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ON THE MINIMUM WEIGHT OF A 3-CONNECTED 1-PLANAR GRAPH

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Abstract. A graph is called 1-*planar* if it can be drawn in the Euclidean plane $\mathbb{R}^2$ such that each edge is crossed by at most one other edge. The *weight* of an edge is the sum of degrees of two ends. It is known that every planar graph of minimum degree $\delta \geq 3$ has an edge with weight at most 13. In the present paper, we show the existence of edges with weight at most 25 in 3-connected 1-planar graphs.

1. Introduction

All graphs considered in this paper are finite, simple, undirected and connected. The notations and terminology used but undefined here can be found in the book of Bondy and Murty [1].

Let G be a graph, and denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. We denote the degree of a vertex $v \in V(G)$ by $\deg(v)$. For a positive integer k, we say that a vertex $v \in V(G)$ is a *k-vertex*, *k^+-vertex* and *k^--vertex* if $\deg(v) = k$, $\deg(v) \geq k$ and $\deg(v) \leq k$, respectively. For positive integers a and b, if $xy \in E(G)$ with $\deg(x) = a$ and $\deg(y) = b$, then we say that xy is of *type* (a, b) or xy is an *(a, b)-edge*, and say x is an *a-neighbour* of y. For a tuple denoted type of an edge, we sometimes use a^+ and a^- for some entry in the tuple if the corresponding vertex is of degree $\geq a$ and $\leq a$, respectively.

For an edge $xy \in E(G)$, its *weight* is the sum of degrees of two ends, denoted by $w(xy)$. If $\min_{e\in E(G)} w(e) = w$, then we say that G has the *minimum weight* w, and say the edges with weight w are *light edges* of G. (In some earlier papers, "light edge" was defined as an edge with weight at most 13. But in [8], the meaning of "light edge" was changed, and in the present paper, we use the definition in [8].)

The interesting for light edges stemmed from the famous Kotzig's Theorem [10]. It states that the minimum weight of every 3-dimension polyhedral graph (i.e., 3-connected planar graph) is at most 13, and if the graph has no 3-vertices

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then the minimum weight of it is at most 11. Furthermore these bounds are sharp. Afterward, this theorem is developed by many graph-theorists. According to Grünbaum (see [7]), Erdős conjectured that Kotzig's conclusion holds for every planar graph of minimum degree at least 3, which was proved by Barnette (but never published, see [7]) and by Borodin [3] in 1989 independently. Readers may consult [9] for more results on this topic.

This paper focuses on light edges of 1-planar graphs. A graph G is called 1-*planar* if it can be drawn in the plane such that each edge is crossed by at most one other edge, while the drawing is called a 1-*planar drawing* of G and a crossing point is called by a *crossing* for short. Note that we assume that the interiors of any two edges are not tangent and any three distinct edges do not intersect at a crossing in common throughout this paper.

The conception of 1-planar graphs was introduced by Ringle [2] in the solution of simultaneous vertex-face coloring problem. Since then, 1-planar graphs have been studied extensively and lots of interesting results have appeared on acyclic coloring [4], decomposition [5], light subgraphs [11] and edge coloring [12, 13]. Especially, Fabrici and Madaras [6] investigated the local structure of 1-planar graph and they showed the following result which implies that each light edge in a 3-connected 1-planar graph has weight at most 40.

Theorem 1.1 ([6]). *Every* 3*-connected* 1*-planar graph* G *contains an edge with both ends of degree at most* 20 *in* G*. The bound* 20 *is the best possible.*

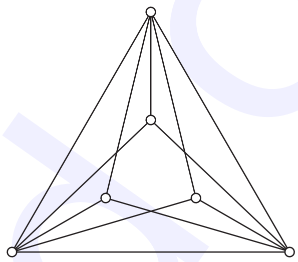


Fig. 1

In [6], the authors gave an example to show the sharpness of the bound 20 as follows: for each triangle face f of the icosahedron, insert three new vertices in the interior of f, add 9 edges joining the new vertices and the vertices of f, see Fig. 1. Then the resulting graph has only edges of type $(3, 20)$ and $(20, 20)$. This example also indicates that 40 might not be the best bound of the minimum weight of 3-connected 1-planar graphs.

In 2012, Hudák and Šugerek [8] proved the following theorem.

Theorem 1.2 ([8]). *Every* 1*-planar graph* G *of minimum degree* $\delta \geq 4$ *contains an edge of type* $(4, 13^-)$, $(5, 9^-)$, $(6, 8^-)$ *or* $(7, 7)$*. In particular, the minimum weight of* G *is at most* 17*, and it is at most* 14 *when* $\delta > 4$.

Based on the example mentioned above, the authors of [8] proposed the following conjecture.

Conjecture 1.3 ([8])**.** *Every* 1*-planar graph of minimum degree* $\delta \geq 3$ *contains an edge of type* $(3, 20^-)$, $(4, 13^-)$, $(5, 9^-)$, $(6, 8^-)$ *or* $(7, 7)$.

Motivated by this conjecture, we prove the following theorem in the present paper.

Theorem 1.4. *Every* 3*-connected* 1*-planar graph* G *contains an edge of type* $(3, 22^-)$, $(4, 13^-)$, $(5, 9^-)$, $(6, 8^-)$ *or* $(7, 7)$*. In particular, the minimum weight of* G *is at most* 25.

2. Proof of Theorem 1.4

Suppose Theorem 1.4 does not hold. Let G be a counterexample to Theorem 1.4 with n vertices, such that G has the largest number of edges among all counterexamples with n vertices.

Define a function ϕ on $\{3, 4, 5, \ldots\}$ such that $\phi(\cdot)$ satisfies the following table.

d	3	4	5	6	≥ 7
$\phi(d)$	23	14	10	9	8

Noting G is a counterexample and the minimum degree of G is at least 3 since G is 3-connected, the following observation holds clearly.

Observation 2.1. *For every edge* $uv \in E(G)$*, if* $\deg(u) = d \leq 7$*, then* $\deg(v) \geq \phi(d)$*, i.e., every edge of* G *is of type* $(3, 23^+)$, $(4, 14^+)$, $(5, 10^+)$, $(6, 9^+)$ *or* $(7^+, 8^+)$.

Note that a 1-planar graph may have different 1-planar drawings. We use $\mathcal{D}(G)$ to denote the set of 1-planar drawings of G with the least number of crossings. Take $D \in \mathcal{D}(G)$. Then it is easy to see that no edge is self-crossing and adjacent edges (i.e., edges with a common end) do not cross in D. By the above assumptions, G and D has the following properties.

- (I) G is a 3-connected 1-planar n-order graph of the minimum degree $\delta \geq 3$;
- (II) every edge of G is of type $(3, 23^+)$, $(4, 14^+)$, $(5, 10^+)$, $(6, 9^+)$ or $(7^+, 8^+)$;
- (III) for all graphs satisfying above (I) and (II), the number of edges of G is maximum;
- (IV) D is a 1-planar drawing of G and has as few crossings as possible;
- (V) no edge is self-crossing and adjacent edges do not cross in D.

For $D \in \mathcal{D}(G)$, we can get a plane graph, denoted by $D^\times$ and called *associated plane graph* of D, by replacing every crossing by a new 4-vertex. In $D^\times$, a vertex is called *false* if it corresponds to a crossing of D, and an edge or face is called *false* if it is incident with some false vertex. A vertex, edge or face is called *true* if it is not false.

Denote by $F(D^\times)$ the set of faces of $D^\times$. Since G is 3-connected, it is easy to see $D^\times$ is at least 2-connected. Then for every $f \in F(D^\times)$, the boundary of

f is a cycle, denoted by ∂f. The length of ∂f is called the *degree* of f, denoted by $\deg(f)$. We say a face f is an r-face, r^--face and r^+-face if $\deg(f) = r$, $\deg(f) \leq r$ and $\deg(f) \geq r$, respectively.

Let f be an r-face with vertices $v_1, v_2, \ldots, v_r$ in a cyclic order. Then we write $f = [v_1 v_2 \cdots v_r]$. Furthermore, if $\deg(v_i) = d_i$, then we say that f is of *type* $(d_1, d_2, \ldots, d_r)$ or f is a $(d_1, d_2, \ldots, d_r)$-*face*. In a tuple denoting the type of a face, we sometimes use a^+ and a^- for some entry in the tuple if the corresponding vertex is of degree $\geq a$ and $\leq a$, respectively. For a false face or false edge, in the type tuple of it, we always write an entry as the symbol $\otimes$ if its corresponding vertex is false.

Next assign charge on the vertices and faces of $D^\times$. Define the initial charge function $ch_0(\cdot)$ as follows:

$$ch_0(x) = \deg(x) - 4 \quad \text{for } x \in V(D^\times) \cup F(D^\times).$$

By Euler's Formula, we have

$$\sum_{v \in V(D^\times)} (\deg(v) - 4) + \sum_{f \in F(D^\times)} (\deg(f) - 4) = -8.$$

Thus

$$\sum_{x \in V(D^\times) \cup F(D^\times)} ch_0(x) = -8.$$

Next we use a two-step discharging process to finish our proof. Denote by $ch_i(x)$ the charge of x after ith discharging where $i = 1, 2$. We shall show that $ch_2(x) \geq 0$ for every $x \in V(D^\times) \cup F(D^\times)$.

2.1. First-step

Some work in the first-step is similar with [8]. But some difference exists between [8] and the present paper. For completeness, we shall write this part as follows.

The discharging rules of the first-step:

Rule F0: The charge of every 4^+-face and every 4-vertex is not changed.

Rule F1: Let $d \in [5, 8]$ be an integer. Assume that v is a d-vertex and f is an incident 3-face of v.

- **Subrule F1.1:** If f is false, then move $\frac{d-4}{2 \cdot \lfloor d/2 \rfloor}$ units charge from v to f.
- **Subrule F1.2:** If f is true and $d \in [5, 7]$, then move no charge from v to f.
- **Subrule F1.3:** If f is true and $d = 8$, then move $\frac{1}{2}$ unit charge from v to f.

Rule F2: Assume that v is a 9-vertex and f is an incident 3-face of v.

- **Subrule F2.1:** If f is of type $(9, 6, \otimes)$, then move $\frac{2}{3}$ units charge from v to f.

- **Subrule F2.2:** If f is of type $(9, 7^+, \otimes)$, then move $\frac{1}{2}$ unit charge from v to f.
- **Subrule F2.3:** If f is of type $(9, 9^+, 9^+)$, then move $\frac{1}{3}$ unit charge from v to f.
- **Subrule F2.4:** If f is true but not of type $(9, 9^+, 9^+)$, then move $\frac{1}{2}$ unit charge from v to f.

Rule F3: Let $d \geq 10$ be an integer. Assume that v is a d-vertex and f is an incident 3-face of v.

- **Subrule F3.1:** If f is of type $(d, 3, \otimes)$, then move 1 unit charge from v to f.
- **Subrule F3.2:** If f is of type $(d, 4, \otimes)$, then move 1 unit charge from v to f.
- **Subrule F3.3:** If f is of type $(d, 5, \otimes)$, then move $\frac{3}{4}$ units charge from v to f.
- **Subrule F3.4:** If f is of type $(d, 6, \otimes)$, then move $\frac{2}{3}$ units charge from v to f.
- **Subrule F3.5:** If f is of type $(d, 7^+, \otimes)$, then move $\frac{1}{2}$ unit charge from v to f.
- **Subrule F3.6:** If f is true, then move $\frac{1}{2}$ unit charge from v to f.

Lemma 2.2. *Let f be a true 3-face of $D^\times$. Then $ch_1(f) \geq 0$.*

Proof. Assume that $f = [u_1u_2u_3]$ and $\deg(u_1) \leq \deg(u_2) \leq \deg(u_3)$. Denote $d_i = \deg(u_i)$ for $i = 1, 2, 3$.

Case 1. Assume $3 \leq d_1 \leq 7$. Then $d_2, d_3 \geq \phi(d_1)$ by Observation 2.1. If $3 \leq d_1 \leq 5$, then $d_2, d_3 \geq 10$, thus by Subrule F3.6, $\frac{1}{2}$ is moved from u_2 and u_3 to f, respectively, and it follows that $ch_1(f) = (3-4) + \frac{1}{2} + \frac{1}{2} = 0$. If $d_1 = 6$, then $d_2, d_3 \geq 9$, thus by Subrules F2.4 and F3.6, $\frac{1}{2}$ is moved from u_2 and u_3 to f, respectively, and it follows that $ch_1(f) = (3-4) + \frac{1}{2} + \frac{1}{2} = 0$. If $d_1 = 7$, then $d_2, d_3 \geq 8$, thus by Subrules F1.3, F2.4 and F3.6, $\frac{1}{2}$ is moved from u_2 and u_3 to f, respectively, and it follows that $ch_1(f) = (3-4) + \frac{1}{2} + \frac{1}{2} = 0$.

Case 2. Assume $d_1 \geq 8$. If $d_1 = 8$, then $d_2, d_3 \geq 8$, thus by Subrules F1.3, F2.4 and F3.6, $\frac{1}{2}$ is moved from u_2 and u_3 to f, respectively, and it follows that $ch_1(f) \geq (3-4) + 1 = 0$. If $d_1 = 9$, then $d_2, d_3 \geq 9$, thus by Subrules F2.3 and F3.6, at least $\frac{1}{3}$ is moved from each u_i to f for $i = 1, 2, 3$, and it follows that $ch_1(f) \geq (3-4) + 3 \cdot \frac{1}{3} = 0$. If $d_1 \geq 10$, then $d_2, d_3 \geq 10$, thus by Subrule F3.6, $\frac{1}{2}$ is moved from each u_i to f for $i = 1, 2, 3$, and it follows that $ch_1(f) \geq (3-4) + 3 \cdot \frac{1}{2} > 0$. □

Lemma 2.3. *Let f be a false 3-face of $D^\times$. Then $ch_1(f) \geq 0$.*

Proof. Assume that $f = [u_1u_2u_3]$. Since f is false and D is a 1-planar drawing, assume that u_1 is false and u_2 and u_3 are true. Denote $d_i = \deg(u_i)$ for $i = 2, 3$ and assume $d_2 \leq d_3$.

Case 1. Assume $3 \le d_2 \le 7$. Then $d_3 \ge \phi(d_2)$ by Observation 2.1. If $d_2 = 3$, then $d_3 \ge 23$ and by Subrule F3.1, u_3 sends 1 to f, thus $ch_1(f) = (3-4)+1 = 0$. If $d_2 = 4$, then $d_3 \ge 14$ and by Subrule F3.2, u_3 sends 1 to f, thus $ch_1(f) = (3-4)+1 = 0$. If $d_2 = 5$, then $d_3 \ge 10$, thus u_3 sends $\frac{3}{4}$ to f by Subrule F3.3 and u_2 sends $\frac{1}{4}$ to f by Subrule F1.1, and it follows that $ch_1(f) \ge (3-4) + \frac{3}{4} + \frac{1}{4} = 0$. If $d_2 = 6$, then $d_3 \ge 9$, thus u_2 sends $\frac{1}{3}$ to f by Subrule F1.1 and u_3 sends $\frac{2}{3}$ to f by Subrules F2.1 and F3.4, and it follows that $ch_1(f) \ge (3-4) + \frac{1}{3} + \frac{2}{3} = 0$. If $d_2 = 7$, then $d_3 \ge 8$, thus u_2 sends $\frac{1}{2}$ to f by Subrule F1.1 and u_3 sends $\frac{1}{2}$ to f by Subrules F1.1, F2.2 and F3.5, and it follows that $ch_1(f) \ge (3-4) + \frac{1}{2} + \frac{1}{2} = 0$.

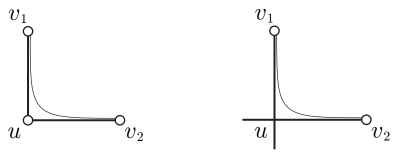
Case 2. Assume $d_2 \ge 8$. If $d_2 = 8$, then $d_3 \ge 8$, thus u_2 sends $\frac{1}{2}$ to f by Subrule F1.1 and u_3 sends $\frac{1}{2}$ to f by Subrules F1.1, F2.2 and F3.5, and it follows that $ch_1(f) \ge (3-4) + \frac{1}{2} + \frac{1}{2} = 0$. If $d_2 = 9$, then $d_3 \ge 9$, thus u_2 sends $\frac{1}{2}$ to f by Subrule F2.2 and u_3 sends $\frac{1}{2}$ to f by Subrules F2.2 and F3.5, and it follows that $ch_1(f) \ge (3-4) + \frac{1}{2} + \frac{1}{2} = 0$. If $d_2 \ge 10$, then $d_3 \ge 10$, thus by Subrule F3.5, u_2 and u_3 send $\frac{1}{2}$ to f, respectively, and it follows that $ch_1(f) \ge (3-4) + \frac{1}{2} + \frac{1}{2} = 0$. □

Lemma 2.4. *Let v be a d-vertex of $D^\times$ where $4 \le d \le 8$. Then $ch_1(v) \ge 0$.*

Proof. If $d = 4$, then the charge of v is not changed by Rule F0, thus $ch_1(v) = ch_0(v) = 4-4 = 0$. Assume $5 \le d \le 8$. By Rule F0, v does not send any charge to any incident 4^+-face. Thus it is sufficient to consider the incident 3-faces of v. If $d = 8$, then v sends $\frac{1}{2}$ to every incident 3-face by Subrules F1.1 and F1.3, thus $ch_1(v) = (8-4) - 8 \cdot \frac{1}{2} = 0$. Assume $5 \le d \le 7$. By Subrules F1.1 and F1.2, v sends $\frac{d-4}{2 \cdot \lfloor d/2 \rfloor}$ to every incident false 3-face and does not send any charge to any true incident 3-face. Since D is a 1-planar drawing, v has at most $2 \cdot \lfloor \frac{d}{2} \rfloor$ incident false 3-faces. It follows that $ch_1(v) \ge (d-4) - \left(2 \cdot \lfloor \frac{d}{2} \rfloor\right) \cdot \frac{d-4}{2 \cdot \lfloor d/2 \rfloor} = 0$. □

Lemma 2.5. *Let $u \in V(D^\times)$ and $uv_1, uv_2 \in E(D^\times)$ such that no edge of $D^\times$ incident with u lies between uv_1 and uv_2 (in a cyclic order). Denote $d_i = \deg(v_i)$ for $i = 1, 2$ and assume $d_1 \ge d_2$. If $d_1 + 1 \ge \phi(d_2+1)$, then $v_1v_2 \in E(G)$ without crossing and uv_1v_2u bounds a face of $D^\times$.*

Proof. Suppose that v_1 is not adjacent to v_2 in G. Add a new edge to G joining v_1 and v_2, and draw this edge along a route closed enough to the simple curve formed by v_1u and uv_2, see the thin curve in Fig. 2. Denote the resulting graph and drawing by G_1 and D_1, respectively. Note that uv_1 and uv_2 are not crossed and no edge incident with u lies between uv_1 and uv_2 in $D^\times$. Then the new edge v_1v_2 has no crossing in D_1. Thus D_1 is a 1-planar drawing and G_1 is a 1-planar graph. Since $d_1 + 1 \ge \phi(d_2 + 1)$, the new edge v_2v_1 is of type $(d_2 + 1, \phi(d_2 + 1)^+)$ in G_1, thus G_1 still is a counterexample to Theorem 1.4. But G_1 has more one edge than G, which contradicts the maximality of G. Thus $v_1v_2 \in E(G)$.



u is true $\qquad$ u is false

FIG. 2

Consider the closed simple curve formed by v_1u, uv_2 and v_2v_1, denoted by C. Suppose some edge e of G crossing v_1v_2 in D. Since no edge incident with u lies between uv_1 and uv_2, e is not adjacent to u. Thus e has an end w located in the interior of C. Redrawing v_1v_2 along a route closed enough to the simple curve formed by v_1u and uv_2. Then we get a 1-planar drawing which has less crossings than D, a contradiction. Thus v_1v_2 has no crossing.

Considering stereographic projection, assume that there is some true vertex outside C. Suppose some true vertex lies inside C. Note that uv_1, uv_2 and v_1v_2 are not crossed and no edge incident with u lies between uv_1 and uv_2. If remove v_1 and v_2 then the resulting graph is not connected, which contradicts the 3-connectivity of G. Thus no true vertex lies inside C. It follows that no false vertex lies inside C since D is 1-planar. Since uv_1, uv_2 and v_1v_2 are not crossed, then no edge of G crosses C. Thus C bounds a face of $D^{\times}$. □

Considering stereographic projection, in this paper, we always assume that the face bounded by uv_1v_2u is an inner-face.

Take an integer $d_0 \in [3,7]$. Let $u \in V(G)$ with $\deg(u) \geq \phi(d_0)$. Denote by $F(u)$ the set of incident faces of u. Define

$$F_1(u,d_0) = \{f \in F(u) \mid f \text{ is of type } (\deg(u), d, \otimes) \text{ for every } d \in [d_0,7]\}$$

and

$$F_2(u,d_0) = \{f \in F(u) \mid f \text{ is of type } (\deg(u), \phi(d)^+, \otimes) \text{ for every } d \in [d_0,7]\}.$$

Corollary 2.6. *Let $d_0 \in [3,7]$ and $u \in V(G)$ with $\deg(u) \geq \phi(d_0)$. For every $f \in F_1(u,d_0)$, there is exactly one $f' \in F_2(u,d_0)$ neighbouring f; and for every $f' \in F_2(u,d_0)$, there is at most one face $f \in F_1(u,d_0)$ neighbouring f'.*

Proof. Take $d \in [d_0,7]$. Assume $f = [uvx] \in F_1(u,d_0)$ where v and x are d- and false neighbour of u in $D^{\times}$, respectively. Assume that vx is contained in an edge vw of G in D. Since $d, d_0 \leq 7$, $\deg(w) \geq \phi(d) \geq 8$ and $\deg(u) \geq \phi(d_0) \geq 8$ by Observation 2.1. Thus, by Lemma 2.5, $uw \in E(D^{\times})$ and cycle $uxwu$ bounds a face, denoted by f'. Clearly, $f' \in F_2(u,d_0)$. Since G is simple, the neighbour of f sharing vx cannot incident with u, thus it is not a member of $F_2(u,d_0)$. Noting $d \leq 7 < 8 \leq \phi(d')$ for every $d' \in [d_0,7]$, the neighbour of f sharing uv is not a member of $F_2(u,d_0)$. Thus there is exactly one $f' \in F_2(u,d_0)$ neighbouring f. Similarly, for every $f' \in F_2$, there is at most one face $f \in F_1(u,d_0)$ neighbouring f'. □

Lemma 2.7. *Let* $d_0 \in [3,7]$ *and* $u \in V(G)$ *with* $\deg(u) = r \geq \phi(d_0)$. *If* u *has exactly* s *incident* 4^+*-faces, then* $|F_1(u,d_0)| \leq \lfloor \frac{r-s}{2} \rfloor$. *Further, if* $s = 0$ *and* $r \equiv 2 (\text{mod } 4)$, *then* $|F_1(u,d_0)| \leq \frac{r}{2} - 1$.

Proof. By Corollary 2.6, $|F_1(u,d_0)| \leq |F_2(u,d_0)|$. Noting $F_1(u,d_0) \cap F_2(u,d_0) = \emptyset$, then $r = \deg(u) \geq |F_1(u,d_0)| + |F_2(u,d_0)| + s \geq 2|F_1(u,d_0)| + s$. Thus $|F_1(u,d_0)| \leq \lfloor \frac{r-s}{2} \rfloor$.

Assume that $s = 0$ and $r = 4k+2$. Then $|F_1(u,d_0)| \leq 2k+1$. Suppose $|F_1(u,d_0)| = 2k+1$. Then $|F_2(u,d_0)| \geq |F_1(u,d_0)| = 2k+1$. But $\deg(u) = r = 4k+2$, thus $|F_2(u,d_0)| = 2k+1$. Then $F(u) = F_1(u,d_0) \cup F_2(u,d_0)$. Take a face $f = [uvx] \in F_1(u,d_0)$ where v and x are d-neighbour ($d \in [d_0,7]$) and false neighbour of u in $D^\times$, respectively. Denote by f'' the neighbour of f sharing uv. Then $f'' \notin F_2(u,d_0)$ by Corollary 2.6. But $F(u) = F_1(u,d_0) \cup F_2(u,d_0)$, thus $f'' \in F_1(u,d_0)$. It follows that for every $f \in F_1(u,d_0)$, there is exactly one $f'' \in F_1(u,d_0)$ neighbouring f and sharing a true edge. Thus $|F_1(u,d_0)|$ is even, which contradicts $|F_1(u,d_0)| = 2k+1$. Hence $|F_1(u,d_0)| < 2k+1$, i.e., $|F_1(u,d_0)| \leq \frac{r}{2} - 1$. □

Lemma 2.8. *Let* u *be a* 9*-vertex. Then* $ch_1(u) \geq 0$.

Proof. Let a_1, a_2, a_3, a_4 and a_5 denote the number of incident 4^+-faces, incident $(9,6,\otimes)$-faces, incident $(9,7^+,\otimes)$-faces, incident $(9,9^+,9^+)$-faces and the other true incident 3-faces of u, respectively. First we show that

$$-3a_1 + a_2 - a_4 \leq 3. \tag{1}$$

Take $d_0 = 6$. By Lemma 2.7, $a_2 \leq |F_1(u,6)| \leq \lfloor \frac{9-a_1}{2} \rfloor$. Then $-3a_1 + a_2 - a_4 \leq -3a_1 + \lfloor \frac{9-a_1}{2} \rfloor - a_4$. If $a_1 \geq 1$, then $-3a_1 + a_2 - a_4 \leq -3 + \lfloor \frac{8}{2} \rfloor - a_4 \leq 1 < 3$. Assume $a_1 = 0$. Then $a_2 \leq \lfloor \frac{9}{2} \rfloor = 4$. If $a_2 \leq 3$, then $-3a_1 + a_2 - a_4 \leq 0 + 3 - a_4 \leq 3$. Next assume $a_1 = 0$ and $a_2 = 4$. Denote by $e_1, e_2, \ldots, e_9$ the nine edges of $D^\times$ incident with u (do not consider the order). Note that one $(9,6,\otimes)$-face cannot be a neighbour of another $(9,6,\otimes)$-face by sharing a $(9,\otimes)$-edge (otherwise there is an $(6,6)$-edge of G, which contradicts the choice of G). Then there are four $(9,\otimes)$-edges incident with u since $a_2 = 4$, and assume that e_1, e_2, e_3 and e_4 are of type $(9,\otimes)$. If one of $e_5, e_6, \ldots, e_9$ is false, then u has an incident face f with two false vertices, but $\deg(f) = 3$ since $a_1 = 0$, which is impossible by the 1-planarity of D. Thus $e_5, e_6, \ldots, e_9$ are true. It follows that u has a true incident 3-face g (note $a_1 = 0$). By Corollary 2.6, there are four incident $(9,9^+,\otimes)$-faces of u, which are neighbours of the four incident $(9,6,\otimes)$-faces of u, respectively. Thus g is a $(9,9^+,9^+)$-face. So $a_4 = 1$ and $-3a_1 + a_2 - a_4 = 0 + 4 - 1 = 3$. Then (1) holds.

Note that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 9. \tag{2}$$

By $(1) + 3 \cdot (2)$, we have

$$4a_2 + 3a_3 + 2a_4 + 3a_5 \leq 30.$$

Then by Rule F2, $ch_1(u) = 9 - 4 - \frac{2}{3}a_2 - \frac{1}{2}a_3 - \frac{1}{3}a_4 - \frac{1}{2}a_5 = 5 - \frac{1}{6}(4a_2 + 3a_3 + 2a_4 + 3a_5) \geq 5 - \frac{30}{6} = 0$. □

Lemma 2.9. *Let u be an r-vertex where $10 \leq r \leq 13$. Then $ch_1(u) \geq 0$.*

Proof. Let a_1, a_2, a_3, a_4 and a_5 denote the number of incident 4^+-faces, incident $(r, 5, \otimes)$-faces, incident $(r, 6, \otimes)$-faces, incident $(r, 7^+, \otimes)$-faces and incident true 3-faces of u, respectively. First we show that

$$-2a_1 + a_2 + a_3 \leq 2r - 16. \tag{3}$$

Take $d_0 = 5$. By Lemma 2.7, $a_2 + a_3 \leq |F_1(u, 5)| \leq \left\lfloor \frac{r-a_1}{2} \right\rfloor$. If $a_1 \geq 1$, then $-2a_1 + a_2 + a_3 \leq -2a_1 + \frac{r-a_1}{2} = \frac{r-5a_1}{2} \leq \frac{r-5}{2} \leq 2r - 16$ since $r \geq 10$. Assume that $a_1 = 0$. If $r \geq 11$, then $-2a_1 + a_2 + a_3 \leq 0 + \frac{r}{2} \leq 2r - 16$. Next consider the case of $r = 10$. Since $10 \equiv 2 (\mathrm{mod}\ 4)$, $a_2 + a_3 \leq \frac{r}{2} - 1 = 4$ by Lemma 2.7. Thus $-2a_1 + a_2 + a_3 \leq 4 = 2r - 16$. Then (3) holds. Note that

$$a_1 + a_2 + a_3 + a_4 + a_5 = r. \tag{4}$$

By (3) + 2 · (4), we have

$$3a_2 + 3a_3 + 2a_4 + 2a_5 \leq 4r - 16.$$

Then by Rule F3, $ch_1(u) = r - 4 - \frac{3}{4}a_2 - \frac{2}{3}a_3 - \frac{1}{2}a_4 - \frac{1}{2}a_5 \geq r - 4 - \frac{3}{4}a_2 - \frac{3}{4}a_3 - \frac{1}{2}a_4 - \frac{1}{2}a_5 = r - 4 - \frac{1}{4}(3a_2 + 3a_3 + 2a_4 + 2a_5) \geq r - 4 - \frac{1}{4}(4r - 16) = 0$. □

Lemma 2.10. *Let u be an r-vertex with $14 \leq r \leq 22$. Then $ch_1(u) \geq 0$.*

Proof. Let a_1, a_2, a_3, a_4, a_5 and a_6 denote the number of incident 4^+-faces, incident $(r, 4, \otimes)$-faces, incident $(r, 5, \otimes)$-faces, incident $(r, 6, \otimes)$-faces, incident $(r, 7^+, \otimes)$-faces and incident true 3-faces of u, respectively. First we show that

$$-a_1 + a_2 + a_3 + a_4 \leq r - 8. \tag{5}$$

Take $d_0 = 4$. By Lemma 2.7, $a_2 + a_3 + a_4 \leq |F_1(u, 4)| \leq \left\lfloor \frac{r-a_1}{2} \right\rfloor$. Thus $-a_1 + a_2 + a_3 + a_4 \leq \frac{r-3a_1}{2}$. If $a_1 \geq 1$, then $-a_1 + a_2 + a_3 + a_4 \leq \frac{r-3}{2} \leq r - 8$ since $r \geq 14$. Next assume that $a_1 = 0$. If $r \geq 16$, then $-a_1 + a_2 + a_3 + a_4 \leq 0 + \frac{r}{2} \leq r - 8$. If $r = 15$, then $-a_1 + a_2 + a_3 + a_4 \leq 0 + \left\lfloor \frac{15}{2} \right\rfloor = 7 = r - 8$. Next consider the case of $r = 14$. Since $14 \equiv 2 (\mathrm{mod}\ 4)$, $a_2 + a_3 + a_4 \leq \frac{r}{2} - 1 = 6$ by Lemma 2.7. Thus $-a_1 + a_2 + a_3 + a_4 \leq 6 = r - 8$. Then (5) holds. Note that

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = r. \tag{6}$$

By (5) + (6), we have

$$2a_2 + 2a_3 + 2a_4 + a_5 + a_6 \leq 2r - 8.$$

Then by Rule F3, $ch_1(u) = r - 4 - a_2 - \frac{3}{4}a_3 - \frac{2}{3}a_4 - \frac{1}{2}a_5 - \frac{1}{2}a_6 \geq r - 4 - a_2 - a_3 - a_4 - \frac{1}{2}a_5 - \frac{1}{2}a_6 = r - 4 - \frac{1}{2}(2a_2 + 2a_3 + 2a_4 + a_5 + a_6) \geq r - 4 - \frac{1}{2}(2r - 8) = 0$. □

Let $f \in F(D^\times)$. If $\deg(f) = 3$, then $ch_1(f) \geq 0$ by Lemmas 2.2 and 2.3. If $\deg(f) \geq 4$, then $ch_1(f) = ch_0(f) = \deg(f) - 4 \geq 0$ by Rule F0. Hence, $ch_1(f) \geq 0$ for every $f \in F(D^\times)$. Let $v \in V(D^\times)$. If $4 \leq \deg(v) \leq 8$, then $ch_1(v) \geq 0$ by Lemma 2.4. If $\deg(v) = 9$, then $ch_1(v) \geq 0$ by Lemma 2.8. If $10 \leq \deg(v) \leq 13$, then $ch_1(v) \geq 0$ by Lemma 2.9. If $14 \leq \deg(v) \leq 22$, then $ch_1(v) \geq 0$ by Lemma 2.10. Hence, $ch_1(v) \geq 0$ when $4 \leq \deg(v) \leq 22$. In summary, when we finish the first-step discharging, we have the following table.

degree of faces	$ch_1(\cdot)$	degree of vertices	$ch_1(\cdot)$
3	≥ 0	3	-1
$d \geq 4$	$d - 4 \geq 0$	$4 \leq d \leq 22$	≥ 0

2.2. Bad 3-vertices

Lemma 2.11. *Let $f = [v_1 v_2 \cdots v_r]$ $(r \geq 4)$ be a face of $D^\times$. If $\deg(v_1) \geq 13$, then $r = 4$, v_3 is false and v_2 and v_4 are true.*

Proof. Suppose that v_j is true for some $3 \leq j \leq r - 1$. We claim that $v_1 v_j \in E(G)$. Suppose that v_1 and v_j are not adjacent. Then add a new edge to D joining v_1 and v_j in the interior of the face f of $D^\times$. Since $\deg(v_1) \geq 13$ and $\delta \geq 3$, the resulting graph is still a counterexample with n vertices but has more edges, which contradicts the maximality of G. Thus $v_1 v_j \in E(G)$. Since f is a face, $v_1 v_j$ is located outside f in $D^\times$. Further, if $v_1 v_j$ has a crossing, then we can redraw $v_1 v_j$ inside f, and lose a crossing, but D has the minimum crossings, a contradiction. Let C and C' be the cycles $v_1 v_2 \cdots v_j v_1$ and $v_j v_{j+1} \cdots v_r v_1 v_j$ of $D^\times$, respectively. Since $v_1 v_j$ has two drawings, either v_2 lies inside C' or v_r inside C. Considering stereographic projection, assume that v_2 lies inside C'. Then v_r locates outside C, and further, since $v_1 v_j$ has no crossing, f is a face of $D^\times$ and adjacent edges do not cross, there is some true vertex located outside C whether v_r is true or not, denoted by u. For $1 < i < j$, if some vertex v_i is true, then every path of G from v_i to u must meet v_1 or v_j since $v_1 v_j$ has no crossing and f is a face of $D^\times$, thus $\{v_1, v_j\}$ is a 2-cut of G, which contradicts 3-connectivity of G. It follows that every v_i $(1 < i < j)$ is false. But no false vertices are adjacent in $D^\times$ since D is a 1-planar drawing. Since $3 \leq j \leq r-1$, we have $j = 3$, and thus v_2 is false. By the property (V) on Page 4, there are two true neighbours of v_2 inside C. Denote by w one of them. Then every path of G from w to u must meet v_1 or v_j since $v_1 v_j$ has no crossing and f is a face of $D^\times$, thus $\{v_1, v_j\}$ is a 2-cut of G, which contradicts 3-connectivity of G, again. Hence every v_j is false for $3 \leq j \leq r - 1$, in particular, v_3 is false.

Since D is a 1-planar drawing, no false vertices are adjacent. Thus $r \leq 4$. But by the assumption of this lemma, $r \geq 4$, thus $r = 4$. Since v_3 is false, v_2 and v_4 are true. □

Say an r-face f of $D^\times$ is *bad* if f is incident with at least $(r-3)$ 3-vertices. A face is *good* if it is not bad. For bad faces, we have some easy properties as follows.

Lemma 2.12. (1) *Every* 3*-face is bad.*

(2) *A bad face has degree* 3, 4, *or* 6.

(3) *A bad* 6*-face is of type* $(3, \otimes, 3, \otimes, 3, \otimes)$.

Proof. By the definition, (1) holds clearly. Let f be a bad r-face with $r \geq 5$. By the property (II) on Page 3, any two 3-vertices are not adjacent. Thus f has at most $\lfloor \frac{r}{2} \rfloor$ incident 3-vertices. It follows that $\frac{r}{2} \geq r - 3$ since f is bad, thus $r \leq 6$. Since $r \geq 5$, $r = 5$ or 6. If $r = 5$, then f has at most $\lfloor \frac{5}{2} \rfloor = 2$ incident 3-vertices; on the other hand, f has at least $5 - 3 = 2$ incident 3-vertices since f is bad, thus f has exactly two incident 3-vertices. Similarly, if $r = 6$, then f has exactly three incident 3-vertices. In a word, f has exactly $(r - 3)$ incident 3-vertices, and the other three vertices are 23^+- or false vertices. By Lemma 2.11, f has no 23^+-vertex. Hence, if $r = 5$, f is incident with three false vertices, which is impossible by 1-planarity of D, and since $r \leq 6$, (2) holds; if $r = 6$, f is incident with three 3-vertices and three false vertices, thus (3) holds. □

Lemma 2.13. *Let* $f = [v_1x_1v_2x_2v_3x_3]$ *be a bad* 6*-face where* v_i*'s and* x_i*'s are* 3*- and false vertices, respectively. Then* $N_G(v_1) = N_G(v_2) = N_G(v_3)$*. Moreover, we can label the three neighbours by* u_1, u_2 *and* u_3*, such that*

(1) v_iu_{i+1} *crosses* $v_{i+1}u_i$ *at* x_i,

(2) v_iu_i *and* u_iu_{i+1} *are not crossed,*

(3) *these* v_i*'s and* x_i*'s are the only six vertices of* $D^\times$ *inside* $u_1u_2u_3u_1$*, where* $i = 1, 2, 3$, $u_4 = u_1$ *and* $v_4 = v_1$*, see Fig. 3.*

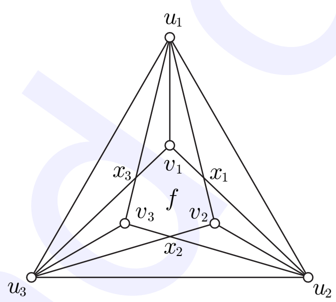


FIG. 3

Proof. Let $N_G(v_1) = \{u_1, u_2, u_3\}$ such that x_1 and x_3 are located on v_1u_2 and v_1u_3, respectively. Let $N_G(v_2) = \{w_1, w_2, w_3\}$ such that x_1 and x_2 are located on v_2w_1 and v_2w_3, respectively. Then x_1 is the crossing of v_1u_2 and v_2w_1.

Since $\deg(v_2) = 3$, $\deg(w_1) \geq 23$. Then by Lemma 2.5, $v_1w_1 \in E(G)$ and has no crossing. Since $\deg(v_1) = 3$, w_1 is coincide to some u_i. By the property (V) on Page 4, $w_1 \neq u_2$. Since v_1w_1 has no crossing but v_1u_3 has a crossing x_3, $v_1w_1 \neq v_1u_3$. It follows that $w_1 \neq u_3$ since G is simple. Thus $w_1 = u_1$ and then v_1u_1 has no crossing. Similarly, $u_2 = w_2$ and v_2u_2 (i.e., v_2w_2) has no crossing.

Since $\deg(v_1) = 3$, $\deg(u_i) \geq 23$ where $i = 1, 2, 3$. By Lemma 2.5, u_1u_2 is an edge of G without crossing and $u_1x_1u_2u_1$, $u_1v_1x_1u_1$ and $u_2v_2x_1u_2$ bound three faces of $D^\times$, respectively. By repeating above argument, this lemma is proved. □

Lemma 2.14. *Let f be a bad 4-face. Then f is of type $(3, \otimes, 12^-, \otimes)$ or $(3, 23^+, 12^-, \otimes)$. If f has two incident 3-vertices, then f is of type $(3, 23^+, 3, \otimes)$.*

Proof. Since f is a bad 4-face, there is a 3-vertex u incident with f. Denote $f = [uxvy]$. Suppose that x and y are true. Then by the property (II) on Page 3, x and y are 23^+-vertices. By Lemma 2.5, $xy \in E(G)$ and has no crossing. Since f is a face of $D^\times$, xy is located outside f. Since xy has no crossing and f is a face, $\{x, y\}$ is a 2-cut of G, which contradicts the 3-connectivity of G. It follows that at least one of x and y is false, and say that y is false. Thus v is true since D is a 1-planar drawing. If $\deg(v) \geq 13$, then by Lemma 2.5, $uv \in E(G)$ and has no crossing, but considering f is a face, we have $\{u, v\}$ is a 2-cut of G, a contradiction again. Thus $\deg(v) \leq 12$. Then f is of type $(3, \otimes, 12^-, \otimes)$ when x is false, or $(3, 23^+, 12^-, \otimes)$ when x is true (by the property (II) on Page 3, $\deg(x) \geq 23$ when x is true).

Assume that f has two incident 3-vertices u and v. Then f is of type $(3, 23^+, 3, \otimes)$ or $(3, \otimes, 3, \otimes)$. Suppose that f is a $(3, \otimes, 3, \otimes)$-face. Then x and y are false. Let uu_1 and vv_1 cross at x and uu_2 and vv_2 cross at y. By the property (II) on Page 3, $\deg(u_i) \geq 23$ and $\deg(v_i) \geq 23$ $(i = 1, 2)$. By Lemma 2.5, for $i = 1, 2$, v_iu and u_iv are edges of G without crossing. Since vu_1 has no crossing but vv_2 has a crossing y, $vu_1 \neq vv_2$. It follows that $u_1 \neq v_2$ since G is simple. Similarly, $u_2 \neq v_1$. Further, $v_1 \neq v_2$ and $u_1 \neq u_2$ since G is simple; $u_1 \neq v_1$ and $u_2 \neq v_2$ by the property (V) on Page 4. Hence u_1, u_2, v_1 and v_2 are distinct pairwise. It follows that u has degree at least 4 since v_iu and uu_i are edges of G, which contradicts $\deg(u) = 3$. Thus f is a $(3, 23^+, 3, \otimes)$-face. □

A 3-vertex is *bad*, if it is incident with three bad faces. Let v be a bad 3-vertex. Assume that $N_G(v) = \{u_1, u_2, u_3\}$ such that vu_1, vu_2 and vu_3 round v in a cyclic order in D. Then $\deg(u_i) \geq 23$ $(i = 1, 2, 3)$. By Lemma 2.5, the following lemma holds.

Lemma 2.15. *Let v be a bad 3-vertex and u_i's keep the assumption above. Assume that v has no false neighbour in $D^\times$. Then G has a cycle $C_1 = u_1u_2u_3u_1$ without crossing. Considering stereographic projection, assume that v lies inside C_1. Then v is the unique vertex of $D^\times$ inside C_1, see Fig. 4.*

We denote by H_1 the subgraph of G bounded by C_1 and fix the drawing (up to stereographic projection) of H_1 shown in Fig. 4. In the present paper, we shall define some H_i's, and when we say a graph H_i, we assume that some drawing of H_i is fixed.

Lemma 2.16. *Let v be a bad 3-vertex and u_i's keep the assumption above. Assume that vu_1 is crossed by xy at w where u_1w, xw, vw and yw round w*

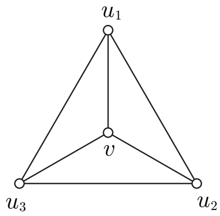


Fig. 4. H_1.

in a cyclic order and vu_2 has no crossing. Assume $x \neq u_2$. Then $xu_2 \in E(G)$ without crossing and $xwvu_2x$ bounds a $(3, 23^+, 12^-, \otimes)$-face of $D^\times$.

Proof. By a similar argument with Lemma 2.5, we can get that $xu_2 \in E(G)$ without crossing. Then $xwvu_2x$ is a cycle of $D^\times$. Considering stereographic projection, assume that u_3 is located outside $xwvu_2x$. Suppose there is a true vertex z of $D^\times$ lying inside $xwvu_2x$. Since $\deg(v) = 3$ and every u_i is not located inside $xwvu_2x$, every path from z to a true vertex outside $xwvu_2x$ must meet x or u_2. Thus $\{x, u_2\}$ is a 2-cut of G. That contradicts the 3-connectivity of G. Thus no true vertex inside $xwvu_2x$. It follows that no false vertex inside $xwvu_2x$ actually. Since xu_2 and vu_2 have no crossing, no edge of G crosses $xwvu_2x$. Thus $xwvu_2x$ bounds a face of $D^\times$, denoted by g. Since v is a bad 3-vertex, g is a bad 4-face. By Lemma 2.14, g is of type $(3, 23^+, 12^-, \otimes)$. □

As Lemma 2.5, in this paper, we always assume that the face bounded by $xwvu_2x$ is an inner-face.

Lemma 2.17. *Let v be a bad 3-vertex and u_i's keep the assumption above. Assume that vu_2 and vu_3 are not crossed but vu_1 is crossed by xy at w where u_1w, xw, vw and yw round w in a cyclic order. Then $x \neq u_2$ or $y \neq u_3$. Next assume $x \neq u_2$.*

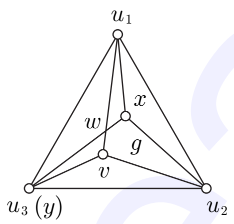


Fig. 5. H_2.

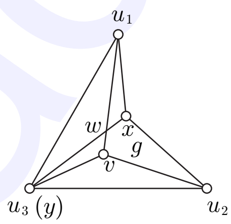


Fig. 6. H_3.

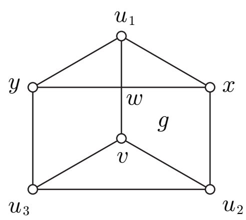


Fig. 7. H_4.

(1) *If $y = u_3$ and $\deg(x) = 3$, then $u_1x, xu_2 \in E(G)$ without crossing, G has a cycle $C_2 = u_1u_2u_3u_1$ without crossing and x is also a bad 3-vertex.*

Considering stereographic projection, assume that v lies inside C_2, then there are exactly three vertices v, w and x of $D^\times$ inside C_2, see Fig. 5.

(2) *If $y = u_3$ and $\deg(x) \geq 4$, then G has a cycle $C_3 = u_1xu_2u_3u_1$ without crossing and $4 \leq \deg(x) \leq 12$. Considering stereographic projection, assume that v lies inside C_3, then there are exactly two vertices v and w of $D^\times$ inside C_3, see Fig. 6.*

(3) *If $y \neq u_3$, then G has a cycle $C_4 = u_1xu_2u_3yu_1$ without crossing, $5 \leq \deg(x) \leq 12$ and $5 \leq \deg(y) \leq 12$. Considering stereographic projection, assume that v lies inside C_4, then there are exactly two vertices v and w of $D^\times$ inside C_4, see Fig. 7.*

Proof. Note $\deg(u_i) \geq 23$. Then by Lemmas 2.5, yu_1, u_1x and u_2u_3 are edges of G without crossing and ywu_1y, xwu_1x and vu_3u_2v bound faces, respectively. Note u_2u_3 has no crossing but xy has a crossing w. Then $xy \neq u_2u_3$. Since G is simple, then $x \neq u_2$ or $y \neq u_3$. Next assume $x \neq u_2$. By Lemma 2.16, $xu_2 \in E(G)$ without crossing, $xwvu_2x$ bounds a face g and g is of type $(3, 23^+, 12^-, \otimes)$. Thus $\deg(x) \leq 12$.

Assume $y = u_3$. By Lemmas 2.5, vwu_3v bounds a face of $D^\times$. If $\deg(x) = 3$, then u_1u_2 is an edge of G without crossing and xu_1u_2x bounds a face by Lemmas 2.5, and since $[xu_1u_2]$ is bad by Lemma 2.12, (1) holds; if $\deg(x) \geq 4$, since $\deg(x) \leq 12$, then (2) holds.

Assume $y \neq u_3$. By Lemma 2.16, yu_3 is an edge of G without crossing and $ywvu_3y$ bounds a $(3, 23^+, 12^-, \otimes)$-face of $D^\times$, thus $\deg(y) \leq 12$. Since $\deg(x) \leq 12$ and $\deg(y) \leq 12$ but $\deg(u_i) \geq 23$, $x, y \notin \{u_1, u_2, u_3\}$. Thus $u_1xu_2u_3yu_1$ is a cycle of $D^\times$ (as Fig. 7). If $\deg(x) \leq 4$ or $\deg(y) \leq 4$, then xy is a $(4^-, 12^-)$-edge of G, which contradicts the property (II) on Page 3. Thus $\deg(x) \geq 5$ and $\deg(y) \geq 5$. Hence (3) holds. □

For $i = 2, 3, 4$, denote by H_i the subgraph of G bounded by C_i and fix the drawing (up to stereographic projection) of H_i shown in Fig. 5, Fig. 6 and Fig. 7, respectively.

Lemma 2.18. *Let v be a bad 3-vertex and u_i's keep the assumption above. Assume that for $i = 1, 2$, vu_i is crossed by x_iy_i at w_i such that u_iw_i, x_iw_i, vw_i and y_iw_i round w_i in a cyclic order and vu_3 has no crossing. Denote by f the face incident with x_1, w_1, v, w_2 and y_2. Then $\deg(f) = 4$ or 6.*

(1) *Assume $\deg(f) = 6$. Then y_1, x_2 and u_3 are coincide, x_1u_2, y_2u_1, u_1x_1, $y_2u_2 \in E(G)$ and x_1 and y_2 are bad 3-vertices. Further x_1u_2 and y_2u_1 intersect at a crossing w_3, u_1x_1 and y_2u_2 are not crossed, and G has a cycle $C_5 = u_1u_2u_3u_1$ without crossing. Considering stereographic projection, assume that v lies inside C_5. Then there are exactly six vertices x_1, w_1, v, w_2, y_2 and w_3 of $D^\times$ inside C_5, see Fig. 8.*

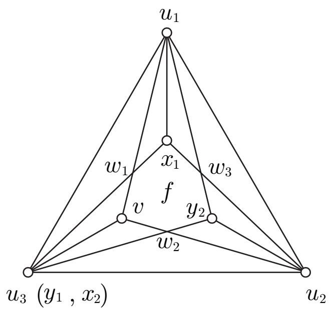


FIG. 8. H_5.

(2) *Assume* $\deg(f) = 4$. *Then* $x_1 = y_2$, *and either* $x_2 \neq u_3$ *or* $y_1 \neq u_3$ *since* G *is simple. Assume that* $x_2 \neq u_3$. *Then* y_1u_1, u_1x_1, x_1u_2, u_2x_2 *and* x_2u_3 *are edges of* G *without crossing.*

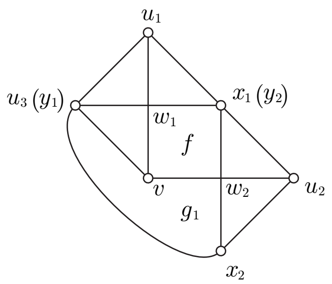


FIG. 9. H_6.

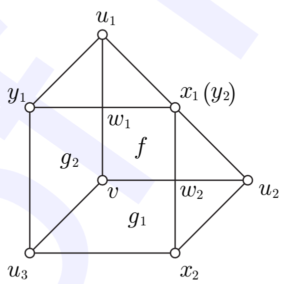


FIG. 10. H_7.

(2a) *If* $y_1 = u_3$, *then* G *has a cycle* $C_6 = u_3u_1x_1u_2x_2u_3$ *without crossing,* $5 \leq \deg(x_1) \leq 12$ *and* $5 \leq \deg(x_2) \leq 12$. *Considering stereographic projection, assume that* v *lies inside* C_6, *then there are exactly three vertices* v, w_1 *and* w_2 *of* $D^\times$ *inside* C_6, *see Fig. 9.*

(2b) *If* $y_1 \neq u_3$, *then* G *has a cycle* $C_7 = u_1x_1u_2x_2u_3y_1u_1$ *without crossing,* $5 \leq \deg(x_i) \leq 12$ *and* $5 \leq \deg(y_i) \leq 12$ *for* $i = 1, 2$. *Considering stereographic projection, assume that* v *lies inside* C_7, *then there are exactly three vertices* v, w_1 *and* w_2 *of* $D^\times$ *inside the cycle* C_7, *see Fig. 10.*

Proof. By the assumption of this lemma, $\deg(f) \geq 4$. Since v is a bad 3-vertex, we have that f is a bad face, thus by Lemma 2.12, f is a 4- or 6-face. Since every 3-face is bad by Lemma 2.12, if f is a 6-face, then (1) holds by Lemma 2.13.

Next assume $\deg(f) = 4$. Then $x_1 = y_2$. Since $\deg(v) = 3$, $\deg(u_i) \geq 23$ by the property (II) on Page 3. By Lemma 2.5, y_1u_1, u_1x_1, x_1u_2 and u_2x_2 are edges of G without crossing and $y_1w_1u_1y_1$, $u_1w_1x_1u_1$, $x_1w_2u_2x_1$ and

$u_2w_2x_2u_2$ bound faces, respectively. Since G is simple, either $x_2 \neq u_3$ or $y_1 \neq u_3$. Assume $x_2 \neq u_3$. By Lemma 2.16, $u_3x_2 \in E(G)$ without crossing and $vw_2x_2u_3v$ bounds a face g_1. Since v is a bad 3-vertex, f and g_1 are bad. By Lemma 2.14, f is of type $(3, \otimes, 12^-, \otimes)$ and g_1 is of type $(3, 23^+, 12^-, \otimes)$, Thus $\deg(x_1) \leq 12$ and $\deg(x_2) \leq 12$. It follows that $\deg(x_2) \geq 5$ and $\deg(x_1) \geq 5$; otherwise x_1x_2 is a $(4^-, 12^-)$-edge, which contradicts the property (II) on Page 3. Thus $5 \leq \deg(x_1) \leq 12$ and $5 \leq \deg(x_2) \leq 12$. Since $\deg(u_i) \geq 23$, $\{x_1, x_2\} \cap \{u_1, u_2, u_3\} = \emptyset$.

Assume $y_1 = u_3$. Then $y_1u_1x_1u_2x_2y_1$ is a cycle of $D^\times$. By Lemma 2.5, vw_1y_1v bounds a face of $D^\times$, then (2a) holds. Assume $y_1 \neq u_3$. By Lemma 2.16, $y_1u_3 \in E(G)$ without crossing and $vw_1y_1u_3$ bounds a $(3, 23^+, 12^-, \otimes)$-face. Thus $\deg(y_1) \leq 12$. Since $\deg(x_1) \leq 12$ and $\deg(y_1) \leq 12$, $\deg(y_1) \geq 5$ by the property (II) on Page 3. Since $\deg(u_i) \geq 23$, $y_1 \notin \{u_1, u_2, u_3\}$. Since G is simple, x_1, x_2 and y_1 are pairwise distinct. Thus $u_1x_1u_2x_2u_3y_1u_1$ is a cycle of G, and (2b) holds. □

For $i = 5, 6, 7$, denote by H_i the subgraph of G bounded by C_i and fix the drawing (up to stereographic projection) of H_i shown in Fig. 8, Fig. 9 and Fig. 10, respectively.

Lemma 2.19. *Let v be a bad 3-vertex and u_i's keep the assumption above. Assume that for $i = 1, 2, 3$, vu_i is crossed by x_iy_i at w_i such that u_iw_i, x_iw_i, vw_i and y_iw_i round w_i in a cyclic order. For $i = 1, 2, 3$, denote by f_i the face incident with w_i, v and w_{i+1} ($w_4 = w_1$), then $\deg(f_i) = 4$, $x_i = y_{i+1}$ ($y_4 = y_1$) and $5 \leq \deg(x_i) \leq 12$.*

Moreover, G has a cycle $C_8 = u_1x_1u_2x_2u_3x_3u_1$ without crossing. Considering stereographic projection, assume that v lies inside C_8, then there are exactly four vertices w_1, w_2, w_3 and v of $D^\times$ inside C_8, see Fig. 11.

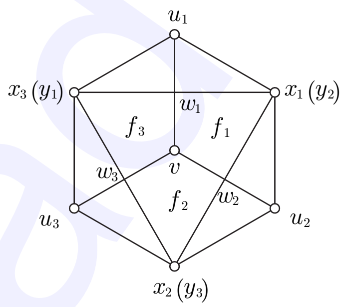


FIG. 11. H_8.

Proof. Since v is a bad 3-vertex, every f_i is bad. Then $\deg(f_i) = 3, 4, 6$ by Lemma 2.12. By the assumption of this lemma, $\deg(f_i) \geq 4$, but by Lemma 2.13, $\deg(f_i) \neq 6$. Thus $\deg(f_i) = 4$ for $i = 1, 2, 3$. Then $x_1 = y_2$, $x_2 = y_3$ and $x_3 = y_1$. For $i = 1, 2, 3$, since every f_i is a bad 4-face, $\deg(x_i) \leq 12$ by Lemma

2.14, and then $\deg(x_i) \geq 5$ by the property (II) on Page 3. Since $\deg(u_i) \geq 23$, by Lemma 2.5, $u_ix_i, x_iu_{i+1} \in E(G)$ without crossing and $u_iw_ix_iu_i$ and $x_iw_iu_{i+1}x_i$ bound faces of D where $u_4 = u_1$, respectively. Thus this lemma holds. □

Denote by H_8 the subgraph of G bounded by C_8 and fix the drawing (up to stereographic projection) of H_8 shown in Fig. 11.

For $i \in [1, 8]$, denote by $\mathcal{H}_i$ the set of subgraphs X of G (under D) such that X is isomorphic to H_i and containing a bad 3-vertex. Then every $X \in \mathcal{H}_i$ keeps the drawing (up to stereographic projection) and the property (Lemmas 2.15, 2.17, 2.18 and 2.19, respectively) of H_i under D.

By Lemmas 2.15, 2.17, 2.18 and 2.19, we have the following corollary.

Corollary 2.20. *For every bad* 3*-vertex* v*, there is a unique* $X \in \mathcal{H}_i$ *for some* $i \in [1, 8]$ *containing* v*.*

For every 23^+-vertex u and $i \in [1, 8]$, denote $\mathcal{H}_i(u) = \{X \in \mathcal{H}_i \mid u \in V(X)\}$. Then for $X \in \mathcal{H}_i(u)$, u is isomorphic to some u_j $(j = 1, 2, 3)$ of H_i. For more convenience, denote $\mathcal{H}_{i,j}(u) = \{X \in \mathcal{H}_i \mid u \in V(X)$ and u is isomorphic to u_j of $H_i\}$ where $i \in \{3, 6\}$ and $j \in [1, 3]$.

Considering stereographic projection, next when we say that $X \in \mathcal{H}_i(u)$, we always assume that X keeps the drawing of H_i and every bad 3-vertices of X is located inside the cycle of X isomorphic to C_i.

2.3. Spanning vertices and enumeration

For a face f of $D^\times$ and a vertex v of G, if v is incident with f or v is incident with an edge e of G such that e contains an incident edge of f in $D^\times$, then call v a *spanning vertex* of f.

Take a 23^+-vertex u_1 and $f_1 \in F(u_1)$. Considering stereographic projection, assume that f_1 is an inner-face. Assume that $\deg(f_1) = 3$ and no spanning vertex of f_1 is a bad 3-vertex. If f_1 is false and denote $f_1 = [u_1v_1w]$ where w is a crossing formed by u_1v_2 and v_1v_3, then $u_1v_3 \in E(G)$ without crossing and cycle $u_1wv_3u_1$ bounds a face of $D^\times$ by Lemma 2.5, see Fig. 12. Denote by H_9 the subgraph (keep the drawing), and denote by C_9 the cycle $u_1v_1v_3u_1$. If f_1 is true and denote $f_1 = [u_1v_1v_2]$ where v_1 and v_2 are true, then we get a triangle $[u_1v_1v_2]$. For convenience, we denote by H_{10} the triangle $[u_1v_1v_2]$, and denote by C_{10} the cycle $u_1v_1v_2u_1$. Note that in H_9 and H_{10}, every v_j is not a bad 3-vertex.

For $i \in \{9, 10\}$, denote by $\mathcal{H}_i$ the set of subgraphs X of G (under D) which is isomorphic to H_i and keep the drawing (up to stereographic projection) and the property of H_i, i.e., no vertex is located inside (or outside, considering stereographic projection) the cycle of X isomorphic to C_i and no vertex of X is a bad 3-vertex. For every 23^+-vertex u, denote $\mathcal{H}_i(u) = \{X \in \mathcal{H}_i \mid u \in V(X)$ and u is isomorphic to u_1 in $H_i\}$ where $i \in \{9, 10\}$.

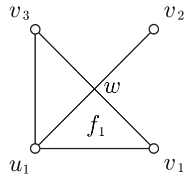


FIG. 12. H_9

Considering stereographic projection, next when we say that $X \in \mathcal{H}_i(u)$ ($i \in \{9, 10\}$), we always assume that X keeps the drawing of H_i and no vertex is located inside the cycle of X isomorphic to C_i.

For a subgraph X of G, if restrict the drawing D in X, then we get a drawing of X, and we denote it by $D|_X$.

Lemma 2.21. *Let u be a 23^+-vertex, $X \in \mathcal{H}_i(u)$ and $Y \in \mathcal{H}_{i'}(u)$ where $1 \le i, i' \le 10$ and $X \ne Y$. If f_X and f_Y are inner-faces incident with u in $(D|_X)^\times$ and $(D|_Y)^\times$, respectively, then we have $f_X \ne f_Y$ in $D^\times$.*

Proof. Since u is isomorphic to u_1 of H_9 or H_{10} by the definition of $\mathcal{H}_9(u)$ and $\mathcal{H}_{10}(u)$, if $i, i' \in \{9, 10\}$, then the conclusion holds clearly since $X \ne Y$ and f_X and f_Y are incident with u. Next assume that $i \in [1, 8]$. By observing the results of Lemmas 2.15, 2.17, 2.18 and 2.19, we can find that X contains a bad 3-vertex v which is a spanning vertex of f_X. Suppose $f_X = f_Y$ in $D^\times$. Then v is also a spanning bad 3-vertex of f_Y. But H_9 and H_{10} do not contain bad 3-vertex, thus $i' \in [1, 8]$. Note that v is also a spanning bad 3-vertex of f_Y. By observing the results of Lemmas 2.15, 2.17, 2.18 and 2.19, we have $v \in V(Y)$. Then v is a bad 3-vertex of X and Y in common. But by Corollary 2.20, $X = Y$, a contradiction. □

For a 23^+-vertex u, denote $h_i(u) = |\mathcal{H}_i(u)|$ for $i \in [1, 10]$, and denote $h_{i,j}(u) = |\mathcal{H}_{i,j}(u)|$ for $i \in \{3, 6\}$ and $j \in [1, 3]$. By Lemma 2.21, $X \in \mathcal{H}_i(u)$ and $Y \in \mathcal{H}_{i'}(u)$ have no common inner-face incident with u. Then when we enumerate the number of inner-faces incident with u, which are contained in members of $\mathcal{H}_i(u)$ for $i \in [1, 10]$, we get an estimation of the degree of u as the following lemma.

Lemma 2.22. *Let u be a 23^+-vertex. Then*

$$\begin{aligned}\deg(u) \ge\ & 2h_1(u) + 3h_2(u) + 2h_{3,1}(u) + 2h_{3,2}(u) + 3h_{3,3}(u) + 2h_4(u) \\ & + 4h_5(u) + 2h_{6,1}(u) + 2h_{6,2}(u) + 3h_{6,3}(u) + 2h_7(u) + 2h_8(u) \\ & + 2h_9(u) + h_{10}(u). \qquad (7)\end{aligned}$$

2.4. Second-step

Recall that an r-face is bad if it is incident with at least $(r-3)$ 3-vertices, and a 3-vertex is bad if every its incident face is bad; if a face or a 3-vertex is not bad, then say it is good. Next we start the second-step discharging.

The discharging rules of the second-step:

Rule S1: Assume that v is a 3-vertex and f is a good face incident with v. Then we move 1 from f to v.
Rule S2: Assume that v is a 3-vertex and f is a bad 6-face incident with v. Then we move $\frac{2}{3}$ from f to v.

Next (in Rules S3-S10) we assume that v is a bad 3-vertex. Then there is a unique $X \in \mathcal{H}_i$ containing v for some $i \in [1, 8]$ and we identify X and H_i.

Rule S3: If $i = 1$, then we move $\frac{1}{3}$ from every u_i to v where $i = 1, 2, 3$.
Rule S4: If $i = 2$, then move $\frac{1}{3}$ from u_1 to x and from u_3 to v, respectively, and move $\frac{2}{3}$ from u_2 to v and to x, respectively, see Fig. 13.
Rule S5: If $i = 3$, then move $\frac{1}{7}$ from u_3 to v, move $\frac{6}{7}$ from u_2 to v, and move $\frac{1}{14}$ from u_2 to u_1, see Fig. 14.

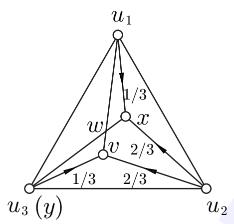


Fig. 13. Rule S4.

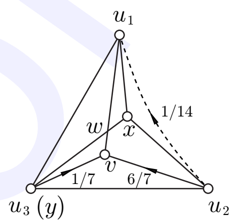


Fig. 14. Rule S5.

Rule S6: If $i = 4$, then move $\frac{1}{2}$ from u_2 to v and from u_3 to v, respectively, and move $\frac{1}{6}$ from u_2 to u_1 and from u_3 to u_1, respectively, see Fig. 15.

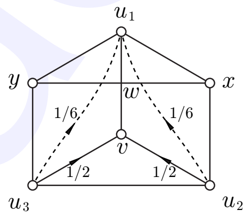


Fig. 15. Rule S6.

Rule S7: If $i = 5$, then move $\frac{1}{3}$ from u_1 to x_1, from u_2 to y_2 and from u_3 to v, respectively.

Rule S8: If $i = 6$, then move $\frac{1}{4}$ from u_1 to v, and move $\frac{3}{4}$ from u_3 to v, see Fig. 16.
Rule S9: If $i = 7$, then move 1 from u_3 to v, and move $\frac{1}{6}$ from u_3 to u_1 and to u_2, respectively, see Fig. 17.

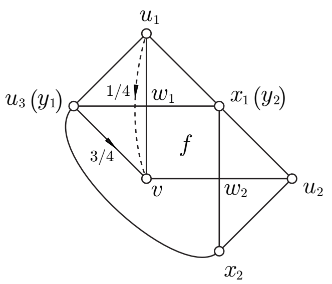


Fig. 16. Rule S8.

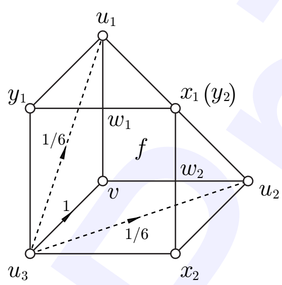


Fig. 17. Rule S9.

Rule S10: If $i = 8$, assume $\deg(x_1) \le \deg(x_2) \le \deg(x_3)$, then move $\frac{1}{4}$ from u_1 to v and from u_2 to v, respectively, and move $\frac{1}{2}$ from u_3 to v, see Fig. 18.

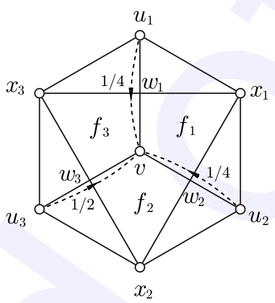


Fig. 18. Rule S10 ($\deg(x_1) \le \deg(x_2) \le \deg(x_3)$).

Lemma 2.23. *If $f \in F(D^\times)$, then $ch_2(f) \ge 0$.*

Proof. Denote $d = \deg(f)$. Assume $d = 3$. Then $ch_1(f) \ge 0$ by Lemmas 2.2 and 2.3. Since $d = 3$, f is bad by the definition of bad faces. Thus Rules S1 and S2 do not change the charge of f. Since Rules S3-S10 do not change the charge of any face, $ch_2(f) = ch_1(f) \ge 0$.

Assume $d \ge 4$. In the first-step discharging, the charge of every 4^+-face is not changed, thus $ch_1(f) = d - 4$. Suppose that f is good. Then there are at most $(d-4)$ 3-vertices incident with f. Thus by Rule S1, $ch_2(f) \ge ch_1(f) - (d-4) \ge 0$. Next assume that f is bad. If d=6, then there are three 3-vertices incident with f by Lemma 2.12. By Rule S2, $ch_2(f) = ch_1(f) - 3 \cdot \frac{2}{3} = 6 - 4 - 2 = 0$. If $d \ne 6$, then the charge of f is not changed in the second-step discharging. Thus $ch_2(f) = ch_1(f) = d - 4 \ge 0$. □

Lemma 2.24. *If* $v \in V(D^\times)$ *and* $3 \le \deg(v) \le 22$*, then* $ch_2(v) \ge 0$.

Proof. If $4 \le \deg(v) \le 22$, then by Lemmas 2.4, 2.8, 2.9 and 2.10, $ch_1(v) \ge 0$. Since no charge of v is lost in the second-step discharging, $ch_2(v) = ch_1(v) \ge 0$. If $\deg(v) = 3$, then by the rules of the first-step, $ch_1(v) = 3 - 4 = -1$. If v is a good 3-vertex, then there is at least one good face f incident with v. Then by Rule S1, f sends 1 to v, thus $ch_2(v) \ge ch_1(v) + 1 = -1 + 1 = 0$.

Next assume that v is a bad 3-vertex. Then there is a unique $X \in H_i$ containing v for some $i \in [1, 8]$ and we identify X and H_i.

If $i = 1$, then by Rule S3, $ch_2(v) = ch_1(v) + 3 \cdot \frac{1}{3} = -1 + 1 = 0$. If $i = 2$, then by Rule S4, $ch_2(v) = ch_1(v) + \frac{1}{3} + \frac{2}{3} = -1 + 1 = 0$. (Note that x is a bad 3-vertex too. Symmetrically, we have $ch_2(x) \ge 0$ too.) If $i = 3$, then by Rule S5, $ch_2(v) = ch_1(v) + \frac{1}{7} + \frac{6}{7} = -1 + 1 = 0$. If $i = 4$, then by Rule S6, $ch_2(v) = ch_1(v) + 2 \cdot \frac{1}{2} = -1 + 1 = 0$. If $i = 5$, then by Rule S2, f sends $\frac{2}{3}$ to v, and by Rule S7, u_3 sends $\frac{1}{3}$ to v, thus $ch_2(v) = ch_1(v) + \frac{1}{3} + \frac{2}{3} = -1 + 1 = 0$. (Note that both x_1 and y_2 are bad 3-vertices too. And symmetrically, $ch_2(x_1) \ge 0$ and $ch_2(y_2) \ge 0$ too.) If $i = 6$, then by Rule S8, $ch_2(v) = ch_1(v) + \frac{1}{4} + \frac{3}{4} = -1 + 1 = 0$. If $i = 7$, then by Rule S9, $ch_2(v) = ch_1(v) + 1 = -1 + 1 = 0$. If $i = 8$, then by Rule S10, $ch_2(v) = ch_1(v) + 2 \cdot \frac{1}{4} + \frac{1}{2} = -1 + 1 = 0$. □

Consider H_i where $1 \le i \le 10$. Define the *net-losing-charge* of u_j in H_i, denoted by $\Delta_i(u_j)$, as the value of losing-charge minus getting-charge of u_j ($1 \le j \le 3$ when $1 \le i \le 8$; $j = 1$ when $i = 9, 10$) restricted in one H_i after the two discharging steps. For example, in H_3, see Fig. 6, assume that $\deg(x) = 4$, then u_1 sends $\frac{1}{2}$ to the face $[u_1 w u_3]$ by Subrule F3.5, sends 1 to the face $[u_1 x w]$ by Subrule F3.2 and gets $\frac{1}{14}$ from u_2 by Rule S5, thus $\Delta_3(u_1) = \frac{1}{2} + 1 - \frac{1}{14} = \frac{10}{7}$.

Lemma 2.25. *For subgraphs (keep the drawings under D) $H_1, H_2, \ldots, H_{10}$ where v is a bad 3-vertex, we have the following results.*

(1) $\Delta_1(u_j) = \frac{4}{3}$ *for* $j = 1, 2, 3$.
(2) $\Delta_2(u_j) = \frac{7}{3}$ *for* $j = 1, 2, 3$.
(3) $\Delta_3(u_j) \le \frac{10}{7}$ *for* $j = 1, 2$ *and* $\Delta_3(u_3) = \frac{15}{7}$.
(4) $\Delta_4(u_j) \le \frac{7}{6}$ *for* $j = 1, 2, 3$.
(5) $\Delta_5(u_j) = \frac{10}{3}$ *for* $j = 1, 2, 3$.
(6) $\Delta_6(u_j) \le \frac{3}{2}$ *for* $j = 1, 2$ *and* $\Delta_6(u_3) = \frac{9}{4}$.
(7) $\Delta_7(u_j) \le \frac{4}{3}$ *for* $j = 1, 2, 3$.
(8) $\Delta_8(u_j) \le \frac{3}{2}$ *for* $j = 1, 2, 3$.
(9) $\Delta_9(u_1) \le \frac{3}{2}$.
(10) $\Delta_{10}(u_1) = \frac{1}{2}$.

Proof. Since $\deg(v) = 3$, every u_j ($j = 1, 2, 3$) is a 23^+-vertex by Observation 2.1.

(1) Consider H_1, see Fig. 4. By Subrule F3.6, u_j sends $\frac{1}{2}$ to faces $[u_ju_{j+1}v]$ and $[u_ju_{j-1}v]$ for $j = 1, 2, 3$ ($u_4 = u_1$ and $u_0 = u_3$), respectively. By Rule S3, u_j sends $\frac{1}{3}$ to v for $j = 1, 2, 3$. Thus $\Delta_1(u_j) = 2 \cdot \frac{1}{2} + \frac{1}{3} = \frac{4}{3}$.

(2) Consider H_2, see Fig. 5. By Subrule F3.5, u_1 sends $\frac{1}{2}$ to the face $[u_1wu_3]$. By Subrule F3.1, u_1 sends 1 to $[u_1wx]$. By Subrule F3.6, u_1 sends $\frac{1}{2}$ to the face $[u_1xu_2]$. By Rule S4, u_1 sends $\frac{1}{3}$ to x. Thus $\Delta_2(u_1) = \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} = \frac{7}{3}$. By Subrule F3.6, u_2 sends $\frac{1}{2}$ to faces $[u_2xu_1]$ and $[u_2vu_3]$, respectively. Since v is a bad 3-vertex, $[xwvu_2]$ is a bad 4-face, thus u_2 neither sends any charge to it, nor gets any charge from it. By Rule S4, u_2 sends $\frac{2}{3}$ to v and to x, respectively. Thus $\Delta_2(u_2) = 2 \cdot \frac{1}{2} + 2 \cdot \frac{2}{3} = \frac{7}{3}$. Symmetrically, $\Delta_2(u_3) = \frac{7}{3}$.

(3) Consider H_3, see Fig. 6. By Subrule F3.5, u_1 sends $\frac{1}{2}$ to the face $[u_1wu_3]$. By Subrules F3.2-F3.5, u_1 sends at most 1 to $[u_1wx]$. By Rule S5, u_1 gets $\frac{1}{14}$ from u_2. Note that u_1 does not send any charge to v or x. Thus $\Delta_3(u_1) \leq \frac{1}{2} + 1 - \frac{1}{14} = \frac{10}{7}$. By Subrule F3.6, u_2 sends $\frac{1}{2}$ to the face $[u_2vu_3]$. Since v is bad, $[u_2vwx]$ is a bad 4-face, thus u_2 neither sends any charge to it, nor gets any charge from it. By Rule S5, u_2 sends $\frac{6}{7}$ to v, and sends $\frac{1}{14}$ to v_1. Thus $\Delta_3(u_2) = \frac{1}{2} + \frac{6}{7} + \frac{1}{14} = \frac{10}{7}$. By Subrule F3.5, u_3 sends $\frac{1}{2}$ to the face $[u_3wu_1]$. By Subrule F3.1, u_3 sends 1 to $[u_3wv]$. By Subrule F3.6, u_3 sends $\frac{1}{2}$ to the face $[u_3vu_2]$. By Rule S5, u_3 sends $\frac{1}{7}$ to v. Thus $\Delta_3(u_3) = \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{7} = \frac{15}{7}$.

(4) Consider H_4, see Fig. 7. By Lemma 2.17, $5 \leq \deg(x), \deg(y) \leq 12$. Then by Subrules F3.3-F3.5, u_1 sends at most $\frac{3}{4}$ to faces $[u_1wy]$ and $[u_1wx]$, respectively. By Rule S6, u_1 gets $\frac{1}{6}$ from u_2 and u_3, respectively. Thus $\Delta_4(u_1) \leq 2 \cdot \frac{3}{4} - 2 \cdot \frac{1}{6} = \frac{7}{6}$. By Subrule F3.6, u_j ($j = 2, 3$) sends $\frac{1}{2}$ to the face $[u_2vu_3]$. By Rule S6, u_j sends $\frac{1}{6}$ to u_1, and sends $\frac{1}{2}$ to v. Thus $\Delta_4(u_j) \leq \frac{1}{2} + \frac{1}{6} + \frac{1}{2} = \frac{7}{6}$.

(5) Consider H_5, see Fig. 8. By Subrule F3.5, u_1 sends $\frac{1}{2}$ to faces $[u_1w_1u_3]$ and $[u_1w_3u_2]$, respectively. By Subrule F3.1, u_1 sends 1 to faces $[u_1x_1w_1]$ and $[u_1x_1w_3]$, respectively. By Rule S7, u_1 sends $\frac{1}{3}$ to x_1. Thus $\Delta_5(u_1) = 2 \cdot \frac{1}{2} + 2 \cdot 1 + \frac{1}{3} = \frac{10}{3}$. Symmetrically, $\Delta_5(u_2) = \Delta_5(u_3) = \frac{10}{3}$.

(6) Consider H_6, see Fig. 9. By (2a) of Lemma 2.18, $5 \leq \deg(x_1) \leq 12$. Then by Subrules F3.3-F3.5, u_1 sends at most $\frac{3}{4}$ to the face $[u_1w_1x_1]$. By Subrule F3.5, u_1 sends $\frac{1}{2}$ to the face $[u_1w_1u_3]$. By Rule S8, u_1 sends $\frac{1}{4}$ to v. Thus $\Delta_6(u_1) \leq \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{2}$. By (2a) of Lemma 2.18, $5 \leq \deg(x_1), \deg(x_2) \leq 12$. Then by Subrules F3.3-F3.5, u_2 sends at most $\frac{3}{4}$ to faces $[u_2w_2x_1]$ and $[u_2w_2x_2]$, respectively. Thus $\Delta_6(u_2) \leq 2 \cdot \frac{3}{4} = \frac{3}{2}$. By Subrule F3.5, u_3 sends $\frac{1}{2}$ to the face $[u_1w_1u_3]$. By Subrule F3.1, u_3 sends 1 to the face $[u_3w_1v]$. By Rule S8, u_1 sends $\frac{3}{4}$ to v. Thus $\Delta_6(u_3) = \frac{1}{2} + 1 + \frac{3}{4} = \frac{9}{4}$.

(7) Consider H_7, see Fig. 10. By (2b) of Lemma 2.18, $5 \leq \deg(x_1), \deg(y_1) \leq 12$. Then by Subrules F3.3-F3.5, u_1 sends at most $\frac{3}{4}$ to faces $[u_1w_1y_1]$ and $[u_1w_1x_1]$, respectively. By Rule S9, u_1 gets $\frac{1}{6}$ from u_3. Thus $\Delta_7(u_1) \leq 2 \cdot \frac{3}{4} - \frac{1}{6} = \frac{4}{3}$. Similarly, $\Delta_7(u_2) \leq \frac{4}{3}$. Since v is a bad 3-vertex, $[u_3y_1w_1v]$ and $[u_3x_2w_2v]$ are bad 4-faces. Then u_3 neither sends any charge to them, nor gets any charge

from them. By Rule S9, u_3 sends $\frac{1}{6}$ to u_1 and u_2, respectively, and sends 1 to v. Thus $\Delta_7(u_3) \le 2 \cdot \frac{1}{6} + 1 = \frac{4}{3}$.

(8) Consider H_8, see Fig. 11. Assume that $\deg(x_1) \le \deg(x_2) \le \deg(x_3)$. If $\deg(x_1) \ge 8$, then $\deg(x_3) \ge \deg(x_2) \ge 8$; otherwise, $\deg(x_1) \le 7$, then by Observation 2.1, we also have $\deg(x_3) \ge \deg(x_2) \ge 8$. Further, by Lemma 2.19, $5 \le \deg(x_1) \le 12$ and $8 \le \deg(x_2), \deg(x_3) \le 12$. Since $\deg(x_1) \ge 5$, by Subrules F3.3-F3.5, we have that u_1 sends at most $\frac{3}{4}$ to the face $[u_1w_1x_1]$ and u_2 sends at most $\frac{3}{4}$ to the face $[u_2w_2x_1]$. Since $\deg(x_2) \ge 8$, by Subrule F3.5, we have that u_2 sends at most $\frac{1}{2}$ to the face $[u_2w_2x_2]$ and u_3 sends at most $\frac{1}{2}$ to the face $[u_3w_3x_2]$. Similarly, since $\deg(x_3) \ge 8$, by Subrule F3.5, we have that u_3 sends at most $\frac{1}{2}$ to the face $[u_3w_3x_3]$ and u_1 sends at most $\frac{1}{2}$ to the face $[u_1w_1x_3]$. By Rule S10, u_1 and u_2 send $\frac{1}{4}$ to v, respectively, and u_3 sends $\frac{1}{2}$ to v. Thus $\Delta_8(u_1) \le \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{2}$, $\Delta_8(u_2) \le \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{2}$ and $\Delta_8(u_3) \le \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$.

(9) Consider H_9, see Fig 12. Assume that $\deg(v_1) \le \deg(v_3)$. If $\deg(v_1) \ge 8$, then $\deg(v_3) \ge 8$; otherwise $\deg(v_1) \le 7$, then by Observation 2.1, we also have $\deg(v_3) \ge 8$. By Subrules F3.1-F3.5, u_1 sends at most 1 to the face $[u_1wv_1]$. By Subrule F3.5, u_1 sends at most $\frac{1}{2}$ to the face $[u_1wv_3]$ since $\deg(v_3) \ge 8$. Since none of v_1, v_2 and v_3 is a bad 3-vertex, we have that u_1 does not send any charge to v_1, v_2 or v_3. Thus $\Delta_i(u_1) \le \frac{1}{2} + 1 = \frac{3}{2}$.

(10) Consider H_{10}. By Subrule F3.6, u_1 sends $\frac{1}{2}$ to f, thus $\Delta_{10}(u_1) = \frac{1}{2}$. □

Define the *total net-losing-charge* of a 23^+-vertex u as the value of losing-charge minus getting-charge of u. Recall the definition of spanning vertices. A spanning vertex of a face f of $D^\times$ is a vertex of G, which is incident with f, or is incident with an edge e of G such that e contains an incident edge of f in $D^\times$. We have the following lemma.

Lemma 2.26. *Let u be a 23^+-vertex. Then*

$$\begin{aligned}\Delta(u) \le\ & \frac{4}{3}h_1(u) + \frac{7}{3}h_2(u) + \frac{10}{7}h_{3,1}(u) + \frac{10}{7}h_{3,2}(u) + \frac{15}{7}h_{3,3}(u) + \frac{7}{6}h_4(u) \\ & + \frac{10}{3}h_5(u) + \frac{3}{2}h_{6,1}(u) + \frac{3}{2}h_{6,2}(u) + \frac{9}{4}h_{6,3}(u) + \frac{4}{3}h_7(u) + \frac{3}{2}h_8(u) \\ & + \frac{3}{2}h_9(u) + \frac{1}{2}h_{10}(u). \end{aligned} \tag{8}$$

Proof. Let f be an incident face of u. Assume that f has a spanning bad 3-vertex v'. Then by Lemmas 2.15, 2.17, 2.18 and 2.19, there is a unique $X \in \mathcal{H}_i$ for $i \in [1, 8]$, which contains v' and f under D. Since u is incident with f, X contains u. Thus, since $\deg(u) \ge 23$, $uv' \in E(G)$ by Lemmas 2.15, 2.17, 2.18 and 2.19, and then $X \in \mathcal{H}_i(u)$. Thus the part of total net-losing-charge of u formed by X can be checked by (1)-(8) of Lemma 2.25. Assume that no spanning vertex of f is a bad 3-vertex. If $\deg(f) \ge 4$, then u does not lose charge to f in the two steps of discharging, thus we do not consider this case. If $\deg(f) = 3$, then there is a unique $Y \in \mathcal{H}_{i'}(u)$ for $i' \in \{9, 10\}$ containing f

under D (note for $Y \in \mathcal{H}_{i'}(u)$, u is isomorphic to u_1 of $H_{i'}$). Thus the part of total net-losing-charge of u formed by Y can be checked by (9) and (10) of Lemma 2.25. Hence, by Lemma 2.25, this lemma holds. □

Lemma 2.27. *Let u be a 23^+-vertex. Then $ch_2(u) \geq 0$.*

Proof. Denote $d(u) = \deg(u)$. By inequality (8) (Lemma 2.26), we have

$$\begin{aligned}\Delta(u) \leq \frac{5}{6}\big[&2h_1(u) + 3h_2(u) + 2h_{3,1}(u) + 2h_{3,2}(u) + 3h_{3,3}(u) + 2h_4(u) \\ &+ 4h_5(u) + 2h_{6,1}(u) + 2h_{6,2}(u) + 3h_{6,3}(u) + 2h_7(u) + 2h_8(u) \\ &+ 2h_9(u) + h_{10}(u)\big].\end{aligned}$$

Further, by inequality (7) (Lemma 2.22) we have $\Delta(u) \leq \frac{5}{6}d(u)$. Thus $ch_2(u) = (d(u)-4) - \Delta(u) \geq (d(u)-4) - \frac{5}{6}d(u) = \frac{1}{6}d(u) - 4$. When $d(u) \geq 24$, we have $ch_2(u) \geq 0$.

Next assume that $d(u) = 23$. By inequality (8), we have

$$\begin{aligned}\Delta(u) \leq \big(\frac{5}{6} - \frac{7}{9}\big) \cdot 4h_5(u) + \frac{7}{9}\big[&2h_1(u) + 3h_2(u) + 2h_{3,1}(u) + 2h_{3,2}(u) \\ &+ 3h_{3,3}(u) + 2h_4(u) + 4h_5(u) + 2h_{6,1}(u) + 2h_{6,2}(u) + 3h_{6,3}(u) \\ &+ 2h_7(u) + 2h_8(u) + 2h_9(u) + h_{10}(u)\big].\end{aligned}$$

Further, by inequality (7) (Lemma 2.22), we have $\Delta(u) \leq \frac{2}{9}h_5(u) + \frac{7}{9}d(u)$. Since $4h_5(u) \leq d(u) = 23$, we have $h_5(u) \leq 5$. Thus $ch_2(u) = (d(u)-4) - \Delta(u) \geq d(u) - 4 - \frac{2}{9}h_5(u) - \frac{7}{9}d(u) = \frac{2}{9}d(u) - \frac{2}{9}h_5(u) - 4 \geq \frac{2}{9}\cdot(23-5) - 4 = 0$.

Thus this lemma holds. □

By Lemmas 2.23, 2.24 and 2.27, for every $x \in V(D^\times) \cup F(D^\times)$, $ch_2(x) \geq 0$. But

$$\sum_{x \in V(D^\times)\cup F(D^\times)} ch_2(x) = \sum_{x \in V(D^\times)\cup F(D^\times)} ch_0(x) = -8 < 0,$$

a contradiction. Therefore, we prove Theorem 1.4.

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