The (vertex-)monochromatic index of a graph*

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Abstract

A tree T in an edge-colored (vertex-colored) graph H is called a *monochromatic (vertex-monochromatic) tree* if all the edges (internal vertices) of T have the same color. For $S \subseteq V(H)$, a *monochromatic (vertex-monochromatic) S-tree* in H is a monochromatic (vertex-monochromatic) tree of H containing the vertices of S. For a connected graph G and a given integer k with $2 \leq k \leq |V(G)|$, the *k-monochromatic index $mx_k(G)$ (k-monochromatic vertex-index $mvx_k(G)$)* of G is the maximum number of colors needed such that for each subset $S \subseteq V(G)$ of k vertices, there exists a monochromatic (vertex-monochromatic) S-tree. For $k = 2$, Caro and Yuster showed that $mc(G) = mx_2(G) = |E(G)| - |V(G)| + 2$ for many graphs, but it is not true in general. In this paper, we show that for $k \geq 3$, $mx_k(G) = |E(G)| - |V(G)| + 2$ holds for any connected graph G, completely determining the value. However, for the vertex-version $mvx_k(G)$ things will change tremendously. We show that for a given connected graph G, and a positive integer L with $L \leq |V(G)|$, to decide whether $mvx_k(G) \geq L$ is NP-complete for each integer k such that $2 \leq k \leq |V(G)|$. Finally, we obtain some Nordhaus-Gaddum-type results for the k-monochromatic vertex-index.

Keywords: k-monochromatic index, k-monochromatic vertex-index, NP-complete, Nordhaus-Gaddum-type result.

AMS subject classification 2010: 05C15, 05C40, 68Q17, 68Q25, 68R10.

1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty [1]. A path in an edge-colored graph H is a *monochromatic path* if all the edges of the path are colored with the same color. The

*Supported by NSFC No.11371205 and 11531011, "973" program No.2013CB834204, and PCSIRT.

graph H is called *monochromatically connected* if for any two vertices of H there exists a monochromatic path connecting them. An edge-coloring of H is a *monochromatically connecting coloring* (*MC-coloring*) if it makes H monochromatically connected. How colorful can an MC-coloring be? This question is the natural opposite of the well-studied problem of rainbow connecting coloring [4, 6, 10, 12, 13], where in the latter we seek to find an edge-coloring with minimum number of colors so that there is a rainbow path joining any two vertices. For a connected graph G, the *monochromatic connection number* of G, denoted by $mc(G)$, is the maximum number of colors that are needed in order to make G monochromatically connected. An *extremal MC-coloring* is an MC-coloring that uses $mc(G)$ colors. These above concepts were introduced by Caro and Yuster in [5]. They obtained some nontrivial lower and upper bounds for $mc(G)$. Later, Cai et al. in [2] obtained two kinds of Erdős-Gallai-type results for $mc(G)$.

In this paper, we generalize the concept of a monochromatic path to a monochromatic tree. In this way, we can give the monochromatic connection number a natural generalization. A tree T in an edge-colored graph H is called a *monochromatic tree* if all the edges of T have the same color. For an $S \subseteq V(H)$, a *monochromatic S-tree* in H is a monochromatic tree of H containing the vertices of S. Given an integer k with $2 \leq k \leq |V(H)|$, the graph H is called *k-monochromatically connected* if for any set S of k vertices of H, there exists a monochromatic S-tree in H. For a connected graph G and a given integer k such that $2 \leq k \leq |V(G)|$, the *k-monochromatic index* $mx_k(G)$ of G is the maximum number of colors that are needed in order to make G k-monochromatically connected. An edge-coloring of G is called a *k-monochromatically connecting coloring* (*MX_k-coloring*) if it makes G k-monochromatically connected. An *extremal MX_k-coloring* is an MX_k-coloring that uses $mx_k(G)$ colors. When $k = 2$, we have $mx_2(G) = mc(G)$. Obviously, we have $mx_{|V(G)|}(G) \leq \ldots \leq mx_3(G) \leq mc(G)$.

There is a vertex version of the monochromatic connection number, which was introduced by Cai et al. in [3]. A path in a vertex-colored graph H is a *vertex-monochromatic path* if its internal vertices are colored with the same color. The graph H is called *monochromatically vertex-connected*, if for any two vertices of H there exists a vertex-monochromatic path connecting them. For a connected graph G, the *monochromatic vertex-connection number* of G, denoted by $mvc(G)$, is the maximum number of colors that are needed in order to make G monochromatically vertex-connected. A vertex-coloring of G is a *monochromatically vertex-connecting coloring* (*MVC-coloring*) if it makes G

monochromatically vertex-connected. An *extremal MVC-coloring* is an MVC-coloring that uses $mvc(G)$ colors. This k-monochromatic index can also have a natural vertex version. A tree T in a vertex-colored graph H is called a *vertex-monochromatic tree* if its internal vertices have the same color. For an $S \subseteq V(H)$, a *vertex-monochromatic S-tree* in H is a vertex-monochromatic tree of H containing the vertices of S. Given an integer k with $2 \leq k \leq |V(H)|$, the graph H is called *k-monochromatically vertex-connected* if for any set S of k vertices of H, there exists a vertex-monochromatic S-tree in H. For a connected graph G and a given integer k such that $2 \leq k \leq |V(G)|$, the *k-monochromatic vertex-index* $mvx_k(G)$ of G is the maximum number of colors that are needed in order to make G k-monochromatically vertex-connected. A vertex-coloring of G is called a *k-monochromatically vertex-connecting coloring* (*MVX_k-coloring*) if it makes G k-monochromatically vertex-connected. An *extremal MVX_k-coloring* is an MVX_k-coloring that uses $mvx_k(G)$ colors. When $k = 2$, we have $mvx_2(G) = mvc(G)$. Obviously, we have $mvx_{|V(G)|}(G) \leq \ldots \leq mvx_3(G) \leq mvc(G)$.

A *Nordhaus-Gaddum-type result* is a (tight) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The Nordhaus-Gaddum-type is given because Nordhaus and Gaddum [14] first established the following inequalities for the chromatic numbers of graphs: If G and $\overline{G}$ are complementary graphs on n vertices whose chromatic numbers are $\chi(G)$ and $\chi(\overline{G})$, respectively, then $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1$. Since then, many analogous inequalities of other graph parameters are concerned, such as domination number [9], Wiener index and some other chemical indices [15], rainbow connection number [7], and so on.

For $k = 2$, Caro and Yuster [5] showed that $mc(G) = mx_2(G) = |E(G)| - |V(G)| + 2$ for many graphs, but it is not true in general. In this paper, we show that for $k \geq 3$, $mx_k(G) = |E(G)| - |V(G)| + 2$ holds for any connected graph G, completely determining the value. However, for the vertex-version $mvx_k(G)$ things will change tremendously. We show that for a given a connected graph G, and a positive integer L with $L \leq |V(G)|$, to decide whether $mvx_k(G) \geq L$ is NP-complete for each integer k such that $2 \leq k \leq |V(G)|$. Finally, we obtain some Nordhaus-Gaddum-type results for the k-monochromatic vertex-index.

2 Determining $mx_k(G)$

Let G be a connected graph with n vertices and m edges. In this section, we mainly study $mx_k(G)$ for each k with $3 \leq k \leq n$. A straightforward lower bound for $mx_k(G)$ is $m-n+2$. Just give the edges of a spanning tree of G with one color, and give each of the remaining edges a distinct new color. A property of an extremal MX_k-coloring is that the set of edges of each color induces a tree for any k with $3 \leq k \leq n$. In fact, if an MX_k-coloring contains a monochromatic cycle, we can choose any edge of this cycle and give it a new color while still maintaining an MX_k-coloring; if the subgraph induced by the edges with a given color is disconnected, then we can give the edges of one component with a new color while still maintaining an MX_k-coloring for each k with $3 \leq k \leq n$. Then, we use *color tree* T_c to denote the tree consisting of the edges colored with c. The color c is called *nontrivial* if T_c has at least two edges; otherwise c is called *trivial*. We now introduce the definition of *a simple extremal MX_k-coloring*, which is generalized of *a simple extremal MC-coloring* defined in [5].

Call an extremal MX_k-coloring *simple* for a k with $3 \leq k \leq n$, if for any two nontrivial colors c and d, the corresponding T_c and T_d intersect in at most one vertex. The following lemma shows that a simple extremal MX_k-coloring always exists.

Lemma 2.1. *Every connected graph G on n vertices has a simple extremal MX_k-coloring for each k with $3 \leq k \leq n$.*

Proof. Let f be an extremal MX_k-coloring with the most number of trivial colors for each k with $3 \leq k \leq n$. Suppose f is not simple. By contradiction, assume that c and d are two nontrivial colors such that T_c and T_d contain p common vertices with $p \geq 2$. Let $H = T_c \cup T_d$. Then, H is connected. Moreover, $|V(H)| = |V(T_c)| + |V(T_d)| - p$, and $|E(H)| = |V(T_c)| + |V(T_d)| - 2$. Now color a spanning tree of H with c, and give each of the remaining $p-1$ edges of H distinct new colors. The new coloring is also an MX_k-coloring for each k with $3 \leq k \leq n$. If $p > 2$, then the new coloring uses more colors than f, contradicting that f is extremal. If $p = 2$, then the new coloring uses the same number of colors as f but more trivial colors, contracting that f contains the most number of trivial colors. □

By using this lemma, we can completely determine $mx_k(G)$ for each k with $3 \leq k \leq n$.

Theorem 2.2. *Let G be a connected graph with n vertices and m edges, then $mx_k(G) = m - n + 2$ for each k with $3 \le k \le n$.*

Proof. Let f be a simple extremal MX_3-coloring of G. Choose a set S of 3 vertices of G. Then, there exists a monochromatic S-tree in G. Since $|S| = 3$, then this monochromatic S-tree is contained in some nontrivial color tree T_c. Suppose that the color tree T_c is not a spanning tree of G. Choose $v \notin V(T_c)$, and $\{u, w\} \subseteq V(T_c)$. Let $S' = \{v, u, w\}$. Then, there exists a monochromatic S'-tree in G. Since $|S'| = 3$, then this monochromatic S'-tree is contained in some nontrivial color tree T_d. Moreover, since $v \notin V(T_c)$, then $c \neq d$. But now, $\{u, w\} \in V(T_c) \cap V(T_d)$, contracting that f is simple. Then, we have that T_c is a spanning tree of G. Hence, $m - n + 2 \le mx_n(G) \le \ldots \le mx_3(G) \le m - n + 2$. The theorem thus follows. □

3 Hardness results for computing $mvx_k(G)$

Though we can completely determine the value of $mx_k(G)$ for each k with $3 \le k \le n$, for the vertex version it is difficult to compute $mvx_k(G)$ for any k with $2 \le k \le n$. In this section, we will show that given a connected graph $G = (V, E)$, and a positive integer L with $L \le |V|$, to decide whether $mvx_k(G) \ge L$ is NP-complete for each k with $2 \le k \le |V|$.

We first introduce some definitions. A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex not in D has a neighbor in D. If the subgraph induced by D is connected, then D is called a *connected dominating set*. The *dominating number* $\gamma(G)$, and the *connected dominating number* $\gamma_c(G)$, are the cardinalities of a minimum dominating set, and a minimum connected dominating set, respectively. A graph G has a connected dominating set if and only if G is connected. The problem of computing $\gamma_c(G)$ is equivalent to the problem of finding a spanning tree with the most number of leaves, because a vertex subset is a connected dominating set if and only if its complement is contained in the set of leaves of a spanning tree. Let G be a connected graph on n vertices where $n \ge 3$. Note that the problem of computing $mvx_n(G)$ is also equivalent to the problem of finding a spanning tree with the most number of leaves. In fact, let T_{max} be a spanning tree of G with the most number of leaves, and $l(T_{max})$ be the number of leaves in T_{max}. Then, $mvx_n(G) = l(T_{max}) + 1 = n - \gamma_c(G) + 1$ for $n \ge 3$. For convenience, suppose that all the graphs in this section have at least 3 vertices.

Now we introduce a useful lemma. For convenience, call a tree T *with vertex-color* c if the internal vertices of T are colored with c.

Lemma 3.1. *Let G be a connected graph on n vertices with a cut-vertex v_0. Then, $mvc(G) = l(T_0) + 1$, where T_0 is a spanning tree of G with the most number of leaves.*

Proof. Let f be an extremal MVC-coloring of G. Suppose that $f(v)$ is the color of the vertex v, and $f(v_0) = c$. Let $G_1, G_2, \ldots, G_p$ be the components of $G - v_0$ where $p \geq 2$. We construct a spanning tree T_0 of G with vertex-color c as follows. At first, choose any pair $(v_i, v_j) \in (V(G_i), V(G_j))(i \neq j)$. Since v_0 is a cut-vertex, then there must exist a $\{v_i, v_j\}$-path P containing v_0 with vertex-color c. Initially, set $T_0 = P$. Secondly, choose another pair $(v_s, v_t) \in (V(G_s), V(G_t))(s \neq t)$ such that v_s is not in T_0. Similarly, there must exist a $\{v_s, v_t\}$-path P' containing v_0 with vertex-color c. Let x be the first vertex of P' that is also in T_0, and y be the last vertex of P' that is also in T_0. Then, reset $T_0 = T_0 \cup v_s P' x \cup y P' v_t$. Thus, T_0 is still a tree with vertex-color c now. Repeat the above process until all vertices are contained in T_0. Finally, we get a spanning tree T_0 of G with vertex-color c, thus, we have $mvc(G) \leq l(T_0) + 1$ now. However, $mvc(G) \geq mvx_n(G) = l(T_{max}) + 1$, where T_{max} is a spanning tree of G with the most number of leaves. Then, we have $l(T_0) = l(T_{max})$. Hence, it follows that $mvc(G) = l(T_0) + 1$. □

Corollary 3.2. *Let G be a connected graph on n vertices with a cut-vertex. Then, $mvx_k(G) = l(T_{max}) + 1$ for each k with $2 \leq k \leq n$, where T_{max} is a spanning tree of G with the most number of leaves.*

Now, we show that the following Problem 0 is NP-complete.

Problem 0**:** k-monochromatic vertex-index

Instance: Connected graph $G = (V, E)$, a positive integer L with $L \leq |V|$.

Question: Deciding whether $mvx_k(G) \geq L$ for each k with $2 \leq k \leq |V|$.

In order to prove the NP-completeness of Problem 0, we first introduce the following problems.

Problem 1**:** Dominating Set.

Instance: Graph $G = (V, E)$, a positive integer $K \leq |V|$.

Question: Deciding wether there is a dominating set of size K or less.

Problem 2**:** CDS of a connected graph containing a cut-vertex.

Instance: Connected graph $G = (V, E)$ with a cut-vertex, a positive integer K with $K \leq |V|$.

Question: Deciding wether there is a connected dominating set of size K or less.

The NP-completeness of Problem 1 is a known result in [8]. In the following, we will reduce Problem 1 to Problem 2 polynomially.

Lemma 3.3. *Problem* 1 $\preceq$ *Problem* 2*.*

Proof. Given a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E, we construct a graph $G' = (V', E')$ as follows:

$$\begin{aligned}
V' =& V \cup \{u_1, u_2, \ldots, u_n\} \cup \{x, y\} \\
E' =& E \cup E_1 \cup E_2 \\
E_1 =& \{u_i v : \text{if } v = v_i \text{ or } v_i v \text{ is an edge in } G \text{ for } 1 \leq i \leq n\} \\
E_2 =& \{x u_i : 1 \leq i \leq n\} \cup \{xy\}
\end{aligned}$$

It is easy to check that G' is connected with a cut-vertex x. In the following, we will show that G contains a dominating set of size K or less if and only if G' contains a connected dominating set of size $K+1$ or less. On one hand, suppose w.l.o.g that G contains a dominating set $D = \{v_1, v_2, \ldots, v_t\}, t \leq K$. Let $D' = \{u_1, u_2, \ldots, u_t\} \cup \{x\}$. Then, it is easy to check that D' is a connected dominating set of G' and $|D'| \leq K+1$. On the other hand, suppose that G' contains a connected dominating set D' of size $K+1$ or less. Since x is a cut-vertex of G', then $x \in D'$. For $1 \leq i \leq n$, if $u_i \in D'$ or $v_i \in D'$, then put v_i in D. It is easy to check that D is a dominating set of G and $|D| \leq K$. □

Theorem 3.4. *Problem* 0 *is NP-complete.*

Proof. Let $G = (V, E)$ be a connected graph with a cut-vertex, and K a positive integer with $K \leq |V|$. Recall that $\gamma_c(G) \leq K$ if and only if $mvx_k(G) = l(T_{max}) + 1 = |V| - \gamma_c(G) + 1 \geq |V| - K + 1$ for $2 \leq k \leq |V|$, where T_{max} is a spanning tree of G with the most leaves by Corollary 3.2. Then, given a connected graph $G = (V, E)$ with a cut-vertex, and a positive integer L with $L \leq |V|$, to decide whether $mvx_k(G) \geq L$ is NP-complete for each k with $2 \leq k \leq |V|$ by Lemma 3.3. Moreover, Problem 0 is NP-complete. □

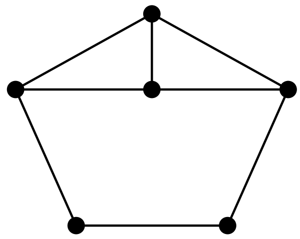

Fig. 1: The graph F_1 with $\gamma_c(F_1)=\gamma_c(\overline{F_1})=3$.

Corollary 3.5. *Let G be a connected graph on n vertices. Then, computing $mvx_k(G)$ is NP-hard for each k with $2\leq k\leq n$.*

4 Nordhaus-Gaddum-type results for mvx_k

Suppose that both G and $\overline{G}$ are connected graphs on n vertices. Now for $n=4$, we have $G=\overline{G}=P_4$. It is easy to check that $mvx_k(P_4)+mvx_k(\overline{P_4})=6$ for each k with $2\leq k\leq 4$. For $k=2$, Cai et al. [3] proved that for $n\geq 5$, $n+3\leq mvc(G)+mvc(\overline{G})\leq 2n$, and the bounds are sharp. Then, in the following we suppose that $n\geq 5$ and $3\leq k\leq n$.

We first consider the lower bound of $mvx_k(G)+mvx_k(\overline{G})$ for each k with $3\leq k\leq n$. Now we introduce some useful lemmas.

Lemma 4.1. *[11] If both G and $\overline{G}$ are connected graphs on n vertices, then $\gamma_c(G)+\gamma_c(\overline{G})=n+1$ if and only if G is the cycle C_5. Moreover, if G is not C_5, then $\gamma_c(G)+\gamma_c(\overline{G})\leq n$ with equality if and only if $\{G,\overline{G}\}=\{C_n,\overline{C_n}\}$ for $n\geq 6$, or $\{G,\overline{G}\}=\{P_n,\overline{P_n}\}$ for $n\geq 4$, or $\{G,\overline{G}\}=\{F_1,\overline{F_1}\}$, where F_1 is the graph represented in Fig.1.*

Lemma 4.2. *[3] Let C_n be a cycle on n vertices. Then,*

$$mvc(C_n)=\begin{cases} n & n\leq 5\\ 3 & n\geq 6.\end{cases}$$

Recall that a vertex-monochromatic S-tree is a vertex-monochromatic tree containing S. For convenience, if the vertex-monochromatic S-tree is a star (with the center v), we use S-star (S_v-star) to denote this vertex-monochromatic S-tree. For two subsets $U,W\subseteq V(G)$, we use $U\sim W$ to denote that any vertex in U is adjacent with any vertex in W. If $U=\{x\}$, we use $x\sim W$ instead of $\{x\}\sim W$.

From Lemma 4.1, we have $mvx_k(C_n) + mvx_k(\overline{C_n}) \geq mvx_n(C_n) + mvx_n(\overline{C_n}) = 2n - (\gamma_c(C_n) + \gamma_c(\overline{C_n})) + 2 \geq n + 2$ for $n \geq 6$ and k with $3 \leq k \leq n$. It is easy to check that $mvx_k(C_n) = 3$ for $n \geq 6$ and k with $3 \leq k \leq n$ by Lemma 4.2. Then, we have $mvx_k(\overline{C_n}) \geq n - 1$ for $n \geq 6$ and k with $3 \leq k \leq n$. Now we introduce the following lemma.

Lemma 4.3. *For $n \geq 6$, if n is odd, then $mvx_k(\overline{C_n}) = n$ for k with $3 \leq k \leq \frac{n-1}{2}$, and $mvx_k(\overline{C_n}) = n - 1$ for k with $\frac{n+1}{2} \leq k \leq n$; if $n = 4t$, then $mvx_k(\overline{C_n}) = n$ for k with $3 \leq k \leq \frac{n}{2} - 1$, and $mvx_k(\overline{C_n}) = n - 1$ for k with $\frac{n}{2} \leq k \leq n$; if $n = 4t + 2$, then $mvx_k(\overline{C_n}) = n$ for k with $3 \leq k \leq \frac{n}{2}$, and $mvx_k(\overline{C_n}) = n - 1$ for k with $\frac{n}{2} + 1 \leq k \leq n$.*

Proof. Suppose that $V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\}$, and the clockwise permutation sequence is $v_0, v_1, \ldots, v_{n-1}, v_0$ in C_n. Let f be an extremal MVX_k-coloring of $\overline{C_n}$ for each k with $3 \leq k \leq n$. Suppose first that n is odd. Let $S = \{v_i : i \equiv 0 \textit{ or } 1 \pmod 4\}$. Then, $|S| = \frac{n+1}{2}$. It is easy to check that there exists no S-star in $\overline{C_n}$. Then, we have $mvx_k(\overline{C_n}) < n$ for k with $\frac{n+1}{2} \leq k \leq n$. Hence, $mvx_k(\overline{C_n}) = n - 1$ for k with $\frac{n+1}{2} \leq k \leq n$. For k with $3 \leq k \leq \frac{n-1}{2}$, we will show that $mvx_k(\overline{C_n}) = n$. In other words, for any set S of k vertices of $\overline{C_n}$, there exists an S-star in $\overline{C_n}$. We first show that $mvx_k(\overline{C_n})$ for $k = \frac{n-1}{2}$. By contradiction, assume that $mvx_k(\overline{C_n}) < n$ for $k = \frac{n-1}{2}$. Suppose that S is a set of k vertices such that there exists no S-star in $\overline{C_n}$. Note that the vertex-induced subgraph $C_n[S]$ consists of some disjoint paths $\{P_{v_{i_1}v_{j_1}}, P_{v_{i_2}v_{j_2}}, \ldots, P_{v_{i_p}v_{j_p}}\}$ where $\{v_{i_q}, v_{j_q}\}$ denote the ends of $P_{v_{i_q}v_{j_q}}$ such that the vertex-sequence v_{i_q} to v_{j_q} along $P_{v_{i_q}v_{j_q}}$ is in clockwise direction in C_n for each q with $1 \leq q \leq p$.

Claim 1: Each $P_{v_{i_q}v_{j_q}}$ contains at least 2 vertices for each q with $1 \leq q \leq p$.

Proof of Claim 1: By contradiction, assume that $P_{v_{i_q}v_{j_q}} = v$ for some $v \in V(C_n)$. Since $\{P_{v_{i_1}v_{j_1}}, P_{v_{i_2}v_{j_2}}, \ldots, P_{v_{i_p}v_{j_p}}\}$ are disjoint paths in C_n, then $v \sim S \setminus \{v\}$ in $\overline{C_n}$. Hence, there exists an S_v-star in $\overline{C_n}$, a contradiction.

Consider $\{P_{v_{i_1}v_{j_1}}, P_{v_{i_2}v_{j_2}}, \ldots, P_{v_{i_p}v_{j_p}}\}$ in C_n. Suppose w.l.o.g that the clockwise permutation sequence of these paths is $P_{v_{i_1}v_{j_1}}, P_{v_{i_2}v_{j_2}}, \ldots, P_{v_{i_p}v_{j_p}}, P_{v_{i_{p+1}}v_{j_{p+1}}} = P_{v_{i_1}v_{j_1}}$ in C_n. For any two successive paths $P_{v_{i_q}v_{j_q}}$ and $P_{v_{i_{q+1}}v_{j_{q+1}}}$ where $1 \leq q \leq p$, we have the following claim.

Claim 2: There are at most 2 vertices between $\{v_{j_q}, v_{i_{q+1}}\}$ in clockwise direction in C_n for each q with $1 \leq q \leq p$.

Proof of Claim 2: By contradiction, assume that there are at least 3 vertices $\{v_{r-1}, v_r, v_{r+1}\}$, where the subscript is subject to modulo n, between $\{v_{j_q}, v_{i_{q+1}}\}$ in clockwise direction in C_n. Now, we have $v_r \sim S$ in $\overline{C_n}$. Then, there exists an S_{v_r}-star in $\overline{C_n}$, a contradiction.

If $n = 4t + 1$, then $k = 2t$. Now, we have $p \leq \lfloor \frac{k}{2} \rfloor = t$ by Claim 1. Then, $|V(C_n)| \leq k+2p \leq n-1 < n$ by Claim 2, a contradiction. If $n = 4t+3$, then $k = 2t+1$. Now, we have $p \leq \lfloor \frac{k}{2} \rfloor = t$ by Claim 1. Then, $|V(C_n)| \leq k+2p \leq n-2 < n$ by Claim 2, a contradiction. Hence, if n is odd, then $n = mvx_{\frac{n-1}{2}}(\overline{C_n}) \leq \ldots mvx_4(\overline{C_n}) \leq mvx_3(\overline{C_n}) \leq n$. The proof for the case $n = 4t$ or $n = 4t + 2$ is similar. We omit their details. □

Theorem 4.4. *Suppose that both G and $\overline{G}$ are connected graphs on n vertices. For $n = 5$, $mvx_k(G) + mvx_k(\overline{G}) \geq 6$ for k with $3 \leq k \leq 5$. For $n = 6$, $mvx_k(G) + mvx_k(\overline{G}) \geq 8$ for k with $3 \leq k \leq 6$. For $n \geq 7$, if n is odd, then $mvx_k(G) + mvx_k(\overline{G}) \geq n + 3$ for k with $3 \leq k \leq \frac{n-1}{2}$, and $mvx_k(G) + mvx_k(\overline{G}) \geq n + 2$ for k with $\frac{n+1}{2} \leq k \leq n$; if $n = 4t$, then $mvx_k(G) + mvx_k(\overline{G}) \geq n+3$ for k with $3 \leq k \leq \frac{n}{2} - 1$, and $mvx_k(G) + mvx_k(\overline{G}) \geq n+2$ for k with $\frac{n}{2} \leq k \leq n$; if $n = 4t + 2$, then $mvx_k(G) + mvx_k(\overline{G}) \geq n + 3$ for k with $3 \leq k \leq \frac{n}{2}$, and $mvx_k(G) + mvx_k(\overline{G}) \geq n + 2$ for k with $\frac{n}{2} + 1 \leq k \leq n$. Moreover, all the above bounds are sharp.*

Proof. For $n = 5$, if $G = \overline{G} = C_5$, then it is easy to check that $2mvx_k(C_5) = 6$ for k with $3 \leq k \leq 5$; if $G \neq C_5$, then $mvx_k(G) + mvx_k(\overline{G}) \geq 7$ for k with $3 \leq k \leq 5$ by Lemma 4.1. For $n \geq 6$, we have $mvx_k(G) + mvx_k(\overline{G}) \geq mvx_n(G) + mvx_n(\overline{G}) = n + 2$ for k with $3 \leq k \leq n$ with equality if and only if $\{G, \overline{G}\} = \{C_n, \overline{C_n}\}$, or $\{G, \overline{G}\} = \{P_n, \overline{P_n}\}$, or $\{G, \overline{G}\} = \{F_1, \overline{F_1}\}$, where F_1 is the graph represented in Fig.1 by Lemma 4.1. For $n \geq 6$, it is easy to check that $mvx_k(C_n) = mvx_k(P_n) = 3$ for k with $3 \leq k \leq n$ by Lemma 4.2. Then, we have $mvx_k(P_n) + mvx_k(\overline{P_n}) \geq mvx_k(C_n) + mvx_k(\overline{C_n})$ for k with $3 \leq k \leq n$. Furthermore, for $n = 6$, it is easy to check that $mvx_k(F_1) + mvx_k(\overline{F_1}) = 8$ for k with $3 \leq k \leq 6$. Thus, the theorem follows for $n \geq 6$ by Lemma 4.3. □

Now we consider the upper bound of $mvx_k(G)+mvx_k(\overline{G})$ for each k with $\lceil \frac{n}{2} \rceil \leq k \leq n$. For convenience, we use $d_G(v)$ and $N_G(v)$ to denote the degree and the neighborhood of a vertex v in G, respectively. For any two vertices $u, v \subseteq V(G)$, we use $d_G(u, v)$ to denote the distance between u and v in G. Note that a straightforward upper bound of $mvx_k(G)$ is that $mvx_k(G) \leq mvc(G) \leq n - diam(G) + 2$ where $diam(G)$ is the diameter of G for each k with $3 \leq k \leq n$. Next we introduce some useful lemmas.

Lemma 4.5. *Let K_{n_1,n_2} be a complete bipartite graph such that $n = n_1 + n_2$, and $n_1, n_2 \geq 2$. Let $G = K_{n_1,n_2} - e$, where e is an edge of K_{n_1,n_2}. Then, $mvx_k(G) + mvx_k(\overline{G}) = 2n - 2$ for $3 \leq k \leq n$.*

Proof. It is easy to check that $diam(G) = 3$, and $diam(\overline{G}) = 3$. Then, we have $mvc(G) + mvc(\overline{G}) \leq 2n - 2$. It is also easy to check that both G and $\overline{G}$ contain a double star as a spanning tree. Then, we have $mvx_n(G) + mvx_n(\overline{G}) \geq 2n - 2$. Hence, the lemma follows by the fact that $mvx_n(G) \leq \ldots \leq mvx_3(G) \leq mvc(G)$. □

Lemma 4.6. *If $k = \lceil \frac{n}{2} \rceil$, then $mvx_k(G) + mvx_k(\overline{G}) \leq 2n - 2$ for $n \geq 5$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Since $\overline{G}$ is connected, then $\Delta(G) \leq n - 2$. Suppose first that $mvx_k = n$, and f is an extremal MVX_k-coloring of G. Then, for any set S of k vertices of G, there exists an S-star in G. This also implies that $\Delta(G) \geq k - 1$.

Case 1: $\Delta(G) \geq n - k + 1$.

Suppose w.l.o.g that $d_G(v_1) = \Delta(G)$, and $N_G(v_1) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Let $S = \{v_1, v_{\Delta+2}, \ldots, v_{n-1}, v_n\}$. Since $|S| = n - \Delta(G) \leq k - 1 < k$, then there exists an S_v-star in G. Moreover, since $v_1 \nsim \{v_{\Delta+2}, \ldots, v_{n-1}, v_n\}$ in G, then $v \in N_G(v_1)$. Suppose w.l.o.g that $v = v_2$. Then, we have $d_{\overline{G}}(v_1, v_2) \geq 3$. Since $d_{\overline{G}}(v_1, v_2) \geq 3$, then $mvx_k(\overline{G}) \leq n - diam(\overline{G}) + 2 \leq n - 1$. Suppose $mvx_k(\overline{G}) = n - 1$. Then, $diam(\overline{G}) = 3$. Let g be an extremal MVX_k-coloring of $\overline{G}$. Note that if $\overline{G}$ is k-monochromatically vertex-connected, it is also monochromatically vertex-connected. Since $mvx_k(\overline{G}) = n - 1$, then there exists a vertex-monochromatic path $P = v_1xyv_2$ of length 3 in $\overline{G}$ such that $x \in \{v_{\Delta+2}, \ldots, v_{n-1}, v_n\}$, and $y \in N_G(v_1) \setminus \{v_2\}$. Suppose w.l.o.g that $P = v_1v_{\Delta+2}v_{\Delta+1}v_2$. This also implies that $v_{\Delta+1} \nsim \{v_2, v_{\Delta+2}\}$ in G. Let $S' = \{v_1, v_{\Delta+1}, v_{\Delta+2}, \ldots, v_n\}$ now. Since $|S'| = n - \Delta(G) + 1 \leq k$, then there exists an $S'_{v'}$-star in G. Moreover, since $v_1 \nsim \{v_{\Delta+2}, \ldots, v_{n-1}, v_n\}$ and $v_{\Delta+1} \nsim \{v_2, v_{\Delta+2}\}$ in G, then $v' \in N_G(v_1) \setminus \{v_2, v_{\Delta+1}\}$. Now, we have $d_{\overline{G}}(v_1, v') = 3$. Since $mvx_k(\overline{G}) = n - 1$, then $\{v_{\Delta+1}, v_{\Delta+2}\}$ are the only two vertices with the same color in $\overline{G}$. But now, since $v' \nsim \{v_{\Delta+1}, v_{\Delta+2}\}$ in $\overline{G}$, then there exists no vertex-monochromatic path connecting $\{v_1, v'\}$ in $\overline{G}$, a contradiction. Hence, we have that $mvx_k(\overline{G}) \leq n - 2$, and $mvx_k(G) + mvx_k(\overline{G}) \leq 2n - 2$.

Case 2: $\Delta(G) \leq n - k$.

Since $k = \lceil \frac{n}{2} \rceil$, and $\Delta(G) \geq k - 1$, then $\lceil \frac{n}{2} \rceil - 1 \leq \Delta(G) \leq n - \lceil \frac{n}{2} \rceil$.

If n is odd, then $\Delta(G) = \frac{n-1}{2} = k - 1$. Suppose w.l.o.g that $d_G(v_1) = \Delta(G)$, and $N_G(v_1) = \{v_2, v_3, \ldots, v_k\}$. Let $S = \{v_1, v_{k+1}, \ldots, v_n\}$. Since $|S| = n - k + 1 = k$, then there exists an S_v-star in G. Moreover, since $v_1 \nsim \{v_{k+1}, \ldots, v_{n-1}, v_n\}$ in G, then v is not in S. But now, $d_G(v) \geq |S| = k > \Delta(G)$, a contradiction.

If n is even, then $\Delta(G) = \frac{n}{2} - 1$ or $\frac{n}{2}$. Suppose w.l.o.g that $d_G(v_1) = \Delta(G)$, and $N_G(v_1) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. If $\Delta(G) = \frac{n}{2} - 1 = k - 1$, then let $S = \{v_1, v_{k+1}, \ldots, v_{n-1}\}$. Since $|S| = n - k = k$, then there exists an S_v-star in G. Moreover, since $v_1 \nsim \{v_{k+1}, \ldots, v_{n-1}\}$ in G, then v is not in S. But now, $d_G(v) \geq |S| = k > \Delta(G)$, a contradiction. If $\Delta(G) = \frac{n}{2} = k$, then let $S = \{v_1, v_{k+2}, \ldots, v_n\}$. Since $|S| = n - k = k$, then there exists an S_v-star in G. Moreover, since $v_1 \nsim \{v_{k+2}, \ldots, v_{n-1}, v_n\}$ in G, then $v \in N_G(v_1)$. Suppose w.l.o.g that $v = v_2$. Then, $d_G(v_2) = k = \Delta(G)$, and $N_G(v_2) = \{v_1, v_{k+2}, \ldots, v_n\}$. If $k \geq 4$, then let $S' = \{v_1, v_2, v_{k+1}, v_{k+2}\}$. Since $|S'| \leq k$, then there exists an $S'_{v'}$-star in G. But now, since $v_1 \nsim v_{k+2}$, and $v_2 \nsim v_{k+1}$ in G, then $v' \in N_G(v_1) \cap N_G(v_2) = \emptyset$, a contradiction. If $k = 3$, then $n = 6$. If $\{v_2, v_3, v_4\} \sim \{v_5, v_6\}$ in G, then G contains a complete bipartite spanning subgraph. But now, $\overline{G}$ is not connected, a contradiction. So, suppose w.l.o.g that $v_4 \nsim v_5$ in G. Similarly consider $S' = \{v_1, v_3, v_5\}$,$\{v_1, v_4, v_5\}$, $\{v_1, v_4, v_6\}$, and $\{v_3, v_5, v_6\}$, respectively. Then, we will have that $v_3 \sim v_5$, $v_3 \sim v_4$, $v_4 \sim v_6$, and $v_5 \sim v_6$ in G, respectively. But now, $\overline{G}$ is contained in a cycle C_6. Then, $mvx_3(\overline{G}) \leq mvx_3(C_6) = 3$. So, for $n = 6$ we have $mvx_3(G) + mvx_3(\overline{G}) \leq n + 3 < 2n - 2$.

Suppose w.l.o.g that $mvx_k(G) \leq n - 1$, and $mvx_k(\overline{G}) \leq n - 1$, respectively. Thus, we also have $mvx_k(G) + mvx_k(\overline{G}) \leq 2n - 2$. □

Theorem 4.7. *Suppose that both G and $\overline{G}$ are connected graphs on $n \geq 5$ vertices. Then, for k with $\lceil \frac{n}{2} \rceil \leq k \leq n$, we have that $mvx_k(G) + mvx_k(\overline{G}) \leq 2n - 2$, and this bound is sharp.*

Proof. For k with $\lceil \frac{n}{2} \rceil \leq k \leq n$, we have $mvx_k(G) \leq mvx_{\lceil \frac{n}{2} \rceil} \leq 2n - 2$ by Lemma 4.6. From Lemma 4.5, this bound is sharp for k with $\lceil \frac{n}{2} \rceil \leq k \leq n$. □

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