More on Borderenergetic Graphs*

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. If a graph G of order n has the same energy as the complete graph K_n, i.e., if $\mathcal{E}(G) = 2(n-1)$, then G is said to be borderenergetic. We obtain three asymptotically tight bounds on the edge number of borderenergetic graphs. Then, by using disconnected regular graphs we construct connected non-complete borderenergetic graphs.

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1 Introduction

All graphs considered in this paper are simple and undirected. Let G be such a graph with m edges, and $V(G) = \{v_1, v_2, \ldots, v_n\}$ its vertex set with $|V(G)| = n$. The complement of G is denoted by $\overline{G}$. The complete graph and the cycle of order n are denoted by K_n and C_n, respectively.

Let $\mathbf{A}(G)$ be an adjacency matrix of G and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $\mathbf{A}(G)$. These eigenvalues form the spectrum G, which is denoted by $Sp(G)$. A graph is said to be integral if all its eigenvalues are integers.

For details on spectral graph theory, see [2].

The energy of the graph G, denoted by $\mathcal{E}(G)$, is defined as [5, 6]

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| \, .$$

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For additional information on graph energy and its applications in chemistry, we refer to [6, 9, 10, 18].

Graphs of order n, whose energy exceeds the energy of the complete graph K_n, i.e., graphs for which $\mathcal{E}(G) > \mathcal{E}(K_n) = 2(n-1)$, have been named *hyperenergetic*; otherwise, graphs of order n with $\mathcal{E}(G) \leq \mathcal{E}(K_n) = 2(n-1)$ are called *non-hyperenergetic*. These graphs were studied in some detail; see e.g. [1, 11, 22]. For further review, see [7]. Graphs with energy less than or equal to the order n were also particularly investigated. Graphs of order n with energy less than n, $n-1$, are called *hypoenergetic* and *strong hypoenergetic*, respectively, and were studied; see e.g. [8, 15, 17]; whereas graphs of order n with energy equal to n were also studied; see e.g. [16].

Recently, Gong et al. [4] proposed the concept of *borderenergetic* graphs, namely graphs of order n satisfying $\mathcal{E}(G) = 2(n-1)$.

In a trivial manner, the complete graph is borderenergetic. We are, of course, interested in borderenergetic species different from K_n. Such graphs exist for all $n \geq 7$ [4]. Their numbers were determined for $n = 7, 8, 9$ [4] and $n = 10, 11$ [19, 21]. In [12], a family of non-regular and non-integral borderenergetic threshold graphs was discovered.

It is interesting to find more borderenergetic graphs, especially, connected and to establish their structural differences. So far, very little is known about such structural properties.

The paper is organized as follows. In Section 2, we obtain three asymptotically tight bounds on the number of edges of borderenergetic graphs. In Section 3, using disconnected regular graphs we construct connected non-complete borderenergetic graphs.

2 Bounds on the number of edges

Examples show [4, 19, 21] that the number of edges of borderenergetic graphs of fixed order n vary significantly. In this section, we offer some results that shed some more light on this phenomenon.

We first state the definition of the r-degree of a vertex and a previously known bound for graph energy, valid for general graphs. For an integer $r \geq 0$, the *r-degree* $d_r(v_i)$ of a vertex $v_i \in G$ is defined as the number of walks of length r starting at v_i. Clearly, one has $d_0(v_i) = 1, d_1(v_i) = d_i$ and $d_{r+1}(v_i) = \sum_{w \in N(v_i)} d_r(w)$, where $N(v_i)$ is the set of all neighbors of the vertex v_i. The following upper bound for graph energy on the r-degree is obtained in [13].

Lemma 1. [13] *Let G be a connected graph with n ($n \geq 2$) vertices and m edges. Then*

$$\mathcal{E}(G) \leq \sqrt{\frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)}} + \sqrt{(n-1)\left(2m - \frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)}\right)}.$$

Equality holds if and only if $G \cong K_n$ or G is a strongly regular graph with two nontrivial eigenvalues both with absolute value $\sqrt{(2m-(2m/n)^2)/(n-1)}$.

From it, we can derive the following result.

Theorem 2. *Let G be a borderenergetic graph. Then*

$$m \geq \left\lceil \frac{1}{2}\frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)} + \frac{1}{2(n-1)}\left(2(n-1) - \sqrt{\frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)}}\right)^2 \right\rceil. \qquad (1)$$

If G is $(n-3)$-regular, then the bound in (1) is asymptotically tight.

Proof. By Lemma 1 and $\mathcal{E}(G) = 2(n-1)$, we have

$$\left(2(n-1) - \sqrt{\frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)}}\right)^2 \leq (n-1)\left(2m - \frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)}\right).$$

So,

$$\frac{1}{2}\frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)} + \frac{1}{2(n-1)}\left(2(n-1) - \sqrt{\frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)}}\right)^2 \leq m\,.$$

As m is a positive integer, Ineq.(1) holds. Moreover, we find that the bound in (1) is asymptotically tight when G is $(n-3)$-regular (when $m = n(n-3)/2$). Then, when G is $(n-3)$-regular, by $d_{r+1}(v_i) = \sum_{w \in N(v_i)} d_r(v_i)$, for each vertex v_i, we have

$$d_r(v_i) = (n-3)^r, \quad d_{r+1}(v_i) = (n-3)^{r+1}$$

and

$$\frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)} = \frac{(n-3)^{2(r+1)}}{(n-3)^{2r}} = (n-3)^2\,.$$

Hence,

$$m \geq \left\lceil \frac{(n-3)^2}{2} + \frac{(n+1)^2}{2(n-1)} \right\rceil.$$

Thus,

$$\lim_{n\to\infty} \left\lceil \frac{n(n-3)}{2} \right\rceil^{-1} \left\lceil \frac{(n+1)^2}{2(n-1)} + \frac{(n-3)^2}{2} \right\rceil = 1\,.$$

□

For simplicity, in the following we replace the notation $d_2(v_i)$ and $d_3(v_i)$ by t_i and σ_i for $v_i \in V(G)$, respectively. Then from Theorem 2, we have the corollaries below directly.

Corollary 3. *Let G be a borderenergetic graph of order n. Then*

$$m \geq \left\lceil \frac{\left[2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\,\right]^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n} \right\rceil . \tag{2}$$

If G is $(n-3)$-regular, then the bound in (2) is asymptotically tight.

Corollary 4. *Let G be a borderenergetic graph. Then*

$$m \geq \left\lceil \frac{1}{2}\sum_{i=1}^{n} t_i^2 \Big/ \sum_{i=1}^{n} d_i^2 + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\sum_{i=1}^{n} t_i^2 \Big/ \sum_{i=1}^{n} d_i^2} \right)^2 \right\rceil . \tag{3}$$

If G is $(n-3)$-regular, then the bound in (3) is asymptotically tight.

Corollary 5. *Let G be a borderenergetic graph. Then*

$$m \geq \left\lceil \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 \Big/ \sum_{i=1}^{n} t_i^2 + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\sum_{i=1}^{n} \sigma_i^2 \Big/ \sum_{i=1}^{n} t_i^2} \right)^2 \right\rceil . \tag{4}$$

If G is $(n-3)$-regular, then the bound in (4) is asymptotically tight.

From Lemma 1, it is easy to verify that equalities in (1), (2) and (3) hold in the case of $G \cong K_n$. From the data given in Table 1, it can be seen that these bounds are reasonably good. Let

$$m^* = \left\lceil \frac{\left(2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\right)^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n} \right\rceil$$

$$\widetilde{m} = \left\lceil \frac{1}{2}\sum_{i=1}^{n} t_i^2 \Big/ \sum_{i=1}^{n} d_i^2 + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\sum_{i=1}^{n} t_i^2 \Big/ \sum_{i=1}^{n} d_i^2} \right)^2 \right\rceil$$

$$\widehat{m} = \left\lceil \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 \Big/ \sum_{i=1}^{n} t_i^2 + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\sum_{i=1}^{n} \sigma_i^2 \Big/ \sum_{i=1}^{n} t_i^2} \right)^2 \right\rceil .$$

A few borderenergetic graphs are listed in Table 1 and depicted in Fig. 1. These have been chosen among those determined in [4,19,21], so as to be connected and have the smallest number of edges.

G_i	n	m	m^*	$\widetilde{m}$	$\widehat{m}$	$m-m^*$	$m-\widetilde{m}$	$m-\widehat{m}$
G_0	7	17	17	17	17	0	0	0
G_1	8	18	17	17	17	1	1	1
G_2^0	9	18	17	17	17	1	1	1
G_2^1	9	18	17	17	17	1	1	1
G_3^0	10	23	22	22	22	1	1	1
G_3^1	10	23	22	22	22	1	1	1
G_3^2	10	23	22	22	22	1	1	1
G_3^3	10	23	22	22	22	1	1	1
G_4^0	11	25	24	24	24	1	1	1
G_4^1	11	25	23	23	23	2	2	2
G_4^2	11	25	24	24	24	1	1	1

Table 1. The parameters m, m^*, $\widetilde{m}$, $m-m^*$, $m-\widetilde{m}$ and $m-\widehat{m}$ of the borderenergetic graphs depicted in Fig. 1.

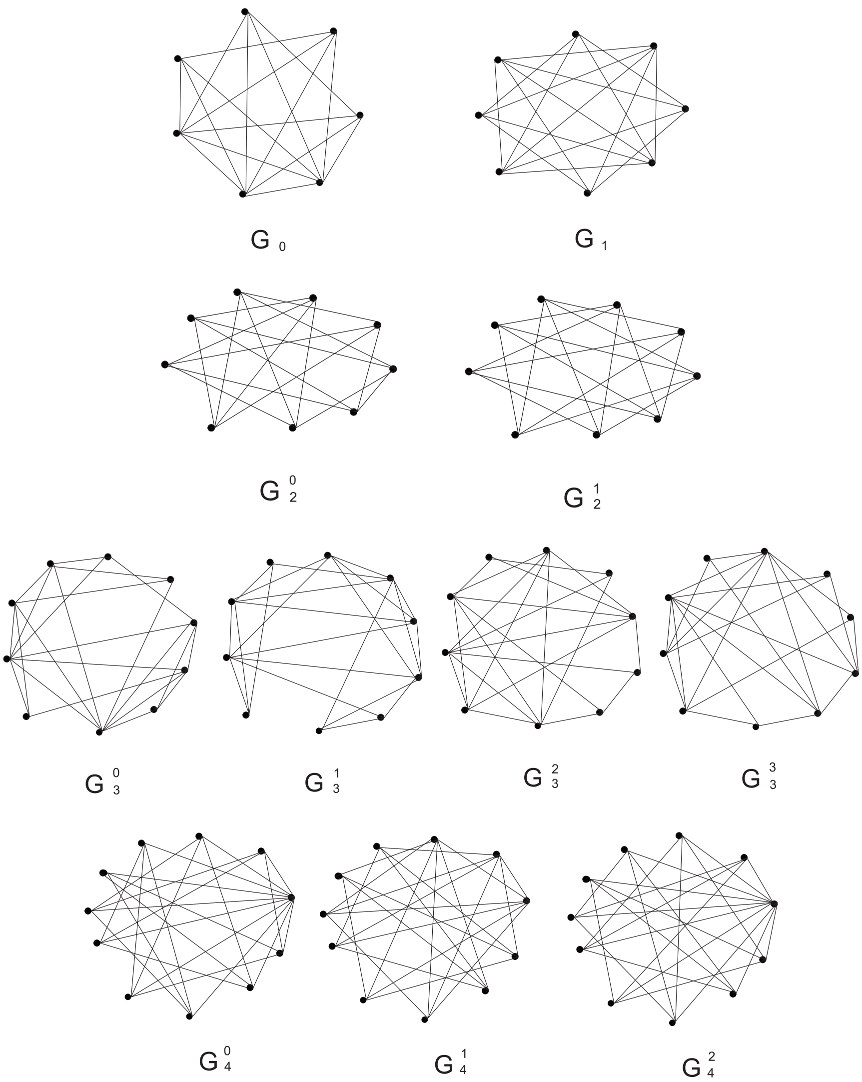

Fig. 1. The connected non-complete borderenergetic graphs listed in Table 1.

3 Constructing regular borderenergetic graphs

A class of non-complete connected $(n-3)$-regular borderenergetic graphs have been found by Gong et al. in [4].

Theorem 6. [4] *Let p, q, and r be non-negative integers, and let $p+q=2$. Then $\overline{pC_4 \cup qC_6 \cup rC_3}$ is borderenergetic.*

Corollary 7. [4] *For each integer n, $n \geq 7$, there exists a connected non-complete borderenergetic graph of order n.*

We now show how to construct connected non-complete $(n-1-k)$-regular $(k>2)$ borderenergetic graphs by using some k-regular graphs of small order.

Lemma 8. [3]. *Let G be a k-regular graph of order n with spectrum $Sp(G) = \{k, \lambda_2, \ldots, \lambda_n\}$. Then $Sp(\overline{G}) = \{n-1-k, -1-\lambda_2, \ldots, -1-\lambda_n\}$.*

Theorem 9. *Let G be a k-regular integral graph of order n with t non-negative eigenvalues. If $\mathcal{E}(G) = 2(n-t+k)$, then $\mathcal{E}(\overline{G}) = 2(n-1)$.*

Proof. Let $N = t-1$. By the above condition, we have

$$\mathcal{E}(G) = 2(n-t+k) = 2n + 2k - 2N - 2\,. \tag{5}$$

From Lemma 8 and $\lambda_1 = k$, the energy of the complement of G is

$$\begin{aligned}
\mathcal{E}(\overline{G}) &= n-1-k+\sum_{j=2}^{n}|1+\lambda_j| = n-1-k+\left(N+\sum_{j=2}^{t}\lambda_j\right)+\sum_{j=t+1}^{n}(-\lambda_j-1) \\
&= n-1-k+N+(n-1-N)(-1)+\left(\sum_{j=2}^{t}|\lambda_j|+\sum_{j=t+1}^{n}|\lambda_j|+k-k\right) \\
&= n-1-k+N+(n-1-N)(-1)+\big[\mathcal{E}(G)-k\big] = 2(n-1)
\end{aligned}$$

where $\mathcal{E}(G)$ is replaced by Eq. (5). □

By Theorem 9, an $(n-1-k)$-regular borderenergetic graph $\overline{G}$ can be constructed from a k-regular graph G. The graph G needs not be connected (see Examples 10 and 11).

Example 10. *G^0 is a connected 3-regular graph with 10 vertices whereas $\overline{G^0}$ is a connected 6-regular borderenergetic graph, see Fig. 2. Note that*

$$Sp(G^0) = \{3,-3,-2,-1,-1,0,0,1,1,2\} \qquad \text{and} \qquad \mathcal{E}(G^0) = 14 = 2(10-6+3)$$

whereas

$$Sp(\overline{G^0}) = \{6, 2, 1, 0, 0, -1, -1, -2, -2, -3\} \qquad \textit{and} \qquad \mathcal{E}(\overline{G^0}) = 2(10-1) = 18\,.$$

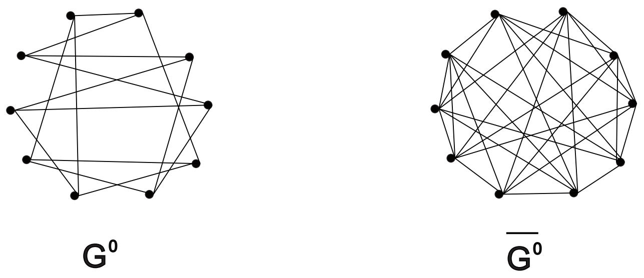

Fig. 2. The graphs from Example 10.

Example 11. *G^1 is a disconnected 2-regular graph with 8 vertices whereas $\overline{G^1}$ is a connected 5-regular borderenergetic graph, see Fig. 3. Note that*

$$Sp(G^1) = \{2, -2, -2, 0, 0, 0, 0, 2\} \qquad \textit{and} \qquad \mathcal{E}(G^1) = 8 = 2(8-6+2)$$

whereas

$$Sp(\overline{G^1}) = \{5, 1, 1, -1, -1, -1, -1, -3\} \qquad \textit{and} \qquad \mathcal{E}(\overline{G^1}) = 2(8-1) = 14\,.$$

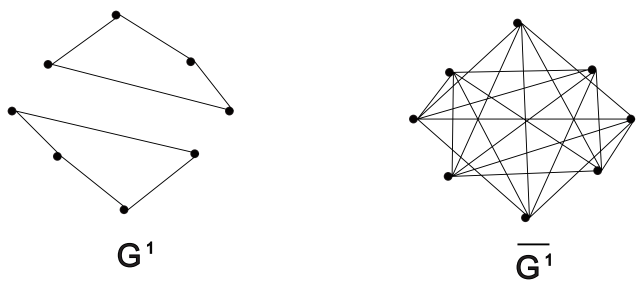

Fig. 3. The graphs from Example 11.

It is easy to find examples of disconnected borderenergetic graphs. A more interesting task is to construct connected non-complete borderenergetic graphs by starting from graphs of small order. Such a construction is achieved by means of the following theorem:

Theorem 12. *Let k be an even integer. Let $G = pG_1 \cup qK_{k+1}$ be a disconnected k-regular graph consisting of p copies of G_1 and q copies of K_{k+1}, where G_1 be a connected k-regular integral graph with $k+2$ vertices, having t_1 non-negative eigenvalues, and satisfying $\mathcal{E}(G_1) = 2k + 4 - 2t_1 + \frac{2k}{p}\,, p|2k\,,\ p \geq 1\,, q \geq 1$. Then $\overline{G}$ is a connected non-complete borderenergetic graph.*

Proof. Since $G = pG_1 \cup qK_{k+1}$ is k-regular, $\overline{G}$ is $\big[p(k+2)+q(k+1)-(k+1)\big]$-regular. Let $k = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k+2}$ be the eigenvalues of G_1 and recall that $Sp(K_{k+1}) = \{k, -1^{(k)}\}$. By Lemma 8, the eigenvalues of $\overline{G}$ are $p(k+2)+q(k+1)-(k+1)$ with multiplicity 1, $-k-1$ with multiplicity $p+q-1$, 0 with multiplicity kq, $-\lambda_2 - 1$ with multiplicity p, $-\lambda_3 - 1$ with multiplicity p, …, and $-\lambda_{k+2} - 1$ with multiplicity p. Thus,

$$
\begin{aligned}
\mathcal{E}(\overline{G}) &= p(k+2)+q(k+1)-(k+1)+(k+1)(p+q-1) \\
&+ \left(\sum_{i=1}^{t_1} |\lambda_i| + t_1\right) p + \left(\sum_{i=t_1+1}^{k+2} |\lambda_i| - k - 2 + t_1)\right) p - (|\lambda_1|+1)p \\
&= p(k+2)+q(k+1)-(k+1)+(k+1)(p+q-1) \\
&+ \big[\mathcal{E}(G_1) + (2t_1 - k - 2)\big]p - (k+1)p = 2(p(k+2)+q(k+1)-1)\,.
\end{aligned}
$$

As G is disconnected, $\overline{G}$ is connected. □

It is obvious that $p = 2$ satisfies the condition $p|2k$. The case of $k = 2$ has been discussed in [4]. In what follows, we separately consider the cases of $k = 4$ and $k = 6$, under the condition $p = 2$.

When $k = 4$, the 4-regular connected graph G_1^0 with 6 vertices is depicted in Fig.4, for which $Sp(G_1^0) = \{-2, -2, 0, 0, 0, 4\}$ and $\mathcal{E}(G_1^0) = 8 = 4 + 2 - 2 + 4$.

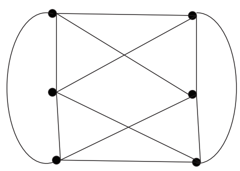

Fig. 4. The 4-regular graph G_1^0

Corollary 13. *For integer n ($n > 12$) satisfying $5|(n-12)$, there exists a connected non-complete $(n-5)$-regular borderenergetic graphs of order n.*

Proof. Let $G_1 = G_1^0$ and $q = \frac{n-12}{5}$. By Theorem 12, the graph $\overline{2G_1^0 \cup qK_5}$ is connected, non-complete, $(n-5)$-regular, and borderenergetic. □

When $k = 6$, the 6-regular connected graph G_1^1 with 8 vertices is depicted in Fig.5, for which $Sp(G_1^1) = \{-2, -2, -2, 0, 0, 0, 0, 6\}$ and $\mathcal{E}(G_1^1) = 12 = 6 + 2 - 2 + 6$.

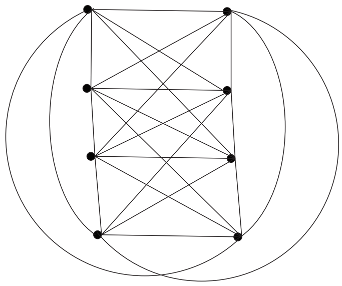

Fig. 5. The 6-regular graph G_1^1

Corollary 14. *For integer n ($n > 16$) satisfying $7|(n-16)$, there exists a connected non-complete $(n-7)$-regular bordernergetic graphs of order n.*

Proof. Let $G_1 = G_1^1$ and $q = \frac{n-12}{7}$. By Theorem 12, the graph $\overline{2G_1^1 \cup qK_7}$ is connected, non-complete, $(n-7)$-regular, and borderenergetic. □

At this point we note that for $k = 3$ or $k = 5$, the above described construction of connected borderenergetic graphs is not possible.

4 Concluding remarks

In one way we have constructed many families of borderenergetic graphs. In another way, we may think about finding some structural properties of non-borderenergetic graphs. This can exclude a lot of graph classes. For examples, we can easily get that any connected non-complete borderenergetic graph G is not a tree, a cycle, or a complete bipartite graph K_{n_1,n_2}, etc.

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