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On ∗-clean group rings over abelian groups

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An associative ring with unity is called clean if each of its elements is the sum of an idempotent and a unit. A clean ring with involution $*$ is called $*$-clean if each of its elements is the sum of a unit and a projection ($*$-invariant idempotent). In a recent paper, Huang, Li and Yuan provided a complete characterization that when a group ring $\mathbb{F}_qC_{p^k}$ is $*$-clean, where $\mathbb{F}_q$ is a finite field and C_{p^k} is a cyclic group of an odd prime power order p^k. They also provided a necessary condition and a few sufficient conditions for $\mathbb{F}_qC_n$ to be $*$-clean, where C_n is a cyclic group of order n. In this paper, we extend the above result of Huang, Li and Yuan from $\mathbb{F}_qC_{p^k}$ to $\mathbb{F}G$ and provide a characterization of $*$-clean group rings $\mathbb{F}G$, where G is a finite abelian group and $\mathbb{F}$ is a field with characteristic not dividing the exponent of G.

Keywords: Group ring; $*$-clean ring; Galois group; idempotent.

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1. Introduction

All rings considered here are associative rings with unity. An element of a ring is called *clean* if it is the sum of an idempotent and a unit, and the ring is called *clean*

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if each of its elements is clean. In 1977, Nicholson [8] introduced the clean rings and related them to exchange rings. After that, many results have been established about clean rings. A clean ring can be regarded as an additive analog of a unit-regular ring in which each element is a product of a unit and an idempotent. Some important examples of clean rings include local rings, semiperfect rings and left (right) Artinian rings.

A ring R is called a $*$-ring (or ring with involution $*$) if there exists an operation $*: R \to R$ such that

$$(x+y)^* = x^* + y^*, \quad (xy)^* = y^* x^* \quad \text{and} \quad (x^*)^* = x$$

for all $x, y \in R$. We call an element p of a $*$-ring R a projection if p is a self-adjoint(or $*$-invariant) idempotent, i.e. $p^* = p = p^2$, and call a $*$-ring R a $*$-clean ring if each element of R is the sum of a unit and a projection. Let R be a ring and G be a group. We denote by RG, the group ring of G over R. It is well-known that for a commutative ring R, the map $*: RG \to RG$ given by $(\sum a_g g)^* = \sum a_g g^{-1}$ is an involution which is called the classical (or standard) involution on RG.

In 2010, Vaš [10] started an investigation of $*$-clean rings and proposed a question of whether there exists a clean ring (with involution $*$) that is not $*$-clean. A year later, Li and Zhou [5] gave a positive answer to the above question. Recently, several interesting results regarding $*$-clean group rings have appeared. In [1], Gao, Chen and Li characterized the $*$-clean group rings RG for a commutative local ring R and some small groups G. Later, Li, Parmenter and Yuan [6] provided a complete characterization of when a group algebra $\mathbb{F}C_p$ is $*$-clean, where $\mathbb{F}$ is a field with char$(\mathbb{F}) \geq 0$ and C_p is the cyclic group of a prime order p, and also gave characterizations of all $*$-clean group rings $RC_n (3 \leq n \leq 6)$ over commutative local rings R. In a recent paper, Huang, Li and Yuan [3] extended the above mentioned result from $\mathbb{F}C_p$ to $\mathbb{F}_q C_{p^k}$, where $\mathbb{F}_q$ is a finite field and C_{p^k} is a cyclic group of an odd prime power order p^k. For the general case, when $G = C_n$ is a cyclic group of order n, they also provided a necessary condition and a few sufficient conditions for $\mathbb{F}_q C_n$ to be $*$-clean. Most recently, Huang, Li and Tang [2] considered the noncommutative case and investigated, when $\mathbb{Q}G$ is $*$-clean, where G are dihedral groups D_{2n} or generalized quaternion groups Q_{2n}.

In this paper, we extend the above result of Huang, Li and Yuan [3] from $\mathbb{F}_q C_{p^k}$ to $\mathbb{F}G$ and provide a characterization of $*$-clean group rings $\mathbb{F}G$, where G is a finite abelian group with exponent n_r and $\mathbb{F}$ is a field with char$(\mathbb{F}) \nmid n_r$. Note that, in this case, let ω be an n_rth primitive root of unity, if $\omega \in \mathbb{F}$, then $\mathbb{F}G$ is not $*$-clean. Let $g \in G$ and the order of g is n_r. Since char$(\mathbb{F}) \nmid n_r$, we have $u = \frac{1}{n_r} \sum_{i=0}^{n_r - 1} (\omega g)^i$ is an element in $\mathbb{F}G$. It can be easily verified that $u^2 = u$, but $u^* \neq u$. Therefore, it remains to consider $\omega \notin \mathbb{F}$.

First, when char$(\mathbb{F}) > 0$, we have the following characterization.

Theorem 1.1. *Let G be a finite abelian group with exponent n_r and $\mathbb{F}$ be a field of characteristic $p > 0$, where $p \nmid n_r$. Let ω be an n_rth primitive root of unity and*

$\omega \notin \mathbb{F}$. Then the group ring $\mathbb{F}G$ is ∗-clean if and only if there exists t such that $p^t \equiv -1 \pmod{n_r}$.

Second, when $\mathrm{char}(\mathbb{F}) = 0$, we give a complete characterization. We also provide other characterizations in certain cases.

Theorem 1.2. *Let G be a finite abelian group with exponent n_r and $\mathbb{F}$ be a field of characteristic 0. Let ω be an n_rth primitive root of unity and $\omega \notin \mathbb{F}$. Then the group ring $\mathbb{F}G$ is ∗-clean if and only if $\mathbb{F}(\omega + \omega^{-1}) \subsetneqq \mathbb{F}(\omega)$. In particular, if $\mathbb{F} \subseteq \mathbb{R}$, then $\mathbb{F}G$ is ∗-clean, where $\mathbb{R}$ denotes the field of real numbers.*

Theorem 1.3. *Let G be a finite abelian group with exponent $n_r = p^k$ or $2p^k$, where p is an odd prime and k is a positive integer, and $\mathbb{F}$ be a field of characteristic 0. Let ω be an n_rth primitive root of unity and $\omega \notin \mathbb{F}$. Then the group ring $\mathbb{F}G$ is ∗-clean if and only if $2 \mid [\mathbb{F}(\omega) : \mathbb{F}]$.*

2. Basic Notions and Terminologies

Throughout this paper, let G be an abelian group written multiplicatively. By the fundamental theorem of finite abelian groups, we have

$$G \cong C_{n_1} \times \cdots \times C_{n_r} \cong \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_r \rangle,$$

where $r = \mathsf{r}(G) \in \mathbb{N}$ is the rank of G, $n_1, \ldots, n_r \in \mathbb{N}$ are integers with $1 < n_1 \mid \ldots \mid n_r$. Moreover, $n_1, \ldots, n_r$ are uniquely determined by G, and n_r is the *exponent* of G. Let n be the order of G, then $n = n_1 \cdots n_r$.

Let $\mathbb{F}$ be a field. By the basic Galois theory, $\mathbb{F}(\omega)$ is a Galois extension of $\mathbb{F}$, where ω is an n_rth primitive root of unity. For an element $a = \sum_{g \in G} a_g g \in \mathbb{F}(\omega)G$, we define that $\sigma(a) = \sum_{g \in G} \sigma(a_g) g$, where $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$. Note that for $a, b \in \mathbb{F}(\omega)G$, we have $\sigma(ab) = \sigma(a)\sigma(b)$ for any $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$.

Let $\mathbb{K}$ be a field. Recall that [9, Definition 2.5.12], if a family $\{e_1, \ldots, e_t\}$ of idempotents in the group ring $\mathbb{K}G$ satisfying $e_i \neq 0$ for $1 \leq i \leq t$, $e_i e_j = 0$ for $1 \leq i \neq j \leq t$, and $\sum_{i=1}^{t} e_i = 1$, we call $\{e_1, \ldots, e_t\}$ a complete family of orthogonal idempotents of $\mathbb{K}G$.

3. Preliminary Lemmas

Lemma 3.1 [5, Theorem 2.2]. *A commutative ∗-ring is ∗-clean if and only if it is clean and every idempotent is a projection.*

Lemma 3.2 [4, Chapter XVIII, Theorem 1.2] (Maschke). *Let G be a finite group and K a field whose characteristic does not divide the order of G. Then the group ring $\mathbb{K}\mathbb{G}$ is semisimple.*

Lemma 3.3 [7, Theorem 3.5]. *Let $m \geq 2$ be a positive integer, then m has a primitive root if and only if $m = 2, 4, p^k$ or $2p^k$, where p is an odd prime and k is a positive integer.*

4. The Idempotents in the Group Ring

In this section, we construct the idempotents in the group ring. Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \nmid n_r$, ω be an n_rth primitive root of unity and $\omega \notin \mathbb{F}$. For $e \in \mathbb{F}G$, we denote $e\mathbb{F}G = \{ef \mid f \in \mathbb{F}G\}$.

Lemma 4.1. *Let $\{e_1, \ldots, e_n\}$ be a complete family of orthogonal idempotents of $\mathbb{F}G$. Then, we have the following decomposition*:

$$\mathbb{F}G \cong e_1\mathbb{F} \oplus e_2\mathbb{F} \oplus \cdots \oplus e_n\mathbb{F}.$$

Moreover, any idempotent $u \in \mathbb{F}G$ can be expressed in an unique way as $u = \sum_{i=1}^n r_i e_i$, with $r_i \in \{0, 1\}$ for $1 \le i \le n$.

Proof. We can view each e_i as a linear operator on $\mathbb{F}G$, which operates by multiplication. Since $\sum_{i=1}^n e_i = 1$, we have $\mathbb{F}G \subseteq e_1\mathbb{F}G + e_2\mathbb{F}G + \cdots + e_n\mathbb{F}G$, so $\mathbb{F}G = e_1\mathbb{F}G + e_2\mathbb{F}G + \cdots + e_n\mathbb{F}G$. If $x \in e_i\mathbb{F}G \cap e_j\mathbb{F}G$, by $e_i^2 = e_i$, $e_ie_j = 0$ for $i \neq j$, we have $x = 0$. That means $\mathbb{F}G \cong e_1\mathbb{F}G \oplus e_2\mathbb{F}G \oplus \cdots \oplus e_n\mathbb{F}G$. Moreover, since $\mathbb{F}G$ is a vector space over $\mathbb{F}$ of dimension n, and $e_i\mathbb{F}G$ contains $e_i\mathbb{F}$ as a vector subspace over $\mathbb{F}$ of dimension 1, we have each term in the direct sum has dimension 1. Consequently, $\mathbb{F}G \cong e_1\mathbb{F} \oplus e_2\mathbb{F} \oplus \cdots \oplus e_n\mathbb{F}$. Moreover, let u be an idempotent and $u = \sum_{i=1}^n r_ie_i$, with all $r_i \in \mathbb{F}$. Since $u^2 = (\sum_{i=1}^n r_ie_i)^2 = \sum_{i=1}^n r_i^2 e_i = \sum_{i=1}^n r_ie_i = u$, we must have $r_i \in \{0, 1\}$ for $1 \le i \le n$. □

For any $e \in \mathbb{F}(\omega)G$, let $H = \{\sigma \mid \sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}), \sigma(e) = e\}$. Then clearly H is a subgroup of $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$. Assume that $|\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})/H| = t$ and $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})/H = \{\sigma_1H, \ldots, \sigma_tH\}$, where $\{\sigma_1, \ldots, \sigma_t\}$ is a transversal to H. We define $\Gamma(e) = \sigma_1(e) + \cdots + \sigma_t(e)$.

Lemma 4.2. *$\Gamma(e)$ is well-defined and $|H|\Gamma(e) = \sum_{\sigma\in\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})} \sigma(e)$. Moreover, for any $\theta \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$, we have $\theta(\Gamma(e)) = \Gamma(e)$. Consequently, $\Gamma(e) \in \mathbb{F}G$ for any $e \in \mathbb{F}(\omega)G$.*

Proof. Let $\{\sigma_1', \ldots, \sigma_t'\}$ be another transversal to H in $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$, where $\sigma_i' \in \sigma_iH$ for $1 \le i \le t$. Since $\sigma_i' = \sigma_ih_i$, where $h_i \in H$, for $1 \le i \le t$, we have $\sigma_i'(e) = \sigma_ih_i(e) = \sigma_i(e)$. That means

$$\sigma_1'(e) + \cdots + \sigma_t'(e) = \sigma_1(e) + \cdots + \sigma_t(e)$$

therefore $\Gamma(e)$ does not depend on the choice of the transversal to H. Since $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}) = \cup_{i=1}^t \sigma_iH$, we have

$$\sum_{\sigma\in\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})} \sigma(e) = \sum_{i=1}^t \sum_{\sigma\in\sigma_iH} \sigma(e) = \sum_{i=1}^t |H|\sigma_i(e) = |H|\Gamma(e).$$

Moreover for any $\theta \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$, $\{\theta\sigma_1, \ldots, \theta\sigma_t\}$ is another transversal to H. Otherwise, if $\theta\sigma_i(\theta\sigma_j)^{-1} \in H$ for some $i \neq j$, $1 \le i, j \le t$, as $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ is abelian, so $\sigma_i\sigma_j^{-1} \in H$, which is a contradiction. Therefore for any $\theta \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$, we have

$$\theta(\Gamma(e)) = \theta\sigma_1(e) + \cdots + \theta\sigma_t(e) = \sigma_1(e) + \cdots + \sigma_t(e) = \Gamma(e)$$

by Galois theory, we have $\Gamma(e) \in \mathbb{F}G$. □

Lemma 4.3. *Let $\{e_1, \ldots, e_n\}$ be a complete family of orthogonal idempotents of $\mathbb{F}(\omega)G$. Then for $1 \le i \le n$, $\Gamma(e_i)$ is an idempotent in $\mathbb{F}G$. Let u be any idempotent in $\mathbb{F}G$, then we have $u = \sum_{j\in J}\Gamma(e_j)$ for some subset J of $\{1, \ldots, n\}$.*

Proof. We claim that $\{e_1, e_2, \ldots, e_n\} = \{\sigma(e_1), \sigma(e_2), \ldots, \sigma(e_n)\}$ for any $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$. Since $\sigma(e_i)$ is also an idempotent for $1 \le i \le n$, by Lemma 4.1 we may assume that

$$\sigma(e_1) = e_{11} + \cdots + e_{1k_1},$$

$$\vdots$$

$$\sigma(e_n) = e_{n1} + \cdots + e_{nk_n},$$

where for $1 \le i \le n$ and $1 \le j_i \le k_i$, e_{ij_i} belongs to $\{e_1, \ldots, e_n\}$. As $\{\sigma(e_1), \sigma(e_2), \ldots, \sigma(e_n)\}$ also satisfies the orthogonal relation, we must have $\{e_{i1}, \ldots, e_{ik_i}\} \cap \{e_{j1}, \ldots, e_{jk_j}\} = \emptyset$. Otherwise, it contradicts the orthogonal relation of $\{e_1, e_2, \ldots, e_n\}$. Therefore, our claim follows.

Let $L_i = \{\sigma \,|\, \sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}), \sigma(e_i) = e_i\}$, assume that for $1 \le i \le n$, $\Gamma(e_i) = e_{i1} + \cdots + e_{it_i}$, where $t_i = |\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})|/|L_i|$, for $1 \le j \le t_i$, e_{ij} belongs to $\{e_1, \ldots, e_n\}$ and $e_{il} \ne e_{ik}$ for $1 \le l \ne k \le t_i$. By the orthogonal relation of $\{e_1, e_2, \ldots, e_n\}$, $\Gamma(e_i)$ is an idempotent for $1 \le i \le n$. By Lemma 4.2, we have $\Gamma(e_i) \in \mathbb{F}G$. Let u be any idempotent in $\mathbb{F}G$, then by Lemma 4.1, we have $u = \sum_{i=1}^n r_i e_i$, where $r_i \in \{0, 1\}$ for $1 \le i \le n$. If e_i appears in the summation of u, as $u \in \mathbb{F}G$, we have $\sigma(e_i)$ also appears in the summation of u for any $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$. Consequently, $\Gamma(e_i)$ appears in the summation of u. Hence, $u = \sum_{j\in J}\Gamma(e_j)$ for some subset J of $\{1, \ldots, n\}$. □

5. An Important Lemma

In this section, let $\mathbb{F}$ be a field of characteristic 0 or $p > 0$ with $p \nmid n_r$, ω be an n_rth primitive root of unity and $\omega \notin \mathbb{F}$.

In $\mathbb{F}(\omega)G$, let

$$e_{ij_i} = \frac{1}{n_i}\sum_{k_i=0}^{n_i-1}(\omega^{\frac{n_r}{n_i}j_i}x_i)^{k_i},$$

where $1 \le i \le r$, $0 \le j_i \le n_i - 1$. It is easy to check that

$$(e_{1j_1}\cdots e_{rj_r})(e_{1j_1'}\cdots e_{rj_r'}) = \delta_{j_1,\ldots,j_r}^{j_1',\ldots,j_r'}(e_{1j_1}\cdots e_{rj_r}),$$

where

$$\delta_{j_1,\ldots,j_r}^{j'_1,\ldots,j'_r} = \begin{cases} 1 & \text{if } j_i = j'_i \text{ for all } i \text{ with } 1 \le i \le r, \\ 0 & \text{otherwise} \end{cases}$$

and $\sum_{j_i=0}^{n_i-1} e_{ij_i} = 1$ for $1 \le i \le r$. Then, we have

$$\left(\sum_{j_1=0}^{n_1-1} e_{1j_1}\right) \cdots \left(\sum_{j_r=0}^{n_r-1} e_{rj_r}\right) = 1.$$

Let $\sum_{j_1,\ldots,j_r} = \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_r=0}^{n_r-1}$. Expanding the above equation, we have

$$\sum_{j_1,\ldots,j_r} e_{1j_1} \cdots e_{rj_r} = 1.$$

Therefore, $\{e_{1j_1} \cdots e_{rj_r} | 0 \le j_i \le n_i - 1 \text{ for } 1 \le i \le r\}$ is a complete family of orthogonal idempotents of $\mathbb{F}(\omega)G$. For $0 \le j \le n_r - 1$, let

$$H_j = \{\sigma \mid \sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}), \sigma(e_{10} \cdots e_{(r-1)0} e_{rj}) = e_{10} \cdots e_{(r-1)0} e_{rj}\}.$$

Assume that $|\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})/H_j| = t_j$ and

$$\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})/H_j = \{\sigma_{j1} H_j, \ldots, \sigma_{jt_j} H_j\},$$

where $\{\sigma_{j1}, \ldots, \sigma_{jt_j}\}$ is a transversal to H_j. By Lemma 4.3,

$$u_j = \Gamma(e_{10} \cdots e_{(r-1)0} e_{rj})$$

is an idempotent in $\mathbb{F}G$. Let $U_r = \{u_j \mid 0 \le j \le n_r - 1\}$.

The following lemma is very important in our proof.

Lemma 5.1. *$\mathbb{F}G$ is $*$-clean if and only if there exists $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \omega^{-1}$.*

Proof. First, we assume that $\mathbb{F}G$ is $*$-clean. By Lemma 3.1, we have $u = u^*$ for all idempotents $u \in \mathbb{F}G$, especially $u = u^*$ for all $u \in U_r$, where U_r is defined above. Let $u_j \in U_r$, where $0 \le j \le n_r - 1$, we have

$$\begin{aligned} u_j^* &= (\Gamma(e_{10} \cdots e_{(r-1)0} e_{rj}))^* \\ &= \Gamma\left(\frac{1}{n} \sum_{k_1=0}^{n_1-1} (x_1^{-1})^{k_1} \cdots \sum_{k_{r-1}=0}^{n_{r-1}-1} (x_{r-1}^{-1})^{k_{r-1}} \sum_{k_r=0}^{n_r-1} (\omega^j x_r^{-1})^{k_r}\right) \\ &= \Gamma\left(\frac{1}{n} \sum_{k_1=0}^{n_1-1} (x_1)^{k_1} \cdots \sum_{k_{r-1}=0}^{n_{r-1}-1} (x_{r-1})^{k_{r-1}} \sum_{k_r=0}^{n_r-1} (\omega^{j(n_r-1)} x_r)^{k_r}\right). \end{aligned}$$

Then the coefficient of x_r in u_j^* is

$$\frac{1}{n} \sum_{i=1}^{t_j} \sigma_{ji}(\omega^{j(n_r-1)}) = \frac{1}{n} \sum_{i=1}^{t_j} \sigma_{ji}((\omega^j)^{n_r-1}).$$

Since $u_j = u_j^*$, comparing the coefficients of x_r, we have for $0 \le j \le n_r - 1$,

$$\sum_{i=1}^{t_j} \sigma_{ji}((\omega^j)^{n_r-1}) = \sum_{i=1}^{t_j} \sigma_{ji}(\omega^j). \tag{5.1}$$

Clearly,

$$|H_j| \sum_{i=1}^{t_j} \sigma_{ji}((\omega^j)^{n_r-1}) = |H_j| \sum_{i=1}^{t_j} \sigma_{ji}(\omega^j). \tag{5.2}$$

By the proof of Lemma 4.2, we have

$$\sum_{\sigma\in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})} \sigma(\omega^j)^{n_r-1} = \sum_{\sigma\in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})} \sigma(\omega^j). \tag{5.3}$$

Assume that $|\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})| = [\mathbb{F}(\omega) : \mathbb{F}] = d$. Let $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}) = \{\sigma_k | 1 \le k \le d\}$ and $\sigma_k(\omega) = \omega^{l_k}$ with $1 \le l_k \le n_r - 1$ for $1 \le k \le d$. We assume that $\sigma_1 = id$, $l_1 = 1$. Then for $0 \le j \le n_r - 1$,

$$\sum_{\sigma\in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})} \sigma(\omega^j)^{n_r-1} = \sum_{k=1}^{d} \sigma_k(\omega^j)^{n_r-1} = \sum_{k=1}^{d} \omega^{jl_k(n_r-1)} = \sum_{k=1}^{d} (\omega^j)^{n_r-l_k}$$

and

$$\sum_{\sigma\in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})} \sigma(\omega^j) = \sum_{k=1}^{d} \sigma_k(\omega^j) = \sum_{k=1}^{d} (\omega^j)^{l_k}.$$

Therefore, we can change (5.3) to be the following one

$$\sum_{k=1}^{d} (\omega^j)^{n_r-l_k} = \sum_{k=1}^{d} (\omega^j)^{l_k}. \tag{5.4}$$

Let

$$g(x) = \sum_{k=1}^{d} x^{n_r-l_k} - \sum_{k=1}^{d} x^{l_k}$$

and $A = \{n_r - l_1, n_r - l_2, \ldots, n_r - l_d\}$, $B = \{l_1, l_2, \ldots, l_d\}$. Then $g(x)$ is a trivial polynomial. Otherwise, from (5.4), we obtain that $g(x) = 0$, for $x = \omega^j$ for $0 \le j \le n_r - 1$, that means $g(x)$ has at least n_r distinct roots, which contradicts the fact that $g(x)$ is a polynomial over field $\mathbb{F}$ of degree $n_r - 1$. Since $g(x)$ is a trivial polynomial, we must have $A = B$, so $n_r - l_1 = n_r - 1 \in B$, that means $n_r - 1 = l_v$ for some $1 \le v \le d$. Consequently, $\sigma_v(\omega) = \omega^{-1}$, and there exists $\sigma = \sigma_v \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \sigma_v(\omega) = \omega^{l_v} = \omega^{n_r-1} = \omega^{-1}$.

Conversely, we assume that there exists $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \omega^{-1}$. By Lemma 3.2, $\mathbb{F}G$ is semisimple, so it is clean. By Lemma 3.1, all we need to prove is that every idempotent is a projection. Let $\{\Gamma(\beta_1), \ldots, \Gamma(\beta_l)\}$ be the set of all distinct elements in $\{\Gamma(e_{1j_1} \cdots e_{rj_r}) \,|\, 0 \le j_i \le n_i - 1 \text{ for } 1 \le i \le r\}$. Let u be any idempotent in $\mathbb{F}G$, by Lemma 4.3, $u = \sum_{j\in J}\Gamma(\beta_j)$, for some subset J of $\{1,\ldots,l\}$. It suffices to prove that $\Gamma(\beta_j) = \Gamma(\beta_j)^*$ for $1 \le j \le l$.

Assume that $\beta_j = e_{1j_1}\cdots e_{rj_r}$ for some $0 \le j_i \le n_i - 1$ and $1 \le i \le r$. Let $K_j = \{\sigma \mid \sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}), \sigma(\beta_j) = \beta_j\}$. Also assume that $|\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})/K_j| = t_j$ and $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})/K_j = \{\theta_1 K_j,\ldots,\theta_{t_j}K_j\}$, where $\{\theta_1,\ldots,\theta_{t_j}\}$ is a transversal to K_j. Note that $\mathbb{F} \subseteq \mathbb{F}(\omega)$ is an abelian extension, so $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ is an abelian group. Since there exists $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \omega^{-1}$, by Lemma 4.2, we have

$$\begin{aligned}
\Gamma(\beta_j) &= \Gamma(e_{1j_1}\cdots e_{rj_r}) = \sum_{i=1}^{t_j}(\theta_i(e_{1j_1}\cdots e_{rj_r})) = \sigma\sum_{i=1}^{t_j}(\theta_i(e_{1j_1}\cdots e_{rj_r}))\\
&= \frac{1}{n}\sum_{i=1}^{t_j}\left(\sum_{k_1=0}^{n_1-1}(\sigma\theta_i(\omega^{\frac{n_r}{n_1}j_1})x_1)^{k_1}\cdots\sum_{k_r=0}^{n_r-1}(\sigma\theta_i(\omega^{\frac{n_r}{n_r}j_r})x_r)^{k_r}\right)\\
&= \frac{1}{n}\sum_{i=1}^{t_j}\left(\sum_{k_1=0}^{n_1-1}(\theta_i\sigma(\omega^{\frac{n_r}{n_1}j_1})x_1)^{k_1}\cdots\sum_{k_r=0}^{n_r-1}(\theta_i\sigma(\omega^{\frac{n_r}{n_r}j_r})x_r)^{k_r}\right)\\
&= \frac{1}{n}\sum_{i=1}^{t_j}\left(\sum_{k_1=0}^{n_1-1}(\theta_i(\omega^{\frac{n_r}{n_1}(n_1-j_1)})x_1)^{k_1}\cdots\sum_{k_r=0}^{n_r-1}(\theta_i(\omega^{\frac{n_r}{n_r}(n_r-j_r)})x_r)^{k_r}\right)\\
&= \frac{1}{n}\sum_{i=1}^{t_j}\left(\sum_{k_1=0}^{n_1-1}(\theta_i(\omega^{\frac{n_r}{n_1}j_1n_1-\frac{n_r}{n_1}j_1})x_1)^{k_1}\cdots\sum_{k_r=0}^{n_r-1}(\theta_i(\omega^{\frac{n_r}{n_r}j_rn_r-\frac{n_r}{n_r}j_r)})x_r)^{k_r}\right)\\
&= \frac{1}{n}\sum_{i=1}^{t_j}\left(\sum_{k_1=0}^{n_1-1}(\theta_i(\omega^{\frac{n_r}{n_1}j_1(n_1-1)})x_1)^{k_1}\cdots\sum_{k_r=0}^{n_r-1}(\theta_i(\omega^{\frac{n_r}{n_r}j_r(n_r-1)})x_r)^{k_r}\right)\\
&= \frac{1}{n}\sum_{i=1}^{t_j}\left(\sum_{k_1=0}^{n_1-1}(\theta_i(\omega^{\frac{n_r}{n_1}j_1})x_1^{-1})^{k_1}\cdots\sum_{k_r=0}^{n_r-1}(\theta_i(\omega^{\frac{n_r}{n_r}j_r})x_r^{-1})^{k_r}\right)\\
&= (\Gamma(e_{1j_1}\cdots e_{rj_r}))^* = (\Gamma(\beta_j))^*.
\end{aligned}$$

Hence $\mathbb{F}G$ is $*$-clean. This completes the proof. □

6. Char($\mathbb{F}$) $= p > 0$

Now, we are going to prove our main results. In this section, let $\mathbb{F}$ be a field of characteristic $p > 0$ with $p \nmid n_r$, ω be an n_rth primitive root of unity and $\omega \notin \mathbb{F}$. Let $\mathbb{F}_p$ denote the finite field of p elements. First, we have the following important lemma.

Lemma 6.1. *The Galois group* $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ *is isomorphic to a subgroup of* $\mathrm{Gal}(\mathbb{F}_p(\omega)/\mathbb{F}_p)$.

Proof. If $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$, since σ fixes $\mathbb{F}$, then clearly the restriction $\sigma|_{\mathbb{F}_p(\omega)}$ fixes $\mathbb{F}_p$. Since $\mathbb{F}_p \subseteq \mathbb{F}_p(\omega)$ is a normal extension, we have $\sigma|_{\mathbb{F}_p(\omega)} \in \mathrm{Aut}(\mathbb{F}_p(\omega))$. This together with the fact that $\sigma|_{\mathbb{F}_p(\omega)}$ fixes $\mathbb{F}_p$ implies that $\sigma|_{\mathbb{F}_p(\omega)} \in \mathrm{Gal}(\mathbb{F}_p(\omega)/\mathbb{F}_p)$. Therefore, we have a homomorphism f from $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ to $\mathrm{Gal}(\mathbb{F}_p(\omega)/\mathbb{F}_p)$. Moreover, if $\sigma|_{\mathbb{F}_p(\omega)}$ is the identity in $\mathrm{Gal}(\mathbb{F}_p(\omega)/\mathbb{F}_p)$ which fixes ω, then σ fixes ω too. Therefore, f is a monomorphism and $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ is isomorphic to a subgroup of $\mathrm{Gal}(\mathbb{F}_p(\omega)/\mathbb{F}_p)$. □

Let $q = p^m$, where m is an arbitrary positive integer, and $\mathbb{F}_q$ be the finite field with q elements. It is known that the group of automorphisms of $\mathbb{F}_q$ is generated by the Frobenius map

$$\Phi : \mathbb{F}_q \to \mathbb{F}_q \quad \text{and} \quad \Phi(x) = x^p.$$

From Lemma 6.1, we know that for any $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$, σ is also generated by the Frobenius map Φ.

Proof of Theorem 1.1. First, we assume that there exists t such that $p^t \equiv -1 \pmod{n_r}$. Since the Galois group $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ is generated by the Frobenius map, there exists $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \omega^{p^t}$, which means $\sigma(\omega) = \omega^{-1}$. By Lemma 5.1, $\mathbb{F}G$ is $*$-clean.

Conversely, assume that $\mathbb{F}G$ is $*$-clean. Then by Lemma 5.1, there exists $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \omega^{-1}$, since σ is generated by the Frobenius map, we have $\sigma(\omega) = \omega^{p^t}$, which means there exists t such that $p^t \equiv -1 \pmod{n_r}$. This completes the proof. □

In [3], Huang, Li and Yuan proved the following theorem.

Theorem 6.2. *Let $\mathbb{F}_q$ be a finite field of order q, $C_n = \langle g \rangle$ be a cyclic group of order $n \geq 3$, and $\gcd(q, n) = 1$. If there exists a positive integer v, such that $q^v \equiv -1 \pmod{m}$ for every positive divisor m of n, then $\mathbb{F}_q C_n$ is $*$-clean.*

Now, we can deduce Theorem 6.2 as a corollary from Theorem 1.1. Since $\mathbb{F}_q$ is in characteristic p for some prime $p \nmid n$, and $q = p^k$ for some positive integer k, if $q^v \equiv -1 \pmod{n}$, then we have $p^{kv} \equiv -1 \pmod{n}$. Moreover $\omega \notin \mathbb{F}_q$, where ω be an nth primitive root of unity. Therefore, there exists $t = kv$ such that $p^t \equiv -1 \pmod{n}$, by Theorem 1.1, $\mathbb{F}_q C_n$ is $*$-clean.

7. Char($\mathbb{F}$) = 0

In this section, let $\mathbb{F}$ be a field of characteristic 0, ω be an n_rth primitive root of unity and $\omega \notin \mathbb{F}$. Let $\mathbb{Q}$ denote the field of rational numbers. We also have the following important lemma.

Lemma 7.1. *The Galois group* $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ *is isomorphic to a subgroup of* $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$.

Proof. If $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$, since σ fixes $\mathbb{F}$, then clearly the restriction $\sigma|_{\mathbb{Q}(\omega)}$ fixes $\mathbb{Q}$. Since $\mathbb{Q} \subseteq \mathbb{Q}(\omega)$ is a normal extension, we have $\sigma|_{\mathbb{Q}(\omega)} \in \mathrm{Aut}(\mathbb{Q}(\omega))$. This together with the fact that $\sigma|_{\mathbb{Q}(\omega)}$ fixes $\mathbb{Q}$ implies that $\sigma|_{\mathbb{Q}(\omega)} \in \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$. Therefore, we have a homomorphism f from $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ to $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$. Moreover, if $\sigma|_{\mathbb{Q}(\omega)}$ is the identity in $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ which fixes ω, then σ fixes ω too. Therefore, f is a monomorphism and $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ is isomorphic to a subgroup of $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$. □

Lemma 7.2. *There exists* $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$, *such that* $\sigma(\omega) = \omega^{-1}$ *if and only if* $\mathbb{F}(\omega + \omega^{-1}) \subsetneqq \mathbb{F}(\omega)$.

Proof. First, we assume that $\mathbb{F}(\omega + \omega^{-1}) \subsetneqq \mathbb{F}(\omega)$. Since ω satisfies the equation $x^2 - (\omega + \omega^{-1})x + 1 = 0$ in $\mathbb{F}(\omega + \omega^{-1})[x]$, we have $[\mathbb{F}(\omega) : \mathbb{F}(\omega + \omega^{-1})] = 2$. If $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}(\omega + \omega^{-1}))$ and $\sigma(\omega) = \omega^k$, then $\sigma(\omega + \omega^{-1}) = \sigma(\omega) + \sigma(\omega)^{-1} = \omega^k + \omega^{-k} = \omega + \omega^{-1}$. That is $\cos\frac{2\pi}{n_r} = \cos\frac{2\pi k}{n_r}$, so

$$\cos\frac{2\pi}{n_r} - \cos\frac{2\pi k}{n_r} = -2\sin\frac{2\pi(k+1)}{n_r}\sin\frac{2\pi(k-1)}{n_r} = 0.$$

It follows that $k = -1 \pmod{n_r}$. So there exists $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}(\omega + \omega^{-1})) \subseteq \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \omega^{-1}$.

Conversely, we assume that $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \omega^{-1}$. Then the subgroup $\langle\sigma\rangle$ of $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ generated by σ fixes $\mathbb{F}(\omega + \omega^{-1})$. Since $\langle\sigma\rangle$ is not a trivial subgroup, by the fundamental theorem of Galois, we have $\mathbb{F}(\omega+\omega^{-1}) \subsetneqq \mathbb{F}(\omega)$. □

Proof of Theorem 1.2. By Lemmas 5.1 and 7.2, $\mathbb{F}G$ is $*$-clean if and only if $\mathbb{F}(\omega+\omega^{-1}) \subsetneqq \mathbb{F}(\omega)$. In particular, if $\mathbb{F} \subseteq \mathbb{R}$, then we clearly have $\mathbb{F}(\omega+\omega^{-1}) \subsetneqq \mathbb{F}(\omega)$. Therefore, we get the desired result. □

Proof of Theorem 1.3. Assume that $2 \mid [\mathbb{F}(\omega) : \mathbb{F}]$. By Lemma 7.1, $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ is isomorphic to a subgroup of $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$. Since $2 \mid [\mathbb{F}(\omega) : \mathbb{F}]$, there exists $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F}) < \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ with $|\sigma| = 2$. Let $\sigma(\omega) = \omega^m$ for some m with $1 < m \leq n_r - 1$. Moreover by Lemma 3.3, when $n_r = p^k, 2p^k$, where p is an odd prime and k is a positive integer, n_r has a primitive root, that means if a has order 2 modulo n_r, then $a \equiv -1 \pmod{n_r}$. Since $\sigma^2 = 1$, we have $\omega = \sigma(\omega)^2 = \omega^{m^2}$. So m has order 2 modulo n_r and $m \equiv -1 \pmod{n_r}$. Therefore σ satisfies $\sigma(\omega) = \omega^{-1}$. By Lemma 5.1, $\mathbb{F}G$ is $*$-clean.

Conversely, assume that $\mathbb{F}G$ is $*$-clean. By Lemma 5.1, there exists $\sigma \in \mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$ such that $\sigma(\omega) = \omega^{-1}$. Therefore σ has order 2 in $\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})$. Hence $2 \mid |\mathrm{Gal}(\mathbb{F}(\omega)/\mathbb{F})| = [\mathbb{F}(\omega) : \mathbb{F}]$. □

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