FAMILIES OF MULTISUMS AS MOCK THETA FUNCTIONS

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Abstract. In view of the Bailey lemma and the relations between Hecke-type sums and Appell-Lerch sums given by Hickerson and Mortenson, we find that many Bailey pairs given by Slater can be used to deduce mock theta functions. Therefore, by constructing generalized Bailey pairs with more parameters, we derive some new families of mock theta functions. Meanwhile, some identities between new mock theta functions and classical ones are established. Furthermore, based on the proofs of the main theorems, many q-hypergeometric transformations are obtained.

1. Introduction

Mock theta functions were first introduced by Ramanujan [19, pp.354-355] in his last letter to G. H. Hardy. Ramanujan listed 17 mock theta functions which were assigned orders 3, 5, and 7. Since then, constructing new mock theta functions has received a great deal of attention. See, for example, [2,5,9,17]. Around the year 2000, connections between mock theta functions and Maass forms brought a new insight for the study of mock theta functions. Based on the work of Zagier [23], Zwegers [25], and Bringmann and Ono [7,8], we know that each of Ramanujan's mock theta functions is the holomorphic part of a weight 1/2 harmonic weak Maass form with a weight 3/2 unary theta function as its shadow. More importantly, Zagier [24] and Zwegers [25] showed that specializations of Appell-Lerch sums give mock theta functions. For these developments, see Ono's memoir [18]. Recently, Hickerson and Mortenson [11] built some relations between Hecke-type sums and Appell-Lerch sums. By means of these relations and the Bailey lemma, Lovejoy and Osburn [13,14,16] and Lovejoy [12] found some new families of mock theta functions. The motivation for this paper is an observation that many Bailey pairs given by Slater [20,21] can be used to construct new mock theta functions. Based on the forms of these Bailey pairs, we derive more families of q-hypergeometric multisums as mock theta functions by establishing generalized Bailey pairs with more parameters.

Here we follow the standard q-series notations

$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a_1,a_2,\ldots,a_m;q)_n := \prod_{j=1}^{m}(a_j;q)_n,$$

where $|q|<1$ and $n \in \mathbb{N}\cup\{\infty\}$.

Date: April 6, 2016.

2010 *Mathematics Subject Classification.* 33D15; 11F03.

Key words and phrases. q-series, Bailey pairs, mock theta functions, Hecke-type sums, Appell-Lerch sums.

Appell-Lerch sums which were first studied by Appell [6] are stated as follows:

$$m(x,q,z):=\frac{1}{j(z;q)}\sum_{r\in\mathbb{Z}}\frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz},$$

where $x,z\in\mathbb{C}^*:=\mathbb{C}/\{0\}$ with neither z nor xz an integral power of q. Here

$$j(z;q):=(z,q/z,q;q)_\infty \quad \text{and} \quad j(z_1,z_2,\cdots,z_m;q):=\prod_{i=1}^{m} j(z_i;q).$$

Notice that

$$j(zq^n;q)=(-1)^n z^{-n} q^{-\binom{n}{2}} j(z;q), \quad n\in\mathbb{Z}. \tag{1.1}$$

For convenience, we also define that for $a\in\mathbb{Z}$ and $m\in\mathbb{N}$,

$$J_{a,m}:=j(q^a;q^m), \quad \overline{J}_{a,m}:=j(-q^a;q^m), \quad \textit{and} \quad J_m:=J_{m,3m}=(q^m;q^m)_\infty.$$

Hecke-type sums which were given by Hecke [10] are defined as

$$f_{a,b,c}(x,y,q):=\sum_{sg(n)=sg(j)} sg(n)(-1)^{n+j}x^n y^j q^{a\binom{n}{2}+bnj+c\binom{j}{2}},$$

where $x,y\in\mathbb{C}^*:=\mathbb{C}/\{0\}$, $sg(n):=1$ for $n\geq 0$, and $sg(n):=-1$ for $n<0$.

The Bailey lemma plays a very important role in the study of mock theta functions. A Bailey pair relative to (a,q) is a pair of sequences $(\alpha_n,\beta_n)_{n\geq 0}$ satisfying

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}. \tag{1.2}$$

In fact, α_n and β_n form a Bailey pair if and only if for all $n\geq 0$ [1, Lemma 3]

$$\alpha_n=(1-aq^{2n})\sum_{j=0}^{n}\frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q)_{n-j}}\beta_j,$$

which can be rewritten as

$$\alpha_n=\frac{(-1)^n q^{\binom{n}{2}}(1-aq^{2n})(aq)_{n-1}}{(q)_n}\sum_{j=0}^{n}(q^{-n})_j(aq^n)_j q^j\beta_j. \tag{1.3}$$

The Bailey lemma says that if (α_n,β_n) is a Bailey pair relative to (a,q), then so is $(\alpha_n^{'},\beta_n^{'})$, where

$$\alpha_n^{'}=\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n \tag{1.4}$$

and

$$\beta_n^{'}=\frac{1}{(aq/\rho_1)_n(aq/\rho_2)_n}\sum_{j=0}^{n}\frac{(\rho_1)_j(\rho_2)_j(aq/\rho_1\rho_2)_{n-j}(aq/\rho_1\rho_2)^j}{(q)_{n-j}}\beta_j \tag{1.5}$$

for $\rho_1,\rho_2\in\mathbb{C}^*$. Substituting (1.4) and (1.5) into (1.2) and letting $n\to\infty$, we have

$$\sum_{n\geq 0}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n=\frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty}\sum_{n\geq 0}\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n. \tag{1.6}$$

Especially, setting $a=1$ and $\rho_1,\rho_2\to\infty$ in (1.6) yields that

$$\sum_{n\geq 0} q^{n^2}\beta_n=\frac{1}{(q)_\infty}\sum_{n\geq 0} q^{n^2}\alpha_n. \tag{1.7}$$

Setting $a=q$, $\rho_1=-q$, and $\rho_2\to\infty$ in (1.6), we get

$$\sum_{n\geq 0}\frac{(-q)_n q^{\binom{n+1}{2}}}{1-q}\beta_n=\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0} q^{\binom{n+1}{2}}\alpha_n. \tag{1.8}$$

Setting $a=1$, $\rho_1=\sqrt{q}$, and $\rho_2=-\sqrt{q}$ in (1.6), we have

$$2\sum_{n\geq 0}(-1)^n(q;q^2)_n\beta_n=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}(-1)^n\alpha_n. \tag{1.9}$$

For more on Bailey pairs and the Bailey lemma, see [3, 4, 22].

In this paper, according to the Bailey pairs given by Slater [20, 21], we construct some generalized Bailey pairs by employing (1.3). Then, with the aid of (1.7)-(1.9) and the relations between Hecke-type sums and Appell-Lerch sums given by Hickerson and Mortenson [11], we deduce some new families of mock theta functions. Following [11], we use "generic" to mean that the parameters do not cause poles in Appell-Lerch sums or in quotients of theta functions. Here we recall two theta functions $\theta_{n,p}(x,y,q)$ and $\theta_{a,b,c}(x,y,q)$ in [11]:

$$\theta_{n,p}(x,y,q):=\sum_{r^*=0}^{p-1}\sum_{s^*=0}^{p-1} q^{n\binom{r-(n-1)/2}{2}+(n+p)(r-(n-1)/2)(s+(n+1)/2)+n\binom{s+(n+1)/2}{2}}(-x)^{r-(n-1)/2}$$
$$\times\frac{(-y)^{s+(n+1)/2}J^3_{p^2(2n+p)}j(-q^{np(s-r)}x^n/y^n;q^{np^2})j(q^{p(2n+p)(r+s)+p(n+p)}x^py^p;q^{p^2(2n+p)})}{j(q^{p(2n+p)r+p(n+p)/2}(-y)^{n+p}/(-x)^n,q^{p(2n+p)s+p(n+p)/2}(-x)^{n+p}/(-y)^n;q^{p^2(2n+p)})},$$

where $r:=r^*+\{(n-1)/2\}$ and $s:=s^*+\{(n-1)/2\}$, with $0\leq\{\alpha\}<1$ denoting the fractional part of α, and x and y are generic.

$$\theta_{a,b,c}(x,y,q):=\sum_{d=0}^{b/c-1}\sum_{e=0}^{b/a-1}\sum_{f=0}^{b/a-1} q^{(b^2/a-c)\binom{d+1}{2}+(b^2/c-a)\binom{e+f+1}{2}+a\binom{f}{2}}j(q^{(b^2/a-c)(d+1)+bf}y;q^{b^2/a})$$
$$\times(-x)^f j(q^{b(b^2/(ac)-1)(e+f+1)-(b^2/a-c)(d+1)+b^3(b-a)/(2a^2c)}(-x)^{b/a}y^{-1};q^{(b^2/a)(b^2/(ac)-1)})$$
$$\times\frac{J^3_{b(b^2/(ac)-1)}j(q^{(b^2/c-a)(e+1)+(b^2/a-c)(d+1)-c\binom{b/c}{2}-a\binom{b/a}{2}}(-x)^{1-b/a}(-y)^{1-b/c};q^{b(b^2/(ac)-1)})}{j(q^{(b^2/c-a)(e+1)-c\binom{b/c}{2}}(-x)(-y)^{-b/c},q^{(b^2/a-c)(d+1)-a\binom{b/a}{2}}(-x)^{-b/a}(-y);q^{b(b^2/(ac)-1)})},$$

where x and y are generic.

The main theorems of this paper are stated as follows.

Theorem 1.1. *For $k_1,k_2,m\in\mathbb{Q}$, (α_n,β_n) is a Bailey pair relative to (a,q^m) with*

$$\alpha_n=\begin{cases}(-1)^n(q^{k_1n^2-k_2n}+q^{k_1n^2+k_2n}), & n>0,\\ 1, & n=0,\end{cases}$$

and $(\alpha_n^{'},\beta_n^{'})$ is obtained by letting $q\to q^t$ $(t\in\mathbb{Q})$ in (α_n,β_n). Then for $l,k_2t\in\mathbb{Z}$, the following are mock theta functions:

$$\begin{aligned}W_1(l,t;q):=\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}&\frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j}\\&\times q^{n^2+\frac{r(r+3)}{2}+3(2l+1)rj+(\frac{3}{2}-k_1t)j^2+(3l+\frac{3}{2})j+mt\binom{j}{2}+mts}\beta_s^{'},\end{aligned}\tag{1.10}$$

where $l\geq 1$, $k_2t\equiv 1,2\pmod 3$, $k_2t\neq\frac{12kl(l+1)}{2l+1}$ for any $k\in\mathbb{Z}$, and l and t are generic for $\theta_{1,2l}(q^3,q^{3l+3-k_2t},q^3)$.

$$\begin{aligned}W_2(l,t;q):=\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}&\frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j}\\&\times q^{n^2+\frac{r(r+3)}{2}+(2l+1)rj+(\frac{3}{2}-k_1t)j^2+(l+\frac{1}{2})j+mt\binom{j}{2}+mts}\beta_s^{'},\end{aligned}\tag{1.11}$$

where $l\equiv 0,2\pmod 3$, $l\geq 2$, $k_2t\neq(2-4k)(l-1)(l+2)$, $k_2t\neq\frac{(12k+4\varepsilon-4)(l-1)(l+2)}{2l+1}$ $(\varepsilon=0,1,2)$ for any $k\in\mathbb{Z}$, and l and t are generic for $\theta_{3,2l-2}(q^3,q^{l+2-k_2t},q)$.

$$\begin{aligned}W_3(l,t;q):=\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}&\frac{(-1)^{r+j}(1-aq^{2mtj})(-q)_n(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r+1}(q^{mt};q^{mt})_j}\\&\times q^{\binom{n+1}{2}+\binom{r+1}{2}+2(2l+k_2t)rj+(1-k_1t)j^2+(2l+k_2t)j+mt\binom{j}{2}+mts}\beta_s^{'},\end{aligned}\tag{1.12}$$

where $l>\frac{1-k_2t}{2}$, and l and t are generic for $\theta_{2,4l+2k_2t,2}(q^2,q^{2l+1},q)$.

Theorem 1.2. *For $k_1,k_2,m\in\mathbb{Q}$, (α_n,β_n) is a Bailey pair relative to (a,q^m) with*

$$\alpha_n=\begin{cases}q^{k_1n^2-k_2n}-q^{k_1n^2+k_2n}, & n>0,\\ 1, & n=0,\end{cases}$$

and $(\alpha_n^{'},\beta_n^{'})$ is obtained by letting $q\to q^t$ $(t\in\mathbb{Q})$ in (α_n,β_n). Then for $l,2k_2t\in\mathbb{Z}$, the following are mock theta functions:

$$\begin{aligned}W_4(l,t;q):=2\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}&\frac{(-1)^{n+r+j}(1-aq^{2mtj})(q;q^2)_n(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j}\\&\times q^{\binom{r+1}{2}+(2l-1+2k_2t)rj+(\frac{1}{2}-k_1t)j^2+(l-\frac{1}{2}+k_2t)j+mt\binom{j}{2}+mts}\beta_s^{'},\end{aligned}$$

where $l>1-k_2t$, $2k_2t\neq\frac{-1\pm\sqrt{1+8(2k-1)(k+l-1)}}{4k-2}-2l+1$, $2k_2t\neq\frac{(1-4k)(2l-1)\pm\sqrt{(2l-1)^2+8k(2k-1)}}{4k-2}$ for any $k\in\mathbb{Z}$, and l and t are generic for $\theta_{1,2l-2+2k_2t}(-q,-q^l,q)$.

Theorem 1.3. *For $k_1,k_2,m\in\mathbb{Q}$, (α_n,β_n) is a Bailey pair relative to (a,q^m) with*

$$\alpha_n=\begin{cases}(-1)^r(q^{k_1r^2-k_2r}+q^{k_1r^2+k_2r}), & n=2r,\\ 0, & n \text{ is odd},\\ 1, & n=0,\end{cases}$$

and $(\alpha_n^{'}, \beta_n^{'})$ *is obtained by letting* $q \to q^t$ $(t \in \mathbb{Q})$ *in* (α_n, β_n)*. Then for* $l, k_2 t \in \mathbb{Z}$*, the following are mock theta functions:*

$$W_5(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j} \times q^{n^2+\frac{r(r+3)}{2}+(3l+\frac{3}{2})rj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{3}{2}l+\frac{3}{4})j+mt\binom{j}{2}+mts}\beta_s^{'},$$

where $l \geq 1$*,* $k_2 t \equiv 1, 2 \pmod 3$*,* $k_2 t \neq \frac{12kl(l+1)}{2l+1}$ *for any* $k \in \mathbb{Z}$*, and* l *and* t *are generic for* $\theta_{1,2l}(q^3, q^{3l+3-k_2t}, q^3)$*.*

$$W_6(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j} \times q^{n^2+\frac{r(r+3)}{2}+(l+\frac{1}{2})rj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{1}{2}l+\frac{1}{4})j+mt\binom{j}{2}+mts}\beta_s^{'},$$

where $l \equiv 0, 2 \pmod 3$*,* $l \geq 2$*,* $k_2 t \neq (2-4k)(l-1)(l+2)$*,* $k_2 t \neq \frac{(12k+4\varepsilon-4)(l-1)(l+2)}{2l+1}$ $(\varepsilon = 0, 1, 2)$ *for any* $k \in \mathbb{Z}$*, and* l *and* t *are generic for* $\theta_{3,2l-2}(q^3, q^{l+2-k_2t}, q)$*.*

$$W_7(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(-q)_n(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r+1}(q^{mt};q^{mt})_j} \times q^{\binom{n+1}{2}+\binom{r+1}{2}+(2l+k_2t)rj+(\frac{1}{4}-\frac{1}{4}k_1t)j^2+(l+\frac{1}{2}k_2t)j+mt\binom{j}{2}+mts}\beta_s^{'},$$

where $l > \frac{1-k_2t}{2}$*, and* l *and* t *are generic for* $\theta_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q)$*.*

Furthermore, we find some identities between the new mock theta functions and the classical ones. Recall that a "2nd order" mock theta function and two "8th order" mock theta functions are defined as

$$\mu(q) = \sum_{n\geq 0} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2},$$

$$S_1(q) = \sum_{n\geq 0} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n},$$

and

$$T_0(q) = \sum_{n\geq 0} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}.$$

Corollary 1.4. *We have*

$$\omega_1(1,4;q) = -\frac{1}{4}q^{-1}\mu(q^6) + M_1(q), \tag{1.13}$$

$$\omega_1(1,2;q) = \frac{1}{2}qS_1(q^3) + M_2(q), \tag{1.14}$$

and

$$\omega_4(1,-1;q) = -4T_0(q) + M_3(q), \tag{1.15}$$

where the mock theta functions $\omega_1(1,4;q)$*,* $\omega_1(1,2;q)$*, and* $\omega_4(1,-1;q)$ *are defined as* (3.19)*,* (3.8)*, and* (3.21)*. In addition,* $M_1(q)$*,* $M_2(q)$*, and* $M_3(q)$ *are (explicit) weakly holomorphic modular forms.*

This paper is organized as follows. In Section 2, we prove Theorems 1.1-1.3. In Section 3, for the Bailey pairs occurring in Slater's list [20, 21], we apply Theorems 1.1-1.3 to derive some examples. Meanwhile, we give a proof of Corollary 1.4. In Section 4, in view of the proofs of Theorems 1.1-1.3, we deduce some q-hypergeometric transformations.

2. Proofs of Theorems 1.1-1.3

First, we recall some required facts from [11]. Define

$$\begin{aligned} &g_{a,b,c}(x,y,q,z_1,z_0)\\ &:=\sum_{t=0}^{a-1}(-y)^t q^{c\binom{t}{2}} j(q^{bt}x;q^a) m\left(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-t(b^2-ac)}\frac{(-y)^a}{(-x)^b}, q^{a(b^2-ac)}, z_0\right)\\ &\quad+\sum_{t=0}^{c-1}(-x)^t q^{a\binom{t}{2}} j(q^{bt}y;q^c) m\left(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-t(b^2-ac)}\frac{(-x)^c}{(-y)^b}, q^{c(b^2-ac)}, z_1\right). \quad (2.1)\end{aligned}$$

The following two results allow us to convert from Hecke-type sums to Appell-Lerch sums.

Theorem 2.1. *([11, Theorem 1.3]). Let n and p be positive integers with $(n,p)=1$. For generic $x, y \in \mathbb{C}^*$*

$$f_{n,n+p,n}(x,y,q)=g_{n,n+p,n}(x,y,q,-1,-1)+\frac{1}{\overline{J}_{0,np(2n+p)}}\theta_{n,p}(x,y,q).$$

Theorem 2.2. *([11, Theorem 1.4]). Let a, b, and c be positive integers with $ac<b^2$ and b divisible by a and c. Then for generic $x, y \in \mathbb{C}^*$*

$$f_{a,b,c}(x,y,q)=h_{a,b,c}(x,y,q,-1,-1)-\frac{1}{\overline{J}_{0,b^2/a-c}\overline{J}_{0,b^2/c-a}}\theta_{a,b,c}(x,y,q),$$

where

$$\begin{aligned} h_{a,b,c}(x,y,q,z_1,z_0) :=& j(x;q^a) m\left(-q^{a\binom{b/a+1}{2}-c}(-y)(-x)^{-b/a}, q^{b^2/a-c}, z_1\right)\\ &+j(y;q^c) m\left(-q^{c\binom{b/c+1}{2}-a}(-x)(-y)^{-b/c}, q^{b^2/c-a}, z_0\right). \end{aligned}$$

To prove Theorem 1.1, we need the following lemma.

Lemma 2.3. *For $k_1, k_2, m \in \mathbb{Q}$, (α_n, β_n) is a Bailey pair relative to (a, q^m) with*

$$\alpha_n=\begin{cases}(-1)^n(q^{k_1n^2-k_2n}+q^{k_1n^2+k_2n}), & n>0,\\ 1, & n=0,\end{cases}$$

and (α_n', β_n') is obtained by letting $q\to q^t$ $(t\in\mathbb{Q})$ in (α_n, β_n). Then $(\mathcal{A}_n^{(1)}, \mathcal{B}_n^{(1)})$ is a Bailey pair relative to $(1, q)$, where

$$\mathcal{A}_n^{(1)}=(-1)^n q^{n(n+3)/2}\sum_{j\geq 0} q^{3(2l+1)nj+(\frac{3}{2}-k_1t)j^2+(3l+\frac{3}{2})j}\alpha_j'$$

and

$$\mathcal{B}_n^{(1)} = \sum_{r=0}^{n} \sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j} \times q^{\frac{r(r+3)}{2}+3(2l+1)rj+(\frac{3}{2}-k_1t)j^2+(3l+\frac{3}{2})j+mt\binom{j}{2}+mts}\beta_s'. \tag{2.2}$$

$(\mathcal{A}_n^{(2)},\mathcal{B}_n^{(2)})$ *is a Bailey pair relative to* $(1,q)$, *where*

$$\mathcal{A}_n^{(2)} = (-1)^n q^{n(n+3)/2} \sum_{j\geq 0} q^{(2l+1)nj+(\frac{3}{2}-k_1t)j^2+(l+\frac{1}{2})j}\alpha_j'$$

and

$$\mathcal{B}_n^{(2)} = \sum_{r=0}^{n} \sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j} \times q^{\frac{r(r+3)}{2}+(2l+1)rj+(\frac{3}{2}-k_1t)j^2+(l+\frac{1}{2})j+mt\binom{j}{2}+mts}\beta_s'.$$

$(\mathcal{A}_n^{(3)},\mathcal{B}_n^{(3)})$ *is a Bailey pair relative to* (q,q), *where*

$$\mathcal{A}_n^{(3)} = (-1)^n q^{\binom{n+1}{2}} \sum_{j\geq 0} q^{2(2l+k_2t)nj+(1-k_1t)j^2+(2l+k_2t)j}\alpha_j'$$

and

$$\mathcal{B}_n^{(3)} = \sum_{r=0}^{n} \sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q^2)_{n+r}(q^{mt};q^{mt})_j} \times q^{\binom{r+1}{2}+2(2l+k_2t)rj+(1-k_1t)j^2+(2l+k_2t)j+mt\binom{j}{2}+mts}\beta_s'.$$

Proof. If $(\mathcal{A}_n^{(1)},\mathcal{B}_n^{(1)})$ is a Bailey pair relative to $(1,q)$, then from (1.2), it follows that

$$\begin{aligned}\mathcal{B}_n^{(1)} &= \sum_{r=0}^{n} \frac{A_r^{(1)}}{(q)_{n-r}(q)_{n+r}} \\ &= \sum_{r=0}^{n} \frac{(-1)^r q^{r(r+3)/2}}{(q)_{n-r}(q)_{n+r}} \sum_{j\geq 0} q^{3(2l+1)rj+(\frac{3}{2}-k_1t)j^2+(3l+\frac{3}{2})j}\alpha_j'.\end{aligned} \tag{2.3}$$

Since the Bailey pair (α_j',β_j') is relative to (a,q^{mt}), in view of (1.3), we get

$$\alpha_j' = \frac{(-1)^j q^{mt\binom{j}{2}}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}}{(q^{mt};q^{mt})_j} \sum_{s=0}^{j} (q^{-mtj},aq^{mtj};q^{mt})_s q^{mts}\beta_s'. \tag{2.4}$$

The combination of (2.3) and (2.4) gives (2.2).

In the same manner, we can prove that $(\mathcal{A}_n^{(2)},\mathcal{B}_n^{(2)})$ is a Bailey pair relative to $(1,q)$, and $(\mathcal{A}_n^{(3)},\mathcal{B}_n^{(3)})$ is a Bailey pair relative to (q,q). □

Proof of Theorem 1.1. Substituting the Bailey pair $(\mathcal{A}_n^{(1)},\mathcal{B}_n^{(1)})$ in Lemma 2.3 into (1.7) yields that

$$L.H.S. = \sum_{n\geq 0} q^{n^2}\mathcal{B}_n^{(1)} = W_1(l,t;q)$$

and

$$\begin{aligned}
R.H.S. &= \frac{1}{(q)_\infty}\sum_{n\geq 0} q^{n^2}\mathcal{A}_n^{(1)}\\
&= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}}\sum_{j\geq 0} q^{3(2l+1)nj+(\frac{3}{2}-k_1t)j^2+(3l+\frac{3}{2})j}\alpha_j'\\
&= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}}\left(1+\sum_{j\geq 1} q^{3(2l+1)nj+(\frac{3}{2}-k_1t)j^2+(3l+\frac{3}{2})j}\alpha_j'\right)\\
&= \frac{1}{(q)_\infty}\left(\sum_{n\geq 0, j\geq 0}(-1)^{n+j} q^{\frac{3n(n+1)}{2}+3(2l+1)nj+\frac{3}{2}j^2+(3l+\frac{3}{2}-k_2t)j}\right.\\
&\quad\left.+\sum_{n\geq 0, j\geq 1}(-1)^{n+j} q^{\frac{3n(n+1)}{2}+3(2l+1)nj+\frac{3}{2}j^2+(3l+\frac{3}{2}+k_2t)j}\right).
\end{aligned}$$

After replacing n with $-n-1$ and j with $-j$ in the second sum, we have

$$\begin{aligned}
R.H.S. &= \frac{1}{(q)_\infty}\left(\sum_{n\geq 0, j\geq 0}-\sum_{n<0, j<0}\right)(-1)^{n+j} q^{\frac{3n(n+1)}{2}+3(2l+1)nj+\frac{3}{2}j^2+(3l+\frac{3}{2}-k_2t)j}\\
&= \frac{1}{(q)_\infty}f_{1,2l+1,1}(q^3, q^{3l+3-k_2t}, q^3).
\end{aligned}$$

Then

$$W_1(l,t;q) = \frac{1}{(q)_\infty}f_{1,2l+1,1}(q^3, q^{3l+3-k_2t}, q^3). \tag{2.5}$$

For $l, k_2t \in \mathbb{Z}$, when $l \geq 1$, with the aid of Theorem 2.1 and (2.1), we deduce that

$$\begin{aligned}
&f_{1,2l+1,1}(q^3, q^{3l+3-k_2t}, q^3)\\
&= g_{1,2l+1,1}(q^3, q^{3l+3-k_2t}, q^3, -1, -1) + \frac{1}{\overline{J}_{0,12l(l+1)}}\theta_{1,2l}(q^3, q^{3l+3-k_2t}, q^3)\\
&= j(q^{3l+3-k_2t}; q^3)m(-q^{k_2t(2l+1)}, q^{12l(l+1)}, -1) + \frac{1}{\overline{J}_{0,12l(l+1)}}\theta_{1,2l}(q^3, q^{3l+3-k_2t}, q^3).
\end{aligned}$$

If $k_2t \equiv 1, 2 \pmod 3$, then $3l+3-k_2t \equiv 1, 2 \pmod 3$. Therefore, $W_1(l,t;q)$ can be expressed as a sum of specializations of Appell-Lerch sums and theta functions. More specifically, when $k_2t = 3h+1, h \in \mathbb{Z}$, using the fact that $j(q^{3(l-h)+2}; q^3) = (-1)^{l-h}q^{-3\binom{l-h}{2}-2(l-h)}j(q^2; q^3)$ by (1.1), we obtain

$$\begin{aligned}
W_1(l,t;q) =& (-1)^{l-h}q^{-3\binom{l-h}{2}-2(l-h)}m(-q^{(2l+1)(3h+1)}, q^{12l(l+1)}, -1)\\
&+ \frac{1}{J_1\overline{J}_{0,12l(l+1)}}\theta_{1,2l}(q^3, q^{3l-3h+2}, q^3).
\end{aligned} \tag{2.6}$$

Similarly, when $k_2t = 3h+2, h \in \mathbb{Z}$, we get

$$\begin{aligned}
W_1(l,t;q) =& (-1)^{l-h}q^{-3\binom{l-h}{2}-(l-h)}m(-q^{(2l+1)(3h+2)}, q^{12l(l+1)}, -1)\\
&+ \frac{1}{J_1\overline{J}_{0,12l(l+1)}}\theta_{1,2l}(q^3, q^{3l-3h+1}, q^3).
\end{aligned} \tag{2.7}$$

Meanwhile, according to the definition of Appell-Lerch sums, for any $k \in \mathbb{Z}$, we have

$$k_2t(2l+1) \neq 12kl(l+1).$$

Therefore, for $l, k_2t \in \mathbb{Z}$, when $l \geq 1$, $k_2t \equiv 1, 2 \pmod 3$, $k_2t \neq \frac{12kl(l+1)}{2l+1}$ for any $k \in \mathbb{Z}$, and l and t are generic for $\theta_{1,2l}(q^3, q^{3l+3-k_2t}, q^3)$, $W_1(l,t;q)$ are mock theta functions.

Similarly, substituting the Bailey pair $(\mathcal{A}_n^{(2)}, \mathcal{B}_n^{(2)})$ into (1.7), we have

$$L.H.S. = \sum_{n\geq 0} q^{n^2}\mathcal{B}_n^{(2)} = W_2(l,t;q)$$

and

$$\begin{aligned}
R.H.S. &= \frac{1}{(q)_\infty}\sum_{n\geq 0} q^{n^2}\mathcal{A}_n^{(2)} \\
&= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}} \sum_{j\geq 0} q^{(2l+1)nj+(\frac{3}{2}-k_1t)j^2+(l+\frac{1}{2})j}\alpha_j' \\
&= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}} \left(1+\sum_{j\geq 1} q^{(2l+1)nj+(\frac{3}{2}-k_1t)j^2+(l+\frac{1}{2})j}\alpha_j'\right) \\
&= \frac{1}{(q)_\infty}\left(\sum_{n\geq 0, j\geq 0}(-1)^{n+j} q^{\frac{3n(n+1)}{2}+(2l+1)nj+\frac{3}{2}j^2+(l+\frac{1}{2}-k_2t)j}\right. \\
&\quad \left. + \sum_{n\geq 0, j\geq 1}(-1)^{n+j} q^{\frac{3n(n+1)}{2}+(2l+1)nj+\frac{3}{2}j^2+(l+\frac{1}{2}+k_2t)j}\right).
\end{aligned}$$

Replacing n with $-n-1$ and j with $-j$ in the second sum yields that

$$\begin{aligned}
R.H.S. &= \frac{1}{(q)_\infty}\left(\sum_{n\geq 0, j\geq 0} - \sum_{n<0, j<0}\right)(-1)^{n+j} q^{\frac{3n(n+1)}{2}+(2l+1)nj+\frac{3}{2}j^2+(l+\frac{1}{2}-k_2t)j} \\
&= \frac{1}{(q)_\infty} f_{3,2l+1,3}(q^3, q^{l+2-k_2t}, q).
\end{aligned}$$

Then

$$W_2(l,t;q) = \frac{1}{(q)_\infty} f_{3,2l+1,3}(q^3, q^{l+2-k_2t}, q). \tag{2.8}$$

For $l, k_2t \in \mathbb{Z}$, when $l \equiv 0, 2 \pmod 3$ and $l \geq 2$, applying Theorem 2.1 and (2.1), we obtain

$$\begin{aligned}
& f_{3,2l+1,3}(q^3, q^{l+2-k_2t}, q) \\
&= g_{3,2l+1,3}(q^3, q^{l+2-k_2t}, q, -1, -1) + \frac{1}{J_1\overline{J}_{0,12(l-1)(l+2)}}\theta_{3,2l-2}(q^3, q^{l+2-k_2t}, q) \\
&= \sum_{s=0}^{2}(-q^{l+2-k_2t})^s q^{3\binom{s}{2}} j(q^{(2l+1)s+3}; q^3) \\
&\quad \times m\left(-q^{3\binom{2l+2}{2}-18-4s(l-1)(l+2)}\frac{(-q^{l+2-k_2t})^3}{(-q^3)^{2l+1}}, q^{12(l-1)(l+2)}, -1\right)
\end{aligned}$$

$$+\sum_{s=0}^{2}(-q^3)^s q^{3\binom{s}{2}} j(q^{(2l+1)s+l+2-k_2t};q^3)$$
$$\times m\left(-q^{3\binom{2l+2}{2}-18-4s(l-1)(l+2)}\frac{(-q^3)^3}{(-q^{l+2-k_2t})^{2l+1}}, q^{12(l-1)(l+2)}, -1\right)$$
$$+\frac{1}{J_1\overline{J}_{0,12(l-1)(l+2)}}\theta_{3,2l-2}(q^3,q^{l+2-k_2t},q). \tag{2.9}$$

Then we have

$$\begin{aligned}W_2(l,t;q) =&\frac{1}{(q)_\infty}\big(j(q^3;q^3)m(-q^{6(l-1)(l+2)-3k_2t},q^{12(l-1)(l+2)},-1)\\
&-q^{l+2-k_2t}j(q^{2l+4};q^3)m(-q^{2(l-1)(l+2)-3k_2t},q^{12(l-1)(l+2)},-1)\\
&+q^{2l+7-2k_2t}j(q^{4l+5};q^3)m(-q^{-2(l-1)(l+2)-3k_2t},q^{12(l-1)(l+2)},-1)\\
&+j(q^{l+2-k_2t};q^3)m(-q^{(2l+1)(2l+1+k_2t)-9},q^{12(l-1)(l+2)},-1)\\
&-q^3j(q^{3l+3-k_2t};q^3)m(-q^{k_2t(2l+1)},q^{12(l-1)(l+2)},-1)\\
&+q^9j(q^{5l+4-k_2t};q^3)m(-q^{k_2t(2l+1)-4(l-1)(l+2)},q^{12(l-1)(l+2)},-1)\big)\\
&+\frac{1}{J_1\overline{J}_{0,12(l-1)(l+2)}}\theta_{3,2l-2}(q^3,q^{l+2-k_2t},q).\end{aligned} \tag{2.10}$$

Since $2l+4\equiv 1,2 \pmod 3$, $4l+5\equiv 1,2 \pmod 3$, $j(q;q^3)=j(q^2;q^3)=(q)_\infty$, and $j(q^3;q^3)=0$, by means of (1.1), $W_2(l,t;q)$ can be expressed as a sum of specializations of Appell-Lerch sums and theta functions. Furthermore, by the definition of Appell-Lerch sums and (2.9), we get that $k_2t\neq\frac{(6-4\varepsilon-12k)(l-1)(l+2)}{3}$ ($\varepsilon=0,1,2$) and $k_2t\neq\frac{(12k+4\varepsilon-4)(l-1)(l+2)}{2l+1}$ ($\varepsilon=0,1,2$) for any $k\in\mathbb{Z}$. While we notice that

$$\frac{(6-4\varepsilon-12k)(l-1)(l+2)}{3}=(2-4k)(l-1)(l+2)-\frac{4}{3}\varepsilon(l-1)(l+2).$$

When $l\equiv 0,2 \pmod 3$ and $\varepsilon=1,2$, we have that $\frac{(6-4\varepsilon-12k)(l-1)(l+2)}{3}$ cannot be integers. Therefore, for $l,k_2t\in\mathbb{Z}$, when $l\equiv 0,2 \pmod 3$, $l\geq 2$, $k_2t\neq(2-4k)(l-1)(l+2)$, $k_2t\neq\frac{(12k+4\varepsilon-4)(l-1)(l+2)}{2l+1}$ ($\varepsilon=0,1,2$) for any $k\in\mathbb{Z}$, and l and t are generic for $\theta_{3,2l-2}(q^3,q^{l+2-k_2t},q)$, we conclude that $W_2(l,t;q)$ are mock theta functions.

In the same manner, substituting the Bailey pair $(\mathcal{A}_n^{(3)},\mathcal{B}_n^{(3)})$ into (1.8) yields that

$$L.H.S.=\sum_{n\geq 0}\frac{(-q)_nq^{\binom{n+1}{2}}}{1-q}\mathcal{B}_n^{(3)}=W_3(l,t;q)$$

and

$$\begin{aligned}R.H.S.=&\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}q^{\binom{n+1}{2}}\mathcal{A}_n^{(3)}\\
=&\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}(-1)^nq^{n^2+n}\sum_{j\geq 0}q^{2(2l+k_2t)nj+(1-k_1t)j^2+(2l+k_2t)j}\alpha_j'\end{aligned}$$

$$= \frac{(-q)_\infty}{(q)_\infty} \sum_{n\geq 0} (-1)^n q^{n^2+n} \left(1 + \sum_{j\geq 1} q^{2(2l+k_2t)nj+(1-k_1t)j^2+(2l+k_2t)j} \alpha_j^{'}\right)$$

$$= \frac{(-q)_\infty}{(q)_\infty} \left(\sum_{n\geq 0, j\geq 0} (-1)^{n+j} q^{n^2+n+2(2l+k_2t)nj+j^2+2lj} \right.$$

$$\left. + \sum_{n\geq 0, j\geq 1} (-1)^{n+j} q^{n^2+n+2(2l+k_2t)nj+j^2+(2l+2k_2t)j} \right).$$

After replacing n with $-n-1$ and j with $-j$ in the second sum, we have

$$R.H.S. = \frac{(-q)_\infty}{(q)_\infty} \left(\sum_{n\geq 0, j\geq 0} - \sum_{n<0, j<0} \right) (-1)^{n+j} q^{n^2+n+2(2l+k_2t)nj+j^2+2lj}$$

$$= \frac{(-q)_\infty}{(q)_\infty} f_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q).$$

Hence,

$$W_3(l,t;q) = \frac{(-q)_\infty}{(q)_\infty} f_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q). \tag{2.11}$$

When $l, k_2t \in \mathbb{Z}$ and $l > \frac{1-k_2t}{2}$, applying Theorem 2.2 gives that

$$f_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q)$$
$$= h_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q, -1, -1) - \frac{1}{\overline{J}^2_{0,2(2l+k_2t)^2-2}} \theta_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q)$$
$$= j(q^{2l+1}; q^2) m\left((-1)^{k_2t} q^{k_2t(2l+k_2t)}, q^{2(2l+k_2t)^2-2}, -1\right) - \frac{1}{\overline{J}^2_{0,2(2l+k_2t)^2-2}} \theta_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q).$$

From (1.1), it follows that

$$W_3(l,t;q) = (-1)^l q^{-l^2} m\left((-1)^{k_2t} q^{k_2t(2l+k_2t)}, q^{2(2l+k_2t)^2-2}, -1\right)$$
$$- \frac{1}{J_{1,2}\overline{J}^2_{0,2(2l+k_2t)^2-2}} \theta_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q). \tag{2.12}$$

Based on the definition of Appell-Lerch sums, when k_2t are even integers, -1 and $-q^{k_2t(2l+k_2t)}$ are not integral powers of $q^{2(2l+k_2t)^2-2}$. When k_2t are odd integers, $k_2t(2l+k_2t)$ are odd integers, and $(2(2l+k_2t)^2-2)$ are even integers. Hence, $q^{k_2t(2l+k_2t)}$ are not integral powers of $q^{2(2l+k_2t)^2-2}$. In conclusion, for $l, k_2t \in \mathbb{Z}$, when $l > \frac{1-k_2t}{2}$ and l and t are generic for $\theta_{2,4l+2k_2t,2}(q^2, q^{2l+1}, q)$, $W_3(l,t;q)$ are mock theta functions. □

Lemma 2.4. *For $k_1, k_2, m \in \mathbb{Q}$, (α_n, β_n) is a Bailey pair relative to (a, q^m) with*

$$\alpha_n = \begin{cases} q^{k_1n^2-k_2n} - q^{k_1n^2+k_2n}, & n > 0, \\ 1, & n = 0, \end{cases}$$

and (α_n', β_n') *is obtained by letting* $q \to q^t$ $(t \in \mathbb{Q})$ *in* (α_n, β_n)*. Then* $(\mathcal{A}_n^{(4)}, \mathcal{B}_n^{(4)})$ *is a Bailey pair relative to* $(1, q)$*, where*

$$\mathcal{A}_n^{(4)} = (-1)^n q^{\binom{n+1}{2}} \sum_{j\geq 0} q^{(2l-1+2k_2t)nj+(\frac{1}{2}-k_1t)j^2+(l-\frac{1}{2}+k_2t)j} \alpha_j'$$

and

$$\mathcal{B}_n^{(4)} = \sum_{r=0}^{n} \sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj}, aq^{mtj}; q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j} \times q^{\binom{r+1}{2}+(2l-1+2k_2t)rj+(\frac{1}{2}-k_1t)j^2+(l-\frac{1}{2}+k_2t)j+mt\binom{j}{2}+mts} \beta_s'.$$

Proof. If $(\mathcal{A}_n^{(4)}, \mathcal{B}_n^{(4)})$ is a Bailey pair relative to $(1, q)$, then by (1.2), we have

$$\begin{aligned}\mathcal{B}_n^{(4)} &= \sum_{r=0}^{n} \frac{A_r^{(4)}}{(q)_{n-r}(q)_{n+r}} \\ &= \sum_{r=0}^{n} \frac{(-1)^r q^{\binom{r+1}{2}}}{(q)_{n-r}(q)_{n+r}} \sum_{j\geq 0} q^{(2l-1+2k_2t)rj+(\frac{1}{2}-k_1t)j^2+(l-\frac{1}{2}+k_2t)j} \alpha_j'.\end{aligned}$$

Employing (2.4) in the above identity, we prove the lemma. □

Proof of Theorem 1.2. Substituting the Bailey pair $(\mathcal{A}_n^{(4)}, \mathcal{B}_n^{(4)})$ into (1.9) implies that

$$L.H.S. = 2\sum_{n\geq 0} (-1)^n (q;q^2)_n \mathcal{B}_n^{(4)} = W_4(l,t;q)$$

and

$$\begin{aligned}R.H.S. &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n \mathcal{A}_n^{(4)} \\ &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{\binom{n+1}{2}} \sum_{j\geq 0} q^{(2l-1+2k_2t)nj+(\frac{1}{2}-k_1t)j^2+(l-\frac{1}{2}+k_2t)j} \alpha_j' \\ &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{\binom{n+1}{2}} \left(1 + \sum_{j\geq 1} q^{(2l-1+2k_2t)nj+(\frac{1}{2}-k_1t)j^2+(l-\frac{1}{2}+k_2t)j} \alpha_j'\right) \\ &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \left(\sum_{n\geq 0, j\geq 0} q^{\binom{n+1}{2}+(2l-1+2k_2t)nj+\frac{1}{2}j^2+(l-\frac{1}{2})j} \right. \\ &\quad \left. - \sum_{n\geq 0, j\geq 1} q^{\binom{n+1}{2}+(2l-1+2k_2t)nj+\frac{1}{2}j^2+(l-\frac{1}{2}+2k_2t)j} \right).\end{aligned}$$

Replacing n with $-n-1$ and j with $-j$ in the second sum, we have

$$\begin{aligned}R.H.S. &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \left(\sum_{n\geq 0, j\geq 0} - \sum_{n<0, j<0} \right) q^{\binom{n+1}{2}+(2l-1+2k_2t)nj+\frac{1}{2}j^2+(l-\frac{1}{2})j} \\ &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} f_{1,2l-1+2k_2t,1}(-q, -q^l, q).\end{aligned}$$

Then

$$W_4(l,t;q)=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}f_{1,2l-1+2k_2t,1}(-q,-q^l,q).$$

When $l,2k_2t\in\mathbb{Z}$ and $l>1-k_2t$, from Theorem 2.1 and (2.1), it shows that

$$\begin{aligned}&f_{1,2l-1+2k_2t,1}(-q,-q^l,q)\\&=g_{1,2l-1+2k_2t,1}(-q,-q^l,q,-1,-1)+\frac{1}{\overline{J}_{0,4(l+k_2t)(l-1+k_2t)}}\theta_{1,2l-2+2k_2t}(-q,-q^l,q)\\&=j(-q;q)m(-q^{(l-1+k_2t)(2l-1+2k_2t)+l-1},q^{(2l-1+2k_2t)^2-1},-1)\\&\quad+j(-q^l;q)m(-q^{k_2t(2l-1+2k_2t)},q^{(2l-1+2k_2t)^2-1},-1)\\&\qquad+\frac{1}{\overline{J}_{0,4(l+k_2t)(l-1+k_2t)}}\theta_{1,2l-2+2k_2t}(-q,-q^l,q).\end{aligned}$$

In light of (1.1), we have

$$j(-q^l;q)=q^{-\binom{l}{2}}j(-q;q)=2q^{-\binom{l}{2}}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Then

$$\begin{aligned}W_4(l,t;q)=&2m(-q^{(l-1+k_2t)(2l-1+2k_2t)+l-1},q^{(2l-1+2k_2t)^2-1},-1)\\&+2q^{-\binom{l}{2}}m(-q^{k_2t(2l-1+2k_2t)},q^{(2l-1+2k_2t)^2-1},-1)\\&\quad+\frac{1}{\overline{J}_{1,1}\overline{J}_{0,4(l+k_2t)(l-1+k_2t)}}\theta_{1,2l-2+2k_2t}(-q,-q^l,q).\end{aligned}\tag{2.13}$$

By the definition of Appell-Lerch sums, $2k_2t\neq\frac{-1\pm\sqrt{1+8(2k-1)(k+l-1)}}{4k-2}-2l+1$ and $2k_2t\neq\frac{(1-4k)(2l-1)\pm\sqrt{(2l-1)^2+8k(2k-1)}}{4k-2}$ for any $k\in\mathbb{Z}$. Therefore, for $l,2k_2t\in\mathbb{Z}$, when $l>1-k_2t$, $2k_2t\neq\frac{-1\pm\sqrt{1+8(2k-1)(k+l-1)}}{4k-2}-2l+1$, $2k_2t\neq\frac{(1-4k)(2l-1)\pm\sqrt{(2l-1)^2+8k(2k-1)}}{4k-2}$ for any $k\in\mathbb{Z}$, and l and t are generic for $\theta_{1,2l-2+2k_2t}(-q,-q^l,q)$, $W_4(l,t;q)$ are mock theta functions. □

Lemma 2.5. *For $k_1,k_2,m\in\mathbb{Q}$, (α_n,β_n) is a Bailey pair relative to (a,q^m) with*

$$\alpha_n=\begin{cases}(-1)^r(q^{k_1r^2-k_2r}+q^{k_1r^2+k_2r}), & n=2r,\\ 0, & n \text{ is odd},\\ 1, & n=0,\end{cases}$$

and (α_n',β_n') is obtained by letting $q\to q^t$ $(t\in\mathbb{Q})$ in (α_n,β_n). Then $(\mathcal{A}_n^{(5)},\mathcal{B}_n^{(5)})$ is a Bailey pair relative to $(1,q)$, where

$$\mathcal{A}_n^{(5)}=(-1)^nq^{\frac{n(n+3)}{2}}\sum_{j\geq0}q^{(3l+\frac32)nj+(\frac38-\frac14k_1t)j^2+(\frac32l+\frac34)j}\alpha_j'$$

and

$$\mathcal{B}_n^{(5)}=\sum_{r=0}^{n}\sum_{j\geq s\geq0}\frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j}$$

$$\times\, q^{\frac{r(r+3)}{2}+(3l+\frac{3}{2})rj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{3}{2}l+\frac{3}{4})j+mt\binom{j}{2}+mts}\beta_s'.$$

$(\mathcal{A}_n^{(6)}, \mathcal{B}_n^{(6)})$ *is a Bailey pair relative to* $(1, q)$*, where*

$$\mathcal{A}_n^{(6)} = (-1)^n q^{\frac{n(n+3)}{2}} \sum_{j\geq 0} q^{(l+\frac{1}{2})nj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{1}{2}l+\frac{1}{4})j}\alpha_j'$$

and

$$\mathcal{B}_n^{(6)} = \sum_{r=0}^{n} \sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q)_{n+r}(q^{mt};q^{mt})_j}$$
$$\times\, q^{\frac{r(r+3)}{2}+(l+\frac{1}{2})rj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{1}{2}l+\frac{1}{4})j+mt\binom{j}{2}+mts}\beta_s'.$$

$(\mathcal{A}_n^{(7)}, \mathcal{B}_n^{(7)})$ *is a Bailey pair relative to* (q, q)*, where*

$$\mathcal{A}_n^{(7)} = (-1)^n q^{\binom{n+1}{2}} \sum_{j\geq 0} q^{(2l+k_2t)nj+(\frac{1}{4}-\frac{1}{4}k_1t)j^2+(l+\frac{1}{2}k_2t)j}\alpha_j'$$

and

$$\mathcal{B}_n^{(7)} = \sum_{r=0}^{n} \sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1-aq^{2mtj})(aq^{mt};q^{mt})_{j-1}(q^{-mtj},aq^{mtj};q^{mt})_s}{(q)_{n-r}(q^2)_{n+r}(q^{mt};q^{mt})_j}$$
$$\times\, q^{\binom{r+1}{2}+(2l+k_2t)rj+(\frac{1}{4}-\frac{1}{4}k_1t)j^2+(l+\frac{1}{2}k_2t)j+mt\binom{j}{2}+mts}\beta_s'.$$

Proof. The proof is similar to that of Lemma 2.3. The combination of (1.2) and (2.4) gives the proof. □

Proof of Theorem 1.3. Substituting the Bailey pair $(\mathcal{A}_n^{(5)}, \mathcal{B}_n^{(5)})$ into (1.7), we have

$$L.H.S. = \sum_{n\geq 0} q^{n^2}\mathcal{B}_n^{(5)} = W_5(l,t;q)$$

and

$$\begin{aligned}
R.H.S. &= \frac{1}{(q)_\infty}\sum_{n\geq 0} q^{n^2}\mathcal{A}_n^{(5)} \\
&= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}} \sum_{j\geq 0} q^{(3l+\frac{3}{2})nj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{3}{2}l+\frac{3}{4})j}\alpha_j' \\
&= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}} \left(1+\sum_{j\geq 1} q^{(3l+\frac{3}{2})nj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{3}{2}l+\frac{3}{4})j}\alpha_j'\right) \\
&= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}} \left(1+\sum_{j\geq 1} q^{3(2l+1)nj+(\frac{3}{2}-k_1t)j^2+(3l+\frac{3}{2})j}\alpha_{2j}'\right) \\
&= \frac{1}{(q)_\infty}\left(\sum_{n\geq 0, j\geq 0}(-1)^{n+j} q^{\frac{3n(n+1)}{2}+3(2l+1)nj+\frac{3}{2}j^2+(3l+\frac{3}{2}-k_2t)j}\right. \\
&\qquad \left. +\sum_{n\geq 0, j\geq 1}(-1)^{n+j} q^{\frac{3n(n+1)}{2}+3(2l+1)nj+\frac{3}{2}j^2+(3l+\frac{3}{2}+k_2t)j}\right).
\end{aligned}$$

After replacing n with $-n-1$ and j with $-j$ in the second sum, we get

$$\begin{aligned} R.H.S &= \frac{1}{(q)_\infty}\left(\sum_{n\geq 0, j\geq 0} - \sum_{n<0, j<0}\right)(-1)^{n+j} q^{\frac{3n(n+1)}{2}+3(2l+1)nj+\frac{3}{2}j^2+(3l+\frac{3}{2}-k_2t)j} \\ &= \frac{1}{(q)_\infty} f_{1,2l+1,1}(q^3, q^{3l+3-k_2t}, q^3). \end{aligned}$$

Then

$$W_5(l,t;q) = \frac{1}{(q)_\infty} f_{1,2l+1,1}(q^3, q^{3l+3-k_2t}, q^3). \tag{2.14}$$

Notice that $W_5(l,t;q)$ and $W_1(l,t;q)$ have the same expression in terms of Hecke-type sums by (2.5). Therefore, for $l, k_2t \in \mathbb{Z}$, when $l \geq 1$, $k_2t \equiv 1, 2 \pmod 3$, $k_2t \neq \frac{12kl(l+1)}{2l+1}$ for any $k \in \mathbb{Z}$, and l and t are generic for $\theta_{1,2l}(q^3, q^{3l+3-k_2t}, q^3)$, $W_5(l,t;q)$ are mock theta functions.

Substituting the Bailey pair $(\mathcal{A}_n^{(6)}, \mathcal{B}_n^{(6)})$ into (1.7) yields that

$$L.H.S. = \sum_{n\geq 0} q^{n^2}\mathcal{B}_n^{(6)} = W_6(l,t;q)$$

and

$$\begin{aligned} R.H.S. &= \frac{1}{(q)_\infty}\sum_{n\geq 0} q^{n^2}\mathcal{A}_n^{(6)} \\ &= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}} \sum_{j\geq 0} q^{(l+\frac{1}{2})nj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{1}{2}l+\frac{1}{4})j}\alpha_j' \\ &= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}} \left(1+\sum_{j\geq 1} q^{(l+\frac{1}{2})nj+(\frac{3}{8}-\frac{1}{4}k_1t)j^2+(\frac{1}{2}l+\frac{1}{4})j}\alpha_j'\right) \\ &= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{3n(n+1)}{2}} \left(1+\sum_{j\geq 1} q^{(2l+1)nj+(\frac{3}{2}-k_1t)j^2+(l+\frac{1}{2})j}\alpha_{2j}'\right) \\ &= \frac{1}{(q)_\infty}\left(\sum_{n\geq 0, j\geq 0}(-1)^{n+j} q^{\frac{3n(n+1)}{2}+(2l+1)nj+\frac{3}{2}j^2+(l+\frac{1}{2}-k_2t)j}\right. \\ &\quad \left. + \sum_{n\geq 0, j\geq 1}(-1)^{n+j} q^{\frac{3n(n+1)}{2}+(2l+1)nj+\frac{3}{2}j^2+(l+\frac{1}{2}+k_2t)j}\right). \end{aligned}$$

Replacing n with $-n-1$ and j with $-j$ in the second sum implies

$$\begin{aligned} R.H.S &= \frac{1}{(q)_\infty}\left(\sum_{n\geq 0, j\geq 0} - \sum_{n<0, j<0}\right)(-1)^{n+j} q^{\frac{3n(n+1)}{2}+(2l+1)nj+\frac{3}{2}j^2+(l+\frac{1}{2}-k_2t)j} \\ &= \frac{1}{(q)_\infty} f_{3,2l+1,3}(q^3, q^{l+2-k_2t}, q). \end{aligned}$$

Then

$$W_6(l,t;q) = \frac{1}{(q)_\infty} f_{3,2l+1,3}(q^3, q^{l+2-k_2t}, q). \tag{2.15}$$

Similar to the proof of $W_2(l,t;q)$ in Theorem 1.1, we prove that $W_6(l,t;q)$ are mock theta functions under the same conditions.

Substituting the Bailey pair $(\mathcal{A}_n^{(7)},\mathcal{B}_n^{(7)})$ into (1.8), we have

$$L.H.S. = \sum_{n\geq 0}\frac{(-q)_n q^{\binom{n+1}{2}}}{1-q}\mathcal{B}_n^{(7)} = W_7(l,t;q)$$

and

$$\begin{aligned}
R.H.S. &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0} q^{\binom{n+1}{2}}\mathcal{A}_n^{(7)}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{n^2+n}\sum_{j\geq 0} q^{(2l+k_2t)nj+(\frac{1}{4}-\frac{1}{4}k_1t)j^2+(l+\frac{1}{2}k_2t)j}\alpha_j^{'}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{n^2+n}\left(1+\sum_{j\geq 1} q^{(2l+k_2t)nj+(\frac{1}{4}-\frac{1}{4}k_1t)j^2+(l+\frac{1}{2}k_2t)j}\alpha_j^{'}\right)\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{n^2+n}\left(1+\sum_{j\geq 1} q^{2(2l+k_2t)nj+(1-k_1t)j^2+(2l+k_2t)j}\alpha_{2j}^{'}\right)\\
&= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{n\geq 0,j\geq 0}(-1)^{n+j}q^{n^2+n+2(2l+k_2t)nj+j^2+2lj}\right.\\
&\qquad\left.+\sum_{n\geq 0,j\geq 1}(-1)^{n+j}q^{n^2+n+2(2l+k_2t)nj+j^2+(2l+2k_2t)j}\right).
\end{aligned}$$

After replacing n with $-n-1$ and j with $-j$ in the second sum, we get

$$\begin{aligned}
R.H.S. &= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{n\geq 0,j\geq 0}-\sum_{n<0,j<0}\right)(-1)^{n+j}q^{n^2+n+2(2l+k_2t)nj+j^2+2lj}\\
&= \frac{(-q)_\infty}{(q)_\infty}f_{2,4l+2k_2t,2}(q^2,q^{2l+1},q).
\end{aligned}$$

Then

$$W_7(l,t;q) = \frac{(-q)_\infty}{(q)_\infty}f_{2,4l+2k_2t,2}(q^2,q^{2l+1},q). \tag{2.16}$$

Similar to the proof of $W_3(l,t;q)$ in Theorem 1.1, we prove that $W_7(l,t;q)$ are mock theta functions under the same conditions. □

3. Examples

In this section, we focus on the Bailey pairs in Slater's list [20, 21] to derive some new mock theta functions by using the main theorems.

Theorem 3.1. *The following are mock theta functions:*

$$\omega_1(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{tj})(q^{-tj},q^{tj};q^t)_s(q^{\frac{t}{2}};q^{\frac{t}{2}})_{2s}}{(q)_{n-r}(q)_{n+r}(q^t;q^t)_{2s}}$$

$$\times q^{n^2+\frac{r(r+3)}{2}+3(2l+1)rj+\frac{3}{2}j^2+(3l+\frac{3}{2}-\frac{1}{2}t)j+ts}, \tag{3.1}$$

where $l \in \mathbb{N}$, $t \equiv 2,4 \pmod 6$, $t \neq \frac{24kl(l+1)}{2l+1}$ *for any* $k \in \mathbb{Z}$, *and* l *and* t *are generic for* $\theta_{1,2l}(q^3, q^{3l+3-\frac{1}{2}t}, q^3)$.

$$\omega_2(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{tj})(q^{-tj},q^{tj};q^t)_s(q^{\frac{t}{2}};q^{\frac{t}{2}})_{2s}}{(q)_{n-r}(q)_{n+r}(q^t;q^t)_{2s}}$$
$$\times q^{n^2+\frac{r(r+3)}{2}+(2l+1)rj+\frac{3}{2}j^2+(l+\frac{1}{2}-\frac{1}{2}t)j+ts}, \tag{3.2}$$

where $\frac{1}{2}t \in \mathbb{Z}$, $l \equiv 0,2 \pmod 3$, $l \geq 2$, $\frac{1}{2}t \neq (2-4k)(l-1)(l+2)$, $\frac{1}{2}t \neq \frac{(12k+4\varepsilon-4)(l-1)(l+2)}{2l+1}$ $(\varepsilon = 0,1,2)$ *for any* $k \in \mathbb{Z}$, *and* l *and* t *are generic for* $\theta_{3,2l-2}(q^3, q^{l+2-\frac{1}{2}t}, q)$.

$$\omega_3(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{tj})(-q)_n(q^{-tj},q^{tj};q^t)_s(q^{\frac{t}{2}};q^{\frac{t}{2}})_{2s}}{(q)_{n-r}(q)_{n+r+1}(q^t;q^t)_{2s}}$$
$$\times q^{\binom{n+1}{2}+\binom{r+1}{2}+(4l+t)rj+j^2+2lj+ts}, \tag{3.3}$$

where $l, \frac{1}{2}t \in \mathbb{Z}$, $l > \frac{1}{2} - \frac{1}{4}t$, *and* l *and* t *are generic for* $\theta_{2,4l+t,2}(q^2, q^{2l+1}, q)$.

$$\omega_4(l,t;q) := 2\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{n+r+j}(1+q^{tj})(q;q^2)_n(q^{-tj},q^{tj};q^t)_s}{(q)_{n-r}(q)_{n+r}(q^t;q^t)_s(q^t;q^t)_s}$$
$$\times q^{\binom{r+1}{2}+(2l-2t-1)rj+(\frac{1}{2}-\frac{1}{2}t)j^2+(l-\frac{1}{2}-\frac{3}{2}t)j+2ts}, \tag{3.4}$$

where $l, 2t \in \mathbb{Z}$, $l > 1+t$, $2t \neq \frac{1\pm\sqrt{1+8(2k-1)(k+l-1)}}{4k-2}+2l-1$, $2t \neq \frac{(4k-1)(2l-1)\pm\sqrt{(2l-1)^2+8k(2k-1)}}{4k-2}$ *for any* $k \in \mathbb{Z}$, *and* l *and* t *are generic for* $\theta_{1,2l-2-2t}(-q, -q^l, q)$.

$$\omega_5(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{tj})(q^{-tj},q^{tj};q^t)_s}{(q)_{n-r}(q)_{n+r}(q^t;q^{2t})_s(q^t;q^t)_s}$$
$$\times q^{n^2+\frac{r(r+3)}{2}+(3l+\frac{3}{2})rj+(\frac{3}{8}-\frac{1}{4}t)j^2+(\frac{3}{2}l+\frac{3}{4}-\frac{1}{2}t)j+ts}, \tag{3.5}$$

where $l \in \mathbb{N}$, $t \equiv 1,2 \pmod 3$, $t \neq \frac{12kl(l+1)}{2l+1}$ *for any* $k \in \mathbb{Z}$, *and* l *and* t *are generic for* $\theta_{1,2l}(q^3, q^{3l+3-t}, q^3)$.

$$\omega_6(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{tj})(q^{-tj},q^{tj};q^t)_s}{(q)_{n-r}(q)_{n+r}(q^t;q^{2t})_s(q^t;q^t)_s}$$
$$\times q^{n^2+\frac{r(r+3)}{2}+(l+\frac{1}{2})rj+(\frac{3}{8}-\frac{1}{4}t)j^2+(\frac{1}{2}l+\frac{1}{4}-\frac{1}{2}t)j+ts}, \tag{3.6}$$

where $t \in \mathbb{Z}$, $l \equiv 0,2 \pmod 3$, $l \geq 2$, $t \neq (2-4k)(l-1)(l+2)$, $t \neq \frac{(12k+4\varepsilon-4)(l-1)(l+2)}{2l+1}$ $(\varepsilon = 0,1,2)$ *for any* $k \in \mathbb{Z}$, *and* l *and* t *are generic for* $\theta_{3,2l-2}(q^3, q^{l+2-t}, q)$.

$$\omega_7(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{tj})(-q)_n(q^{-tj},q^{tj};q^t)_s}{(q)_{n-r}(q)_{n+r+1}(q^t;q^{2t})_s(q^t;q^t)_s}$$
$$\times q^{\binom{n+1}{2}+\binom{r+1}{2}+(2l+t)rj+(\frac{1}{4}-\frac{1}{4}t)j^2+lj+ts}, \tag{3.7}$$

where $l, t \in \mathbb{Z}$, $l > \frac{1-t}{2}$, *and* l *and* t *are generic for* $\theta_{2,4l+2t,2}(q^2, q^{2l+1}, q)$.

Proof. Consider the Bailey pair (α_n, β_n) relative to $(1, q)$ in [21, M(1)]

$$\alpha_n = \begin{cases} (-1)^n q^{\frac{1}{2}n^2}(q^{-\frac{n}{2}} + q^{\frac{n}{2}}), & n > 0, \\ 1, & n = 0, \end{cases}$$

and

$$\beta_n = \frac{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2n}}{(q)_{2n}}.$$

Setting $a = 1$, $m = 1$, and $k_1 = k_2 = \frac{1}{2}$, and applying Theorem 1.1, we obtain (3.1)-(3.3), respectively.

For the Bailey pair (α_n, β_n) relative to $(1, q)$ in [20]

$$\alpha_n = \begin{cases} q^{n^2}(q^n - q^{-n}), & \text{if } n > 0, \\ 1, & \text{if } n = 0, \end{cases}$$

and

$$\beta_n = \frac{q^n}{(q)_n (q)_n},$$

Applying Theorem 1.2, we obtain (3.4) by setting $a = 1$, $m = 1$, $k_1 = 1$, and $k_2 = -1$.

For the Bailey pair relative to $(1, q)$ in [20, C(6)]

$$\alpha_n = \begin{cases} (-1)^r q^{3r^2 - r}(1 + q^{2r}), & \text{if } n = 2r > 0, \\ 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n = 0, \end{cases}$$

and

$$\beta_n = \frac{1}{(q; q^2)_n (q)_n},$$

setting $a = 1$, $m = 1$, $k_1 = 3$, and $k_2 = 1$, and applying Theorem 1.3, we derive (3.5)-(3.7), respectively. □

More specifically, we give some explicit values for l and t in (3.1)-(3.7) to show some examples as mock theta functions.

Set $l = 1$ and $t = 2$ in (3.1). Then resorting to (2.6), we have

$$\begin{aligned} \omega_1(1, 2; q) &= \sum_{n \geq r \geq 0} \sum_{j \geq s \geq 0} \frac{(-1)^{r+j}(1 + q^{2j})(q^{-2j}, q^{2j}; q^2)_s (q)_{2s}}{(q)_{n-r}(q)_{n+r}(q^2; q^2)_{2s}} q^{n^2 + \frac{r(r+3)}{2} + 9rj + \frac{3}{2}j^2 + \frac{7}{2}j + 2s} \\ &= -q^{-2} m(-q^3, q^{24}, -1) + \frac{1}{J_1 \overline{J}_{0,24}} \theta_{1,2}(q^3, q^5, q^3). \end{aligned} \tag{3.8}$$

Set $l = 2$ and $t = 2$ in (3.2). Then by (2.10), we obtain

$$\begin{aligned} \omega_2(2, 2; q) &= \sum_{n \geq r \geq 0} \sum_{j \geq s \geq 0} \frac{(-1)^{r+j}(1 + q^{2j})(q^{-2j}, q^{2j}; q^2)_s (q)_{2s}}{(q)_{n-r}(q)_{n+r}(q^2; q^2)_{2s}} q^{n^2 + \frac{r(r+3)}{2} + 5rj + \frac{3}{2}j^2 + \frac{3}{2}j + 2s} \\ &= -2q^{-4} m(-q^5, q^{48}, -1) + 2q^{-13} m(-q^{-11}, q^{48}, -1) + \frac{1}{J_1 \overline{J}_{0,48}} \theta_{3,2}(q^3, q^3, q). \end{aligned} \tag{3.9}$$

Set $l=1$ and $t=2$ in (3.3). Then applying (2.12) yields that

$$\begin{aligned}\omega_3(1,2;q)=&\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{2j})(-q)_n(q^{-2j},q^{2j};q^2)_s(q)_{2s}}{(q)_{n-r}(q)_{n+r+1}(q^2;q^2)_{2s}}q^{\binom{n+1}{2}+\binom{r+1}{2}+6rj+j^2+2j+2s}\\ =&-q^{-1}m(-q^3,q^{16},-1)-\frac{1}{J_{1,2}\overline{J}_{0,16}^2}\theta_{2,6,2}(q^2,q^3,q).\end{aligned} \tag{3.10}$$

Set $l=3$ and $t=1$ in (3.4). Then by (2.13), we obtain

$$\begin{aligned}\omega_4(3,1;q)=&2\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{n+r+j}(1+q^{j})(q;q^2)_n(q^{-j},q^{j};q)_s}{(q)_{n-r}(q)_{n+r}(q)_s(q)_s}q^{\binom{r+1}{2}+3rj+j+2s}\\ =&2m(-q^5,q^8,-1)+2q^{-3}m(-q^{-3},q^8,-1)+\frac{1}{\overline{J}_{1,1}\overline{J}_{0,8}}\theta_{1,2}(-q,-q^3,q).\end{aligned} \tag{3.11}$$

Set $l=1$ and $t=2$ in (3.5). Then the combination of (2.5), (2.7), and (2.14) gives that

$$\begin{aligned}\omega_5(1,2;q)=&\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{2j})(q^{-2j},q^{2j};q^2)_s}{(q)_{n-r}(q)_{n+r}(q^2;q^4)_s(q^2;q^2)_s}q^{n^2+\frac{r(r+3)}{2}+\frac{9}{2}rj-\frac{1}{8}j^2+\frac{5}{4}j+2s}\\ =&-q^{-1}m(-q^6,q^{24},-1)+\frac{1}{J_1\overline{J}_{0,24}}\theta_{1,2}(q^3,q^4,q^3).\end{aligned} \tag{3.12}$$

Set $l=2$ and $t=2$ in (3.6). Then from (2.8), (2.10), and (2.15), it follows that

$$\begin{aligned}\omega_6(2,2;q)=&\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{2j})(q^{-2j},q^{2j};q^2)_s}{(q)_{n-r}(q)_{n+r}(q^2;q^4)_s(q^2;q^2)_s}q^{n^2+\frac{r(r+3)}{2}+\frac{5}{2}rj-\frac{1}{8}j^2+\frac{1}{4}j+2s}\\ =&-q^{-5}m(-q^2,q^{48},-1)+q^{-15}m(-q^{-14},q^{48},-1)+m(-q^{26},q^{48},-1)\\ &-q^{-2}m(-q^{10},q^{48},-1)+\frac{1}{J_1\overline{J}_{0,48}}\theta_{3,2}(q^3,q^2,q).\end{aligned} \tag{3.13}$$

Set $l=1$ and $t=2$ in (3.7). Then in view of (2.11), (2.12), and (2.16), we derive that

$$\begin{aligned}\omega_7(1,2;q)=&\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{2j})(-q)_n(q^{-2j},q^{2j};q^2)_s}{(q)_{n-r}(q)_{n+r+1}(q^2;q^4)_s(q^2;q^2)_s}q^{\binom{n+1}{2}+\binom{r+1}{2}+4rj-\frac{1}{4}j^2+j+2s}\\ =&-q^{-1}m(q^8,q^{30},-1)-\frac{1}{J_{1,2}\overline{J}_{0,30}^2}\theta_{2,8,2}(q^2,q^3,q).\end{aligned} \tag{3.14}$$

Notice that the Appell-Lerch sums and the theta functions involved in (3.8)-(3.14) are well defined.

In fact, resorting to Theorems 1.1-1.3, we find that many other Bailey pairs given by Slater [20, 21] can be used to construct mock theta functions. Here we give the following tables to show the details.

Remark 3.2. *In Table 1, in each case $\alpha_0=1$. Meanwhile, the second and the forth Bailey pairs can only be applied to* (1.11) *and* (1.12) *in Theorem 1.1.*

Table 1. Applications of Theorems 1.1-1.2

	α_n $(n\geq 1)$	β_n	(a,m,k_1,k_2)	Theorem	Reference
1	$(-1)^nq^{\frac{3}{2}n^2}(q^{-\frac{1}{2}n}+q^{\frac{1}{2}n})$	$1/(q)_n$	$(1,1,\frac{3}{2},\frac{1}{2})$	Theorem 1.1	[20, B(1)]
2	$(-1)^nq^{\frac{3}{2}n^2}(q^{-\frac{3}{2}n}+q^{\frac{3}{2}n})$	$q^n/(q)_n$	$(1,1,\frac{3}{2},\frac{3}{2})$	Theorem 1.1	[20, B(2)]
3	$(-1)^nq^{n^2}(q^{-\frac{1}{2}n}+q^{\frac{1}{2}n})$	$1/(-q^{\frac{1}{2}})_n(q)_n$	$(1,1,1,\frac{1}{2})$	Theorem 1.1	[20]
4	$(-1)^nq^{-\frac{1}{2}n^2}(q^{-\frac{3}{2}n}+q^{\frac{3}{2}n})$	$(-1)^n/q^{n(n+3)/2}(q)_n$	$(1,1,-\frac{1}{2},\frac{3}{2})$	Theorem 1.1	[20]
5	$(-1)^nq^{-\frac{1}{2}n^2}(q^{-\frac{1}{2}n}+q^{\frac{1}{2}n})$	$(-1)^n/q^{n(n+1)/2}(q)_n$	$(1,1,-\frac{1}{2},\frac{1}{2})$	Theorem 1.1	[20]
6	$(-1)^n(q^{-n}+q^n)$	$(-1)^n/q^n(q)_n(-q)_n$	$(1,1,0,1)$	Theorem 1.1	[20]
7	$q^{-n}-q^n$	$1/q^n(q)_n(q)_n$	$(1,1,0,1)$	Theorem 1.2	[20]
8	$q^{\frac{1}{2}n^2}(q^{-\frac{1}{2}n}-q^{\frac{1}{2}n})$	$2/(q)_n(q)_n(1+q^n)$	$(1,1,\frac{1}{2},\frac{1}{2})$	Theorem 1.2	[20]
9	$q^{\frac{1}{2}n^2}(q^{\frac{1}{2}n}-q^{-\frac{1}{2}n})$	$2q^n/(q)_n(q)_n(1+q^n)$	$(1,1,\frac{1}{2},-\frac{1}{2})$	Theorem 1.2	[20]

Table 2. Applications of Theorem 1.3

	α_{2n} $(n\geq 1)$	β_n	(a,m,k_1,k_2)	Theorem	Reference
1	$(-1)^nq^{n^2}(q^{-n}+q^n)$	$q^{n(n-1)/2}/(q;q^2)_n(q)_n$	$(1,1,1,1)$	Theorem 1.3	[20, C(5)]
2	$(-1)^nq^{2n^2}(q^{-n}+q^n)$	$(-q^2;q^2)_{n-1}/(q;q^2)_n(q)_n(-q)_{n-1}$	$(1,1,2,1)$	Theorem 1.3	[21, I(14)]

Remark 3.3. *In Table 2, in each case $\alpha_0=1$ and $\alpha_{2r+1}=0$. Meanwhile, in the second Bailey pair $\beta_0=0$.*

To prove Corollary 1.4, we need the following transformation in [11].

Proposition 3.4. *([11, Theorem 3.3], [15]) For generic $x,z\in\mathbb{C}^*$,*

$$m(x,q,z_1)=m(x,q,z_0)+\triangle(x,q,z_1,z_0), \tag{3.15}$$

where

$$\triangle(x,q,z_1,z_0):=\frac{z_0J_1^3j(z_1/z_0;q)j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}.$$

Proof of Corollary 1.4. Equations (5.3), (5.36), and (5.37) in [11] are stated as

$$\mu(q)=4m(-q,q^4,-1)-\frac{J_{2,4}^4}{J_1^3}, \tag{3.16}$$

$$S_1(q)=-2q^{-1}m(-q,q^8,-1)+\frac{\overline{J}_{3,8}J_{2,8}^2}{qJ_{1,8}^2}, \tag{3.17}$$

and

$$T_0(q)=-m(-q^3,q^8,q^2). \tag{3.18}$$

Letting $l=1$ and $t=4$ in (3.1), we have

$$\omega_1(1,4;q)=\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{4j})(q^{-4j},q^{4j};q^4)_s(q^2;q^2)_{2s}}{(q)_{n-r}(q)_{n+r}(q^4;q^4)_{2s}}q^{n^2+\frac{r(r+3)}{2}+9rj+\frac{3}{2}j^2+\frac{5}{2}j+4s}. \tag{3.19}$$

From (2.7), it follows that

$$\omega_1(1,4;q) = -q^{-1}m(-q^6,q^{24},-1) + \frac{1}{J_1\overline{J}_{0,24}}\theta_{1,2}(q^3,q^4,q^3). \tag{3.20}$$

Comparing (3.16) with (3.20), we establish (1.13), where

$$M_1(q) := \frac{1}{J_1\overline{J}_{0,24}}\theta_{1,2}(q^3,q^4,q^3) - \frac{J_{12,24}^4}{4qJ_6^3}.$$

Similarly, the combination of (3.8) and (3.17) yields (1.14), where

$$M_2(q) := \frac{1}{J_1\overline{J}_{0,24}}\theta_{1,2}(q^3,q^5,q^3) - \frac{\overline{J}_{9,24}J_{6,24}^2}{2q^2J_{3,24}^2}.$$

Furthermore, letting $l=1$ and $t=-1$ in (3.4), we get

$$\omega_4(1,-1;q) = 2\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{n+r+j}(1+q^{-j})(q;q^2)_n(q^{-j},q^j;q^{-1})_s}{(q)_{n-r}(q)_{n+r}(q^{-1};q^{-1})_s(q^{-1};q^{-1})_s}q^{\binom{r+1}{2}+3rj+j^2+2j-2s}.$$

Applying (2.13) yields that

$$\omega_4(1,-1;q) = 4m(-q^3,q^8,-1) + \frac{1}{\overline{J}_{1,1}\overline{J}_{0,8}}\theta_{1,2}(-q,-q,q). \tag{3.21}$$

Together with (3.15), (3.18) and (3.21), we derive (1.15), where

$$M_3(q) := \frac{1}{\overline{J}_{1,1}\overline{J}_{0,8}}\theta_{1,2}(-q,-q,q) + 4\Delta(-q^3,q^8,-1,q^2).$$

We complete the proof. □

4. Applications

For two different Bailey pairs with the parameters (a,m,k_1,k_2) and (a',m',k_1',k_2'), applying Theorems 1.1-1.3, we get mock theta functions $W_i(l,t;q)$ and $W_i'(l',t';q)$, where $i=1,2,\ldots,7$. When $l=l'$ and $k_2t=k_2't'$, based on the expressions in terms of Hecke-type sums for $W_i(l,t;q)$ in the proofs of Theorems 1.1-1.3, we can establish transformations between $W_i(l,t;q)$ and $W_i'(l',t';q)$. Furthermore, comparing the proof of Theorem 1.1 with that of Theorem 1.3, we find that $W_1(l,t;q)$ and $W_5(l,t;q)$ have the same expression in terms of Hecke-type sums. In the same manner, we can obtain transformations between $W_1(l,t;q)$ and $W_5(l,t;q)$. Notice that $W_2(l,t;q)$ (resp. $W_3(l,t;q)$) and $W_6(l,t;q)$ (resp. $W_7(l,t;q)$) also have the same expression in terms of Hecke-type sums. Therefore, transformations can be built from these two pairs. In conclusion, for different Bailey pairs and different values for l and t, plenty of transformations can be obtained. Here we show two examples in the following theorem.

Theorem 4.1. *We have*

$$\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0}\frac{(-1)^{r+j}(1+q^{4j})(q^{-4j},q^{4j};q^4)_s(q^2;q^2)_{2s}}{(q)_{n-r}(q)_{n+r}(q^4;q^4)_{2s}}q^{n^2+\frac{r(r+3)}{2}+9rj+\frac{3}{2}j^2+\frac{5}{2}j+4s}$$

$$= \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1+q^{4j})(q^{-4j},q^{4j};q^4)_s}{(q)_{n-r}(q)_{n+r}(q^4;q^4)_s} q^{n^2+\frac{r(r+3)}{2}+9rj-5\binom{j}{2}+4s} \tag{4.1}$$

and

$$\sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1+q^{2j})(q^{-2j},q^{2j};q^2)_s(q)_{2s}}{(q)_{n-r}(q)_{n+r}(q^2;q^2)_{2s}} q^{n^2+\frac{r(r+3)}{2}+9rj+\frac{3}{2}j^2+\frac{7}{2}j+2s}$$
$$= \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1+q^{j})(q^{-j},q^{j};q)_s}{(q)_{n-r}(q)_{n+r}(q;q^2)_s(q)_s} q^{n^2+\frac{r(r+3)}{2}+\frac{9}{2}rj+\frac{1}{8}j^2+\frac{7}{4}j+s}. \tag{4.2}$$

Proof. Substituting the first Bailey pair in Table 1 into (1.10) yields that

$$\omega_1'(l,t;q) := \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1+q^{tj})(q^{-tj},q^{tj};q^t)_s}{(q)_{n-r}(q)_{n+r}(q^t;q^t)_s} \times q^{n^2+\frac{r(r+3)}{2}+3(2l+1)rj+(\frac{3}{2}-t)j^2+(3l+\frac{3}{2}-\frac{1}{2}t)j+ts},$$

where $l \in \mathbb{N}$, $t \equiv 2,4 \pmod 6$, $t \neq \frac{24kl(l+1)}{2l+1}$ for any $k \in \mathbb{Z}$, and l and t are generic for $\theta_{1,2l}(q^3, q^{3l+3-\frac{1}{2}t}, q^3)$. Then, setting $l = 1$ and $t = 4$ in the above functions, we have

$$\omega_1'(1,4;q) = \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1+q^{4j})(q^{-4j},q^{4j};q^4)_s}{(q)_{n-r}(q)_{n+r}(q^4;q^4)_s} q^{n^2+\frac{r(r+3)}{2}+9rj-5\binom{j}{2}+4s}.$$

Together with (2.5) and (3.19), we derive that

$$\omega_1(1,4;q) = \omega_1'(1,4;q) = \frac{1}{(q)_\infty} f_{1,3,1}(q^3,q^4,q^3).$$

Therefore, we prove (4.1).

Setting $l = 1$ and $t = 1$ in (3.5) yields that

$$\omega_5(1,1;q) = \sum_{n\geq r\geq 0}\sum_{j\geq s\geq 0} \frac{(-1)^{r+j}(1+q^{j})(q^{-j},q^{j};q)_s}{(q)_{n-r}(q)_{n+r}(q;q^2)_s(q)_s} q^{n^2+\frac{r(r+3)}{2}+\frac{9}{2}rj+\frac{1}{8}j^2+\frac{7}{4}j+s}.$$

Together with (2.5), (2.14), and (3.8), it follows that

$$\omega_1(1,2;q) = \omega_5(1,1;q) = \frac{1}{(q)_\infty} f_{1,3,1}(q^3,q^5,q^3).$$

Hence, we get (4.2). □

Acknowledgements: The authors would like to thank the referees for valuable comments. This work was supported by the National Natural Science Foundation of China and the PCSIRT Project of the Ministry of Education.

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