On parabolic Kazhdan-Lusztig R-polynomials for the symmetric group

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Abstract

Parabolic R-polynomials were introduced by Deodhar as parabolic analogues of ordinary R-polynomials defined by Kazhdan and Lusztig. In this paper, we are concerned with the computation of parabolic R-polynomials for the symmetric group. Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$, and let $S = \{s_i \,|\, 1 \leq i \leq n-1\}$ be the generating set of S_n, where for $1 \leq i \leq n-1$, s_i is the adjacent transposition. For a subset $J \subseteq S$, let $(S_n)_J$ be the parabolic subgroup generated by J, and let $(S_n)^J$ be the set of minimal coset representatives for $S_n/(S_n)_J$. For $u \leq v \in (S_n)^J$ in the Bruhat order and $x \in \{q, -1\}$, let $R^{J,x}_{u,v}(q)$ denote the parabolic R-polynomial indexed by u and v. Brenti found a formula for $R^{J,x}_{u,v}(q)$ when $J = S \setminus \{s_i\}$, and obtained an expression for $R^{J,x}_{u,v}(q)$ when $J = S \setminus \{s_{i-1}, s_i\}$. In this paper, we provide a formula for $R^{J,x}_{u,v}(q)$, where $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$ and i appears after $i-1$ in v. It should be noted that the condition that i appears after $i-1$ in v is equivalent to that v is a permutation in $(S_n)^{S\setminus\{s_{i-2},s_i\}}$. We also pose a conjecture for $R^{J,x}_{u,v}(q)$, where $J = S \setminus \{s_k, s_{k+1}, \ldots, s_i\}$ with $1 \leq k \leq i \leq n-1$ and v is a permutation in $(S_n)^{S\setminus\{s_k,s_i\}}$.

Keywords: parabolic Kazhdan-Lusztig R-polynomial, the symmetric group, Bruhat order

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1 Introduction

Parabolic R-polynomials for a Coxeter group were introduced by Deodhar [5] as parabolic analogues of ordinary R-polynomials defined by Kazhdan and Lusztig [8]. In this paper, we consider the computation of parabolic R-polynomials for the symmetric group. Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$, and let $S = \{s_1, s_2, \ldots, s_{n-1}\}$ be the generating set of S_n, where for $1 \leq i \leq n-1$, s_i is the adjacent transposition that interchanges the elements i and $i+1$. For a subset $J \subseteq S$, let $(S_n)_J$ be the parabolic subgroup generated by J, and let $(S_n)^J$ be the set of minimal coset representatives of $S_n/(S_n)_J$. Assume that u and v are two permutations in $(S_n)^J$ such that $u \leq v$ in the Bruhat order. For $x \in \{q, -1\}$, let $R^{J,x}_{u,v}(q)$ denote the parabolic R-polynomial indexed by u and v. When $J = S \setminus \{s_i\}$, Brenti [2] found a formula for $R^{J,x}_{u,v}(q)$. Recently, Brenti [3] obtained an expression for $R^{J,x}_{u,v}(q)$ for $J = S \setminus \{s_{i-1}, s_i\}$.

In this paper, we consider the case $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$. We introduce a statistic on pairs of permutations in $(S_n)^J$ and then we give a formula for $R^{J,x}_{u,v}(q)$, where v is restricted to a permutation in $(S_n)^{S\setminus\{s_{i-2},s_i\}}$. Notice that $v \in (S_n)^{S\setminus\{s_{i-2},s_i\}}$ is equivalent to that $v \in (S_n)^J$

and i appears after $i-1$ in v. It should be noted that there does not seem to exist an explicit formula for the case when $v \in (S_n)^J$ and i appears before $i-1$ in v.

We also conjecture a formula for $R^{J,x}_{u,v}(q)$, where $J = S \setminus \{s_k, s_{k+1}, \ldots, s_i\}$ with $1 \leq k \leq i \leq n-1$ and $v \in (S_n)^{S\setminus\{s_k,s_i\}}$. Notice also that $v \in (S_n)^{S\setminus\{s_k,s_i\}}$ can be equivalently described as the condition that $v \in (S_n)^J$ and the elements $k+1, k+2, \ldots, i$ appear in increasing order in v. This conjecture contains Brenti's formulas and our result as special cases. When $k=1$ and $i = n-1$, it becomes a conjecture for a formula of the ordinary R-polynomials $R_{u,v}(q)$, where v is a permutation in S_n such that $2, 3, \ldots, n-1$ appear in increasing order in v.

Let us begin with some terminology and notation. For a Coxeter group W with a generating set S, let $T = \{wsw^{-1} \,|\, w \in W,\, s \in S\}$ be the set of reflections of W. For $w \in W$, the length $\ell(w)$ of w is defined as the smallest k such that w can be written as a product of k generators in S. For $u, v \in W$, we say that $u \leq v$ in the Bruhat order if there exists a sequence $t_1, t_2, \ldots, t_r$ of reflections such that $v = ut_1t_2\cdots t_r$ and $\ell(ut_1\cdots t_i) > \ell(ut_1\cdots t_{i-1})$ for $1 \leq i \leq r$.

For a subset $J \subseteq S$, let W_J be the parabolic subgroup generated by J, and let W^J be the set of minimal right coset representatives of W/W_J, that is,

$$W^J = \{w \in W \,|\, \ell(sw) > \ell(w), \text{ for all } s \in J\}. \tag{1.1}$$

We use $D_R(w)$ to denote the set of right descents of w, that is,

$$D_R(w) = \{s \in S \,|\, \ell(ws) < \ell(w)\}. \tag{1.2}$$

For $u, v \in W^J$, the parabolic R-polynomial $R^{J,x}_{u,v}(q)$ can be recursively determined by the following property.

Theorem 1.1 (Deodhar [5]) *Let (W,S) be a Coxeter system and J be a subset of S. Then, for each $x \in \{q, -1\}$, there is a unique family $\{R^{J,x}_{u,v}(q)\}_{u,v\in W^J}$ of polynomials with integer coefficients such that for all $u, v \in W^J$,*

(i) *if $u \not\leq v$, then $R^{J,x}_{u,v}(q) = 0$;*

(ii) *if $u = v$, then $R^{J,x}_{u,v}(q) = 1$;*

(iii) *if $u < v$, then for any $s \in D_R(v)$,*

$$R^{J,x}_{u,v}(q) = \begin{cases} R^{J,x}_{us,\,vs}(q), & \text{if } s \in D_R(u), \\ qR^{J,x}_{us,\,vs}(q) + (q-1)R^{J,x}_{u,\,vs}(q), & \text{if } s \notin D_R(u) \text{ and } us \in W^J, \\ (q-1-x)R^{J,x}_{u,\,vs}(q), & \text{if } s \notin D_R(u) \text{ and } us \notin W^J. \end{cases}$$

Notice that when $J = \emptyset$, the parabolic R-polynomial $R^{J,x}_{u,v}(q)$ reduces to an ordinary R-polynomial $R_{u,v}(q)$, see, for example, Björner and Brenti [1, Chapter 5] or Humphreys [7, Chapter 7]. The parabolic R-polynomials $R^{J,x}_{u,v}(q)$ for $x = q$ and $x = -1$ satisfy the following relation, so that we only need to consider the computation for the case $x = q$.

Theorem 1.2 (Deodhar [6, Corollary 2.2]) *For $u, v \in W^J$ with $u \leq v$,*

$$q^{\ell(v)-\ell(u)} R^{J,q}_{u,v}\left(\frac{1}{q}\right) = (-1)^{\ell(v)-\ell(u)} R^{J,-1}_{u,v}(q).$$

There is no known explicit formula for $R_{u,v}^{J,x}(q)$ for a general Coxeter system (W, S), and even for the symmetric group. When $W = S_n$, Brenti [2, 3] found formulas for $R_{u,v}^{J,x}(q)$ for certain subsets J, namely, $J = S \setminus \{s_i\}$ or $J = S \setminus \{s_{i-1}, s_i\}$. To describe the formulas for the parabolic R-polynomials obtained by Brenti [2, 3], we recall some statistics on pairs of permutations in $(S_n)^J$ with $J = S \setminus \{s_i\}$ or $J = S \setminus \{s_{i-1}, s_i\}$.

A permutation $u = u_1 u_2 \cdots u_n$ in S_n is also considered as a bijection on $\{1, 2, \ldots, n\}$ such that $u(i) = u_i$ for $1 \leq i \leq n$. For $u, v \in S_n$, the product uv of u and v is defined as the bijection such that $uv(i) = u(v(i))$ for $1 \leq i \leq n$. For $1 \leq i \leq n-1$, the adjacent transposition s_i is the permutation that interchanges the elements i and $i+1$. The length of a permutation $u \in S_n$ can be interpreted as the number of inversions of u, that is,

$$\ell(u) = |\{(i, j) \mid 1 \leq i < j \leq n,\ u(i) > u(j)\}|. \tag{1.3}$$

By (1.2) and (1.3), the right descent set of a permutation $u \in S_n$ is given by

$$D_R(u) = \{s_i \mid 1 \leq i \leq n-1,\ u(i) > u(i+1)\}.$$

When $J = S \setminus \{s_i\}$, it follows from (1.1) and (1.3) that a permutation $u \in S_n$ belongs to $(S_n)^J$ if and only if the elements $1, 2, \ldots, i$ as well as the elements $i+1, i+2, \ldots, n$ appear in increasing order in u, or equivalently,

$$u^{-1}(1) < u^{-1}(2) < \cdots < u^{-1}(i) \quad \text{and} \quad u^{-1}(i+1) < u^{-1}(i+2) < \cdots < u^{-1}(n).$$

For $n \geq 1$, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For $J = S \setminus \{s_i\}$ and $u, v \in (S_n)^J$, let

$$D(u, v) = v^{-1}([i]) \setminus u^{-1}([i]).$$

For $1 \leq j \leq n$, let

$$a_j(u, v) = |\{r \in u^{-1}([i]) \mid r < j\}| - |\{r \in v^{-1}([i]) \mid r < j\}|.$$

It is known that $u \leq v$ in the Bruhat order if and only if $a_j(u, v) \geq 0$ for all $1 \leq j \leq n$. Brenti [2] obtained the following formula for $R_{u,v}^{J,x}(q)$, where $J = S \setminus \{s_i\}$.

Theorem 1.3 (Brenti [2, Corollary 3.2]) *Let $J = S \setminus \{s_i\}$, and let $u, v \in (S_n)^J$ with $u \leq v$. Then*

$$R_{u,v}^{J,q}(q) = (-1)^{\ell(v)-\ell(u)} \prod_{j \in D(u,v)} \left(1 - q^{a_j(u,v)}\right).$$

We now turn to the case $J = S \setminus \{s_{i-1}, s_i\}$. In this case, it can be seen from (1.1) and (1.3) that a permutation $u \in S_n$ belongs to $(S_n)^J$ if and only if

$$u^{-1}(1) < u^{-1}(2) < \cdots < u^{-1}(i-1) \quad \text{and} \quad u^{-1}(i+1) < u^{-1}(i+2) < \cdots < u^{-1}(n).$$

For $u, v \in (S_n)^J$, let

$$\widetilde{D}(u, v) = v^{-1}([i-1]) \setminus u^{-1}([i-1]).$$

For $1 \leq j \leq n$, let

$$\widetilde{a}_j(u, v) = |\{r \in u^{-1}([i-1]) \mid r < j\}| - |\{r \in v^{-1}([i-1]) \mid r < j\}|.$$

The following formula is due to Brenti [3].

Theorem 1.4 (Brenti [3, Theorem 3.1]) *Let $J = S \setminus \{s_{i-1}, s_i\}$, and let $u, v \in (S_n)^J$ with $u \leq v$. Then*

$$R_{u,v}^{J,q}(q) = \begin{cases} (-1)^{\ell(v)-\ell(u)} \left(1 - q + cq^{1+a_{v^{-1}(i)}(u,v)}\right) \prod_{j \in D(u,v)} \left(1 - q^{a_j(u,v)}\right), & \text{if } u^{-1}(i) \geq v^{-1}(i), \\ (-1)^{\ell(v)-\ell(u)} \left(1 - q + cq^{1+\widetilde{a}_{v^{-1}(i)}(u,v)}\right) \prod_{j \in \widetilde{D}(u,v)} \left(1 - q^{\widetilde{a}_j(u,v)}\right), & \text{if } u^{-1}(i) \leq v^{-1}(i), \end{cases}$$

where $c = \delta_{u^{-1}(i), v^{-1}(i)}$ is the Kronecker delta function.

It should be noted that the sets $(S_n)^J$ for $J = S \setminus \{s_i\}$ and $J = S \setminus \{s_{i-1}, s_i\}$ are called tight quotients of S_n by Stembridge [10] in the study of the Bruhat order of Coxeter groups. Therefore, combining Theorem 1.3 and Theorem 1.4 leads to an expression for the parabolic R-polynomials for tight quotients of the symmetric group.

2 A formula for $R_{u,v}^{J,q}(q)$ with $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$

In this section, we present a formula for $R_{u,v}^{J,q}(q)$, where $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$ and v is a permutation in $(S_n)^{S \setminus \{s_{i-2}, s_i\}}$. It is clear that $v \in (S_n)^{S \setminus \{s_{i-2}, s_i\}}$ is equivalent to that $v \in (S_n)^J$ and i appears after $i-1$ in v. We also give a conjectured formula for $R_{u,v}^{J,q}(q)$, where $J = S \setminus \{s_k, s_{k+1}, \ldots, s_i\}$ with $1 \leq k \leq i \leq n-1$ and $v \in (S_n)^{S \setminus \{s_k, s_i\}}$.

For $u, v \in (S_n)^J$ with $u \leq v$, our formula for $R_{u,v}^{J,q}(q)$ relies on a vector of statistics on (u, v), denoted $(a_1(u, v), a_2(u, v), \ldots, a_n(u, v))$. Notice that a permutation $u \in S_n$ belongs to $(S_n)^J$ if and only if the elements $1, 2, \ldots, i-2$ as well as the elements $i+1, i+2, \ldots, n$ appear in increasing order in u. To define $a_j(u, v)$, we need to consider the positions of the elements $i-1$ and i in u and v. For convenience, let $u^{-1} = p_1 p_2 \cdots p_n$ and $v^{-1} = q_1 q_2 \cdots q_n$, that is, t appears in position p_t in u, and appears in position q_t in v. The following set $A(u, v)$ is defined based on the relations $p_{i-1} \geq q_{i-1}$ and $p_i \geq q_i$. More precisely, $A(u, v)$ is a subset of $\{i-1, i\}$ such that $i-1 \in A(u, v)$ if and only if $p_{i-1} \geq q_{i-1}$, and $i \in A(u, v)$ if and only if $p_i \geq q_i$. Set

$$B(u, v) = \{1, 2, \ldots, i-2\} \cup A(u, v).$$

For $1 \leq j \leq n$, we define $a_j(u, v)$ to be the number of elements of $B(u, v)$ that are contained in $\{u_1, \ldots, u_{j-1}\}$ minus the number of elements of $B(u, v)$ that are contained in $\{v_1, \ldots, v_{j-1}\}$, that is,

$$a_j(u, v) = |\{r \in u^{-1}(B(u, v)) \,|\, r < j\}| - |\{r \in v^{-1}(B(u, v)) \,|\, r < j\}|. \tag{2.1}$$

For example, let $n = 9$ and $i = 5$, so that $J = S \setminus \{s_3, s_4, s_5\}$. Let

$$u = 416273859 \quad \text{and} \quad v = 671489253 \tag{2.2}$$

be two permutations in $(S_9)^J$. Then we have $A(u, v) = \{5\}$, $B(u, v) = \{1, 2, 3, 5\}$, and

$$(a_1(u, v), \ldots, a_9(u, v)) = (0, 0, 1, 0, 1, 1, 2, 1, 1). \tag{2.3}$$

The following theorem gives a formula for $R_{u,v}^{J,q}(q)$.

Theorem 2.5 *Let $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$, and let v be a permutation in $(S_n)^{S \setminus \{s_{i-2}, s_i\}}$. Let*

$$D(u, v) = v^{-1}(B(u, v)) \setminus u^{-1}(B(u, v)). \tag{2.4}$$

Then, for any $u \in (S_n)^J$ with $u \leq v$, we have

$$R_{u,v}^{J,q}(q) = (-1)^{\ell(v)-\ell(u)} \left(1 - q + \delta_{u^{-1}(i-1),v^{-1}(i-1)} q^{1+a_{v^{-1}(i-1)}(u,v)}\right)$$
$$\left(1 - q + \delta_{u^{-1}(i),v^{-1}(i)} q^{1+a_{v^{-1}(i)}(u,v)}\right) \prod_{j \in D(u,v)} \left(1 - q^{a_j(u,v)}\right). \qquad (2.5)$$

Remark. It should be noted that Theorem 2.5 does not imply a formula for $R_{u,v}^{J',q}(q)$ with $J' = S \setminus \{s_{i-2}, s_i\}$, since, by definition, the parabolic R-polynomial $R_{u,v}^{J,q}(q)$ depends heavily on the choice of the subset J.

Let us give an example for Theorem 2.5. Assume that u and v are the permutations as given in (2.2). Then we have $D(u,v) = \{3, 7, 9\}$. In view of (2.3), formula (2.5) gives

$$R_{u,v}^{J,q}(q) = (1-q)^3(1-q^2)(1-q+q^2).$$

To prove the above theorem, we need a criterion for the relation of two permutations in $(S_n)^J$ with respect to the Bruhat order. Let $u, v \in (S_n)^J$, for $h = 1, 2, 3$ and $1 \leq j \leq n$, define

$$b_{h,j}(u,v) = \left|\{r \in u^{-1}([i+h-3]) \,|\, r < j\}\right| - \left|\{r \in v^{-1}([i+h-3]) \,|\, r < j\}\right|. \qquad (2.6)$$

The following proposition, which follows easily from Corollary 2.2.5 and Theorem 2.6.3 of [1], shows that we can use $b_{h,j}(u,v)$ with $h = 1, 2, 3$ and $1 \leq j \leq n$ to determine whether $u \leq v$ in the Bruhat order.

Proposition 2.6 *Let $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$, and let $u, v \in (S_n)^J$. Then, $u \leq v$ if and only if $b_{h,j}(u,v) \geq 0$ for $h = 1, 2, 3$ and $1 \leq j \leq n$.*

We are now in a position to present a proof of Theorem 2.5.

Proof of Theorem 2.5. Assume that $J = S \setminus \{s_{i-2}, s_{i-1}, s_i\}$, and u and v are two permutations in $(S_n)^J$ such that $u \leq v$. Write $u^{-1} = p_1 p_2 \cdots p_n$ and $v^{-1} = q_1 q_2 \cdots q_n$. By the definitions of $(a_1(u,v), \ldots, a_n(u,v))$ and $D(u,v)$, we consider the following four cases:

$$p_{i-1} \geq q_{i-1} \quad \text{and} \quad p_i \geq q_i, \qquad (2.7)$$

$$p_{i-1} \geq q_{i-1} \quad \text{and} \quad p_i < q_i, \qquad (2.8)$$

$$p_{i-1} < q_{i-1} \quad \text{and} \quad p_i \geq q_i, \qquad (2.9)$$

$$p_{i-1} < q_{i-1} \quad \text{and} \quad p_i < q_i. \qquad (2.10)$$

We conduct induction on $\ell(v)$. When $\ell(v) = 0$, formula (2.5) is easy to check. Assume that $\ell(v) > 0$ and formula (2.5) is true for $\ell(v) - 1$. We proceed to prove (2.5) for $\ell(v)$. We shall only provide a proof for the case in (2.8). The other cases can be justified by using similar arguments. By (2.1) and (2.8), we see that for $1 \leq k \leq n$,

$$a_k(u,v) = |\{r \in u^{-1}([i-1]) \,|\, r < k\}| - |\{r \in v^{-1}([i-1]) \,|\, r < k\}|. \qquad (2.11)$$

Note that $a_j(u,v) = b_{2,j}(u,v)$ for all $1 \leq j \leq n$. Moreover, by (2.4) and (2.8) we find that

$$D(u,v) = v^{-1}([i-1]) \setminus u^{-1}([i-1]). \qquad (2.12)$$

1	$v(j) > i$ and $v(j+1) = i$
2	$v(j) > i$ and $v(j+1) = i-1$
3	$v(j) > i$ and $v(j+1) < i-1$
4	$v(j) = i$ and $v(j+1) < i-1$
5	$v(j) = i-1$ and $v(j+1) < i-1$

Table 2.1: The choices of $v(j)$ and $v(j+1)$ in v.

Let $s = s_j \in D_R(v)$ be a right descent of v, that is, $v(j) > v(j+1)$, where $1 \leq j \leq n-1$. Keep in mind that i appears after $i-1$ in v, namely, $q_i > q_{i-1}$, and that the elements $1, 2, \ldots, i-2$ as well as the elements $i+1, i+2, \ldots, n$ appear in increasing order in v. So we get all possible choices of $v(j)$ and $v(j+1)$ as listed in Table 2.1.

According to whether s is a right descent of u, we have the following two cases.

Case 1: $s \in D_R(u)$, that is, $u(j) > u(j+1)$. Since the elements $1, 2, \ldots, i-2$ as well as the elements $i+1, i+2, \ldots, n$ appear in increasing order in u, the possible choices of $u(j)$ and $u(j+1)$ are as given in Table 2.2.

1	$u(j) > i$ and $u(j+1) = i$
2	$u(j) > i$ and $u(j+1) = i-1$
3	$u(j) > i$ and $u(j+1) < i-1$
4	$u(j) = i$ and $u(j+1) = i-1$
5	$u(j) = i$ and $u(j+1) < i-1$
6	$u(j) = i-1$ and $u(j+1) < i-1$

Table 2.2: The choices of $u(j)$ and $u(j+1)$ in u in Case 1.

We only give proofs for the cases when v satisfies Condition 1 in Table 2.1 and u satisfies Conditions 2 and 5 in Table 2.2, and for the cases when v satisfies Condition 5 in Table 2.1 and u satisfies Conditions 1, 2, 3, and 6 in Table 2.2. The remaining cases can be dealt with in the same manner.

Subcase 1. $v(j) > i, v(j+1) = i$ and $u(j) > i, u(j+1) = i-1$. In this case, it is easy to see that $B(u,v) = B(us, vs) = [i-1]$. By (2.1), we have

$$a_{j+1}(u,v) = a_{j+1}(us, vs) - 1, \text{ and } a_k(u,v) = a_k(us, vs) \text{ for } k \neq j+1.$$

Moreover, by (2.4), we find that

$$D(u,v) = D(us, vs) \text{ and } j+1 \notin D(u,v).$$

Thus by the induction hypothesis,

$$\begin{aligned} R_{u,v}^{J,q}(q) &= R_{us,vs}^{J,q}(q) \\ &= (-1)^{\ell(vs)-\ell(us)}(1-q)^2 \prod_{k \in D(us,vs)} \left(1 - q^{a_k(us,vs)}\right) \end{aligned}$$

$$= (-1)^{\ell(v)-\ell(u)}(1-q)^2 \prod_{k\in D(u,v)} \left(1-q^{a_k(u,v)}\right),$$

as desired.

Subcase 2. $v(j) > i, v(j+1) = i$ and $u(j) = i, u(j+1) < i-1$. It is easy to check that $B(u,v) = [i-1], B(us, vs) = [i]$. By (2.1) and (2.4), we have

$$a_j(u,v) = a_j(us,vs) \text{ for } 1 \le j \le n$$

and

$$D(u,v) = D(us,vs).$$

Then by the induction hypothesis,

$$R^{J,q}_{u,v}(q) = R^{J,q}_{us,vs}(q) = (-1)^{\ell(v)-\ell(u)}(1-q)^2 \prod_{k\in D(u,v)} \left(1-q^{a_k(u,v)}\right).$$

Subcase 3. $v(j) = i-1, v(j+1) < i-1$ and $u(j) > i, u(j+1) = i$. Since $B(u,v) = B(us,vs)$, by (2.1), it is easy to check that for $1 \le k \le n$,

$$a_k(us,vs) = a_k(u,v).$$

Moreover, it follows from (2.4) that

$$D(us,vs) = D(u,v).$$

By the induction hypothesis, we deduce that

$$R^{J,q}_{u,v}(q) = R^{J,q}_{us,vs}(q) = (-1)^{\ell(v)-\ell(u)}(1-q)^2 \prod_{k\in D(u,v)} \left(1-q^{a_k(u,v)}\right).$$

Subcase 4. $v(j) = i-1, v(j+1) < i-1$ and $u(j) > i, u(j+1) = i-1$. Notice that in this case us and vs satisfy the relation in (2.10). So we have $B(u,v) = [i-1] = B(us,vs) \cup \{i-1\}$. By (2.1) and (2.4), it is easily verified that for $1 \le k \le n$,

$$a_k(us,vs) = a_k(u,v),$$

and

$$\begin{aligned} D(us,vs) =& (vs)^{-1}([i-2]) \setminus (us)^{-1}([i-2]) \\ =& v^{-1}([i-1]) \setminus u^{-1}([i-1]) \\ =& D(u,v). \end{aligned}$$

By the induction hypothesis, we get

$$R^{J,q}_{u,v}(q) = R^{J,q}_{us,vs}(q) = (-1)^{\ell(v)-\ell(u)}(1-q)^2 \prod_{k\in D(u,v)} \left(1-q^{a_k(u,v)}\right).$$

Subcase 5. $v(j) = i-1, v(j+1) < i-1$ and $u(j) > i, u(j+1) < i-1$. We find that $B(us,vs) = B(u,v) = [i-1]$. By (2.1) and (2.4), we have

$$a_{j+1}(us,vs) = a_{j+1}(u,v) + 1 \;\text{ and }\; a_k(us,vs) = a_k(u,v), \;\text{ for } k \ne j+1,$$

and

$$D(us,vs) = (D(u,v) \setminus \{j\}) \cup \{j+1\}.$$

Thus, the induction hypothesis yields that

$$\begin{aligned}
R^{J,q}_{u,v}(q) =& R^{J,q}_{us,vs}(q) \\
=& (-1)^{\ell(vs)-\ell(us)}(1-q)^2 \prod_{k\in D(us,vs)} \left(1-q^{a_k(us,vs)}\right) \\
=& (-1)^{\ell(v)-\ell(u)}(1-q)^2 \frac{1-q^{a_{j+1}(us,vs)}}{1-q^{a_j(u,v)}} \prod_{k\in D(u,v)} \left(1-q^{a_k(u,v)}\right),
\end{aligned}$$

which reduces to (2.5), since

$$a_{j+1}(us,vs) = a_j(u,v).$$

Subcase 6. $v(j) = i-1, v(j+1) < i-1$ and $u(j) = i-1, u(j+1) < i-1$. For $1 \le k \le n$, we have

$$a_k(us,vs) = a_k(u,v)$$

and

$$B(us,vs) = B(u,v) \text{ and } D(us,vs) = D(u,v).$$

By the induction hypothesis, we find that

$$\begin{aligned}
R^{J,q}_{u,v}(q) &= R^{J,q}_{us,vs}(q) \\
&= (-1)^{\ell(vs)-\ell(us)} \left(1-q+q^{1+a_{j+1}(us,vs)}\right)(1-q) \prod_{k\in D(us,vs)} \left(1-q^{a_k(us,vs)}\right). \qquad (2.13)
\end{aligned}$$

Noticing the following relation

$$a_{j+1}(us,vs) = a_j(u,v),$$

formula (2.13) can be rewritten as

$$R^{J,q}_{u,v}(q) = (-1)^{\ell(v)-\ell(u)} \left(1-q+q^{1+a_j(u,v)}\right)(1-q) \prod_{k\in D(u,v)} \left(1-q^{a_k(u,v)}\right),$$

as required.

Case 2: $s \notin D_R(u)$, that is, $u(j) < u(j+1)$. The possible choices of $u(j)$ and $u(j+1)$ are given in Table 2.3.

We shall provide proofs for three subcases: (i) v satisfies Condition 1 in Table 2.1 and u satisfies Condition 7 in Table 2.3; (ii) v satisfies Condition 3 in Table 2.1 and u satisfies Condition 3 in Table 2.3; (iii) v satisfies Condition 5 in Table 2.1 and u satisfies Condition 3 in Table 2.3. The verifications in other situations are similar or relatively easier.

Subcase (i): $v(j) > i$, $v(j+1) = i$, $i = u(j) < u(j+1)$. By Theorem 1.1, we have

$$R^{J,q}_{u,v}(q) = qR^{J,q}_{us,vs}(q) + (q-1)R^{J,q}_{u,vs}(q). \qquad (2.14)$$

We need to compute $R^{J,q}_{us,vs}(q)$ and $R^{J,q}_{u,vs}(q)$. We first compute $R^{J,q}_{u,vs}(q)$. Notice that u and vs satisfy the relation in (2.7). Since $A(u,vs) = \{i-1,i\}$ and $B(u,vs) = [i]$, by (2.1), we obtain that for $1 \le k \le n$,

$$a_k(u,vs) = \left|\{r \in u^{-1}([i]) \,|\, r < k\}\right| - \left|\{r \in (vs)^{-1}([i]) \,|\, r < k\}\right|$$

1	$u(j) < u(j+1) < i-1$
2	$u(j) < u(j+1) = i-1$
3	$u(j) < i-1$ and $u(j+1) = i$
4	$u(j) < i-1$ and $u(j+1) > i$
5	$u(j) = i-1$ and $u(j+1) = i$
6	$u(j) = i-1$ and $u(j+1) > i$
7	$i = u(j) < u(j+1)$
8	$i < u(j) < u(j+1)$

Table 2.3: The choices of $u(j)$ and $u(j+1)$ in u in Case 2.

$$= \left|\{r \in u^{-1}([i-1]) \,|\, r < k\}\right| - \left|\{r \in v^{-1}([i-1]) \,|\, r < k\}\right|$$
$$= a_k(u,v).$$

Moreover, by (2.4) we have

$$\begin{aligned} D(u, vs) &= (vs)^{-1}([i]) \setminus u^{-1}([i]) \\ &= v^{-1}([i-1]) \setminus u^{-1}([i-1]) \\ &= D(u,v). \end{aligned}$$

By the induction hypothesis, we deduce that

$$\begin{aligned} R^{J,q}_{u,vs}(q) &= (-1)^{\ell(vs)-\ell(u)} \left(1 - q + \delta_{u^{-1}(i-1),(vs)^{-1}(i-1)} q^{1+a_{(vs)^{-1}(i-1)}(u,vs)}\right) \\ &\quad \left(1 - q + q^{1+a_j(u,vs)}\right) \prod_{k\in D(u,vs)} \left(1 - q^{a_k(u,vs)}\right) \\ &= (-1)^{\ell(v)-\ell(u)-1} \left(1 - q + \delta_{u^{-1}(i-1),v^{-1}(i-1)} q^{1+a_{v^{-1}(i-1)}(u,v)}\right) \\ &\quad \left(1 - q + q^{1+a_j(u,v)}\right) \prod_{k\in D(u,v)} \left(1 - q^{a_k(u,v)}\right). \end{aligned} \tag{2.15}$$

To compute $R^{J,q}_{us,vs}(q)$, we consider two cases according to whether $us \leq vs$. First, we assume that $us \leq vs$. Since us and vs satisfy the relation in (2.7), and $A(us,vs) = \{i-1, i\}$, $B(us,vs) = [i] = B(u,v) \cup \{i\}$, by (2.1) we see that

$$a_{j+1}(us,vs) = a_{j+1}(u,v) - 1 \quad \text{and} \quad a_k(us,vs) = a_k(u,v), \quad \text{for } k \neq j+1. \tag{2.16}$$

Moreover, by (2.4) we get

$$\begin{aligned} D(us,vs) &= (vs)^{-1}([i]) \setminus (us)^{-1}([i]) \\ &= D(u,v) \cup \{j\}. \end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17) and applying the induction hypothesis, we deduce that

$$R^{J,q}_{us,vs}(q) = (-1)^{\ell(vs)-\ell(us)} \left(1 - q + \delta_{(us)^{-1}(i-1),(vs)^{-1}(i-1)} q^{1+a_{(vs)^{-1}(i-1)}(us,vs)}\right)$$

$$\begin{aligned}
&(1-q)\prod_{k\in D(us,vs)}\left(1-q^{a_k(us,vs)}\right)\\
&=(-1)^{\ell(v)-\ell(u)}\left(1-q+\delta_{u^{-1}(i-1),v^{-1}(i-1)}q^{1+a_{v^{-1}(i-1)}(u,v)}\right)\\
&(1-q)\left(1-q^{a_j(u,v)}\right)\prod_{k\in D(u,v)}\left(1-q^{a_k(u,v)}\right). \qquad (2.18)
\end{aligned}$$

Substituting (2.15) and (2.18) into (2.14), we obtain that

$$\begin{aligned}
R^{J,q}_{u,v}(q) &= qR^{J,q}_{us,vs}(q)+(q-1)R^{J,q}_{u,vs}(q)\\
&=(-1)^{\ell(v)-\ell(u)}\left(q\left(1-q^{a_j(u,v)}\right)+\left(1-q+q^{1+a_j(u,v)}\right)\right)\\
&\quad(1-q)\left(1-q+\delta_{u^{-1}(i-1),v^{-1}(i-1)}q^{1+a_{v^{-1}(i-1)}(u,v)}\right)\prod_{k\in D(u,v)}\left(1-q^{a_k(u,v)}\right)\\
&=(-1)^{\ell(v)-\ell(u)}(1-q)\left(1-q+\delta_{u^{-1}(i-1),v^{-1}(i-1)}q^{1+a_{v^{-1}(i-1)}(u,v)}\right)\\
&\quad\prod_{k\in D(u,v)}\left(1-q^{a_k(u,v)}\right).
\end{aligned}$$

We now consider the case $us \not\leq vs$. In this case, we claim that

$$a_j(u,v)=0. \qquad (2.19)$$

In fact, by (2.6), it can be checked that for $1\leq k\leq n$,

$$b_{1,k}(us,vs)=b_{1,k}(u,v) \quad \text{and} \quad b_{2,k}(us,vs)=b_{2,k}(u,v),$$

and

$$b_{3,j+1}(us,vs)=b_{3,j+1}(u,v)-2 \quad \text{and} \quad b_{3,k}(us,vs)=b_{3,k}(u,v), \quad \text{for } k\neq j+1.$$

Since $us \not\leq vs$, by Proposition 2.6, we see that $b_{3,j+1}(u,v)-2<0$. On the other hand, since $j+1\in v^{-1}([i])$ but $j+1\notin u^{-1}([i])$, we have $b_{3,j+1}(u,v)>0$. So we get $b_{3,j+1}(u,v)=1$. Therefore,

$$a_j(u,v)=b_{2,j}(u,v)=b_{3,j+1}(u,v)-1=0.$$

This proves the claim in (2.19).

Combining (2.15) and (2.19), we obtain that

$$\begin{aligned}
R^{J,q}_{u,v}(q)&=(q-1)R^{J,q}_{u,vs}(q)\\
&=(-1)^{\ell(v)-\ell(u)}(1-q)\left(1-q+\delta_{u^{-1}(i-1),v^{-1}(i-1)}q^{1+a_{v^{-1}(i-1)}(u,v)}\right)\prod_{k\in D(u,v)}\left(1-q^{a_k(u,v)}\right).
\end{aligned}$$

Subcase (ii): $v(j)>i$, $v(j+1)<i-1$, $u(j)<i-1$ and $u(j+1)=i$. By Theorem 1.1, we have

$$R^{J,q}_{u,v}(q)=qR^{J,q}_{us,vs}(q)+(q-1)R^{J,q}_{u,vs}(q). \qquad (2.20)$$

We need to compute $R^{J,q}_{us,vs}(q)$ and $R^{J,q}_{u,vs}(q)$. We first compute $R^{J,q}_{u,vs}(q)$. Since $B(u,vs)=B(u,v)=[i-1]$, using (2.1), we get

$$a_{j+1}(u,vs)=a_{j+1}(u,v)-1 \quad \text{and} \quad a_k(u,vs)=a_k(u,v), \quad \text{for } k\neq j+1.$$

Moreover, by (2.4) we have

$$D(u, vs) = D(u, v) \setminus \{j + 1\}.$$

By the induction hypothesis, we deduce that

$$\begin{aligned}
R^{J,q}_{u,vs}(q) &= (-1)^{\ell(vs)-\ell(u)} \left(1 - q + \delta_{u^{-1}(i-1),(vs)^{-1}(i-1)} q^{1+a_{(vs)^{-1}(i-1)}(u,vs)}\right) \\
&\quad (1-q) \prod_{k \in D(u,vs)} \left(1 - q^{a_k(u,vs)}\right) \\
&= (-1)^{\ell(v)-\ell(u)-1} \left(1 - q + \delta_{u^{-1}(i-1),v^{-1}(i-1)} q^{1+a_{v^{-1}(i-1)}(u,v)}\right) (1-q) \\
&\quad \frac{1}{1 - q^{a_{j+1}(u,v)}} \prod_{k \in D(u,v)} \left(1 - q^{a_k(u,v)}\right). \qquad (2.21)
\end{aligned}$$

To compute $R^{J,q}_{us,vs}(q)$, we consider two cases according to whether $us \leq vs$. First, we assume that $us \leq vs$. Since $B(us, vs) = B(u, v) = [i - 1]$, in view of (2.1), it is easy to check that

$$a_{j+1}(us, vs) = a_{j+1}(u, v) - 2 \quad \text{and} \quad a_k(us, vs) = a_k(u, v), \quad \text{for } k \neq j + 1.$$

Moreover, it follows from (2.4) that

$$D(us, vs) = (D(u, v) \setminus \{j + 1\}) \cup \{j\}.$$

By the induction hypothesis, we obtain that

$$\begin{aligned}
R^{J,q}_{us,vs}(q) &= (-1)^{\ell(vs)-\ell(us)} \left(1 - q + \delta_{(us)^{-1}(i-1),(vs)^{-1}(i-1)} q^{1+a_{(vs)^{-1}(i-1)}(us,vs)}\right) \\
&\quad (1-q) \prod_{k \in D(us,vs)} \left(1 - q^{a_k(us,vs)}\right) \\
&= (-1)^{\ell(v)-\ell(u)} \left(1 - q + \delta_{u^{-1}(i-1),v^{-1}(i-1)} q^{1+a_{v^{-1}(i-1)}(u,v)}\right) (1-q) \\
&\quad \frac{1 - q^{a_j(us,vs)}}{1 - q^{a_{j+1}(u,v)}} \prod_{k \in D(u,v)} \left(1 - q^{a_k(u,v)}\right). \qquad (2.22)
\end{aligned}$$

Substituting (2.21) and (2.22) into (2.20) and noticing the following relation

$$a_j(us, vs) = a_{j+1}(u, v) - 1,$$

we are led to formula (2.5).

We now consider the case $us \not\leq vs$. In this case, we claim that

$$a_{j+1}(u, v) = 1. \qquad (2.23)$$

By (2.6), it is easily seen that

$$b_{1,j+1}(us, vs) = b_{1,j+1}(u, v) - 2 \quad \text{and} \quad b_{1,k}(us, vs) = b_{1,k}(u, v), \quad \text{for } k \neq j + 1, \qquad (2.24)$$

$$b_{2,j+1}(us, vs) = b_{2,j+1}(u, v) - 2 \quad \text{and} \quad b_{2,k}(us, vs) = b_{2,k}(u, v), \quad \text{for } k \neq j + 1, \qquad (2.25)$$

$$b_{3,j+1}(us, vs) = b_{3,j+1}(u, v) - 1 \quad \text{and} \quad b_{3,k}(us, vs) = b_{3,k}(u, v), \quad \text{for } k \neq j + 1. \qquad (2.26)$$

It is clear that $a_{j+1}(u,v) = b_{2,j+1}(u,v)$. So the claim in (2.23) reduces to

$$b_{2,j+1}(u,v) = 1.$$

Since $j \notin v^{-1}([i-1])$ but $j \in u^{-1}([i-1])$, we have $b_{2,j+1}(u,v) > 0$. Suppose to the contrary that $b_{2,j+1}(u,v) > 1$. In the notation $u^{-1} = p_1p_2\cdots p_n$ and $v^{-1} = q_1q_2\cdots q_n$, we have the following two cases.

Case (a): $p_{i-1} < j$. By (2.8), we see that $q_{i-1} < j$ and

$$b_{1,j+1}(u,v) = b_{2,j+1}(u,v) > 1.$$

On the other hand, since $j \notin v^{-1}([i])$ but $j \in u^{-1}([i])$, we have $b_{3,j+1}(u,v) > 0$. Hence we conclude that $b_{h,k}(us,vs) \geq 0$ for $h = 1,2,3$ and $1 \leq k \leq n$. By Proposition 2.6, we get $us \leq vs$, contradicting the assumption $us \not\leq vs$.

Case (b): $p_{i-1} > j$. In this case, we find that if $q_{i-1} > j$, then

$$b_{1,j+1}(u,v) = b_{2,j+1}(u,v) > 1,$$

whereas if $q_{i-1} < j$, then

$$b_{1,j+1}(u,v) > b_{2,j+1}(u,v) > 1.$$

Note that in Case (a), we have shown that $b_{3,j+1}(u,v) > 0$. So, we obtain that $b_{h,k}(us,vs) \geq 0$ for $h = 1,2,3$ and $1 \leq k \leq n$. Thus we have $us \leq vs$, contradicting the assumption $us \not\leq vs$. This proves the claim in (2.23). Substituting (2.23) into (2.21), we arrive at (2.5).

Subcase (iii): $v(j) = i-1$, $v(j+1) < i-1$, $u(j) < i-1$ and $u(j+1) = i$. By Theorem 1.1, we have

$$R^{J,q}_{u,v}(q) = qR^{J,q}_{us,vs}(q) + (q-1)R^{J,q}_{u,vs}(q). \tag{2.27}$$

We need to compute $R^{J,q}_{us,vs}(q)$ and $R^{J,q}_{u,vs}(q)$. Since $B(u,v) = B(u,vs) = [i-1]$, by (2.1), we see that for $1 \leq k \leq n$,

$$a_k(u,vs) = a_k(u,v).$$

Moreover, by (2.4) we have

$$D(u,vs) = D(u,v).$$

By the induction hypothesis, we obtain that

$$\begin{aligned} R^{J,q}_{u,vs}(q) &= (-1)^{\ell(vs)-\ell(u)}(1-q)^2 \prod_{k\in D(u,vs)} \left(1-q^{a_k(u,vs)}\right) \\ &= (-1)^{\ell(v)-\ell(u)-1}(1-q)^2 \prod_{k\in D(u,v)} \left(1-q^{a_k(u,v)}\right). \end{aligned} \tag{2.28}$$

To compute $R^{J,q}_{us,vs}(q)$, we claim that $us \leq vs$. By (2.6), we see that

$$b_{1,j+1}(us,vs) = b_{1,j+1}(u,v) - 2 \ \text{ and } \ b_{1,k}(us,vs) = b_{1,k}(u,v), \ \text{ for } k \neq j+1, \tag{2.29}$$

$$b_{2,j+1}(us,vs) = b_{2,j+1}(u,v) - 1 \ \text{ and } \ b_{2,k}(us,vs) = b_{2,k}(u,v), \ \text{ for } k \neq j+1, \tag{2.30}$$

$$b_{3,k}(us,vs) = b_{3,k}(u,v), \ \text{ for } 1 \leq k \leq n. \tag{2.31}$$

Since $j+1 \in v^{-1}([i-1])$ but $j+1 \notin u^{-1}([i-1])$, we have $b_{2,j+1}(u,v) > 0$, which implies that

$$b_{2,j+1}(us,vs) = b_{2,j+1}(u,v) - 1 \geq 0. \tag{2.32}$$

Moreover, since $p_{i-1} \geq q_{i-1} = j$, we have $p_{i-1} > j$. So, we deduce that

$$b_{1,j+1}(u,v) = b_{2,j+1}(u,v) + 1 > 1,$$

and hence

$$b_{1,j+1}(us,vs) = b_{1,j+1}(u,v) - 2 \geq 0. \tag{2.33}$$

Therefore, for $h = 1,2,3$ and $1 \leq j \leq n$,

$$b_{h,j}(us,vs) \geq 0,$$

which together with Proposition 2.6 yields that $us \leq vs$. This proves the claim.

Since $B(us,vs) = B(u,v) = [i-1]$, by (2.1) and (2.4), it is easily verified that

$$a_{j+1}(us,vs) = a_{j+1}(u,v) - 1 \quad \text{and} \quad a_k(us,vs) = a_k(u,v), \quad \text{for } k \neq j+1$$

and

$$D(us,vs) = (D(u,v) \setminus \{j+1\}) \cup \{j\}.$$

By the induction hypothesis, we deduce that

$$\begin{aligned} R^{J,q}_{us,vs}(q) &= (-1)^{\ell(vs)-\ell(us)}(1-q)^2 \prod_{k \in D(us,vs)} \left(1 - q^{a_k(us,vs)}\right) \\ &= (-1)^{\ell(v)-\ell(u)}(1-q)^2 \frac{1-q^{a_j(us,vs)}}{1-q^{a_{j+1}(u,v)}} \prod_{k \in D(u,v)} \left(1 - q^{a_k(u,v)}\right). \end{aligned} \tag{2.34}$$

Since $a_j(us,vs) = a_{j+1}(u,v)$, formula (2.34) becomes

$$R^{J,q}_{us,vs}(q) = (-1)^{\ell(v)-\ell(u)}(1-q)^2 \prod_{k \in D(u,v)} \left(1 - q^{a_k(u,v)}\right) \tag{2.35}$$

Substituting (2.28) and (2.35) into (2.27), we are led to (2.5). This completes the proof. ∎

We conclude this paper by giving a conjecture for a formula of $R^{J,q}_{u,v}(q)$, where

$$J = S \setminus \{s_k, s_{k+1}, \ldots, s_i\}$$

with $1 \leq k \leq i \leq n-1$ and v is a permutation in $(S_n)^{S \setminus \{s_k, s_i\}}$. By (1.1) and (1.3), a permutation $u \in S_n$ belongs to $(S_n)^J$ if and only if the elements $1,2,\ldots,k$ as well as the elements $i+1, i+2, \ldots, n$ appear in increasing order in u. On the other hand, as we have mentioned in Introduction, $v \in (S_n)^{S \setminus \{s_k, s_i\}}$ is equivalent to the condition that $v \in (S_n)^J$ and $k+1, k+2, \ldots, i$ appear in increasing order in v. Let u, v be two permutations in $(S_n)^J$. Write $u^{-1} = p_1 p_2 \cdots p_n$ and $v^{-1} = q_1 q_2 \cdots q_n$. Let

$$A(u,v) = \{t \,|\, k+1 \leq t \leq i,\, p_t \geq q_t\}.$$

Set $B(u,v)$ to be the union of $\{1,2,\ldots,k\}$ and $A(u,v)$. Based on the set $B(u,v)$, we define $a_j(u,v)$ and $D(u,v)$ in the same way as in (2.1) and (2.4), respectively.

The following conjecture has been verified for $n \leq 8$.

Conjecture 2.7 *Let $J = S \setminus \{s_k, s_{k+1}, \ldots, s_i\}$, and v is a permutation in $(S_n)^{S \setminus \{s_k, s_i\}}$. Then, for any $u \in (S_n)^J$ with $u \leq v$, we have*

$$R^{J,q}_{u,v}(q) = (-1)^{\ell(v)-\ell(u)} \prod_{t=k+1}^{i} \left(1 - q + \delta_{u^{-1}(t),v^{-1}(t)} q^{1+a_{v^{-1}(t)}(u,v)}\right) \prod_{j \in D(u,v)} \left(1 - q^{a_j(u,v)}\right).$$

Conjecture 2.7 contains Theorems 1.3, 1.4 and 2.5 as special cases. When $i = n-1$ and $k = 1$, we have $J = \emptyset$ and $(S_n)^J = S_n$, and thus Conjecture 2.7 becomes a conjectured formula for ordinary R-polynomials $R_{u,v}(q)$, that is, for $u \in S_n$ and $v \in (S_n)^{S\backslash\{s_1,s_{n-1}\}}$ with $u \le v$,

$$R_{u,v}(q) = (-1)^{\ell(v)-\ell(u)} \prod_{t=2}^{n-1} \left(1 - q + \delta_{u^{-1}(t),v^{-1}(t)} q^{1+a_{v^{-1}(t)}(u,v)}\right) \prod_{j \in D(u,v)} \left(1 - q^{a_j(u,v)}\right). \quad (2.36)$$

It should be mentioned that Theorem 4.2 of [9] also gives a combinatorial express for (2.36) based on reduced expressions of u and v. We also remark that for $J = S\backslash\{s_1, s_{n-1}\}$, the quotient $(S_n)^J$ is the quasi-minuscule quotient of S_n, and the corresponding parabolic R-polynomials $R^{J,q}_{u,v}(q)$ have been computed by Brenti, Mongelli and Sentinelli [4, Corollary 2].

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