# Log-concavity of the Fennessey-Larcombe-French Sequence

Arthur L.B. Yang[1] and James J.Y. Zhao[2]

[1]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

[2]Center for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

Email: [1]yang@nankai.edu.cn, [2]jjyzhao@tju.edu.cn

**Abstract.** We prove the log-concavity of the Fennessey-Larcombe-French sequence based on its three-term recurrence relation, which was recently conjectured by Zhao. The key ingredient of our approach is a sufficient condition for log-concavity of a sequence subject to certain three-term recurrence.

*AMS Classification 2010:* Primary 05A20

*Keywords:* Log-concavity, the Fennessey-Larcombe-French sequence, three-term recurrence.

## 1 Introduction

The objective of this paper is to prove the log-concavity conjecture of the Fennessey-Larcombe-French sequence, which was posed by Zhao [17] in the study of log-balancedness of combinatorial sequences.

Let us begin with an overview of Zhao's conjecture. Recall that a sequence $\{a_k\}_{k\geq 0}$ is said to be log-concave if

$$a_k^2 \geq a_{k+1}a_{k-1}, \quad \text{for } k \geq 1,$$

and it is log-convex if

$$a_k^2 \leq a_{k+1}a_{k-1}, \quad \text{for } k \geq 1.$$

We say that $\{a_k\}_{k\geq 0}$ is log-balanced if the sequence itself is log-convex while $\{\frac{a_k}{k!}\}_{k\geq 0}$ is log-concave.

The Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$ can be given by the following three-term recurrence relation [9]

$$n(n+1)^2V_{n+1} = 8n(3n^2+5n+1)V_n - 128(n-1)(n+1)^2V_{n-1}, \quad \text{for } n \geq 1, \tag{1.1}$$

with the initial values $V_0 = 1$ and $V_1 = 8$. This sequence was introduced by Larcombe, French and Fennessey [8], in connection with a series expansion of the complete elliptic integral of the second kind, precisely,

$$\int_0^{\pi/2} \sqrt{1-c^2\sin^2\theta}\, d\theta = \frac{\pi\sqrt{1-c^2}}{2}\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{1-c^2}}{16}\right)^n V_n.$$

The Fennessey-Larcombe-French sequence is closely related to the Catalan-Larcombe-French sequence, which was first studied by E. Catalan [1] and later examined and clarified by Larcombe and French [7]. Let $\{P_n\}_{n\geq 0}$ denote the Catalan-Larcombe-French sequence, and the following three-term recurrence relation holds:

$$(n+1)^2 P_{n+1} = 8(3n^2+3n+1)P_n - 128n^2 P_{n-1}, \quad \text{for } n \geq 1,$$

with $P_0 = 1$ and $P_1 = 8$. As a counterpart of $V_n$, the numbers $P_n$ appear as coefficients in the series expansion of the complete elliptic integral of the first kind, precisely,

$$\int_0^{\pi/2} \frac{1}{\sqrt{1-c^2\sin^2\theta}}\, d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \left( \frac{1-\sqrt{1-c^2}}{16} \right)^n P_n.$$

Many interesting properties have been found for the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence, and the reader may consult references [5, 6, 7, 8, 9, 15].

Recently, there has arisen an interest in the study of the log-behavior of the Catalan-Larcombe-French sequence. For instance, Xia and Yao [13] obtained the log-convexity of the Catalan-Larcombe-French sequence, and confirmed a conjecture of Sun [11]. By using a log-balancedness criterion due to Došlić [4], Zhao [16] proved the log-balancedness of the Catalan-Larcombe-French sequence.

Zhao further studied the log-behavior of the Fennessey-Larcombe-French sequence, and obtained the following result.

**Theorem 1.1** ([17])**.** *Both $\{nV_n\}_{n\geq 1}$ and $\{\frac{V_n}{(n-1)!}\}_{n\geq 1}$ are log-concave.*

She also made the following conjecture.

**Conjecture 1.2.** *The Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 1}$ is log-concave.*

Note that the Hadamard product of two log-concave sequences without internal zeros is still log-concave, see [14, Proposition 2]. Since both $\{n\}_{n\geq 1}$ and $\{\frac{1}{(n-1)!}\}_{n\geq 1}$ are log-concave, Conjecture 1.2 implies Theorem 1.1.

In this paper, we obtain a sufficient condition for proving the log-concavity of a sequence satisfying a three-term recurrence. Then we give an affirmative answer to Conjecture 1.2 by using this criterion. By further employing a result of Wang and Zhu [12, Theorem 2.1], we derive the monotonicity of the sequence $\{\sqrt[n]{V_{n+1}}\}_{n\geq 1}$ from the log-concavity of $\{V_n\}_{n\geq 1}$.

# 2 Log-concavity derived from three-term recurrence

The aim of this section is to prove the log-concavity of the Fennessey-Larcombe-French sequence based on its three-term recurrence relation.

We first give a sufficient condition for log-concavity of a positive sequence subject to certain three-term recurrence. It should be mentioned that the log-behavior of sequences satisfying three-term recurrences has been extensively studied, see Liu and Wang [10], Chen and Xia [3], Chen, Guo and Wang [2], and Wang and Zhu [12]. However, most of these studies have focused on the log-convexity of such sequences instead of their log-concavity. Our criterion for determining the log-concavity of a sequence satisfying a three-term recurrence is as follows.

**Proposition 2.1.** *Let $\{S_n\}_{n\geq 0}$ be a positive sequence satisfying the following recurrence relation:*

$$a(n)S_{n+1} + b(n)S_n + c(n)S_{n-1} = 0, \quad \text{for } n \geq 1, \tag{2.2}$$

*where $a(n), b(n)$ and $c(n)$ are real functions in $n$. Suppose that there exists an integer $n_0$ such that for any $n > n_0$,*

*(i) it holds $a(n) > 0$, and*

*(ii) either $b^2(n) < 4a(n)c(n)$ or $\frac{S_n}{S_{n-1}} \geq \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)}$.*

*Then the sequence $\{S_n\}_{n\geq n_0}$ is log-concave, namely, $S_n^2 \geq S_{n+1}S_{n-1}$ for any $n > n_0$.*

*Proof.* Let $r(n) = \frac{S_n}{S_{n-1}}$. It suffices to show that $r(n) \geq r(n+1)$ for any $n > n_0$. On one hand, the conditions $(i)$ and $(ii)$ imply that

$$a(n)r^2(n) + b(n)r(n) + c(n) \geq 0, \quad \text{for } n > n_0.$$

Since $\{S_n\}_{n\geq 0}$ is a positive sequence, so is $\{r_n\}_{n\geq 1}$. Thus, the above inequality is equivalent to the following

$$a(n)r(n) + b(n) + \frac{c(n)}{r(n)} \geq 0, \quad \text{for } n > n_0. \tag{2.3}$$

On the other hand, dividing both sides of (2.2) by $S_n$, we obtain

$$a(n)r(n+1) + b(n) + \frac{c(n)}{r(n)} = 0. \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$a(n)r(n+1) \leq a(n)r(n), \quad \text{for } n > n_0.$$

By the condition $(i)$, we have $r(n+1) \leq r(n)$ for any $n > n_0$. This completes the proof. □

We are now able to give the main result of this section, which offers an affirmative answer to Conjecture 1.2.

**Theorem 2.2.** *Let $\{V_n\}_{n\geq 0}$ be the Fennessey-Larcombe-French sequence given by (1.1). Then, for any $n \geq 2$, we have $V_n^2 \geq V_{n-1}V_{n+1}$.*

*Proof.* By the recurrence relation (1.1), we have $V_1 = 8, V_2 = 144, V_3 = 2432$ and $V_4 = 40000$. It is easy to verify that $V_2^2 \geq V_1V_3$ and $V_3^2 \geq V_2V_4$.

We proceed to use Proposition 2.1 to prove that $V_n^2 > V_{n-1}V_{n+1}$ for $n > 3$, namely taking $n_0 = 3$. For the sequence $\{V_n\}_{n\geq 0}$, the corresponding polynomials $a(n), b(n), c(n)$ appearing in Proposition 2.1 are as follows:

$$\begin{aligned} a(n) &= n(n+1)^2, \\ b(n) &= -8n(3n^2+5n+1), \\ c(n) &= 128(n-1)(n+1)^2. \end{aligned}$$

It is clear that $a(n) > 0$ for any $n > 3$. By a routine computation, we get

$$b^2(n) - 4a(n)c(n) = 64(n^6+6n^5+15n^4+26n^3+25n^2+8n) > 0, \quad \text{for } n > 3.$$

It suffices to show that

$$\frac{V_n}{V_{n-1}} \geq \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)}, \quad \text{for } n > 3. \tag{2.5}$$

This inequality also implies the positivity of $V_n$ since its right-hand side is positive for any $n > 3$. (Note that $b(n)$ is negative.) However, it is difficult to directly prove (2.5). The key idea of our proof is to find an intermediate function $h(n)$ such that

$$\frac{V_n}{V_{n-1}} \geq h(n) \geq \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)}, \quad \text{for } n > 3.$$

Let

$$h(n) = \frac{16(n^3-n^2+1)}{n^3-n^2}, \quad \text{for } n \geq 2, \tag{2.6}$$

and we shall show that this function fulfills our purpose. This will be done in two steps.

First, we need to prove that

$$h(n) - \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)} \geq 0, \quad \text{for } n > 3. \tag{2.7}$$

A straightforward computation shows that the quantity on the left-hand side is equal to

$$\frac{32(4n^6+7n^5+n^4+n^3+9n^2+8n+2)}{(n^4-n^2)(n+1)(n^5+2n^4+n^2+8n+4+(n^2-n)\sqrt{n^6+6n^5+15n^4+26n^3+25n^2+8n})},$$

which is clearly positive for $n > 3$.

Second, we need to prove that

$$\frac{V_n}{V_{n-1}} \geq h(n), \quad \text{for } n > 3. \tag{2.8}$$

For the sake of convenience, let $g(n) = \frac{V_n}{V_{n-1}}$. We use induction on $n$ to prove that $g(n) \geq h(n)$ for $n > 3$. By the recurrence relation (1.1), we have

$$g(n+1) = \frac{8(3n^2+5n+1)}{(n+1)^2} - \frac{128(n-1)}{ng(n)}, \quad n \geq 1, \tag{2.9}$$

with the initial value $g(1) = 8$. It is clear that $g(3) = 152/9 = h(3)$ and $g(4) = 625/38 > 49/3 = h(4)$ by (2.6) and (2.9). Assume that $g(n) > h(n)$, and we proceed to show that $g(n+1) > h(n+1)$. Note that

$$\begin{aligned} g(n+1) - h(n+1) &= \frac{8(3n^2+5n+1)}{(n+1)^2} - \frac{128(n-1)}{ng(n)} - \frac{16(n^3+2n^2+n+1)}{n(n+1)^2} \\ &= \frac{8(n^3+n^2-n-2)}{n(n+1)^2} - \frac{128(n-1)}{ng(n)} \\ &= \frac{8(n^3+n^2-n-2)g(n) - 128(n-1)(n+1)^2}{n(n+1)^2 g(n)}. \end{aligned}$$

By the induction hypothesis, we have $g(n) > h(n) > 0$ and thus

$$\begin{aligned} g(n+1) - h(n+1) &> \frac{8(n^3+n^2-n-2)h(n) - 128(n-1)(n+1)^2}{n(n+1)^2 g(n)} \\ &= \frac{128(2n^2-n-2)}{n^3(n-1)(n+1)^2 g(n)} > 0. \end{aligned}$$

Combining (2.7) and (2.8), we obtained the inequality (2.5). This completes the proof. □

Wang and Zhu [12, Theorem 2.1] showed that if $\{z_n\}_{n\geq 0}$ is a log-concave sequence of positive integers with $z_0 > 1$, then $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is strictly decreasing. Applying their criterion to the Fennessey-Larcombe-French sequence, we obtain immediately the following result.

**Proposition 2.3.** *The sequence* $\{\sqrt[n]{V_{n+1}}\}_{n\geq 1}$ *is strictly decreasing.*

*Proof.* Let $\{z_n\}_{n\geq 0}$ be the sequence given by $z_n = V_{n+1}$. It is clear that $z_0 = V_1 = 8 > 1$. Moreover, by Theorem 2.2, the sequence $\{z_n\}_{n\geq 0}$ is log-concave. Thus, $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is strictly decreasing by [12, Theorem 2.1]. □

**Acknowledgements.** This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education and the National Science Foundation of China.

# References


[1] E. Catalan, Sur les Nombres de Segner, Rend. Circ. Mat. Palermo, 1 (1887) 190–201.

[2] W.Y.C. Chen, J.J.F. Guo and L.X.W. Wang, Infinitely log-monotonic combinatorial sequences, Adv. Appl. Math., 52 (2014) 99-120.

[3] W.Y.C. Chen and E.X.W. Xia, The 2-log-convexity of the Apéry Numbers, Proc. Amer. Math. Soc., 139 (2011) 391–400.

[4] T. Došlić, Log-balanced combinatorial sequences, Int. J. Math. Math. Sci., 4 (2005) 507–522.

[5] A.F. Jarvis, P.J. Larcombe and D.R. French, Linear recurrences between two recent integer sequences, Congr. Numer., 169 (2004) 79–99.

[6] F. Jarvis and H.A. Verrill, Supercongruences for the Catalan-Larcombe-French numbers, Ramanujan J., 22 (2010) 171–186.

[7] P.J. Larcombe and D.R. French, On the "other" Catalan numbers: a historical formulation re-examined, Congr. Numer., 143 (2000) 33–64.

[8] P.J. Larcombe, D.R. French and E.J. Fennessey, The Fennessey-Larcombe-French sequence $\{1, 8, 144, 2432, 40000, ...\}$: formulation and asymptotic form, Congr. Numer., 158 (2002) 179–190.

[9] P.J. Larcombe, D.R. French and E.J. Fennessey, The Fennessey-Larcombe-French sequence $\{1, 8, 144, 2432, 40000, ...\}$: a recursive formulation and prime factor decomposition, Congr. Numer., 160 (2003) 129-137.

[10] L.L. Liu and Y. Wang, On the log-convexity of combinatorial sequences, Adv. Appl. Math., 39 (2007) 453–476.

[11] Z.W. Sun, Conjectures involving arithmetical sequences, in Numbers Theory: Arithmetic in Shangri-La (eds., S. Kanemitsu, H. Li and J. Liu), Proc. 6th China-Japan Seminar (Shanghai, August 15–17, 2011), World Sci., Singapore, 2013, pp. 244–258. arXiv: 1208.2683v9.

[12] Y. Wang and B.X. Zhu, Proofs of some conjectures on monotonicity of number-theoretic and combinatorial sequences, Sci. China Math., 57(11) (2014) 2429–2435.

[13] E.X.W. Xia and O.X.M. Yao, A criterion for the log-convexity of combinatorial sequences, Electron. J. Combin., 20(4) (2013) #P3.

[14] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Ann. New York Acad. Sci., 576 (1989) 500–535.

[15] D. Zagier, Integral solutions of Apéry-like recurrence equations, in: Groups and Symmetries: from Neolithic Scots to John McKay, CRM Proc. Lecture Notes 47, Amer. Math. Soc., Providence, RI, 2009, pp. 349–366.

[16] F.-Z. Zhao, The log-behavior of the Catalan-Larcombe-French sequences, Int. J. Number Theory, 10 (2014) 177–182.

[17] F.-Z. Zhao, The log-balancedness of combinatorial sequences, The 6th National Conference on Combinatorics and Graph Theory, Guangzhou, November 7–10, 2014, China.