Proper connection numbers of complementary graphs*

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Abstract

A path P in an edge-colored graph G is called a proper path if no two adjacent edges of P are colored the same, and G is proper connected if every two vertices of G are connected by a proper path in G. The proper connection number of a connected graph G, denoted by $pc(G)$, is the minimum number of colors that are needed to make G proper connected. In this paper, we investigate the proper connection number of the complement of a graph G according to some constraints of G itself. Also, we characterize the graphs on n vertices that have proper connection number $n-2$. Using this result, we give a Nordhaus-Gaddum-type theorem for the proper connection number. We prove that if G and $\overline{G}$ are both connected, then $4 \leq pc(G) + pc(\overline{G}) \leq n$, and the upper bound holds if and only if G or $\overline{G}$ is the n-vertex tree with maximum degree $n-2$.

Keywords: proper path, proper connection number, complement graph, diameter, Nordhaus-Gaddum-type

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1 Introduction

In this paper we are concerned with simple connected finite graphs. We follow the terminology and the notation of Bondy and Murty [2]. The distance between t-

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wo vertices u and v in a connected graph G, denoted by $dist(u,v)$, is the length of a shortest path between them in G. The eccentricity of a vertex v in G is defined as $ecc_G(v) = \max\{dist(x,v) : x \in V(G)\}$, and the diameter of G denoted by $diam(G)$ is defined as $diam(G) = \max\{ecc_G(v) : x \in V(G)\}$.

An edge coloring of a graph G is an assignment c of colors to the edges of G, one color to each edge of G. If adjacent edges of G are assigned different colors by c, then c is a *proper (edge) coloring*. The minimum number of colors needed in a proper coloring of G is referred to as the *chromatic index* of G and denoted by $\chi'(G)$. A path in an edge-colored graph with no two edges sharing the same color is called a *rainbow path*. An edge-colored graph G is said to be *rainbow connected* if every pair of distinct vertices of G is connected by at least one rainbow path in G. Such a coloring is called a *rainbow coloring* of the graph. The minimum number of colors in a rainbow coloring of G is referred to as the *rainbow connection number* of G and denoted by $rc(G)$. The concept of rainbow coloring was first introduced by Chartrand et al. in [5]. In recent years, the rainbow coloring has been extensively studied and has gotten a variety of nice results, see [4, 6, 11, 12, 14] for examples. For more details we refer to a survey paper [15] and a book [16].

Inspired by rainbow colorings and proper colorings in graphs, Andrews et al. [1] introduce the concept of proper-path colorings. Let G be an edge-colored graph, where adjacent edges may be colored the same. A path P in G is called a *proper path* if no two adjacent edges of P are colored the same. An edge-coloring c is a *proper-path coloring* of a connected graph G if every pair of distinct vertices u, v of G is connected by a proper $u - v$ path in G. A graph with a proper-path coloring is said to be *proper connected*. If k colors are used, then c is referred to as a *proper-path k-coloring*. The minimum number of colors needed to produce a proper-path coloring of G is called the *proper connection number* of G, denoted by $pc(G)$.

Let G be a nontrivial connected graph of order n and size m. Then the proper connection number of G has the following bounds:

$$1 \leq pc(G) \leq \min\{\chi'(G), rc(G)\} \leq m.$$

Furthermore, $pc(G) = 1$ if and only if $G = K_n$ and $pc(G) = m$ if and only if $G = K_{1,m}$ is a star of size m.

Among many interesting problems of determining the proper connection numbers of graphs, it is worth while to study the proper connection number of G according to some constraints of the complementary graph. In [17], the authors considered this kind of question for the rainbow connection number $rc(G)$.

A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name "Nordhaus-Gaddum-type" is given because Nordhaus and Gaddum [18] first established the type of inequalities for the chromatic number of graphs in 1956. They proved that if G and $\overline{G}$ are complementary graphs on n vertices whose chromatic numbers are $\chi(G)$ and $\chi(\overline{G})$, respectively, then $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$. Since then, many analogous inequalities of other graph parameters have been considered, such as diameter [9], domination number [10], rainbow connection number [7, 8], generalized edge-connectivity [13], and so on.

The rest of this paper is organized as follows: In Section 2, we list some important known results on proper connection number. In Section 3, we investigate the proper connection number of the complement of a graph $\overline{G}$ according to some constraints of G. In Section 4, we first characterize the graphs on n vertices that have proper connection number $n-2$. Using this result, we give a Nordhaus-Gaddum-type theorem for the proper connection number. We prove that if G and $\overline{G}$ are both connected, then $4 \leq pc(G) + pc(\overline{G}) \leq n$, and the upper bound holds if and only if G or $\overline{G}$ is the n-vertex tree with maximum degree $n-2$.

2 Preliminaries

At the beginning of this section, we list some fundamental results on proper-path colorings which can be found in [1].

Lemma 2.1. *[1] If G is a connected graph and H is a connected spanning subgraph of G, then $pc(G) \leq pc(H)$. In particular, $pc(G) \leq pc(T)$ for every spanning tree T of G.*

Lemma 2.2. *[1] Let G be a connected graph that contains bridges. If b is the maximum number of bridges incident to a single vertex in G, then $pc(G) \geq b$.*

Lemma 2.3. *[1] If T is a tree with at least two vertices, then $pc(T) = \chi'(T) = \Delta(T)$.*

Given a colored path $P = v_1v_2 \dots v_{s-1}v_s$ between any two vertices v_1 and v_s, we denote by $start(P)$ the color of the first edge in the path, i.e. $c(v_1v_2)$, and by $end(P)$ the last color, i.e. $c(v_{s-1}v_s)$. If P is just the edge v_1v_s then $start(P) = end(P) = c(v_1v_s)$.

Definition 2.1. *Let c be an edge-coloring of G that makes G proper connected. We say G has the strong property if for any pair of vertices u and $v \in V(G)$, there exist two proper paths P_1 and P_2 between them (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

In [3], the authors studied proper-connection numbers in bipartite graphs. Also, they presented a result which improve the upper bound $\Delta(G)$ of $pc(G)$ and this result is best possible whenever the graph G is bipartite and 2-connected.

Lemma 2.4. *[3] Let G be a graph. If a graph G is bipartite and 2-connected then $pc(G) = 2$ and there exists a 2-edge-coloring of G such that G has the strong property.*

Every complete k-partite graph $G = K_{n_1,n_2,\ldots,n_k}$ contains a spanning bipartite subgraph $H = K_{n_1+n_2+\ldots n_{k-1},n_k}$. We know that H is 2-connected if $n_k \geq 2$ and $k \geq 3$. Therefore, we have the following result.

Corollary 2.5. *Every complete k-partite graph G ($k \geq 3$) except for the complete graph K_k has proper connection number two, and there exists a 2-edge-coloring c of G such that G has the strong property.*

For general 2-connected graphs, Borozan et al. [3] gave a tight upper bound for the proper connection number.

Lemma 2.6. *[3] Let G be a graph. If a graph G is 2-connected then $pc(G) \leq 3$ and there exists a 3-edge-coloring c of G such that G has the strong property.*

Lemma 2.7. *Let $H = G \cup \{v_1\} \cup \{v_2\}$ such that H is connected. If there is a proper-path k-coloring c of G such that G has the strong property, then $pc(H) \leq k$.*

Proof. Let $\{1, 2, \ldots, k\}$ be the color set of c. If $v_1v_2 \in E(H)$, since H is connected, then there is a vertex $u \in V(G)$ such that u is adjacent to either v_1 or v_2. Without loss of generality, suppose that $uv_1 \in E(H)$. We extend the coloring c of G to the whole graph H by assigning color 1 to uv_1, and 2 to v_1v_2. To show that H is proper connected, we only need to find a proper path between v_1 and w for any $w \in V(G)$. Since G has the strong property, there exist two proper paths P_1, P_2 between w and u (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$. We can get that at least one of wP_1uv_1 and wP_2uv_1 is a proper path. Then we know that $pc(H) \leq k$. Thus, we may assume that $v_1v_2 \notin E(H)$. Let $u_1 \in N_H(v_1)$ and $u_2 \in N_H(v_2)$. If $u_1 = u_2$, we assign color 1 to u_1v_1, and 2 to u_2v_2. Otherwise, we have that $u_1 \neq u_2$. Since G is proper connected, there exists a proper path P of G connecting u_1 and u_2. We assign a color of c being distinct from $start(P)$ to u_1v_1, and a color of c being distinct from $end(P)$ to u_2v_2. It can be easily checked that H is proper connected. Hence $pc(H) \leq k$ follows correspondingly. $\square$

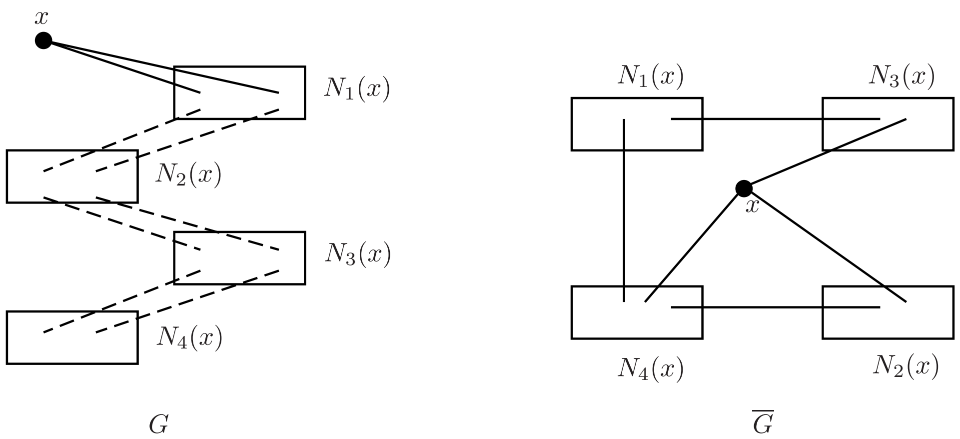

Figure 1: G and $\overline{G}$ with $diam(G) \geq 4$

3 Proper connection number of the complementary graph

We first investigate the proper connection number of $\overline{G}$ if graph G has diameter at least 4.

Theorem 3.1. *If G is a connected graph with $diam(G) \geq 4$, then $pc(\overline{G}) = 2$.*

Proof. First of all, we see that $\overline{G}$ is connected since otherwise $diam(G) \leq 2$, contradicting the condition $diam(G) \geq 4$. We choose a vertex x with $ecc_G(x) = diam(G)$. Let $N_i(x) = \{v : dist(x, v) = i\}$ where $0 \leq i \leq 3$ and $N_4(x) = \{v : dist(x, v) \geq 4\}$. So $N_0 = \{x\}$ and $N_1 = N_G(x)$. In the rest of our paper, we use N_i instead of $N_i(x)$ for convenience. By the definition of N_i, we know that $uv \in E(\overline{G})$ for any $u \in N_i, v \in N_j$ with $|i-j| \geq 2$. Now we give $\overline{G}$ an edge-coloring as follows: we first assign the color 1 to the edges xu for $u \in N_3$, and to all edges between N_1 and N_4; next we give the color 2 to all the remaining edges.

We prove that there is a proper path between any two vertices u and v in $\overline{G}$. It is trivial when $uv \in E(\overline{G})$. Thus we only need to consider the pairs $u, v \in N_i$ or $u \in N_i, v \in N_{i+1}$. As $P = xx_3x_1x_4x_2$ is a proper path where $x_i \in N_i$, one can see that u and v are connected by a proper path for any $u \in N_i, v \in N_{i+1}$. So it suffices to show that for any $u, v \in N_i$, there is a proper path connecting them in $\overline{G}$. For $i = 1$, let $P = ux_3xx_4v$ where $x_3 \in N_3$ and $x_4 \in N_4$. Clearly, P is a proper path. Similarly, there is a proper path connecting any two vertices $u, v \in N_3$ or N_4. For $i = 2$, let $Q = uxx_3x_1x_4v$, where $x_1 \in N_1, x_3 \in N_3$ and $x_4 \in N_4$. One can see that Q is a proper path. Thus $\overline{G}$ is proper connected. Hence we have $pc(\overline{G}) = 2$. $\square$

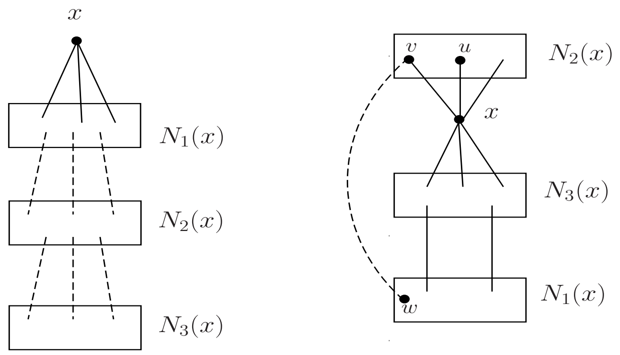

Figure 2: G and $\overline{G}$ with $diam(G) = 3$

Theorem 3.2. *For a connected noncomplete graph G, if $\overline{G}$ does not belong to the following two cases: (i) $diam(\overline{G}) = 2, 3$, (ii) $\overline{G}$ contains exactly two components and one of them is trivial, then $pc(G) = 2$.*

Proof. If $\overline{G}$ is connected, we know that $diam(\overline{G}) \geq 4$. Hence $pc(G) = 2$ clearly holds by Theorem 3.1. Now we may assume that $\overline{G}$ is disconnected. Suppose that $\overline{G}_i$ $(1 \leq i \leq h)$ are the components of $\overline{G}$ with $t_i = |V(\overline{G}_i)|$. Then G contains a spanning subgraph $K_{t_1,t_2,\ldots,t_h}$. By the assumption, $\overline{G}$ has either at least three components or exactly two nontrivial components. Then we have $pc(G) = 2$ from Lemma 2.4 and Corollary 2.5. $\square$

If $diam(G) = 3$, we have the following theorem for the proper connection number of $\overline{G}$.

Theorem 3.3. *Let G be a connected graph with $diam(G) = 3$ and x the vertex of G such that $ecc_G(x) = 3$ (see Fig. 2). Denote by n_i the number of vertices that has distance i to x for $i = 1, 2, 3$. We have $pc(\overline{G}) = 2$ for the two cases (i) $n_1 = n_2 = n_3 = 1$, (ii) $n_2 = 1, n_3 \geq 2$. For the remaining cases, if G is triangle-free, then $pc(\overline{G}) = 2$.*

Proof. If $n_1 = n_2 = n_3 = 1$. Then G is a 4-path P_4, and so $pc(\overline{G}) = pc(P_4) = 2$. Then we consider the case that $n_2 = 1, n_3 \geq 2$. One can see that $\overline{G}[N_0 \cup N_1 \cup N_3]$ contains a spanning subgraph K_{1+n_1,n_3}. By Lemmas 2.1 and 2.4, we know that $pc(\overline{G}[N_0 \cup N_1 \cup N_3]) = 2$. Hence, we can get that $pc(\overline{G}) = 2$ from Lemma 2.7. The remaining cases are: (1) $n_1 > 1, n_2 = n_3 = 1$, and (2) $n_2 \geq 2$.

If G is triangle-free, then N_1 is an independent set in G, and so a clique in $\overline{G}$. We give $\overline{G}$ an edge-coloring as follows: assign color 1 to xx_2 and x_1x_3 for any $x_1 \in N_1, x_2 \in N_2, x_3 \in N_3$ and assign color 2 to all the other edges in $\overline{G}$. Now we prove that this is a proper-path 2-coloring of $\overline{G}$.

For any $u \in N_i$ and $v \in N_j$ with $|i-j| \geq 2$ or $u, v \in N_1$, one have that $uv \in \overline{G}$. Since $P = x_2xx_3x_1$ is a proper path for any $x_i \in N_i$ for $i = 1, 2, 3$, one can see that u and v are connected by a proper path for any $u \in N_i, v \in N_{i+1}$. So we only need to consider the case that for any $u, v \in N_2$ or N_3 with $uv \notin E(\overline{G})$, there is a proper path between them. In fact, as G is triangle-free, if $uv \in E(G)$, one can see that there is a vertex $w \in N_1$ such that $wu \in E(G)$ and $wv \notin E(G)$. Thus $P = uxx_3wv$ is a proper path connecting u and v in $\overline{G}$ where $x_3 \in N_3$. Similarly, we can see that for any $u, v \in N_3$, there is a proper path between them. Thus we have that this coloring is a proper-path 2-coloring. So $pc(\overline{G}) = 2$. $\square$

Remark: If $n_2 = n_3 = 1$ and $n_1 > 1$, let $N_3 = \{x_3\}$, and $n'_1 = |\{v \in N_1 : N_{\overline{G}}(v) \cap N_1 = \emptyset\}|$. One can see that there are n'_1 cut edges in $\overline{G}$ that is adjacent to x_3. By Lemma 2.2, we have that $pc(\overline{G}) \geq n'_1$. If $n_2 \geq 2$, let $n'_2 = |\{v \in N_2 : d_{\overline{G}}(v) = 1\}|$. One can see that there are n'_2 cut edges in $\overline{G}$ that is adjacent to x. By Lemma 2.2, we have that $pc(\overline{G}) \geq n'_2$. Hence, the condition "$G$ is triangle-free" is necessary to determine the proper connection number of $\overline{G}$ in the theorem.

The following corollary clearly holds.

Corollary 3.4. *For any tree T that is not a star, one has that $pc(\overline{T}) = 2$.*

Theorem 3.5. *Let G be a triangle-free graph with $diam(G) = 2$. If $\overline{G}$ is connected, then $pc(\overline{G}) = 2$.*

Proof. We choose a vertex x with $ecc_G(x) = 2$, and $N_i = \{v : dist(x, v) = i\}$ for $i = 0, 1, 2$. One can see that $N_0 = \{x\}, N_1 = N_G(x)$, and $N_2 = V \setminus (N_1 \cup N_0)$. As G is triangle-free, it is obvious that N_1 is a clique in $\overline{G}$. Since $\overline{G}$ is connected, then we have that $|N_1| > 1$ and there is at least one edge $uv \in E(\overline{G})$ such that $u \in N_1$ and $v \in N_2$.

We give $\overline{G}$ an edge-coloring as follows: assign color 1 to the edges between N_1 and N_2, and assign color 2 to all the other edges in $\overline{G}$. Now we prove that this is a proper-path coloring of $\overline{G}$. For any $z \in N_1$, we know that $P = xvuz$ (u and z may coincide) is a proper path. So there are proper paths between x and any other vertices, and there are proper paths between v and vertices in N_1. For any $y \in N_2 \setminus \{v\}$ and $z \in N_1$, if $N_{\overline{G}}(y) \cap N_1 \neq \emptyset$, let $w \in N_{\overline{G}}(y) \cap N_1$. Then ywz is a proper path between y and z. Otherwise, $N_{\overline{G}}(y) \cap N_1 = \emptyset$. We claim that y is adjacent to all the other vertices of N_2 in $\overline{G}$. In fact, for any vertex $w \in N_2 \setminus y$, there exists a vertex $w' \in N_1$ such that $ww' \in E(G)$. Since $yw' \in E(G)$, we know that $yw \in E(\overline{G})$. Especially, we know that $yv \in E(\overline{G})$. Then $yvuz$ is a proper path between y and z. Next consider $x_2, x'_2 \in N_2$ such that $x_2x'_2 \notin E(\overline{G})$. Since $x_2, x'_2 \in N_2$, there are $x_1, x_1' \in N_1$ such that $x_1x_2, x_1'x_2' \in E(G)$. As G is triangle-free, one can see that $x_1 \neq x_1'$ and $x_1x_2', x_2x_1' \in E(\overline{G})$. So we have that $x_2x_1'x_1x_2'$ is a proper path connecting x_1 and x_1'. Hence we have that $pc(\overline{G}) = 2$. ☐

Proposition 3.6. *If G is triangle-free and contains two components one of which is trivial, then $pc(\overline{G}) = 2$.*

Proof. Let G_1 and G_2 be the two components of G such that $V(G_1) = \{v\}$. Then $\overline{G} = \overline{G_1} \vee \overline{G_2}$, where "$\vee$" is the join of two graphs, that is, vertex v is adjacent to all the other vertices in $\overline{G}$. If $\overline{G_2}$ is connected, then $pc(\overline{G_2}) = 2$ from Theorem 3.1, Theorem 3.3 and Theorem 3.5. Hence, we can get that $pc(\overline{G}) = 2$. Otherwise, $\overline{G_2}$ is disconnected. Since G is triangle-free, we know that $\overline{G_2}$ has two components, and both of them are cliques of $\overline{G_2}$. Suppose that H_1 and H_2 are the two component of $\overline{G_2}$, we assign color 1 to all the edges between v and H_1 and assign color 2 to the remaining edges. As $P = x_1vx_2$ is a proper path connecting x_1 and x_2 for any $x_1 \in H_1$ and $x_2 \in H_2$. So we have that $\overline{G}$ is proper connected. Hence $pc(\overline{G}) = 2$. ☐

In conclusion, we can get the following result.

Theorem 3.7. *For a connected noncomplete graph G, if $\overline{G}$ is triangle-free, then $pc(G) = 2$.*

Proof. We consider two cases:

Case 1. $\overline{G}$ is connected. The result holds for the case $diam(\overline{G}) \leq 4$ from Theorem 3.1, the case $diam(\overline{G}) = 3$ from Theorem 3.3 and the case $diam(\overline{G}) = 2$ from Theorem 3.5.

Case 2. $\overline{G}$ is disconnected. The result holds for the case that $\overline{G}$ contains two components with one of them trivial from Proposition 3.6, and holds for the remaining case from Lemma 2.4 and Corollary 2.5. ☐

4 Nordhaus-Gaddum-Type theorem for proper connection number of graphs

Firstly, we characterize the graphs on n vertices that have proper connection number $n-2$. This result is crucial to investigate the Nordhaus-Gaddum-type result for the proper connection number of G. We use C_n, S_n to denote the cycle and the star graph on n vertices, respectively, and use $T(a,b)$ to denote the double star that is obtained by adding an edge between the center vertices of S_a and S_b. For a nontrivial graph G such that $G + uv \cong G + xy$ for every two pairs $\{u, v\}$, $\{x, y\}$ of nonadjacent vertices of G, we use $G + e$ to denote the graph obtained from G by joining two nonadjacent vertices of G.

Theorem 4.1. *Let G be a connected graph on n vertices. Then $pc(G) = n - 2$ if and only if G is one of the following 6 graphs: $T(2, n-2), C_3, C_4, C_4 + e, S_4 + e$, and $S_5 + e$.*

Proof. If G is one of the above 6 graphs, we can easily check that $pc(G) = n - 2$. So it remains to verify the converse of the theorem. Suppose that $pc(G) = n - 2$. If G is acyclic, from Lemma 2.3, we know that $G \cong T(2, n-2)$. So we may assume that G contains cycles. Let G^* be a spanning unicycle subgraph of G such that the cycle C in G^* is the longest cycle in G. Without loss of generality, suppose that $C = v_1v_2 \ldots v_kv_1$ and $d_{G^*}(v_1) \geq d_{G^*}(v_i)$ for $i = 2, 3, \ldots, k$. Note that $pc(C) = 2$ for all $k \geq 4$. Giving C a proper-path 2-coloring and assigning $n - k$ new colors to the remaining $n - k$ edges of G^*, we get a proper-path coloring of G^*. It follows that $pc(G^*) \leq 2 + n - k$. From Lemma 2.1, we know that $pc(G) \leq pc(G^*) \leq 2 + n - k$. Thus we can get that $pc(G) < n - 2$ if $k > 4$, contradicting with the fact that $pc(G) = n - 2$. So we only need to consider that $k = 3$ or $k = 4$.

If $k = 4$, let $G_1 = G^* - v_1v_2$. One can see that G_1 is a spanning tree of G. If $n = 4$, then $G^* \cong C_4$. We can get that $G \cong C_4$ or $G \cong C_4 + e$ since the longest cycle of G is of length 4. So we consider that $n \geq 5$. Since $d_{G^*}(v_1) \geq d_{G^*}(v_i)$ for $i = 2, 3, \ldots, k$ and G^* is unicycle, we see that $\Delta(G_1) \leq n - 3$. So by Lemma 2.1, $pc(G) \leq pc(G_1) \leq n - 3$, contradicting the fact that $pc(G) = n - 2$.

Now we consider the case $k = 3$. Let c be an edge coloring of G^* such that the cut edges are colored by $n - 3$ distinct colors. If $n \geq 6$, that is, G^* has more than three cut edges, choose three colors that have been used on the cut edges, say $1, 2, 3$. Let $c(v_1v_2) = 1$, $c(v_2v_3) = 2$, and $c(v_3v_1) = 3$. We know that G^* is proper connected under edge-coloring c. Hence $pc(G) \leq pc(G^*) \leq n - 3$, contradicting the fact that $pc(G) = n - 2$. So we may assume that $n \leq 5$. If $n = 5$, one can see that $G \cong S_5 + e$ since otherwise there is a spanning P_5 in G, then $pc(G) \leq pc(P_5) = 2$, a contradiction. If $n = 4$, one can see that $G \cong S_4 + e$ since otherwise there exists a cycle of length 4 in G which contradicts the assumption $k = 3$. If $n = 3$, we know that $G \cong C_3$ as $pc(G) = 1$ if and only if G is complete graph. Hence we have that $G \cong C_3$, or $G \cong S_4 + e$, or $G \cong S_5 + e$ when $k = 3$. $\square$

We know that if G is a connected graph with n vertices, then the number of edges in G must be at least $n - 1$. If both G and $\overline{G}$ are connected, then n is at least 4, and $\Delta(G) \leq n - 2$. Therefore we know that $2 \leq pc(G) \leq n - 2$. Similarly, $2 \leq pc(\overline{G}) \leq n - 2$. Hence we can obtain that $4 \leq pc(G) + pc(\overline{G}) \leq 2(n - 2)$. For $n = 4$, we can easily get that $pc(G) + pc(\overline{G}) = 4$ if G and $\overline{G}$ are connected. In the rest of the paper, we always assume that all graphs have at least 5 vertices, and both G and $\overline{G}$ are connected.

Lemma 4.2. *Let G be a graphs with 5 vertices. If both G and $\overline{G}$ are connected, one has that*

$$pc(G) + pc(\overline{G}) = \begin{cases} 5 & \text{if } G \cong T(2,3) \text{ or } \overline{G} \cong T(2,3), \\ 4 & \text{otherwise.} \end{cases}$$

Proof. If $G \cong T(2,3)$ or $\overline{G} \cong T(2,3)$, then it can be easily checked that $pc(G)+pc(\overline{G}) = 5$. From Theorem 4.1, we know that $T(2,3)$ is the only graph on 5 vertices that has proper connection number 3. Since $2 \leq pc(G) \leq n-2 = 3$ and $2 \leq pc(\overline{G}) \leq n-2 = 3$, then all the other graphs considered here on 5 vertices has proper connection number 2. Hence $pc(G) + pc(\overline{G}) = 4$ if $G \ncong T(2,3)$ and $\overline{G} \ncong T(2,3)$. □

Theorem 4.3. *$pc(G) + pc(\overline{G}) \leq n$ for $n \geq 5$, and the equality holds if and only if $G \cong T(2, n-2)$ or $\overline{G} \cong T(2, n-2)$.*

Proof. By Lemma 4.2, we can see that the result holds if $n = 5$. So we consider $n \geq 6$. If $G \cong T(2, n-2)$, $\overline{G}$ contains a spanning subgraph H that is obtained by attaching a pendent edge to the complete bipartite graph $K_{2,n-3}$. Then $pc(\overline{G}) = 2$ by Lemma 2.4 and Lemma 2.7. The result clearly holds. Similarly, we can also get $pc(G) + pc(\overline{G}) = n$ if $\overline{G} \cong T(2, n-2)$. To prove our conclusion, we only need to show that $pc(G)+pc(\overline{G}) < n$ if $G \ncong T(2, n-2)$ and $\overline{G} \ncong T(2, n-2)$. Under this assumption, we know that $2 \leq pc(G) \leq n-3$ and $2 \leq pc(\overline{G}) \leq n-3$ by Theorem 4.1.

Suppose first that both G and $\overline{G}$ are 2-connected. For $n = 6$, we claim that $pc(G) = 2$. Assume that the circumference of G is k. If $k = 6$, one has that $pc(G) \leq pc(C_6) = 2$. If $k = 4$, one can see that G contains a spanning $K_{2,4}$, contradicting the assumption that $\overline{G}$ is 2-connected. Assume that G contains a 5-cycle $C = v_1v_2v_3v_4v_5v_1$, we know that the vertex v_6 is adjacent to two vertices that is nonadjacent in C, say v_1, v_3. Then $P = v_6v_1v_2v_3v_4v_5$ is a hamilton path of G. Hence $pc(G) \leq pc(P) = 2$. So we have that $pc(G) + pc(\overline{G}) \leq 2 + n - 3 < n$. For $n \geq 7$, by Lemma 2.6, we know that $pc(G) \leq 3$ and $pc(\overline{G}) \leq 3$, and so $pc(G) + pc(\overline{G}) \leq 6$, and therefore $pc(G) + pc(\overline{G}) < n$ clearly holds.

Now we consider the case that at least one of G and $\overline{G}$ has cut vertices. Without loss of generality, suppose that G has cut vertices. We distinguish the following three cases.

Case 1. G has a cut vertex u such that $G - u$ has at least three components. Let $G_1, G_2, \ldots, G_k$ $(k \geq 3)$ be the components of $G - u$ and let n_i be the number of vertices of G_i for $1 \leq i \leq k$ with $n_1 \leq n_2 \leq \ldots \leq n_k$. From the definition of $\overline{G}$, we know that $\overline{G} - u$ contains a spanning complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$. Since $\Delta(\overline{G}) \leq n-2$, then

$n_k \geq 2$. From Corollary 2.5, $pc(\overline{G}-u)=2$, and there exists a 2-edge-coloring c of $\overline{G}-u$ that makes it proper connected with the strong property. Hence $pc(\overline{G}) \leq 2$ by Lemma 2.7. Together with the fact that $pc(G) \leq n-3$, we can get the result $pc(G)+pc(\overline{G}) < n$.

Case 2. Each cut vertex u of G satisfies that $G-u$ has only two components. Let G_1, G_2 be the two components of $G-u$, and let n_i be the number of vertices of G_i for $i=1,2$ with $n_1 \leq n_2$.

subcase 2.1. $n_1 \geq 2$, then $\overline{G}-u$ contains a spanning 2-connected bipartite graph K_{n_1,n_2}. From Lemma 2.4, we know that $pc(\overline{G}-u)=2$ and there exists a 2-edge-coloring c of $\overline{G}-u$ that makes it proper connected with the strong property. So by Lemma 2.7, $pc(\overline{G}) \leq 2$. We can get the result that $pc(G)+pc(\overline{G}) < n$.

subcase 2.2. $n_1 = 1$, that is, each cut vertex is incident with a pendent edge. Let $u_1v_1, u_2v_2, \ldots, u_lv_l$ be the pendent edges of G such that v_i is the pendent vertices for $1 \leq i \leq l$. The pendent edges are pairwise disjoint. Let H be the graph obtained from G by deleting all the pendent vertices. Then H must be 2-connected. By Lemma 2.6, we know that $pc(H) \leq 3$ and there exists a 3-edge-coloring c of H that makes it proper connected with the strong property.

If $l \geq 2$, we know that $\overline{G}-\{u_1,u_2\}$ contains a spanning bipartite subgraph $K_{2,n-4}$ with two parts $X=\{v_1,v_2\}$ and $Y=V(G)\setminus\{u_1,v_1,u_2,v_2\}$. Since $v_1u_2, v_2u_1 \notin E(G)$, we know that $v_1u_2, v_2u_1 \in E(\overline{G})$. Then by Lemma 2.4 and Lemma 2.7, we have that $pc(\overline{G}) \leq 2$. By using the fact that $pc(G) \leq n-3$, we have that $pc(G)+pc(\overline{G}) < n$.

If $l=1$, by Lemma 2.6 and Lemma 2.7, one has that $pc(G) \leq pc(H) \leq 3$. Therefore we have $pc(G)+pc(\overline{G}) \leq n$. Now we prove that the equality cannot be attained. Note that $d_{\overline{G}}(v_1)=n-2$. We know that $\overline{G}$ contains T_0 as a proper spanning subgraph. Set $N_{\overline{G}}(v_1)=\{x_1,\cdots,x_{n-2}\}=V(G)\setminus\{u_1,v_1\}$. Without loss of generality, assume that $x_1u_1 \notin E(G)$. So $x_1u_1 \in E(\overline{G})$. If there is a vertex x_j $(2 \leq j \leq n-2)$ that is adjacent to x_1 in $\overline{G}$, assume without loss of generality that $x_1x_2 \in E(\overline{G})$. Let $c(v_1x_1)=1, c(x_1x_2)=2, c(v_1x_2)=c(x_1u_1)=3$ and $c(v_1x_i)=i-2$ for $i=3,4\cdots,n-2$. One can see that $\overline{G}$ is proper connected. If there is a vertex x_j $(2 \leq j \leq n-2)$ that is adjacent to u_1 in $\overline{G}$, assume without loss of generality that $x_2u_2 \in E(\overline{G})$. Let $c(v_1x_i)=i-2$ for $i=3,4\cdots,n-2$ and $c(v_1x_1)=c(u_1x_2)=1, c(v_1x_2)=c(x_1u_1)=2$. One can also see that $\overline{G}$ is proper connected. If there are two vertex x_j, x_k $(2 \leq j < k \leq n-2)$ such that $x_jx_k \in E(\overline{G})$, without loss of generality, assume that $x_2x_3 \in E(\overline{G})$. Let $c(v_1x_i)=i-2$ for $i=4,\cdots,n-2$, $c(v_1x_1)=c(v_1x_2)=1, c(v_1x_3)=c(x_1u_1)=2$ and $c(x_2x_3)=3$. We can check that $\overline{G}$ is proper connected. Hence we have that $pc(\overline{G}) \leq \max\{3,n-4\}$. For $n \geq 7$, we can get that $pc(G)+pc(\overline{G}) \leq 3+n-4=n-1<n$. For $n=6$, as H is a 2-connected graph with 5 vertices, one can see that H contains a spanning

C_5 or a spanning $K_{2,3}$. Hence we can easily get that $pc(G) \leq pc(H) = 2$. So we have $pc(G) + pc(\overline{G}) \leq 2 + 3 = 5 < 6$. $\square$

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