

[5]Department of Foundation Courses
Southwest Jiaotong University
Emeishan, Sichuan 614202, P.R. China

[1]chenyc@tju.edu.cn, [2]aalen@mail.nankai.edu.cn, [3]lguo@nankai.edu.cn
[4]hhuang@cfc.nankai.edu.cn, [5]lyh@cfc.nankai.edu.cn

Abstract

Given a sequence $s = (s_1, s_2, \ldots)$ of positive integers, the notion of inversion sequences with respect to s, or s-inversion sequences, was introduced by Savage and Schuster in their study of lecture hall polytopes. A sequence $(e_1, e_2, \ldots, e_n)$ of nonnegative integers is called an s-inversion sequence of length n if $0 \leq e_i < s_i$ for $1 \leq i \leq n$. Let I_n be the set of s-inversion sequences of length n for $s = (1, 4, 3, 8, 5, 12, \ldots)$, that is, $s_{2i} = 4i$ and $s_{2i-1} = 2i - 1$ for $i \geq 1$, and let P_n be the set of signed permutations on the multiset $\{1^2, 2^2, \ldots, n^2\}$. Savage and Visontai conjectured that the descent number over P_n is equidistributed with the ascent number over I_{2n}. In this paper, we give a proof of this conjecture by using P-partitions of type B. We notice that an independent proof based on recurrence relations was found by Lin. Moreover, we find a set of signed permutations over which the descent number is equidistributed with the ascent number over I_{2n-1}. Let I'_n be the set of s-inversion sequences of length n for $s = (2, 2, 6, 4, 10, 6, \ldots)$, that is, $s_{2i} = 2i$ and $s_{2i-1} = 4i - 2$ for $i \geq 1$. We also find two sets of signed permutations over which the descent number is equidistributed with the ascent number over I'_n, depending on whether n is even or odd.

Keywords: inversion sequence, ascent number, signed permutation, descent number, P-partition of type B

AMS Subject Classifications: 05A05, 05A15

1 Introduction

The notion of s-inversion sequences was introduced by Savage and Schuster [4] in their study of lecture hall polytopes. Let $s = (s_1, s_2, \ldots)$ be a sequence of positive integers. An *inversion sequence* of length n with respect to s, or an s-inversion sequence of length n, is a sequence $e = (e_1, e_2, \ldots, e_n)$ of nonnegative integers such that $0 \leq e_i < s_i$ for $1 \leq i \leq n$. An *ascent* of an s-inversion sequence $e = (e_1, e_2, \ldots, e_n)$ is defined to be an integer $i \in \{0, 1, \ldots, n-1\}$ such that

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}},$$

where we assume that $e_0 = 0$ and $s_0 = 1$. The ascent number $\operatorname{asc}(e)$ of e is meant to be the number of ascents of e.

The generating function of ascent numbers of s-inversion sequences can be viewed as a generalization of the Eulerian polynomial for permutations, since the ascent number over s-inversion sequences of length n for $s = (1, 2, 3, \ldots)$ is equidistributed with the descent number over permutations on $\{1, 2, \ldots, n\}$, see Savage and Schuster [4]. Savage and Visontai [5] showed that for any sequence s of positive integers and any positive integer n, the generating function of ascent numbers of s-inversion sequences of length n has only real roots. In particular, by establishing a relation between the generating function of ascent numbers of s-inversion sequence for $s = (2, 4, 6, ...)$ and the generating function of descent numbers of even-signed permutations, they proved the real-rootedness of the Eulerian polynomial of type D as conjectured by Brenti [1].

Savage and Visontai [5] also found that the descent number over permutations on the multiset $\{1^2, 2^2, \ldots, n^2\}$, where where i^2 stands for two occurrences of i for $1 \leq i \leq n$, is equidistributed with the ascent number over s-inversion sequences of length $2n$ with $s = (1, 1, 3, 2, 5, 3, \ldots)$. They further conjectured that the descent number over signed permutations on $\{1^2, 2^2, \ldots, n^2\}$ is equidistributed with the ascent number over s-inversion sequences of length $2n$ with $s = (1, 4, 3, 8, 5, 12, \ldots)$. Let P_n denote the set of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$, and let I_n denote the set of s-inversion sequences of length n for $s = (1, 4, 3, 8, 5, 12, \ldots)$. Savage and Visontai [5] posed the following conjecture, which implies the real-rootedness of the generating function of descent numbers of signed permutations in P_n.

Conjecture 1.1 (Savage and Visontai [5]) *For $n \geq 1$, the descent number over P_n is equidistributed with the ascent number over I_{2n}.*

In this paper, we give a proof of Conjecture 1.1. Let $P_n(x)$ denote the generating function of descent numbers of signed permutations in P_n, and let $I_n(x)$ denote the generating function of ascent numbers of inversion sequences in I_n. Savage and Schuster [4] deduced that

$$\frac{I_{2n}(x)}{(1-x)^{2n+1}} = \sum_{t \geq 0} (t+1)^n (2t+1)^n x^t. \tag{1.1}$$

Using P-partitions of type B introduced by Chow [2], we show that $P_n(x)$ satisfies the same relation as $I_{2n}(x)$. Thus $P_n(x) = I_{2n}(x)$, and this proves Conjecture 1.1.

It should be noted that Lin [3] found another proof of Conjecture 1.1 by proving that the coefficients of $P_n(x)$ and $I_{2n}(x)$ satisfy the same recurrence relation.

Besides the equidistribution conjectured by Savage and Visontai, we also find a set of signed permutations over which the descent number is equidistributed with the ascent number over I_{2n-1}. Let $U_n(x)$ be the generating function of descent numbers of signed permutations on $\{1^2, 2^2, \ldots, (n-1)^2, n\}$, and let $V_n(x)$ be the generating function of descent numbers of signed permutations on $\{1^2, 2^2, \ldots, (n-1)^2, n\}$ in which n has a minus sign. Similar to relation (1.1) for $I_{2n}(x)$, Savage and Schuster [4] deduced a relation for $I_{2n-1}(x)$. We show that $V_n(x)$ satisfies the same relation as $I_{2n-1}(x)$, and thus we obtain that $I_{2n-1}(x) = V_n(x)$.

Moreover, let I'_n be the set of s-inversion sequences of length n for $s = (2, 2, 6, 4, 10, 6, \ldots)$. We use $I'_n(x)$ to denote the generating function of ascent numbers of inversion sequences in I'_n. Using similar arguments to $I_n(x)$, we obtain that $I'_{2n}(x) = P_n(x)$ and $I'_{2n-1}(x) = U_n(x)$.

2 Proof of Conjecture 1.1

In this section, we present a proof of Conjecture 1.1. For $n \geq 1$, we use F_n to denote the forest consisting of n rooted trees each of which has exactly two vertices. We show that the generating function $P_n(x)$ of descent numbers of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$ equals the generating function $G_n(x)$ of descent numbers of linear extensions of F_n with signed labelings under certain conditions. Using the technique of P-partitions of type B, we deduce that $G_n(x)$ satisfies the same relation (1.1) as $I_{2n}(x)$, which implies that $G_n(x) = I_{2n}(x)$. Thus we reach the conclusion that $P_n(x) = I_{2n}(x)$, and this proves Conjecture 1.1.

Let us begin with an overview of linear extensions of a poset. Let P be a poset on the set $\{v_1, v_2, \ldots, v_n\}$ with order relation $\leq$. As usual, we use the notation $v_i < v_j$ to denote that $v_i \leq v_j$ but $v_i \neq v_j$. A labeling of P is an assignment of positive integers to the elements $v_1, v_2, \ldots, v_n$ such that each positive integer cannot be used more than once. A signed labeling of P is a labeling of P with each label possibly associated with a minus sign. We adopt the notation (P, w) for a signed labeled poset, where w is a signed labeling of P. For a signed labeled poset (P, w) and an element v of P, we use $w(v)$ to denote the label associated with v.

In this paper, we will be concerned only with a special type of posets, namely, labeled forests with each tree consisting of at most two vertices. Such a forest will be called a *simple forest*. When viewed as a poset, a simple forest is endowed with the following order relation. We say that $u < v$ if u is a child of v. For example, Figure 2.1 illustrates a simple forest P along with a signed labeling of P.

(a) (b)

Figure 2.1: A simple forest along with a signed labeling.

Recall that *a linear extension* of a poset P is a permutation $v_{i_1}v_{i_2}\cdots v_{i_n}$ of the elements of P such that $v_{i_j} < v_{i_k}$ only if $j < k$, see Stanley [6]. However, by a linear extension of a signed labeled poset (P, w) we mean a permutation $w(v_{i_1})w(v_{i_2})\cdots w(v_{i_n})$ of the labels associated with the elements of P, where $v_{i_1}v_{i_2}\cdots v_{i_n}$ is a linear extension of P. Let $\mathcal{L}(P, w)$ denote the set of linear extensions of (P, w). For example, for the signed labeled forest in Figure 2.1, we have

$$\mathcal{L}(P, w) = \{3\bar{1}\bar{5}6, 3\bar{5}\bar{1}6, 3\bar{5}6\bar{1}, \bar{5}3\bar{1}6, \bar{5}36\bar{1}, \bar{5}63\bar{1}\},$$

where $\bar{i}$ is identified with $-i$.

In this section, we shall further restrict our attention to simple forests for which each component is a rooted tree with two vertices. More precisely, let F_n denote such a simple forest with n trees $T_1, T_2, \ldots, T_n$, where T_i is rooted at v_i with u_i being the only child. A signed labeling w of F_n is said to be *local* if it satisfies one of the following conditions:

(1) $w(u_i) = 2i - 1$ and $w(v_i) = 2i$;

(2) $w(u_i) = \overline{2i - 1}$ and $w(v_i) = 2i$;

(3) $w(u_i) = 2i - 1$ and $w(v_i) = \overline{2i}$;

(4) $w(u_i) = \overline{2i}$ and $w(v_i) = \overline{2i - 1}$.

We use $L(F_n)$ to denote the set of local signed labelings of F_n. A linear extension of F_n with a local signed labeling becomes a signed permutation on $\{1, 2, \ldots, 2n\}$. A signed permutation on $\{1, 2, \ldots, 2n\}$ is a permutation on $\{1, 2, \ldots, 2n\}$ for which each element is possibly associated with a minus sign. A signed permutation on a multiset $\{1^2, 2^2, \ldots, n^2\}$ can be defined in the same manner. As will be shown in Theorem 2.1, the generating function $P_n(x)$ of descent numbers of signed permutations on the multiset $\{1^2, 2^2, \ldots, n^2\}$ equals the generating function $G_n(x)$ of descent numbers of linear extensions of F_n with local signed labelings.

Recall that the descent set of a signed permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is defined as

$$\{i \,|\, \sigma_i > \sigma_{i+1},\ 1 \le i \le n-1\} \cup \{0 \,|\, \text{if } \sigma_1 < 0\}, \tag{2.1}$$

see Savage and Visontai [5]. However, for the purpose of this paper, we choose the following alternative definition of the descent set of σ:

$$\{i \,|\, \sigma_i > \sigma_{i+1},\ 1 \le i \le n-1\} \cup \{n \,|\, \text{if } \sigma_n > 0\}. \tag{2.2}$$

The descent number $\mathrm{des}_B(\sigma)$ of σ is referred to as the number of elements in the descent set defined by (2.2). In fact, via the bijection

$$\sigma = \sigma_1\sigma_2\cdots\sigma_n \quad \longmapsto \quad \sigma' = (-\sigma_n)(-\sigma_{n-1})\cdots(-\sigma_1),$$

we see that the descent numbers defined by (2.1) and (2.2) are equidistributed over the set of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$.

With the above notation, the generating function $G_n(x)$ can be written as

$$G_n(x) = \sum_{w \in L(F_n)} \sum_{\sigma \in \mathcal{L}(F_n, w)} x^{\mathrm{des}_B(\sigma)}.$$

We have the following equidistribution property.

Theorem 2.1 *For $n \ge 1$, we have*

$$G_n(x) = P_n(x).$$

Proof. Define a map ϕ from the set

$$\bigcup_{w \in L(F_n)} \mathcal{L}(F_n, w) \tag{2.3}$$

of linear extensions of F_n with local signed labelings to the set of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ be a linear extension in $\mathcal{L}(F_n, w)$, where $w \in L(F_n)$. The construction of $\phi(\sigma) = \tau = \tau_1\tau_2\cdots\tau_{2n}$ can be described as follows. For $1 \le i \le 2n$, assume that τ_i has the same sign as σ_i. Moreover, set $|\tau_i| = \frac{|\sigma_i|}{2}$ if $|\sigma_i|$ is even and set $|\tau_i| = \frac{|\sigma_i|+1}{2}$ if $|\sigma_i|$ is odd. Since σ is a signed permutation on $\{1, 2, \ldots, 2n\}$, it can be easily checked that τ is a signed permutation on $\{1^2, 2^2, \ldots, n^2\}$.

To show that ϕ is a bijection, we construct a map ψ from the set of signed permutations on $\{1, 2, \ldots, 2n\}$ to the set in (2.3) and we shall prove that ψ is the inverse of ϕ. Let $\tau = \tau_1\tau_2\cdots\tau_{2n}$ be a signed permutation on $\{1^2, 2^2, \ldots, n^2\}$. Define $\psi(\tau) = \sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ by the following procedure. For each $1 \le i \le n$, assume that a_i and b_i ($a_i < b_i$) are the two positions of τ occupied by i or $\bar{i}$. Moreover, σ_{a_i} and σ_{b_i} are determined according to the following cases:

(1) $\sigma_{a_i} = 2i-1$ and $\sigma_{b_i} = 2i$ if $\tau_{a_i} = \tau_{b_i} = i$;

(2) $\sigma_{a_i} = \overline{2i-1}$ and $\sigma_{b_i} = 2i$ if $\tau_{a_i} = \bar{i}$ and $\tau_{b_i} = i$;

(3) $\sigma_{a_i} = 2i-1$ and $\sigma_{b_i} = \overline{2i}$ if $\tau_{a_i} = i$ and $\tau_{b_i} = \bar{i}$;

(4) $\sigma_{a_i} = \overline{2i}$ and $\sigma_{b_i} = \overline{2i-1}$ if $\tau_{a_i} = \tau_{b_i} = \bar{i}$.

So σ is a signed permutation on $\{1, 2, \ldots, 2n\}$. Let w be a signed labeling of F_n defined by $w(u_i) = \sigma_{a_i}$ and $w(v_i) = \sigma_{b_i}$. It is routine to check that w is a local signed labeling of F_n. It is also straightforward to verify that σ is a linear extension of (F_n, w).

For any linear extension σ of F_n with a local signed labeling, by direct verification we see that $\psi(\phi(\sigma)) = \sigma$. This implies that ψ is the inverse of ϕ, and hence ϕ is a bijection. Finally, by the construction of ϕ, it can be seen that $j \in \{1, 2, \ldots, 2n\}$ is a descent of σ if and only if it is a descent of $\phi(\sigma)$. This completes the proof. ∎

As the simplest example of the bijection ϕ, consider the case $n = 1$. For F_1, there are four local signed labelings and the set of linear extensions of F_1 with local signed labelings is $\{12, \bar{1}2, 1\bar{2}, \bar{2}\,\bar{1}\}$. In this case, we have

$$\phi(12) = 11, \quad \phi(\bar{1}2) = \bar{1}1, \quad \phi(1\bar{2}) = 1\bar{1}, \quad \phi(\bar{2}\,\bar{1}) = \bar{1}\,\bar{1}.$$

The next theorem shows that $G_n(x)$ satisfies the same relation as $I_{2n}(x)$.

Theorem 2.2 *For $n \geq 1$, we have*

$$\frac{G_n(x)}{(1-x)^{2n+1}} = \sum_{t \geq 0} (t+1)^n (2t+1)^n x^t. \tag{2.4}$$

To prove the above theorem, recall the notion of a (P, w)-partition of type B introduced by Chow [2]. Let P be a poset and w be a signed labeling of P. A (P, w)-*partition of type* B is a map f from P to the set of nonnegative integers that satisfies the following conditions:

(1) $f(u) \geq f(v)$ if $u \leq v$;

(2) $f(u) > f(v)$ if $u < v$ and $w(u) > w(v)$;

(3) $f(v) \geq 1$ if $w(v) > 0$.

When w is a labeling with positive integers, a (P, w)-partition of type B reduces to an ordinary (P, w)-partition defined by Stanley [6]. Substituting each element $v \in P$ with its label $w(v)$, a (P, w)-partition of type B can be viewed as a map from the set of labels of P to the set nonnegative integers. Chow [2] showed that (P, w)-partitions of type B can be generated by linear extensions of (P, w). For a linear extension $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of (P, w), a map g from $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ to the set of nonnegative integers is called σ-*compatible* if the following conditions are satisfied:

(1) $g(\sigma_1) \geq g(\sigma_2) \geq \cdots \geq g(\sigma_n)$;

(2) $g(\sigma_i) > g(\sigma_{i+1})$ if $1 \leq i \leq n-1$ and $\sigma_i > \sigma_{i+1}$;

(3) $g(\sigma_n) \geq 1$ if $\sigma_n > 0$.

Notice that for two distinct linear extensions σ and σ' of (P, w), any σ-compatible map is not σ'-compatible.

The following theorem is due to Chow [2], which will be used to establish a relation between the generating function for the number of (P, w)-partitions of type B and the generating function for the descent number of linear extensions of (P, w).

Theorem 2.3 (Chow [2]) *Let P be a poset with a signed labeling w. A map f from P to the set of nonnegative integers is a (P, w)-partition of type B if and only if there exists a linear extension σ of (P, w) such that f is σ-compatible.*

For a nonnegative integer t, let $\Omega_P(w, t)$ denote the number of (P, w)-partitions f of type B such that $f(v) \leq t$ for any $v \in P$. We have the following relation.

Theorem 2.4 *Let P be a poset with n elements, and let w be a signed labeling of P. Then*

$$\frac{1}{(1-x)^{n+1}} \sum_{\sigma \in \mathcal{L}(P,w)} x^{\mathrm{des}_B(\sigma)} = \sum_{t \geq 0} \Omega_P(w, t)\, x^t. \tag{2.5}$$

Proof. The proof is analogous to that of Stanley [6] for the case of an ordinary labeling. For a linear extension $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of (P, w), let $\Omega_\sigma(t)$ denote the number of σ-compatible maps g such that $g(\sigma_i) \leq t$ for $1 \leq i \leq n$. In view of Theorem 2.3, we see that

$$\Omega_P(w, t) = \sum_{\sigma \in \mathcal{L}(P,w)} \Omega_\sigma(t).$$

Thus, to prove (2.5) it suffices to show that

$$\sum_{t \geq 0} \Omega_\sigma(t)\, x^t = \frac{x^{\mathrm{des}_B(\sigma)}}{(1-x)^{n+1}}. \tag{2.6}$$

To count $\Omega_\sigma(t)$, we establish a bijection between the set of σ-compatible maps g with $g(\sigma_i) \leq t$ for $1 \leq i \leq n$ and the set of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \leq t - \mathrm{des}_B(\sigma)$. Recall that a partition is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. For $1 \leq i \leq n$, let d_i denote the number of descents of σ that are greater than or equal to i, that is,

$$d_i = |\{j \,|\, i \leq j \leq n-1,\, \sigma_j > \sigma_{j+1}\} \cup \{n \,|\, \text{if } \sigma_n > 0\}|.$$

Let g be a σ-compatible map with $g(\sigma_i) \leq t$ for $1 \leq i \leq n$. It is easily checked that by setting $\lambda_i = g(\sigma_i) - d_i$ for $1 \leq i \leq n$, we are given a partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \leq t - \mathrm{des}_B(\sigma)$. It can be seen that this procedure is reversible. So we arrive at a bijection. Notice that the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \leq t - \mathrm{des}_B(\sigma)$ is equal to

$$\binom{n + t - \mathrm{des}_B(\sigma)}{n},$$

see Stanley [6]. It follows that

$$\Omega_\sigma(t) = \binom{n + t - \mathrm{des}_B(\sigma)}{n},$$

which implies (2.6). This completes the proof. ∎

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. By Theorem 2.4, we aim to prove the following equivalent form of (2.4):

$$\sum_{w \in L(F_n)} \Omega_{F_n}(w, t) = ((t+1)(2t+1))^n. \tag{2.7}$$

Recall that F_n consists of n components $T_1, T_2, \ldots, T_n$, where T_i is a tree rooted at v_i with u_i being the only child. Keep in mind that the left-hand side of (2.7) equals the number of (F_n, w)-partitions f of type B such that $f(u_i) \leq t$ and $f(v_i) \leq t$, where w is a local signed labeling of F_n. Restricting f to the tree T_i, we obtain a map f_i from T_i to the set nonnegative integers. Similarly, restricting w to T_i gives a signed labeling w_i of T_i. Recall that w_i is given by one of the following assignments:

(1) $w_i(u_i) = 2i - 1$ and $w_i(v_i) = 2i$,

(2) $w_i(u_i) = \overline{2i - 1}$ and $w_i(v_i) = 2i$,

(3) $w_i(u_i) = 2i - 1$ and $w_i(v_i) = \overline{2i}$,

(4) $w_i(u_i) = \overline{2i}$ and $w_i(v_i) = \overline{2i - 1}$.

Clearly, f_i is a (T_i, w_i)-partition of type B satisfying the conditions $f_i(u_i) \leq t$ and $f_i(v_i) \leq t$. Conversely, f can be recovered from $f_1, f_2, \ldots, f_n$.

For a signed labeling w_i of T_i induced by a local signed labeling of F_n, we now compute the number $\Omega_{T_i}(w_i, t)$ of (T_i, w_i)-partitions f_i of type B such that $f_i(u_i) \leq t$ and $f_i(v_i) \leq t$. We consider the above four cases.

Case 1: $w_i(u_i) = 2i - 1$ and $w_i(v_i) = 2i$. It is easily seen that in this case f_i is a (T_i, w_i)-partition of type B if and only if

$$0 < f_i(v_i) \leq f_i(u_i) \leq t.$$

So we have

$$\Omega_{T_i}(w_i, t) = \binom{t+1}{2}.$$

Case 2: $w_i(u_i) = \overline{2i-1}$ and $w_i(v_i) = 2i$. Similarly, in this case, f_i is a (T_i, w_i)-partition of type B if and only if

$$0 < f_i(v_i) \le f_i(u_i) \le t.$$

Thus,

$$\Omega_{T_i}(w_i, t) = \binom{t+1}{2}.$$

Case 3: $w_i(u_i) = 2i-1$ and $w_i(v_i) = \overline{2i}$. We see that f_i is a (T_i, w_i)-partition of type B if and only if

$$0 \le f_i(v_i) < f_i(u_i) \le t.$$

This implies that

$$\Omega_{T_i}(w_i, t) = \binom{t+1}{2}.$$

Case 4: $w_i(u_i) = \overline{2i}$ and $w_i(v_i) = \overline{2i-1}$. In this case, f_i is a (T_i, w_i)-partition of type B if and only if

$$0 \le f_i(v_i) \le f_i(u_i) \le t.$$

Hence,

$$\Omega_{T_i}(w_i, t) = \binom{t+2}{2}.$$

Combining the above four cases, we see that for any $1 \le i \le n$, the number of possible configurations of (T_i, w_i)-partitions of type B equals

$$3\binom{t+1}{2} + \binom{t+2}{2} = (t+1)(2t+1).$$

It follows that

$$\sum_{w \in L(F_n)} \Omega_{F_n}(w, t) = ((t+1)(2t+1))^n,$$

as required. ∎

3 Signed permutations and I_{2n-1}

In the previous section, we proved the conjecture of Savage and Visontai on the equidistribution of the descent number over signed permutations and the ascent number over s-inversion sequences in the set I_{2n}. In this section, we find a set V_n of signed permutations over which the descent number is equidistributed with the ascent number over the set I_{2n-1}. Recall that I_{2n-1} is the set of s-inversion sequences of length $2n-1$ for $s = (1, 4, 3, 8, 5, 12, \ldots)$. For an s-inversion sequence $e = (e_1, e_2, \ldots, e_n)$, an ascent of e is defined as an integer $i \in \{0, 1, \ldots, n-1\}$ such that

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}},$$

where we assume that $e_0 = 0$ and $s_0 = 1$. The ascent number of e is meant to be the number of ascents of e. Define $V_n(x)$ as the generating function of descent numbers of signed permutations in V_n. Recall that $I_{2n-1}(x)$ denotes the generating function of ascent numbers of inversion sequences in I_{2n-1}. Savage and Schuster [4] showed that

$$\frac{I_{2n-1}(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^n (2t+1)^{n-1} x^t. \tag{3.1}$$

We show that $V_n(x)$ satisfies the same relation (3.1) as $I_{2n-1}(x)$. The proof is similar to that of Conjecture 1.1. So we reach the conclusion that $V_n(x) = I_{2n-1}(x)$.

Let U_n be the set of signed permutations on the multiset $\{1^2, 2^2, \ldots, (n-1)^2, n\}$. Define V_n to be the subset of U_n consisting of signed permutations such that the element n carries a minus sign. Set

$$V_n(x) = \sum_{\sigma \in V_n} x^{\mathrm{des}_B(\sigma)}.$$

We have the following equidistribution property.

Theorem 3.1 *For $n \geq 1$, we have $V_n(x) = I_{2n-1}(x)$.*

Proof. In view of (3.1), we aim to show that

$$\frac{V_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^n (2t+1)^{n-1} x^t. \tag{3.2}$$

Let F_n^* be the forest obtained from F_{n-1} by adding a single vertex v_n as a component. For example, Figure 3.2 illustrates the forest F_n^* for $n = 3$. Write $L(F_n^*)$ for the set

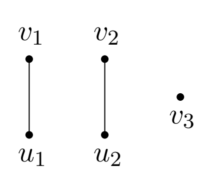

Figure 3.2: The forest F_n^* for $n = 3$.

of signed labelings w of F_n^* such that $w(v_n) = -(2n-1)$ and the labels on F_{n-1} form a local signed labeling of F_{n-1}. Let $Q_n(x)$ denote the generating function of descent numbers of linear extensions of F_n^* with signed labelings $w \in L(F_n^*)$, namely,

$$Q_n(x) = \sum_{w \in L(F_n^*)} \sum_{\sigma \in \mathcal{L}(F_n^*, w)} x^{\mathrm{des}_B(\sigma)}.$$

Analogous to the bijection ϕ in the proof of Theorem 2.1, we can construct a descent preserving map ϕ^* from the set

$$\bigcup_{w \in L(F_n^*)} \mathcal{L}(F_n^*, w)$$

to the set V_n. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_{2n-1}$ be a linear extension in $\mathcal{L}(F_n^*, w)$, where $w \in L(F_n^*)$. Define $\phi^*(\sigma) = \tau = \tau_1\tau_2 \cdots \tau_{2n-1}$ as follows. For $1 \le i \le 2n-1$, assume that τ_i has the same sign as σ_i. Then we set $|\tau_i| = \frac{|\sigma_i|}{2}$ if $|\sigma_i|$ is even and set $|\tau_i| = \frac{|\sigma_i|+1}{2}$ if $|\sigma_i|$ is odd. Clearly, τ is a signed permutation in V_n. Moreover, one can construct the inverse of ϕ^*, which is analogous to the inverse of ϕ. This proves that ϕ^* is a bijection. So we obtain that

$$V_n(x) = Q_n(x). \tag{3.3}$$

Thus (3.2) is equivalent to

$$\frac{Q_n(x)}{(1-x)^{2n}} = \sum_{t \ge 0} (t+1)^n (2t+1)^{n-1} x^t. \tag{3.4}$$

By Theorem 2.4, the left-hand side of (3.4) can be written as

$$\frac{Q_n(x)}{(1-x)^{2n}} = \sum_{t \ge 0} \sum_{w \in L(F_n^*)} \Omega_{F_n^*}(w, t)\, x^t.$$

Hence (3.4) is equivalent to

$$\sum_{w \in L(F_n^*)} \Omega_{F_n^*}(w, t) = (t+1)^n (2t+1)^{n-1}. \tag{3.5}$$

The proof of (3.5) is similar to that for (2.7). For completeness, a detailed proof is presented. Recall that F_{n-1} contains $n-1$ components $T_1, T_2, \ldots, T_{n-1}$, where T_i is a tree rooted at v_i with u_i being the only child. Let T_n^* denote the component consisting of the single vertex v_n. By definition, the left-hand side of (3.5) equals the total number of (F_n^*, w)-partitions f of type B such that $f(v) \le t$ for any vertex v of F_n^*, where w is a signed labeling belonging to $L(F_n^*)$. Restricting f to F_{n-1}, we obtain a map f' from F_{n-1} to the set nonnegative integers. While, restricting f to T_n^*, we are led to a map f'' from T_n^* to the set nonnegative integers. On the other hand, restricting w to F_{n-1} gives a local signed labeling w' of F_{n-1}, whereas restricting w to T_n^* gives a signed labeling w'' of T_n^* such that $w''(v_n) = -(2n-1)$. Obviously, f' is a (F_{n-1}, w')-partition of type

B satisfying the condition that $f'(v) \leq t$ for any vertex v of F_{n-1}, and f'' is a (T_n^*, w'')-partition of type B such that $f''(v_n) \leq t$. It can be seen that the above procedure is reversible. Hence we get

$$\sum_{w \in L(F_n^*)} \Omega_{F_n^*}(w, t) = \Omega_{T_n^*}(w'', t) \sum_{w' \in L(F_{n-1})} \Omega_{F_{n-1}}(w', t). \tag{3.6}$$

In the proof of Theorem 2.2, it has been shown that

$$\sum_{w' \in L(F_{n-1})} \Omega_{F_{n-1}}(w', t) = ((t+1)(2t+1))^{n-1}. \tag{3.7}$$

To compute $\Omega_{T_n^*}(w'', t)$, we see that f'' is a (T_n^*, w'')-partition of type B if and only if

$$0 \leq f''(v_n) \leq t.$$

Thus

$$\Omega_{T_n^*}(w'', t) = t + 1. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we are led to (3.5). This completes the proof. ∎

4 Signed permutations and I'_n

In this section, we consider equidistributions of the descent number over signed permutations and the ascent number over s-inversions sequences for $s = (2, 2, 6, 4, 10, 6, \ldots)$. Recall that the set of such s-inversion sequences of length n is denoted by I'_n. It turns out that we need to distinguish the parity of n.

First, we consider the case for I'_{2n}. Let $I'_{2n}(x)$ be the generating function of ascent numbers of inversion sequences in I'_{2n}. Savage and Schuster [4] obtained a relation for the generating function of ascent numbers of s-inversion sequences for $s = (1, 1, 3, 2, 5, 3, \ldots)$, that is, $s_{2i} = i$ and $s_{2i-1} = 2i - 1$ for $i \geq 1$. This leads to a relation satisfied by $I'_{2n}(x)$. As will be seen, this relation coincides with the relation (1.1) for $I_{2n}(x)$, and so we get $I'_{2n}(x) = I_{2n}(x)$. Since $I_{2n}(x)$ equals the generating function $P_n(x)$ of descent numbers of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$, we are led to the equidistribution as stated below.

Theorem 4.1 *For $n \geq 1$, we have $P_n(x) = I'_{2n}(x)$.*

To prove the above theorem, we recall two formulas of Savage and Schuster [4] on the generating function of ascent numbers of s-inversion sequences of length n. For any sequence $s = (s_1, s_2, \ldots)$ of positive integers, let $f_n^{(s)}(t)$ denote the number of sequences $(a_1, a_2, \ldots, a_n)$ of nonnegative integers such that

$$0 \leq \frac{a_1}{s_1} \leq \frac{a_2}{s_2} \leq \cdots \leq \frac{a_n}{s_n} \leq t. \tag{4.1}$$

Savage and Schuster [4] deduced that

$$\frac{1}{(1-x)^{n+1}}\sum_{e} x^{\mathrm{asc}(e)} = \sum_{t\geq 0} f_n^{(s)}(t)\, x^t, \tag{4.2}$$

where e ranges over s-inversion sequences of length n. For the sequence

$$s = (1, 1, 3, 2, 5, 3, \ldots),$$

Savage and Schuster [4] showed that

$$f_n^{(s)}(t) = (t+1)^{\lceil \frac{n}{2} \rceil} \left(\frac{t+2}{2}\right)^{\lfloor \frac{n}{2} \rfloor}. \tag{4.3}$$

Proof of Theorem 4.1. Let

$$s = (2, 2, 6, 4, 10, 6, \ldots)$$

and

$$s' = s/2 = (1, 1, 3, 2, 5, 3, \ldots).$$

By (4.1), we see that

$$f_n^{(s)}(t) = f_n^{(s')}(2t).$$

Applying (4.3) to s', we get

$$f_n^{(s)}(t) = (t+1)^{\lfloor \frac{n}{2} \rfloor} (2t+1)^{\lceil \frac{n}{2} \rceil}. \tag{4.4}$$

Let $I'_n(x)$ be the generating function of ascent numbers of inversion sequences in I'_n. By (4.2) and (4.4), we obtain that

$$\frac{I'_n(x)}{(1-x)^{n+1}} = \sum_{t\geq 0} (t+1)^{\lfloor \frac{n}{2} \rfloor}(2t+1)^{\lceil \frac{n}{2} \rceil}\, x^t. \tag{4.5}$$

Replacing n with $2n$ in (4.5), we arrive at

$$\frac{I'_{2n}(x)}{(1-x)^{2n+1}} = \sum_{t\geq 0} (t+1)^n(2t+1)^n\, x^t. \tag{4.6}$$

Comparing (4.6) with (1.1), we see that $I'_{2n}(x)$ satisfies the same relation as $I_{2n}(x)$. This implies that $I'_{2n}(x) = I_{2n}(x)$. Since $P_n(x) = I_{2n}(x)$, we conclude that $P_n(x) = I'_{2n}(x)$. This completes the proof. ∎

We now consider the case for I'_{2n-1}. Recall that U_n is the set of signed permutations on $\{1^2, 2^2, \ldots, (n-1)^2, n\}$. Let $U_n(x)$ be the generating function of descent numbers of signed permutations in U_n. Replacing n with $2n-1$ in (4.5), we find that

$$\frac{I'_{2n-1}(x)}{(1-x)^{2n}} = \sum_{t\geq 0} (t+1)^{n-1}(2t+1)^n\, x^t. \tag{4.7}$$

It will be shown that $U_n(x)$ also satisfies relation (4.7). So we have the following equidistribuion property for I'_{2n-1}.

Theorem 4.2 *For $n \geq 1$, we have $U_n(x) = I'_{2n-1}(x)$.*

Proof. We proceed to show that

$$\frac{U_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^{n-1}(2t+1)^n \, x^t. \tag{4.8}$$

As defined in the proof of Theorem 3.1, F_n^* denotes the forest obtained from F_{n-1} by adding a single vertex v_n as a component T_n^*. We use $L'(F_n^*)$ to stand for the set of signed labelings w of F_n^* such that $w(v_n) = 2n-1$ or $w(v_n) = -(2n-1)$, and the restriction of w to F_{n-1} forms a local signed labeling of F_{n-1}. Let

$$H_n(x) = \sum_{w \in L'(F_n^*)} \sum_{\sigma \in \mathcal{L}(F_n^*, w)} x^{\mathrm{des}_B(\sigma)}. \tag{4.9}$$

Analogous to the construction of ϕ^* in the proof of Theorem 3.1, we can establish a descent preserving bijection from the set

$$\bigcup_{w \in L'(F_n^*)} \mathcal{L}(F_n^*, w)$$

to the set U_n. This yields that

$$H_n(x) = U_n(x).$$

Therefore, (4.8) is equivalent to

$$\frac{H_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^{n-1}(2t+1)^n \, x^t. \tag{4.10}$$

By Theorem 2.4, for each signed laleling $w \in L'(F_n^*)$,

$$\frac{1}{(1-x)^{2n}} \sum_{\sigma \in \mathcal{L}(F_n^*, w)} x^{\mathrm{des}_B(\sigma)} = \sum_{t \geq 0} \Omega_{F_n^*}(w, t) \, x^t.$$

It follows that

$$\frac{1}{(1-x)^{2n}} \sum_{w \in L'(F_n^*)} \sum_{\sigma \in \mathcal{L}(F_n^*, w)} x^{\mathrm{des}_B(\sigma)} = \sum_{t \geq 0} \sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) \, x^t.$$

In view of the definition of $G_n(x)$ as given in (4.9), we obtain that

$$\frac{H_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} \sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) \, x^t.$$

Hence (4.10) is equivalent to the following relation

$$\sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) = (t+1)^{n-1}(2t+1)^n. \tag{4.11}$$

The proof of (4.11) is similar to that of (3.5). Let w_1 be the signed labeling of T_n^* such that $w_1(v_n) = -(2n-1)$, and let w_2 be the signed labeling of T_n^* such that $w_2(v_n) = 2n-1$. Then

$$\sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w,t) = \big(\Omega_{T_n^*}(w_1,t) + \Omega_{T_n^*}(w_2,t)\big) \sum_{w \in L(F_{n-1})} \Omega_{F_{n-1}}(w,t). \tag{4.12}$$

In the proof of Theorem 2.2, we have shown that

$$\sum_{w \in L(F_{n-1})} \Omega_{F_{n-1}}(w,t) = ((t+1)(2t+1))^{n-1}, \tag{4.13}$$

whereas in the proof of Theorem 3.1, we deduced that

$$\Omega_{T_n^*}(w_1,t) = t+1. \tag{4.14}$$

Clearly, a map f from T_n^* to the set of nonnegative integers is a (T_n^*, w_2)-partition of type B if and only if $0 < f(v_n) \le t$. Thus

$$\Omega_{T_n^*}(w_2,t) = t. \tag{4.15}$$

Substituting (4.13), (4.14) and (4.15) into (4.12), we arrive at (4.11). This completes the proof. ∎

Acknowledgments. We wish to thank the referees for valuable suggestions. This work was supported by the 973 Project and the National Science Foundation of China.

References

[1] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, European J. Combin. 15 (1994), 417–441.

[2] C.-O. Chow, Noncommutative symmetric functions of type B, Ph.D. Thesis, MIT, 2001.

[3] Z. Lin, On the descent polynomial of signed multipermutations, Proc. Amer. Math. Soc. 143 (2015), 3671–3685.

[4] C.D. Savage and M.J. Schuster, Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences, J. Combin. Theory Ser. A 119 (2012), 850–870.

[5] C.D. Savage and M. Visontai, The s-Eulerian polynomials have only real roots, Trans. Amer. Math. Soc. 367 (2015), 1441–1466.

[6] R.P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge University Press, Cambridge, 2011.