Rainbow connection number and independence number of a graph*

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Abstract

A path in an edge-colored graph is called rainbow if any two edges of the path have distinct colors. An edge-colored graph is called rainbow connected if there exists a rainbow path between every two vertices of the graph. For a connected graph G, the minimum number of colors that are needed to make G rainbow connected is called the rainbow connection number of G, denoted by $\mathrm{rc}(G)$. In this paper, we investigate the relation between the rainbow connection number and the independence number of a graph. We show that if G is a connected graph without pendant vertices, then $\mathrm{rc}(G) \leq 2\alpha(G) - 1$. An example is given showing that the upper bound $2\alpha(G) - 1$ is equal to the diameter of G, and so the upper bound is sharp since the diameter of G is a lower bound of $\mathrm{rc}(G)$.

Keywords: rainbow coloring, rainbow connection number, independence number, connected dominating set

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. The following notation and terminology are needed in the sequel. Let $u, v \in V$ be two distinct vertices of a graph $G = (V, E)$ with vertex set V and edge set E. The *distance between* u *and* v in G, denoted by $d(u, v)$, is the length of a shortest path connecting them in G. Let P be a path of G. We use $P_G[u, v]$ to denote the segment of P with u and v as its end-vertices.

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Let $E[U, W]$ denote the set of edges of G with one end in U and the other end in W, and let $e(U, W) = |E[U, W]|$. As usual, $G[U]$ denotes the subgraph of G induced by U. The following notions were introduced in [11]. A set $D \subseteq V(G)$ is called a *connected dominating set* of G, if $G[D]$ is connected and every vertex in $G \setminus D$ is at a distance 1 from D. A set $D \subseteq V(G)$ is called a *k-step connected dominating set* of G, if $G[D]$ is connected and every vertex in $G \setminus D$ is at a distance at most k from D. The *k-step open neighborhood of a set D* is $N^k(D) := \{x \in V(G) | d(x, D) = k\}$ and $k \in N$. We use $e(G)$ to denote the number of edges in a graph G and $|G|$ to denote the order of G. For undefined terminology and notation, we refer to [1].

A *k-edge-coloring* of a graph G is a mapping $c : E(G) \rightarrow \{1, 2, \ldots, k\}$ the set of colors. In [6], Chartrand et al. introduced a new concept relating to both the connectivity and the coloring of a graph. A path P of an edge-colored graph is called *rainbow* if every edge of P is colored by a distinct color. We say that an edge-colored graph is rainbow connected if, for every pair of vertices of the graph, there is a rainbow path connecting them. For a connected graph G, the *rainbow connection number* $\mathrm{rc}(G)$ is the smallest number of colors that are needed to make G rainbow connected. An edge-coloring of G is called *a rainbow coloring* if it makes G rainbow connected. From the definition of rainbow connection number, we can see that for any connected graph G, $\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq e(G)$. For more background on the rainbow connection, we refer to [13, 14].

In [4], Chakraborty et al. showed that given a graph G, deciding if $\mathrm{rc}(G) = 2$ is NP-complete, in particular, computing $\mathrm{rc}(G)$ is NP-hard, which were conjectured by Caro et al. [3]. There they also conjectured that if G is a connected graph with n vertices and $\delta(G) \geq 3$, then $\mathrm{rc}(G) < \frac{3}{4}n$. Schiermeyer [16] confirmed the conjecture and showed that if G is a connected graph with n vertices and $\delta(G) \geq 3$, then $\mathrm{rc}(G) \leq \frac{3n-1}{4}$. In [11], Krivelevich and Yuster showed that if G is a connected graph of order n with minimum degree $\delta(G)$, then $\mathrm{rc}(G) < \frac{20n}{\delta(G)}$, the result simplifies the relation between the rainbow connection number and the minimum degree of a graph. Later in [5], Chandran et al. showed that if G is a connected graph with minimum degree $\delta(G) \geq 2$ and D is a connected dominating set of G, then $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + 3$; furthermore, they showed that if G is a connected graph of order n with minimum degree $\delta(G)$, then $\mathrm{rc}(G) \leq 3n/(\delta(G) + 1) + 3$, and the bound is tight up to addictive factors. Then, Dong and Li in [9, 8] studied the relation between the rainbow connection number and the minimum degree sum. They showed that if G is a graph with k independent vertices, then $\mathrm{rc}(G) \leq \frac{3kn}{\sigma_k(G)+k} + 6k - 3$. In [2], Basavaraju et al. investigated the relation between the rainbow connection number and the radius of a bridgeless graph. They showed that for every bridgeless graph G with radius $\mathrm{rad}(G)$, $\mathrm{rc}(G) \leq \mathrm{rad}(G)(\mathrm{rad}(G) + 2)$, and gave an example showing that the bound is tight. Then, Li et al. in [12] and Ekstein et al. in [10] showed that if G is a 2-connected graph of order n ($n \geq 3$), then $\mathrm{rc}(G) \leq \lceil \frac{n}{2} \rceil$, and the upper bound is tight for $n \geq 4$, respectively. Furthermore, Li et al. [12] obtained the following result: for every $\kappa \geq 1$, if G is a κ-connected graph of order n, then for every $\epsilon \in (0, 1)$, $\mathrm{rc}(G) \leq (\frac{2+\epsilon}{\kappa})n + \frac{23}{\epsilon^2}$. The bound is not tight. They conjectured that for every $\kappa \geq 1$, if G is a κ-connected graph of order n, then $\mathrm{rc}(G) \leq \frac{n}{\kappa} + C$, where C is a constant. Schiermeyer [15] obtained a relation between the rainbow connection number of a graph G and the chromatic number of the complement of G, i.e., $\mathrm{rc}(G) \leq 2\chi(\bar{G}) - 1$.

This paper intends to give a relation between the rainbow connection number and the independence number of a graph. Recall that an *independent set* of a graph G is a set of vertices such that any two of these vertices are non-adjacent in G, and the *independence number* $\alpha(G)$ of G is the cardinality of a maximum independent set of G. Our result is stated as follows.

Theorem 1 *If G is a connected graph with $\delta(G) \geq 2$, then $rc(G) \leq 2\alpha(G) - 1$, and the bound is sharp.*

We give an example where the bound $2\alpha(G) - 1$ is exactly equal to the diameter of G, and therefore the bound is sharp since the diameter of G is a lower bound of $\mathrm{rc}(G)$.

Example : Let $P_{2t} = v_1 v_2 v_3 \cdots v_{2t-1} v_{2t}$ be a path of length $2t-1$, and let $G_1, G_2, \cdots, G_t$ be t $(t \geq 2)$ complete graphs with $|G_1| = 2$ and $|G_i| = s$ (a positive integer) for i with $2 \leq i \leq t$. For every i with $1 \leq i \leq t$, we join each vertex of G_i to every vertex of v_{2i-1} and v_{2i}. The obtained graph is denoted by G. One can see that G is connected with $\delta(G) = 3$, and $I(G) = \{v_2, v_4, v_6, \cdots, v_{2t}\}$ is a maximum independent set, that is, $\alpha(G) = t$. We also know that the distance $d(v_1, v_{2t}) = 2t - 1$. So, we can get that $\mathrm{rc}(G) \geq 2t - 1$. Now we use $2t - 1$ distinct colors to give G an edge-coloring. Let $1, 2, \cdots, 2t - 1$ be $2t - 1$ distinct colors. We use the $2t-1$ colors to color all the edges of P_{2t} with mutually distinct color. Then, we use color $2i - 1$ to color every edge of $E[V(G_i), \{v_{2i-1}, v_{2i}\}]$. Finally, we use color 1 to color every edge of $G[V(G_i)]$; see Figure 1.

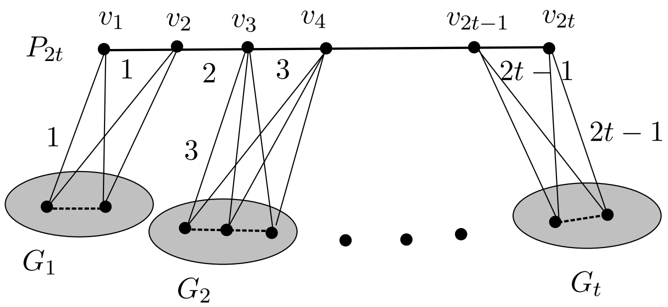

Figure 1: The graph for the Example.

One can show that G is rainbow connected. For each pair $(u, v) \in G_i \times P_{2t}$, either the edge uv_{2i} together with the path in P_{2t} connecting v_{2i} and v forms a rainbow path, or the edge uv_{2i-1} together with the path in P_{2t} connecting v_{2i-1} and v forms a rainbow path. For each pair $(u, v) \in G_i \times G_j$ with $1 \leq i < j \leq t$, the edges uv_{2i} and vv_{2j-1} together with the path in P_{2t} connecting v_{2i} and v_{2j-1} form a rainbow path. So, the graph G is rainbow connected and we can get $\mathrm{rc}(G) \leq 2t - 1$, and, since $\mathrm{diam}(G) = 2t - 1$, we can get $\mathrm{rc}(G) = 2t - 1 = 2\alpha(G) - 1$. Note that $\mathrm{rad}(G) = t$, $\delta(G) = 3$, and for any $v \in V(G) \setminus (\{v_1, v_2\} \cup V(G_1))$, the degree of v is at least $s + 1$.

2 Proof of Theorem 1

Proof of Theorem 1: If G is a complete graph, then $\alpha(G) = 1$ and $\mathrm{rc}(G) = 1$, and the statement of this theorem follows. Now assume that G is an incomplete graph with $\delta(G) \geq 2$.

We will perform the following procedure to obtain a tree T whose vertex set D is a connected dominating set of G. Let $y_0 \in V(G)$ with $d(y_0) = \delta(G)$. Since G is an incomplete graph, $N^2(y_0) \neq \emptyset$. We look at the following procedure:

Procedure 1

$D = \{y_0\}, T = y_0, X = \emptyset, Y = \{y_0\}$, X, Y partition D at any given step.
While $N^2(D) \neq \emptyset$
take any vertex $v \in N^2(D)$, let $P = vhu$ be a path of length 2,
where $h \in N^1(D)$ and $u \in D$. Let $D = D \cup V(P)$,
$T = T \cup P$, $X = X \cup \{h\}, Y = Y \cup \{v\}$.

At any given step, the set $N^2(D)$ does not contain any neighbor of Y, and if $u \in X$, we call u an X-knot vertex. When the above procedure ends, the algorithm has run $|X|$ rounds. Thus, we get $V(G) = D \cup N^1(D)$, where D is a connected dominating set. Note that Y is an independent set and $|Y| = |X| + 1$. So, $|Y| \leq \alpha(G)$ and $|D| = |Y| + |X| = 2|Y| - 1$. Note that T is a spanning tree of $G[D]$ and the pendant vertices of T are all in Y.

From [5], we know that $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + 3 \leq |D| + 2 = 2|Y| + 1$. So Procedure 1 directly implies that $\mathrm{rc}(G) \leq 2|Y| + 1 \leq 2\alpha(G) + 1$. Whenever we can show that either Y is not a maximum independent set, or $\mathrm{rc}(G) \leq |D|$, we are able to get that $\mathrm{rc}(G) \leq 2\alpha(G) - 1$.

In the following, the sets D, T, Y and X are always the same as those obtained in the above algorithm. We need the following claims in order to continue this proof.

Claim 1. If there exists a vertex $w \in N^1(D)$ such that $e(w, Y) = 0$, then $\mathrm{rc}(G) \leq 2\alpha(G) - 1$.

Proof. Let $I = Y \cup \{w\}$. Then I is an independent set and $|I| = |Y| + 1$. So, $|Y| = |I| - 1 \leq \alpha(G) - 1$. By $\mathrm{rc}(G) \leq rc(G[D]) + 3$, we can get that $\mathrm{rc}(G) \leq |D| - 1 + 3 \leq |D| + 2 = 2|Y| + 1$. Hence, $\mathrm{rc}(G) \leq 2(\alpha(G) - 1) + 1 = 2\alpha(G) - 1$. ∎

Note that from the proof of Claim 1, we can conclude that if we can find a larger independent set than Y, then $\mathrm{rc}(G) \leq 2\alpha(G) - 1$.

Claim 2. If $G[D] = T$, $\{y, y'\} \subset Y$ and $w, w' \in N^1(D)$ with $ww' \notin E(G)$, then

(1) If $e(w, Y) = 1$, $e(w', Y) = 1$, $e(w, X) = 0$ and $e(w', X) = 0$, then $\mathrm{rc}(G) \leq 2\alpha(G) - 1$.

(2) If $N(w_1) \cap D = N(w_2) \cap D = \{y, y'\}$, then $\mathrm{rc}(G) \leq 2\alpha(G) - 1$.

Proof. (1) Let $y, y' \in Y$, and $wy, w'y' \in E(G)$. Since $G[D] = T$, there is a unique path connecting y and y' in T, denoted by $P_T[y, y']$. If there do not exist two successive vertices of X on $P_T[y, y']$, then the following three parts form an independent set larger than Y: the first part is $\{w, w'\}$, the second part is the set of vertices of X on $P_T[y, y']$, and the third part is the set of vertices of Y except for the vertices on $P_T[y, y']$. Note that the vertices in each part are independent, and the vertices of these three parts are independent mutually. So we get an independent set larger than Y. From the proof of Claim 1, if we can find an independent set larger than Y, then we can get $\mathrm{rc}(G) \leq 2\alpha(G) - 1$. If there are two successive vertices of X on $P_T[y, y']$, by the structure of T we can conclude that one of these two vertices is an X-knot vertex; otherwise, from the structure of T we can get that the vertices of X and the vertices of Y appear alternately in T, a contradiction to the assumption that there are two successive vertices of X on $P_T[y, y']$. Then there is a segment on $P_T[y, y']$, without loss of generality, say $P_T[y, x] \subset P_T[y, y']$, such that x is an X-knot vertex, and there is a vertex x' of X on $P_T[y, x]$ adjacent to x, with x', x being the only two successive vertices on $P_T[y, x]$. Then the following three parts form an independent set larger than Y: the first part is $\{w\}$, the second part is set of vertices of X on $P_T[y, x)$ ($P_T[y, x) = P_T[y, x] \setminus \{x\}$), and the third part is the set of vertices of Y except for the vertices on $P_T[y, x)$. Note that the vertices in each part are independent, and the vertices of these three parts are independent mutually. So we get an independent set larger than Y. By the proof of Claim 1, we can get $\mathrm{rc}(G) \leq 2\alpha(G) - 1$.

(2) Since $G[D] = T$, there is a unique path connecting y and y' in T, denoted by $P_T[y, y']$. If there is no pair of successive vertices of X on $P_T[y, y']$, similarly as in the proof of (1), we get $\mathrm{rc}(G) \leq 2\alpha(G) - 1$. If there are two successive vertices of X on $P_T[y, y']$, similarly as in the proof of (1), the following three parts form an independent set larger than Y: the first part is $\{w, w'\}$, the second part is the set of vertices of X on $P_T[y, x)$, and the third part is the set of vertices of Y except for the vertices on $P_T[y, x)$. So, $\mathrm{rc}(G) \leq 2\alpha(G) - 1$. ∎

Let $N^1(D) = A \cup B$ and $A \cap B = \phi$, where $w \in A$ if and only if $e(w, D) \geq 2$, and $w \in B$ if and only if $e(w, D) = 1$. By Claim 1, we can assume that every vertex $w \in B$ satisfies $e(w, Y) = 1$. By Claim 2, we can assume that $G[B]$ is a complete subgraph. In the following text we distinguish two cases to complete the proof of Theorem 1.

Case 1. $e(G[D]) \geq e(T) + 1$.

Let $a_1a_2 \in E(G[D])$ and $a_1a_2 \notin E(T)$. Then $T \cup a_1a_2$ contains a cycle, say C and $a_1a_2 \in E(C)$. Let $G' = T \cup a_1a_2$. Since G' is a spanning subgraph of $G[D]$, let $V = V(G') = V(G[D])$, for any two vertices u, v of V, the number of paths in $G[D]$ passing u, v is not less than the number of paths in G' passing u, v, so $\mathrm{rc}(G[D]) \leq rc(G')$. Noticing that $\mathrm{rc}(G') \leq e(T) - (|C| - 1) + rc(C)$ and $\mathrm{rc}(C) \leq \lceil \frac{|C|}{2} \rceil$ when $|C| \geq 4$, we can get

$$rc(G') \leq \begin{cases} e(T) - \frac{|C|}{2} + 1, & |C| \text{ is even} \\ e(T) - \frac{|C|-3}{2}, & |C| \text{ is odd and } |C| \neq 3 \\ e(T) - 1, & |C| = 3 \end{cases}$$

Hence, $\mathrm{rc}(G[D]) \leq rc(G') \leq e(T) - 1$.

If $e(G[D]) \geq e(T)+2$, then $G[D])$ has at least two cycles, and from the above inequality we can get $\mathrm{rc}(G[D]) \leq e(T) - 2$. Thus, by Lemma 1 we have $\mathrm{rc}(G) \leq rc(G[D]) + 3 \leq e(T) + 1 = |D| = 2|Y| - 1 \leq 2\alpha(G) - 1$, and the statement of the theorem is true.

Next we show that if $e(G[D]) = e(T) + 1$, then $\mathrm{rc}(G) \leq 2\alpha(G) - 1$.

Suppose that the edge $aa' \in E(G[D])$ but $aa' \notin E(T)$. Let $D \setminus \{a'\} = D_1 \cup D_2$, where D_1 and D_2 induce connected components with $a \in D_2$. Since Y is an independent set, vertices a, a' cannot be in Y at the same time, and so one of them is in X, without loss of generality, we assume $a' \in X$. Let B_1 denote the subset of B such that $N(B_1) \cap D_2 = \emptyset$, and let B_2 denote the subset of B such that $N(B_2) \cap D_1 = \emptyset$. Note that B_1 or B_2 may be empty. Thus, by Claim 1, $B_1 \cap B_2 = \emptyset$ and $B = B_1 \cup B_2$. By Claim 2, we assume that both subgraphs $G[B_1]$ and $G[B_2]$ are complete graphs.

Now we color every edge of G and show that G is rainbow connected. First, we use $\mathrm{rc}(G[D])$ distinct colors to rainbow color $G[D]$. Then, let c', c'' be two fresh colors. For any vertex $w \in A$, let $w', w'' \in D$ with $ww', ww'' \in E(G)$, set $c(ww') = c'$ and $c(ww'') = c''$; for any edge $e \in E[B_1, D]$, set $c(e) = c''$; for any edge $e \in E[B_2, D]$, set $c(e) = c'$; see Figure 2.

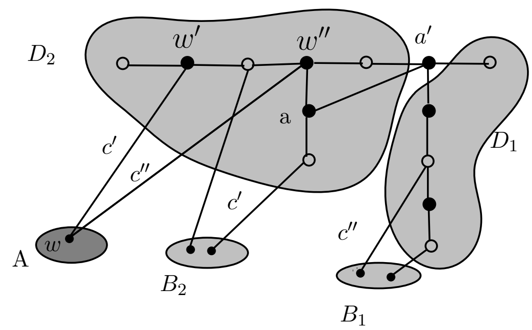

Figure 2: The graph for Case 1.

For the remaining uncolored edges of $E(G)$, we use a previously used color to color them. Thus, we have colored all the edges of G. We will show that G is rainbow connected. For each pair $(u, v) \in N(D) \times D$, the edge uu' together with the path in G' connecting u' and v forms a rainbow path, where $c(uu') = c'$ and $u' \in D$. For each pair $(u, v) \in A \times A$, the edges uu' and vv'' together with the path in G' connecting u' and v'' form a rainbow path, where $c(uu') = c'$ and $c(vv'') = c''$. For each pair $(u, v) \in A \times B_1$, the edges uu' and vv' together with the path in G' connecting u' and v' form a rainbow path, where $c(uu') = c'$. For each pair $(u, v) \in A \times B_2$, the edges uu' and vv' together with the path in G' connecting u'' and v' form a rainbow path, where $c(uu'') = c''$. For each pair $(u, v) \in B_1 \times B_2$, the edges uu' and vv' together with the path in G' connecting u' and v' form a rainbow path. Thus, we have showed that G is rainbow connected. In the above edge-coloring, we used at most $\text{rc}(G[D]) + 2 \leq e(T) + 1$ colors. Hence, $\text{rc}(G) \leq e(T) + 1$, that is, $\text{rc}(G) \leq |D|$. Since $|D| = 2|Y| - 1 \leq 2\alpha(G) - 1$, we get $\text{rc}(G) \leq 2\alpha(G) - 1$ and the theorem is true. ∎

Case 2. $e(G[D]) = e(T)$.

Choose a longest path P in the graph $G[D]$ such that the two ends of P are pendant vertices. We know that the two pendant vertices belong to Y, and $|P| \geq 3$. Let $P = y_1z_1z_2 \cdots z_ky_2$, where $z_1, z_2, \cdots, z_k \subsetneqq Y \cup X$. We distinguish two subcases to show that G is rainbow connected.

Subcase 2.1. $V(P) \subsetneqq D$.

Since P is a longest path and $V(P) \subsetneqq D$, we have $|P| \geq 4$. In T, we choose a pendant edge not in P, say y_3x. Let P' be a path in T passing through y_3x, and $V(P) \cap V(P') = \{z'\}$. Without loss of generality, let $|P[y_1, z']| \geq 3$.

We divide A into four disjoint subsets A_1, A_2, A_3 and A_4, and these four subsets satisfy the following conditions: vertex $w_1 \in A_1$ if and only if $w_1y_1 \in E(G)$ and w_1 is adjacent to only one vertex of $D \setminus \{y_1, y_2\}$; vertex $w_2 \in A_2$ if and only if $w_2y_2 \in E(G)$ and w_2 is adjacent to only one vertex of $D \setminus \{y_1, y_2\}$; vertex $w_3 \in A_3$ if and only if $w_3y_1 \in E(G)$, $w_3y_2 \in E(G)$ and $e(w_3, D) = 2$; vertex $w_4 \in A_4$ if and only if w_4 is adjacent to at least two vertices w'_4 and w''_4 of $D \setminus \{y_1, y_2\}$, note that any vertex of A_4 may be adjacent to vertex y_1 or y_2. Assume that the distance between w'_4 and y_1 in T is not more than the distance between w''_4 and y_1 in T. We divide B into three disjoint subsets B_1, B_2 and B_3, and the three subsets satisfy the following conditions: vertex $b_1 \in B_1$ if and only if b_1 is only adjacent to y_1; vertex $b_2 \in B_2$ if and only if b_2 is only adjacent to y_2; vertex $b_3 \in B_3$ if and only if b_3 is only adjacent to some vertex of $Y \setminus \{y_1, y_2\}$.

We use $e(T)$ colors to color all the edges of $G[D]$, and let $1, 2, c_1, c_2$ be four colors from the above $e(T)$ colors, and a be a new color. In the following we use a to color each edge of $E[B, D]$, and use c_1 to color each edge of graph $G[B]$. Set $c(y_1z_1) = 1$, $c(z_1z_2) = c_1$, $c(z_ky_2) = 2$ and $c(xy_3) = c_2$. For any vertex $w_1 \in A_1$, set $c(w_1y_1) = c_2$ and $c(w_1w'_1) = 2$ where $w'_1 \in D$; for any vertex $w_2 \in A_2$, set $c(w_2y_2) = c_2$ and $c(w_2w'_2) = 1$, where $w'_2 \in D$; for any vertex $w_3 \in A_3$, set $c(w_3y_1) = 2$ and $c(w_3y_2) = 1$; for any vertex $w_4 \in A_4$, set $c(w_4w'_4) = a$ and $c(w_4w''_4) = 1$. Then, we give the remaining uncolored edges a previously used color. Thus, we used $e(T) + 1$ colors finishing the edge-coloring of G; see Figure 3. From Claim 2 we can assume that both $G[A_3]$ and $G[B]$ are complete subgraphs.

In the following we show that when $B_1 \neq \emptyset$, $B_2 \neq \emptyset$ and $B_3 \neq \emptyset$, the graph G is rainbow connected.

We will show that any vertex of $N(D)$ is rainbow connected to every vertex of D. Here and in what follows, a vertex is rainbow connected to another vertex means that

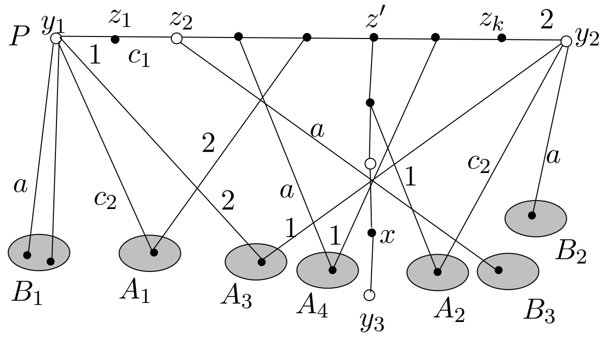

Figure 3: The graph for Subcase 2.1.

there is a rainbow path connecting them. For each pair $(u, v) \in B_1 \times D$, the edge uy_1 together with the path in T connecting y_1 and v form a rainbow path between u and v. For each pair $(u, v) \in B_2 \times D$, the edge uy_2 together with the path in T connecting y_2 and v form a rainbow path between u and v; for each pair $(u, v) \in B_3 \times D$, the edge uz_2 together with the path in T connecting z_2 and v form a rainbow path between u and v; for each pair $(u, v) \in A_1 \times y_3$, the edge ud together with the path in T connecting d and y_3 form a rainbow path between u and y_3, where $d \in D$ and $c(ud) = 2$; for each pair $(u, v) \in A_1 \times (D \setminus y_3)$, the edge uy_1 together with the path in T connecting y_1 and v form a rainbow path between u and v; for each pair $(u, y_3) \in A_2 \times y_3$, the edge ud together with the path in T connecting d and y_3 form a rainbow path between u and y_3, where $d \in D$ and $c(ud) = 1$; for each pair $(u, v) \in A_2 \times (D \setminus y_3)$, the edge uy_2 together with the path in T connecting y_2 and v form a rainbow path between u and v; for each pair $(u, v) \in A_3 \times (D \setminus y_2)$, the edge uy_1 together with the path in T connecting y_1 and v form a rainbow path between u and v, and uy_2 is a rainbow path; for each pair $(u, v) \in A_4 \times D$, the edge uy_1 together with the path in T connecting y_1 and v form a rainbow path between u and v. Thus we show that every vertex of $N(D)$ is rainbow connected to every vertex of D.

Now, we show that there exists a rainbow path connecting every two vertices of A_1, and the internal vertex of the rainbow path is not a vertex of B. For each pair $(u, v) \in A_1 \times A_1$, let $u', v' \in D$ with $uu', vv' \in E(G)$. If $u' \neq v'$, without loss of generality, we then assume that the path in T from v' to y_1 does not contain the edge y_3x. Thus, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path between u and v. If $u' = v'$ and $vy_2 \in E(G)$, then the edges uy_1 and vy_2 together with the path P form a rainbow path between u and v. If $u' = v'$ and $uy_2 \in E(G)$, similarly there is a rainbow path between them. If $u' = v'$ and assume that $vy_2 \notin E(G)$ and $uy_2 \notin E(G)$, then from Claim 2, we can get $uv \in E(G)$. So, for any two vertices of A_1 there is a rainbow path connecting them. Similarly, we can show that there is a rainbow path connecting any two vertices of A_2 or A_4, and the internal vertex of the rainbow path is not a vertex of B.

Now we show that for any vertex $u \in A_1$, there is a rainbow path connecting it to every vertex of $A_2 \cup A_3 \cup A_4 \cup B$. For each pair $(u, v) \in A_1 \times (A_2 \cup A_4)$, the edges uu' and vv'

together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 2$ and $c(vv') = 1$. For each pair $(u, v) \in A_1 \times A_3$, the path uy_1v is rainbow. For each pair $(u, v) \in A_1 \times B_1$, the path uy_1v is rainbow. For each pair $(u, v) \in A_1 \times B_2$, the edges uy_1 and vy_2 together with the path P form a rainbow path. For each pair $(u, v) \in A_1 \times B_3$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 2$ and $c(vv') = a$.

Next, we show that for any vertex $u \in A_2$, there is a rainbow path connecting it to every vertex of $A_3 \cup A_4 \cup B$. For each pair $(u, v) \in A_2 \times A_4$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$. For each pair $(u, v) \in A_2 \times A_3$, the path uy_2v is rainbow. For each pair $(u, v) \in A_2 \times B_1$, the edges uy_2 and vy_1 together with the path P form a rainbow path. For each pair $(u, v) \in A_2 \times B_2$, the path uy_2v is rainbow. For each pair $(u, v) \in A_2 \times B_3$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$.

Then, we show that for any vertex $u \in A_4$, there is a rainbow path connecting it to every vertex of B. For each pair $(u, v) \in A_3 \times A_4$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path, where $c(uy_1) = 2$ and $c(vv') = a$. For each pair $(u, v) \in A_3 \times B_1$, the path uy_1v is rainbow. For each pair $(u, v) \in A_3 \times B_2$, the path uy_2v is rainbow. For each pair $(u, v) \in A_3 \times B_3$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path.

Finally, we show that for any vertex $u \in A_4$, there is a rainbow path connecting it to every vertex of B. For each pair $(u, v) \in A_4 \times B_1$, the edges uu', vb_2 and b_2y_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = 1$ and $b_2 \in B_2$. For each pair $(u, v) \in A_4 \times B_2$, the edges uu' and vy_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = 1$. For each pair $(u, v) \in A_4 \times B_3$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$. So, when $B_1 \neq \emptyset$, $B_2 \neq \emptyset$ and $B_3 \neq \emptyset$, the graph G is rainbow connected.

From the proof above, we can see the following facts: for any vertex of A there is a rainbow path connecting it to every vertex of G, and the internal vertex of the rainbow path is not a vertex of B; for any vertex of B_2, there is a rainbow path connecting it to every vertex of G, and the rainbow path does not contain any vertex of $B_1 \cup B_3$; for any vertex of B_3, there is a rainbow path connecting it to every vertex of G, and the rainbow path does not contain any vertex of $B_1 \cup B_2$.

Hence, in the following we can assume that $B_3 = \emptyset$ and $B_2 = \emptyset$. When $B_1 = \emptyset$, it is not difficult to show that G is rainbow connected. When $B_1 \neq \emptyset$, we still color the edges of G in the above way except for setting $c(w_4w_4') = a$ and $c(w_4w_4'') = 2$. Thus, we only need to show that for any vertex of A_4, there is a rainbow path connecting it to every vertex of G. We will give the proof as follows. For each pair $(u, v) \in A_4 \times A_4$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where

$c(uu') = a$ and $c(vv') = 2$. For each pair $(u, v) \in A_4 \times A_3$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in A_4 \times A_2$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = a$ and $c(vv') = 1$. For each pair $(u, v) \in A_4 \times A_1$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = a$ and $c(vv') = 2$. For each pair $(u, v) \in A_4 \times B_1$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = 2$. So, when $B_1 = \emptyset$ or $B_1 \neq \emptyset$, we have showed that G is rainbow connected. ∎

Thus, we have showed that, when $V(P) \subsetneqq D$, the graph G is rainbow connected.

Subcase 2.2. $V(P) = D$.

Since $V(P) = D$ and $|Y| = |X| + 1$, the path P is (Y, X)-alternate and $|P|$ is odd. Let A_1, A_2, A_3, A_4, B_1, B_2 and B_3 be the above mentioned subsets.

If $|P| = 3$, we can get that $A_4 = \emptyset$, $B_3 = \emptyset$, and $G[A_1 \cup B_1]$ and $G[A_2 \cup B_2]$ are complete subgraphs. Let $P = y_1x_1y_2$. We can easily show that G is rainbow connected. In fact, we use color 1 to color the edge y_1x_1 and use color 2 to color the edge y_2x_1. For any vertices $w_1 \in A_1$ and $w_2 \in A_2$, set $c(w_1y_1) = a$, $c(w_1z_1) = 1$, $c(w_2y_2) = a$ and $c(w_2z_1) = 2$. It is obvious that for any vertex of $A \cup B$, there is a rainbow path connecting it to every vertex of P. For each pair $(u, v) \in A_1 \times A_2$, the path ux_1v is rainbow. For each pair $(u, v) \in A_1 \times A_3$, the path uy_1v is rainbow. For each pair $(u, v) \in A_1 \times B_2$, the path ux_1y_2v is rainbow. For each pair $(u, v) \in A_2 \times A_3$, the path uy_2v is rainbow. For each pair $(u, v) \in A_2 \times B_1$, the path ux_1y_1v is rainbow. For each pair $(u, v) \in A_3 \times B_1$, the path uy_1v is rainbow. For each pair $(u, v) \in A_3 \times B_2$, the path uy_2v is rainbow. So, the graph G is rainbow connected.

So, we can assume $|P| \geq 5$. Set $c(y_1z_1) = 1$, $c(z_1z_2) = c_1$, $c(y_2z_k) = 2$ and $c(z_kz_{k-1}) = c_2$. We color the edges of G in the following way: use a to color each edge of $E[B, D]$, and use c_1 to color each edge of $G[B]$. For any vertex $w_1 \in A_1$, set $c(w_1y_1) = 2$ and $c(w_1w_1') = a$, where $w_1' \in D$; for any vertex $w_2 \in A_2$, set $c(w_2y_2) = 1$ and $c(w_2w_2') = a$, where $w_2' \in D$; for any vertex $w_3 \in A_3$, set $c(w_3y_1) = 2$ and $c(w_3y_2) = 1$; for any vertex $w_4 \in A_4$, assume that the distance between w_4' and y_1 in P is not more than the distance between w_4'' and y_1 in P, and set $c(w_4w_4') = a$ and $c(w_4w_4'') = 1$, where $w_4', w_4'' \in D$; see Figure 4.

In the following we distinguish three cases to continue the proof of Theorem 1.

Subcase 2.2.1. $B_1 \neq \emptyset$, $B_2 \neq \emptyset$ and $B_3 \neq \emptyset$.

It is easy to check that for any vertex of $A \cup B$, there is a rainbow path connecting it to every vertex of P. For each pair $(u, v) \in A_1 \times A_1$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path, where $c(uy_1) = 2$ and $c(vv') = a$. For each pair $(u, v) \in A_2 \times A_2$, the edges uy_2 and vv' together with the path in T connecting y_2 and v' form a rainbow path, where $c(uy_2) = 1$ and $c(vv') = a$. For each pair $(u, v) \in A_4 \times A_4$, the edges uu' and vv' together with the path in T connecting

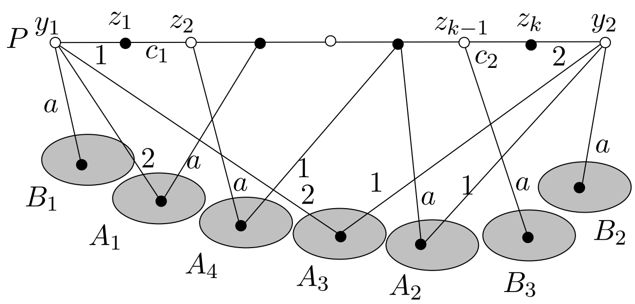

Figure 4: The graph for subcase 2.2.

u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$.

For each pair $(u, v) \in A_1 \times A_2$, the edges uu' and vy_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in A_1 \times A_3$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in A_1 \times A_4$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path, where $c(vv') = a$. For each pair $(u, v) \in A_1 \times B_1$, the path uy_1v is rainbow. For each pair $(u, v) \in A_1 \times B_2$, the path uy_1b_1v is rainbow, where $b_1 \in B_1$. For each pair $(u, v) \in A_1 \times B_3$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path. So, for any vertex of A_1 there is a rainbow path connecting it to every vertex of $A_2 \cup A_3 \cup A_4 \cup B$.

For each pair $(u, v) \in A_2 \times A_3$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in A_2 \times A_4$, the edges uy_2 and vv' together with the path in T connecting y_2 and v' form a rainbow path, where $c(vv') = a$. For each pair $(u, v) \in A_2 \times B_1$, the path uy_2b_2v is rainbow, where $b_2 \in B_2$. For each pair $(u, v) \in A_2 \times B_2$, the path uy_2v is rainbow. For each pair $(u, v) \in A_2 \times B_3$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path. So, for any vertex of A_2 there is a rainbow path connecting it to every vertex of $A_3 \cup A_4 \cup B$.

For each pair $(u, v) \in A_3 \times A_4$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path, where $c(vv') = a$. For each pair $(u, v) \in A_3 \times B_1$, the path uy_1v is rainbow. For each pair $(u, v) \in A_3 \times B_2$, the path uy_2v is rainbow. For each pair $(u, v) \in A_3 \times B_3$, the edges uy_1 and vv' together with the path in T connecting y_1 and v' form a rainbow path. So, for any vertex of A_3 there is a rainbow path connecting it to every vertex of $A_4 \cup B$.

For each pair $(u, v) \in A_4 \times B_1$, the edges uu', vb_2 and b_2y_2 together with the path in T connecting y_2 and u' form a rainbow path, where $c(uu') = 1$. For each pair $(u, v) \in A_4 \times B_2$, the edges uu' and vy_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = 1$. For each pair $(u, v) \in A_4 \times B_3$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$. So, for any vertex of A_4 there is a rainbow path connecting it to every vertex of B. Thus, the graph G is rainbow connected.

From the proof above, we can see the following facts: for any vertex of A there is a rainbow path connecting it to every vertex of G, and the internal vertex of the rainbow path is not a vertex of B; for any vertex of B_3, there is a rainbow path connecting it to every vertex of G, and the rainbow path does not contain any vertex of $B_1 \cup B_2$. So, in the following proof we can assume $B_3 = \emptyset$.

Subcase 2.2.2 $B_1 = \emptyset$ and $B_2 \neq \emptyset$

We still make use of the above way of coloring except for the edges of $E[A_1, D]$. We now color the edges of $E[A_1, D]$ in the following ways. For any vertex $w_1 \in A_1$, if $w_1z_1 \in E(G)$ then set $c(w_1y_1) = a$ and $c(w_1z_1) = 1$; if $w_1z_1 \notin E(G)$, let $w_1' \in D \setminus \{y_1, z_1, y_2\}$ with $w_1w_1' \in E(G)$, and let $P[y_1, w_1']$ be a subpath of P, $z \in V(P[y_1, w_1'])$ with $zw_1' \in E(G)$, then set $c(w_1y_1) = c(zw_1')$ and $c(w_1w_1') = 1$. From the edge-coloring, one can easily check that there is a rainbow path connecting every two vertices of A_1. For each pair $(u, v) \in A_1 \times (A_2 \cup A_4)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 1$ and $c(vv') = a$. For each pair $(u, v) \in A_1 \times A_3$, the path uy_1v is rainbow. For each pair $(u, v) \in A_1 \times B_2$, the edges uu' and vy_2 together with the path in T connecting u' and y_2 form a rainbow path, where $c(uu') = 1$. So, there is a rainbow path connecting any vertex of A_1 to every vertex of $A_2 \cup A_3 \cup A_4 \cup B$. Thus, the graph G is rainbow connected.

Subcase 2.2.3. $B_1 \neq \emptyset$ and $B_2 = \emptyset$.

We still make use of the above way of coloring except for the edges of $E[A_2, D]$ and the edges of $E[A_4, D]$. For any vertex $w_4 \in A_4$, set $c(w_4w_4') = a$ and $c(w_4w_4'') = 2$; for any vertex $w_2 \in A_2$, we will color the edges of $E[A_2, D]$ in the following way: if $w_2x_2 \in E(G)$ then set $c(w_2y_2) = a$ and $c(w_2x_2) = 2$; if $w_2x_2 \notin E(G)$, let $w_2' \in D \setminus \{y_1, x_2, y_2\}$ with $w_2w_2' \in E(G)$, and let $P[y_2, w_2']$ be a subpath of P, $z' \in V(P[y_2, w_2'])$ with $z'w_2' \in E(G)$, then set $c(w_2y_2) = c(z'w_2')$. One can easily check that there are rainbow paths connecting every two vertices of A_2 and A_4, respectively. For each pair $(u, v) \in A_2 \times (A_1 \cup A_4)$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 2$ and $c(vv') = a$. For each pair $(u, v) \in A_2 \times A_3$, the path uy_2v is rainbow. For each pair $(u, v) \in A_2 \times B_1$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = 2$. So, there is a rainbow path connecting any vertex of A_2 to every vertex of $A_1 \cup A_3 \cup A_4 \cup B$. For each pair $(u, v) \in A_4 \times A_1$, the edges uu' and vv' together with the path in T connecting u' and v' form a rainbow path, where $c(uu') = 2$ and $c(vv') = a$. For each pair $(u, v) \in A_4 \times A_3$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = a$. For each pair $(u, v) \in A_4 \times B_1$, the edges uu' and vy_1 together with the path in T connecting u' and y_1 form a rainbow path, where $c(uu') = 2$. So, for any vertex of A_4 there is a rainbow path connecting it to every vertex of $A_1 \cup A_3 \cup B$. Thus, the graph G is rainbow connected.

In the above coloring, we used $e(T)+1$ colors. So, we get $\mathrm{rc}(G) \leq e(T)+1$, and hence we have $\mathrm{rc}(G) \leq 2\alpha(G) - 1$. Combining the above Cases 1 and 2, we have completed the proof of Theorem 1. ∎

For a graph G, we can partition it into cliques, which means that the vertex-set of G is partitioned into a set of disjoint subsets $V_1, V_2, \ldots, V_p$ such that each V_i induces a clique of G. We call it a p-clique-partition of G if the number of cliques in a partition is p. Then, from the definition of the independence number $\alpha(G)$ of G we know that $\alpha(G) \leq p$ for any p-clique-partition of G. On the other hand, since the color-classes of any proper vertex-coloring of the complement $\bar{G}$ of G form a partition of the vertex-set of G that corresponds to a clique-partition of G, then a proper vertex-coloring of $\bar{G}$ with $\chi(\bar{G})$ colors will correspond to a $\chi(\bar{G})$-clique-partition of G, and hence $\alpha(G) \leq \chi(\bar{G})$. Therefore, we can get the following corollary, which is Theorem 10 of [15].

Corollary 1 *(Theorem 10, [15]) Let G be a connected graph with chromatic number $\chi(G)$. Then $rc(G) \leq 2\chi(\bar{G}) - 1$.*

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