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Double-sum Rogers–Ramanujan type identities

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As the q-analog of Chebyshev polynomials, q-Hermite polynomials form a cornerstone in the family of q-orthogonal polynomials, which play a fundamental role in quantum algebra and mathematical physics. Recently, Andrews obtained a series of Rogers–Ramanujan type identities by constructing Bailey pairs from Chebyshev polynomials. In this paper, by applying the expansion formula of Chebyshev polynomials in terms of q-Hermite polynomials and using the orthogonality relations, we derive a series of Rogers–Ramanujan type identities on double sums, which further generalized the known results due to Andrews, Shi, Sun and Yao.

Keywords: Chebyshev polynomials; q-Hermite polynomials; Rogers–Ramanujan type identities.

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1. Introduction

The famous Rogers–Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}, \tag{1.2}$$

were first discovered and proved in 1894 by Rogers [27], and were rediscovered independently by Ramanujan [29], which have profound applications in combinatorics,

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number theory, modular forms, statistical mechanics and representation theory. Thereby the series–product identities similar to the above form are called Rogers–Ramanujan type identities, which were systematically studied and generalized by Bailey [10, 11]. Later, Slater [33, 34] compiled and discovered a list of 130 identities of Rogers–Ramanujan type applying various Bailey pairs, which is widely known as "Slater's list". Sills *et al.* [20, 32] further generalized Slater's results and obtained a more complete list of Rogers–Ramanujan type identities.

MacMahon [21] and Schur [31] also gave the combinatorial versions of the two Rogers–Ramanujan identities. In 1961, Gordon [17] provided a combinatorial interpretation of the generalized Rogers–Ramanujan identities. Then in 1974, Andrews [2] expressed Gordon's result in terms of the following q-identity:

$$\sum_{n_1\geq\cdots\geq n_{k-1}\geq 0}\frac{q^{n_1^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}=\frac{(q^i,q^{2k-i+1},q^{2k+1};q^{2k+1})_\infty}{(q)_\infty},\tag{1.3}$$

with $k\geq 2$ and $1\leq i\leq k$ being two integers, which is the well-known Andrews–Gordon identity.

In 2019, Kanade and Russell [19] searched for Rogers–Ramanujan type identities and conjectured several new ones which are related to level 2 characters of the affine Lie algebra $A_9^{(2)}$. Since then numerous new Rogers–Ramanujan type identities have been derived, especially in the form of double sums. By using the q-Zeilberger algorithm to get the recurrence relation, Andrews and Uncu [8] proved a double-sum identity of Rogers–Ramanujan type as follows:

$$\sum_{m,n\geq 0}\frac{(-1)^n q^{\binom{3n+1}{2}+m^2+3mn}}{(q)_m(q^3;q^3)_n}=\frac{1}{(q;q^3)_\infty}.$$

Note that the right-hand sides of the Rogers–Ramanujan identities (1.1) and (1.2) can be seen as modular forms, which is not easy to be observed from their sum-sides. Nahm [23–25] considered a specific class of q-series as follows, which is known as Nahm sum or Nahm series:

$$f_{A,B,C}(q)=\sum_{n=(n_1,\ldots,n_r)^T\in\mathbb{N}^r}\frac{q^{\frac{1}{2}n^TAn+n^TB+C}}{(q;q)_{n_1}\cdots(q;q)_{n_r}},$$

where r is a positive integer, A is a real positive definite symmetric $r\times r$ matrix, B is an r-dimensional column vector, and C is a rational number. In [25], Nahm proposed a conjecture that characterizes the necessary and sufficient conditions on the matrix component of a modular triple (A,B,C) to ensure that the associated series is a modular function. Then, Wang [37] generalized the identities on Nahm series by indices. Later, Cao and Wang [12] applying the contour integral method to establish a number of multi-sum Rogers–Ramanujan type identities including

$$\sum_{i,j\geq 0}\frac{(-1)^{i+j}q^{2a(i-j)}q^{(i^2-i+j^2-j+4a(i-j)^2)/2}}{(q)_i(q)_j}=\frac{(q,q^{4a},q^{4a+1};q^{4a+1})_\infty}{(q)_\infty}.\tag{1.4}$$

In this paper, based on the orthogonality relations of trigonometric functions and the most classical q-orthogonal polynomials, that is, the (continuous) q-Hermite polynomials, we obtain several double-sum Rogers–Ramanujan type identities.

The q-orthogonal polynomials play an important role in the study of q-series, which were introduced by Rogers and Szegö [26, 27, 36]. Based on these works, in 1985, Askey and Wilson [9] derived one of the most general class of q-orthogonal polynomials, which is named Askey–Wilson polynomials. As its special case, the (continuous) q-Hermite polynomials is defined as follows:

$$H_n(x\,|\,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}e^{-i(n-2k)\theta}, \tag{1.5}$$

where $x=\cos\theta$ and $H_0(x\,|\,q)=1$.

The orthogonality relation of q-Hermite polynomials is

$$\frac{(q)_\infty}{2\pi}\int_0^\pi H_n(x\,|\,q)H_m(x\,|\,q)(e^{2i\theta},e^{-2i\theta})_\infty d\theta=\delta_{mn}(q)_n, \tag{1.6}$$

where δ is the Kronecker delta function such that $\delta_{mn}=1$, if $m=n$, otherwise $\delta_{mn}=0$.

In 1999, Garrett *et al.* [15] applied the orthogonality relations of the q-Hermite and q-ultraspherical polynomials to study the Rogers–Ramanujan identities. In 2012, Andrews [5] demonstrated the applications of q-orthogonal polynomials in the study of mock theta functions. Many other q-orthogonal polynomials have been studied over the years by Al-Salam *et al.* in [1, 13, 18].

As a limit case of q-Hermite polynomials, Chebyshev polynomials of the second kind can be expressed as $U_n(x)=H_n(x|0)$. Garrett *et al.* [15] gave the following expansion formula:

$$U_n(x)=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\begin{bmatrix}n-j\\j\end{bmatrix}(-1)^jq^{\binom{j+1}{2}}H_{n-2j}(x\,|\,q), \tag{1.7}$$

which bridges the study of classical orthogonal polynomials and q-orthogonal polynomials. Recently, Andrews [5, 6] applied orthogonal polynomials to study various problems in q-series, especially to Rogers–Ramanujan type identities. Andrews obtained the following two identities in [6]:

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q)_{2n+1}}=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{3\binom{n+1}{2}}V_n(x), \tag{1.8}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}\prod_{j=0}^{n-1}(1+2xq^j+q^{2j})}{(q)_{2n}}=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1})W_n(x), \tag{1.9}$$

where $V_n(x)$ and $W_n(x)$ are Chebyshev polynomials of the third and fourth kinds, respectively.

Inspired by Andrews' work, Sun [35] and Yao [39] further studied Rogers–Ramanujan type identities by constructing Bailey pairs involving Chebyshev polynomials, which leads to the following identities [39, (3.1); 35, (6.2)]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n} \prod_{j=1}^{n} (1 + 2xq^j + q^{2j})}{(1+q^{n+1})(q)_{2n+1}}$$

$$= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} (1 - q^{n+1}) q^{n(3n+5)/2} U_n(x), \tag{1.10}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-1;q^2)_n \prod_{i=1}^{n} (1 + 2xq^{2i-1} + q^{4i-2})}{(q^2;q^2)_{2n}}$$

$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{2n^2+n}}{(-q^2;q^2)_n} \big(V_n(x) + V_{n-1}(x)\big). \tag{1.11}$$

From these identities on Chebyshev polynomials, by expanding $U_n(x)$ using formula (1.7) and then applying the orthogonality relation (1.6) of q-Hermite polynomials, we obtain some identities of Rogers–Ramanujan type on double sums. For example, from (1.10) we get

$$\sum_{n \geq k \geq 0} \frac{q^{n(n+2)+k(k+1)}}{(q)_k (q)_{n-k} (q^{n+1})_{n+1} (1+q^{n+1})} = \frac{(q, q^{12}, q^{13}; q^{13})_\infty}{(q)_\infty}. \tag{1.12}$$

Note that identity (1.12) can be seen as a new form of the Andrews–Gordon identity (1.3) with $k = 6, i = 1$, in which it takes the form of a quintuple sum. It also coincides with (1.4) with $a = 3$.

This paper is organized as follows. In Sec. 2, we introduce some basic definitions and notations for q-series and orthogonal polynomials. In Sec. 3, we obtain a series of integral identities on q-Hermite polynomials based on the q-binomial theorem. In Sec. 4, from identities on Chebyshev polynomials, by using the integral identities obtained in Sec. 3, we derive Rogers–Ramanujan type identities on double sums such as (1.12) and also an identity on Appell–Lerch series. In Sec. 5, by transforming Chebyshev polynomials $V_n(x)$ and $W_n(x)$ into $U_n(x)$, we also obtain triple-sum identities with a free variable ℓ. When $\ell = 0$, it leads to more Rogers–Ramanujan type identities on double sums.

2. Preliminaries

Throughout this paper, we adopt standard notations and terminologies for q-series [16] and we assume that $|q| < 1$. The q-shifted factorials are given by

$$(a)_n = (a;q)_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=1}^{n}(1-aq^{j-1}) & \text{if } n \geq 1, \end{cases} \tag{2.1}$$

and

$$(a)_\infty = (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

There are more compact notations for the multiple q-shifted factorials

$$(a_1, a_2, \ldots, a_m)_n = (a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n,$$

$$(a_1, a_2, \ldots, a_m)_\infty = (a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

The q-binomial coefficients, or Gaussian polynomials are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where subscript "q" is usually omitted if no confusion arises. The q-binomial theorem is known as [4, (3.3.6)]

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}} z^j = (z)_n. \tag{2.2}$$

The well-known Jacobi's triple product identity [16, (II.28)] is

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n = (z, q/z, q)_\infty. \tag{2.3}$$

Recall that Chebyshev polynomials $T_n(x), U_n(x), V_n(x), W_n(x)$ of the first, second, third and fourth kinds are defined as follows:

$$T_n(x) = \cos(n\theta), \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

$$V_n(x) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta}, \quad W_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta},$$

where $x = \cos\theta$ and the initial conditions are $T_0(x) = U_0(x) = V_0(x) = W_0(x) = 1$. For convenience, we also set $U_n(x) = V_n(x) = W_n(x) = 0$ for $n < 0$. In fact, these four polynomials satisfy the same recurrence relation:

$$p_n(x) = 2xp_{n-1}(x) - p_{n-2}(x), \quad (n \geq 2),$$

with $p_0(x) = 1$, and $p_1(x) = x, 2x, 2x-1, 2x+1$, respectively. They are also closely related to each other. As given in [22] for $n \geq 1$, we see that

$$V_n(x) = U_n(x) - U_{n-1}(x), \tag{2.4}$$

and

$$W_n(x) = U_n(x) + U_{n-1}(x). \tag{2.5}$$

3. Integral Identities on q-Hermite Polynomials

In this section, based on the q-binomial theorem, we obtain a series of integral identities on q-Hermite polynomials, which will be used to derive Rogers–Ramanujan type identities.

By following Rogers' way [28] to treat q-polynomials in terms of Fourier series, we replace x by $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$. As pointed out by Andrews [6], the following formula is a finite version of the result due to Rogers [28]:

$$(-qe^{i\theta}, -qe^{-i\theta})_n = \prod_{j=1}^{n}(1 + 2q^j\cos\theta + q^{2j}). \tag{3.1}$$

From the orthogonality relations of trigonometric functions, it is obvious that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im\theta}d\theta = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

By combining the orthogonality relation of q-Hermite polynomials, we derive the following integral identity.

Lemma 3.1. *We have*

$$\begin{aligned} &\frac{(q)_\infty}{2\pi}\int_{-\pi}^{\pi} H_{\ell_1}(x\,|\,q)H_{\ell_2}(x\,|\,q)(-qe^{i\theta}, -qe^{-i\theta})_n(e^{2i\theta}, e^{-2i\theta})_\infty\, d\theta \\ &\quad = \sum_{\substack{k_i=0 \\ i=1,2}}^{\ell_i}\sum_{m=-\infty}^{\infty}(-1)^m q^{\binom{2m+2k_1+2k_2-\ell_1-\ell_2+1}{2}+\binom{m+1}{2}} \\ &\qquad \times \frac{(1+q^{n+1})(1-q^{2m+2k_1+2k_2-\ell_1-\ell_2+1})}{1-q^{n+2m+2k_1+2k_2-\ell_1-\ell_2+2}} \\ &\qquad \cdot \begin{bmatrix} \ell_1 \\ k_1 \end{bmatrix}\begin{bmatrix} \ell_2 \\ k_2 \end{bmatrix}\begin{bmatrix} 2n+1 \\ n-2m-2k_1-2k_2+\ell_1+\ell_2 \end{bmatrix}, \end{aligned} \tag{3.3}$$

where m ranges over all the integers such that the q-binomial coefficients in the above sum is not equal to zero.

Proof. By applying the q-binomial theorem (2.2), we have that

$$\begin{aligned}(-qe^{i\theta},-qe^{-i\theta})_n &= (-qe^{i\theta})_n\ q^{\binom{n+1}{2}}e^{-in\theta}(-q^{-n}e^{i\theta})_n\\ &= \frac{q^{\binom{n+1}{2}}}{(1+e^{i\theta})e^{in\theta}}(-q^{-n}e^{i\theta})_{2n+1}\\ &= \frac{1}{1+e^{i\theta}}\sum_{j=0}^{2n+1}\begin{bmatrix}2n+1\\ j\end{bmatrix}q^{\binom{n+1}{2}+\binom{j}{2}-nj}e^{i(j-n)\theta}.\end{aligned}$$

Substituting it into the left-hand side of (3.3), using the definition of q-Hermite polynomials (1.5) and then applying Jacobi's triple product identity (2.3), it becomes

$$\begin{aligned}&\frac{(q)_\infty}{2\pi}\sum_{\substack{k_i=0\\ i=1,2}}^{\ell_i}\sum_{j=0}^{2n+1}q^{\binom{n+1}{2}+\binom{j}{2}-nj}\begin{bmatrix}2n+1\\ j\end{bmatrix}\begin{bmatrix}\ell_1\\ k_1\end{bmatrix}\begin{bmatrix}\ell_2\\ k_2\end{bmatrix}\\ &\quad\cdot\int_{-\pi}^{\pi}(1-e^{i\theta})\ e^{(2k_1+2k_2-\ell_1-\ell_2+j-n)i\theta}(qe^{2i\theta},e^{-2i\theta})_\infty\ d\theta\\ &= \sum_{\substack{k_i=0\\ i=1,2}}^{\ell_i}\sum_{j=0}^{2n+1}\sum_{m=-\infty}^{\infty}(-1)^m q^{\binom{n+1}{2}+\binom{m+1}{2}+\binom{j}{2}-nj}\begin{bmatrix}\ell_1\\ k_1\end{bmatrix}\begin{bmatrix}\ell_2\\ k_2\end{bmatrix}\begin{bmatrix}2n+1\\ j\end{bmatrix}\\ &\quad\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-e^{i\theta})e^{(2m+2k_1+2k_2-\ell_1-\ell_2+j-n)i\theta}\ d\theta.\end{aligned}$$

By using (3.2) to compute the integral and simplifying, it completes the proof. □

Now by taking $\ell_1=0$ or $\ell_2=0$, and noting $H_0(x\,|\,q)=1$, the above lemma reduces to the following result.

Corollary 3.2. *We have*

$$\begin{aligned}&\frac{(q)_\infty}{2\pi}\int_{-\pi}^{\pi}H_\ell(x\,|\,q)\ (-qe^{i\theta},-qe^{-i\theta})_n\ (e^{2i\theta},e^{-2i\theta})_\infty\ d\theta\\ &= \sum_{m=-\infty}^{\infty}\sum_{k=0}^{\ell}(-1)^m q^{\binom{2m+2k-\ell+1}{2}+\binom{m+1}{2}}\begin{bmatrix}\ell\\ k\end{bmatrix}\\ &\quad\cdot\left(\begin{bmatrix}2n+1\\ n-2m-2k+\ell\end{bmatrix}-q^{2m+2k-\ell+1}\begin{bmatrix}2n+1\\ n-2m-2k+\ell-1\end{bmatrix}\right).\end{aligned}$$

Proof. By taking $\ell_2=0$ and substituting ℓ_1 by ℓ in Lemma 3.1, it turns to be

$$\begin{aligned}&\sum_{k=0}^{\ell}\sum_{m=-\infty}^{\infty}(-1)^m q^{\binom{2m+2k-\ell+1}{2}+\binom{m+1}{2}}\frac{(1+q^{n+1})(1-q^{2m+2k-\ell+1})}{1-q^{n+2m+2k-\ell+2}}\\ &\quad\cdot\begin{bmatrix}\ell\\ k\end{bmatrix}\begin{bmatrix}2n+1\\ n-2m-2k+\ell\end{bmatrix}\end{aligned}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\ell} (-1)^m q^{\binom{2m+2k-\ell+1}{2}+\binom{m+1}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix} \begin{bmatrix} 2n+1 \\ n-2m-2k+\ell \end{bmatrix}$$

$$- \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\ell} (-1)^m q^{\binom{2m+2k-\ell+2}{2}+\binom{m+1}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix} \begin{bmatrix} 2n+1 \\ n-2m-2k+\ell-1 \end{bmatrix},$$

which completes the proof by simplifying. □

Then by taking $\ell = 0$ in Corollary 3.2 and simplifying, we get the result as follows.

Corollary 3.3. *We have*

$$\frac{(q)_\infty}{2\pi} \int_0^\pi (-qe^{i\theta}, -qe^{-i\theta})_n \ (e^{2i\theta}, e^{-2i\theta})_\infty \ d\theta$$

$$= \sum_{m=-\lfloor \frac{n+1}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (-1)^m q^{m(5m+3)/2} \begin{bmatrix} 2n+1 \\ n-2m \end{bmatrix}. \tag{3.4}$$

Proof. Taking $\ell = 0$ in Corollary 3.2, and by noting $H_0(x \,|\, q) = 1$, it reduces to

$$\frac{(q)_\infty}{2\pi} \int_{-\pi}^{\pi} (-qe^{i\theta}, -qe^{-i\theta})_n (e^{2i\theta}, e^{-2i\theta})_\infty \ d\theta$$

$$= \sum_{m=-\infty}^{\infty} (-1)^m q^{\binom{2m+1}{2}+\binom{m+1}{2}} \left(\begin{bmatrix} 2n+1 \\ n-2m \end{bmatrix} - q^{2m+1} \begin{bmatrix} 2n+1 \\ n-2m-1 \end{bmatrix} \right).$$

Letting $m \mapsto -m-1$ in the second term on the right-hand side of the above identity, and noting the integrand on the left-hand side is an even function, the result is proved. □

Note that by further taking $n \to \infty$ on the right-hand side of Corollary 3.3, it leads to the right-hand side of the second Rogers–Ramanujan identity (1.2).

Lemma 3.4. *We have*

$$\frac{(q)_\infty}{2\pi} \int_{-\pi}^{\pi} H_\ell(x \,|\, q) \ (-qe^{i\theta}, -qe^{-i\theta}; q^2)_n \ (e^{2i\theta}, e^{-2i\theta})_\infty \ d\theta$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\ell} (-1)^m q^{(2m+2k-\ell)^2+\binom{m+1}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}$$

$$\cdot \left(\begin{bmatrix} 2n \\ n-2m-2k+\ell \end{bmatrix}_{q^2} - q^{4(2m+2k-\ell+1)} \begin{bmatrix} 2n \\ n-2m-2k+\ell-2 \end{bmatrix}_{q^2} \right). \tag{3.5}$$

Proof. It is easy to see that

$$(z, q/z)_n = (-1)^n q^{\binom{n+1}{2}} z^{-n} (zq^{-n}; q)_{2n}.$$

By using the q-binomial theorem (2.2), we obtain that

$$(z, q/z)_n = \sum_{j=0}^{2n} (-1)^{j+n} z^{j-n} q^{\binom{n+1}{2}+\binom{j}{2}-nj} \begin{bmatrix} 2n \\ j \end{bmatrix}. \tag{3.6}$$

By taking $q \mapsto q^2, z \mapsto -qe^{i\theta}$ in (3.6), and substituting it into the left-hand side of (3.5), then following the similar procedures as given in the proof of Corollary 3.2, we obtain the result. □

Now by taking $\ell = 0$ in Corollary 3.4, it leads to the following identity.

Corollary 3.5. *We have*

$$\frac{(q)_\infty}{2\pi} \int_0^\pi (-qe^{i\theta}, -qe^{-i\theta}; q^2)_n \ (e^{2i\theta}, e^{-2i\theta})_\infty \ d\theta$$

$$= \sum_{m=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (-1)^m q^{m(9m+1)/2} \begin{bmatrix} 2n \\ n-2m \end{bmatrix}_{q^2}. \tag{3.7}$$

Lemma 3.6. *We have*

$$\frac{(q)_\infty}{2\pi} \int_{-\pi}^{\pi} H_\ell(x \,|\, q) \ (-e^{i\theta}, -e^{-i\theta}; q^a)_n \ (e^{2i\theta}, e^{-2i\theta})_\infty \ d\theta$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\ell} (-1)^m q^{a\binom{2m+2k-\ell}{2}+\binom{m+1}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}$$

$$\cdot \left(\begin{bmatrix} 2n-1 \\ n-2m-2k+\ell \end{bmatrix}_{q^a} + q^{a(2m+2k-\ell)} \begin{bmatrix} 2n-1 \\ n-2m-2k+\ell-1 \end{bmatrix}_{q^a} \right.$$

$$- q^{2a(2m+2k-\ell)+a} \cdot \begin{bmatrix} 2n-1 \\ n-2m-2k+\ell-2 \end{bmatrix}_{q^a} - q^{3a(2m+2k-\ell+1)}$$

$$\left. \cdot \begin{bmatrix} 2n-1 \\ n-2m-2k+\ell-3 \end{bmatrix}_{q^a} \right). \tag{3.8}$$

Proof. By using the q-binomial theorem (2.2), we have that

$$(-z, -1/z; q^a)_n = (1+z) \sum_{j=0}^{2n-1} q^{a\left(\binom{j}{2}+\binom{n}{2}-j(n-1)\right)} z^{j-n} \begin{bmatrix} 2n-1 \\ j \end{bmatrix}_{q^a}. \tag{3.9}$$

Taking $z = e^{i\theta}$ in (3.9) and following the similar procedures as given in the proof of Corollary 3.2, the proof is complete. □

When we take $a = 1$ in Lemma 3.6, it reduces to the following result.

Corollary 3.7. *We have*

$$\begin{aligned}
&\frac{(q)_\infty}{2\pi}\int_{-\pi}^{\pi} H_\ell(x\,|\,q)\ (-e^{i\theta},-e^{-i\theta})_n\ (e^{2i\theta},e^{-2i\theta})_\infty\ d\theta\\
&\quad=\sum_{m=-\infty}^{\infty}\sum_{k=0}^{\ell}(-1)^m q^{\binom{2m+2k-\ell}{2}+\binom{m+1}{2}}\begin{bmatrix}\ell\\ k\end{bmatrix}\\
&\qquad\cdot\left(\begin{bmatrix}2n-1\\ n-2m-2k+\ell\end{bmatrix}+q^{2m+2k-\ell}\begin{bmatrix}2n-1\\ n-2m-2k+\ell-1\end{bmatrix}-q^{2(2m+2k-\ell)+1}\right.\\
&\qquad\left.\cdot\begin{bmatrix}2n-1\\ n-2m-2k+\ell-2\end{bmatrix}-q^{3(2m+2k-\ell+1)}\begin{bmatrix}2n-1\\ n-2m-2k+\ell-3\end{bmatrix}\right).
\end{aligned}\tag{3.10}$$

By taking $a=2$ in Lemma 3.6, it gives the following identity.

Corollary 3.8. *We have*

$$\begin{aligned}
&\frac{(q)_\infty}{2\pi}\int_{-\pi}^{\pi} H_\ell(x\,|\,q)\ (-e^{i\theta},-e^{-i\theta};q^2)_n\ (e^{2i\theta},e^{-2i\theta})_\infty\ d\theta\\
&\quad=\sum_{m=-\infty}^{\infty}\sum_{k=0}^{\ell}(-1)^m q^{2\binom{2m+2k-\ell}{2}+\binom{m+1}{2}}\begin{bmatrix}\ell\\ k\end{bmatrix}\\
&\qquad\cdot\left(\begin{bmatrix}2n-1\\ n-2m-2k+\ell\end{bmatrix}_{q^2}+q^{2(2m+2k-\ell)}\begin{bmatrix}2n-1\\ n-2m-2k+\ell-1\end{bmatrix}_{q^2}\right.\\
&\qquad-q^{4(2m+2k-\ell)+2}\cdot\begin{bmatrix}2n-1\\ n-2m-2k+\ell-2\end{bmatrix}_{q^2}-q^{6(2m+2k-\ell+1)}\\
&\qquad\left.\times\begin{bmatrix}2n-1\\ n-2m-2k+\ell-3\end{bmatrix}_{q^2}\right).
\end{aligned}\tag{3.11}$$

Setting $\ell=0$ in Corollary 3.8, we are led to the following identity.

Corollary 3.9. *We have*

$$\begin{aligned}
&\frac{(q)_\infty}{2\pi}\int_{0}^{\pi} (-e^{i\theta},-e^{-i\theta};q^2)_n\ (e^{2i\theta},e^{-2i\theta})_\infty\ d\theta\\
&\quad=\sum_{m=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}(-1)^m q^{m(9m-5)/2}(1+q^m)\begin{bmatrix}2n-1\\ n-2m\end{bmatrix}_{q^2}.
\end{aligned}\tag{3.12}$$

4. Double-Sum Rogers–Ramanujan Type Identities

By using the theory of Bailey's transform, Sun [35] and Yao [39] obtained several identities involving Chebyshev polynomials. In this section, based on these identities, combining the expansion formula (1.7) of Chebyshev polynomials with

the orthogonality relation of q-Hermite polynomials (1.6), we derive some Rogers–Ramanujan type identities on double sums and also an identity on Appell–Lerch series. Recall that the Appell–Lerch series has the following form:

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n+1)/2} b^n}{1-aq^n},$$

which plays an important role in the theory of q-series, partitions, and modular forms.

First, we illustrate the detailed procedures by showing the double-sum identity (1.12) of the Rogers–Ramanujan type.

Proof of identity (1.12)**.** The first step is that from identity (1.10) due to Yao [39, (3.1)], expanding $U_n(x)$ by using formula (1.7) and noting (3.1), it turns to be

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-qe^{i\theta},-qe^{-i\theta})_n}{(1+q^{n+1})(q)_{2n+1}}\\ &\quad= \frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(1-q^{n+1})q^{n(3n+5)/2}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\begin{bmatrix} n-j \\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}} H_{n-2j}(x\,|\,q).\end{aligned}$$

Next, multiplying both sides by

$$\frac{(q)_\infty}{2\pi}(e^{2i\theta},e^{-2i\theta})_\infty,$$

and then calculating the integrals from 0 to π, we are led to

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(1+q^{n+1})(q)_{2n+1}}\cdot\frac{(q)_\infty}{2\pi}\int_0^\pi (-qe^{i\theta},-qe^{-i\theta})_n(e^{2i\theta},e^{-2i\theta})_\infty\,d\theta\\ &\quad= \frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(1-q^{n+1})q^{n(3n+5)/2}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\begin{bmatrix} n-j \\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}}\\ &\qquad\cdot\frac{(q)_\infty}{2\pi}\int_0^\pi H_{n-2j}(x\,|\,q)(e^{2i\theta},e^{-2i\theta})_\infty\,d\theta.\end{aligned}$$

Then, for the left-hand side, it can be directly simplified by Corollary 3.3 and turns into

$$\sum_{n=0}^{\infty}\sum_{m=-\lfloor \frac{n+1}{2}\rfloor}^{\lfloor \frac{n}{2}\rfloor}\frac{(-1)^m q^{n(n+2)+m(5m+3)/2}}{(1+q^{n+1})(q)_{2n+1}}\begin{bmatrix}2n+1\\ n-2m\end{bmatrix}, \tag{4.1}$$

by noting the identity given by Warnaar in [38, p. 7]

$$\sum_{m\in\mathbb{Z}}(-1)^m q^{m(5m+3)/2}\begin{bmatrix}2n+1\\ n-2m\end{bmatrix}=\sum_{k=0}^{\infty}q^{k(k+1)}\begin{bmatrix}n\\ k\end{bmatrix}, \tag{4.2}$$

the identity (4.1) can be rewritten as

$$\sum_{n\geq k\geq 0}\frac{q^{n(n+2)+k(k+1)}}{(q)_k(q)_{n-k}(q^{n+1})_{n+1}(1+q^{n+1})}.$$

Furthermore, by using the orthogonality relation of q-Hermite polynomials (1.6), it implies that the right-hand side is equal to

$$\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^j q^{\binom{j+1}{2}+j(6j+5)}(1-q^{2j+1}).$$

Finally, by applying Jacobi's triple product identity (2.3), the proof is complete. □

Note that the identity (4.2) given by Warnaar by using the Burge transform is a finite form of the second Rogers–Ramanujan identity (1.2).

By using the similar procedures as given above, from the identities due to [39, (3.14) and (3.22)]

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q)_n\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q)_{2n+1}}$$
$$=\frac{(q^2;q^2)_\infty}{(q)_\infty^2}\sum_{n=0}^{\infty}q^{n^2+n}\big(U_n(x)-U_{n-1}(x)\big),$$
$$\sum_{n=0}^{\infty}\frac{q^{n^2}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q)_{2n+1}}$$
$$=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1+q^{2n+1})\big(U_n(x)-U_{n-1}(x)\big),$$

we obtain the following identities of Rogers–Ramanujan type.

Theorem 4.1. *We have*

$$\sum_{n\geq k\geq 0}\frac{q^{n(n+1)/2+k(k+1)}}{(q)_k(q)_{n-k}(q;q^2)_{n+1}}=\frac{(-q)_\infty(q^2,q^7,q^9;q^9)_\infty}{(q)_\infty},\tag{4.3a}$$

$$\sum_{n\geq k\geq 0}\frac{q^{n^2+k(k+1)}}{(q)_k(q)_{n-k}(q^{n+1})_{n+1}}=\frac{1}{(q)_\infty}\big((q^5,q^8,q^{13};q^{13})_\infty+q(q,q^{12},q^{13};q^{13})_\infty\big).\tag{4.3b}$$

The identity (4.3a) yields a new form of the Andrews–Gordon-type identity with $k=5, i=2$, which is also studied by Sang and Shi in [30, (1.4)].

Based on the identities [35, (5.1)]; [39, (3.29) and (3.42)], respectively,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n\prod_{i=1}^{n}(1+2xq^{2i-1}+q^{4i-2})}{(q^2;q^2)_{2n}}$$
$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{2n^2}\big(V_n(x)+V_{n-1}(x)\big),$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}\prod_{j=1}^{n}(1+2xq^{2j-1}+q^{4j-2})}{(1+q^{2n+1})(q^2;q^2)_{2n}}$$
$$=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(1-q^{2n+1})q^{3n^2+2n}\big(U_n(x)+U_{n-1}(x)\big),$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n\prod_{j=1}^{n}(1+2xq^{2j-1}+q^{4j-2})}{(1+q^{2n+1})(q^2;q^2)_{2n}}$$
$$=\frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2}\sum_{n=0}^{\infty}(1-q^{2n+1})q^{2n^2+n}\big(U_n(x)+U_{n-1}(x)\big),$$

and by using Corollary 3.5, we derive the following identities.

Theorem 4.2. *We have*

$$\sum_{n=0}^{\infty}\sum_{m=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m q^{n^2+m(9m+1)/2}}{(q;q^2)_n(q^4;q^4)_n}\begin{bmatrix}2n\\n-2m\end{bmatrix}_{q^2}=\frac{(-q;q^2)_\infty(q^8,q^9,q^{17};q^{17})_\infty}{(q^2;q^2)_\infty},$$

$$\sum_{n=0}^{\infty}\sum_{m=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m q^{2n(n+1)+m(9m+1)/2}}{(1+q^{2n+1})(q^2;q^2)_{2n}}\begin{bmatrix}2n\\n-2m\end{bmatrix}_{q^2}$$
$$=\frac{1}{(q^2;q^2)_\infty}\big((q^8,q^{17},q^{25};q^{25})_\infty-q(q^4,q^{21},q^{25};q^{25})_\infty\big),$$

$$\sum_{n=0}^{\infty}\sum_{m=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m q^{n(n+1)+m(9m+1)/2}}{(q)_{2n}(-q;q^2)_{n+1}}\begin{bmatrix}2n\\n-2m\end{bmatrix}_{q^2}$$
$$=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}((q^6,q^{11},q^{17};q^{17})_\infty-q(q^2,q^{15},q^{17};q^{17})_\infty).$$

By noting the identity given by Yao in [39, (3.60)]

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}\prod_{j=0}^{n-1}(1+2xq^j+q^{2j})}{(q;q)_{2n}}$$
$$=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{n(3n-1)/2}(1-q^{4n+2})\big(U_n(x)+U_{n-1}(x)\big),$$

and applying Corollary 3.9, we obtain the following identity.

Theorem 4.3. *We have*

$$1+\sum_{n=1}^{\infty}\sum_{m=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m q^{n^2+m(5m-3)/2}(1+q^m)}{(q)_{2n}}\begin{bmatrix}2n-1\\ n-2m\end{bmatrix}$$
$$=\frac{1}{(q)_\infty}\big((q^6,q^7,q^{13};q^{13})_\infty+q(q,q^{12},q^{13};q^{13})_\infty\big). \tag{4.4}$$

Based on (1.11) which is given by Sun in [35, (6.2)], we also derive an identity on Appell–Lerch series.

Theorem 4.4. *We have*

$$\sum_{n=0}^{\infty}\sum_{m=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m q^{n(n+1)+m(9m+1)/2}(-1;q^2)_n}{(q^2;q^2)_{2n}}\begin{bmatrix}2n\\ n-2m\end{bmatrix}_{q^2}$$
$$=\frac{2(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(17n+5)/2}}{1+q^{4n}}.$$

Proof. By substituting (2.4) into identity (1.11), it yields that

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-1;q^2)_n(-qe^{i\theta},-qe^{-i\theta};q^2)_n}{(q^2;q^2)_{2n}}$$
$$=\frac{2(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{n(2n+1)}\left(\frac{1}{1+q^{2n}}-\frac{q^{8n+10}}{1+q^{2n+4}}\right)U_n(x).$$

Then following the similar procedures as given in the proof of identity (1.12), and applying Corollary 3.5, we complete the proof. □

5. Triple-Sum Identities with ℓ

In this section, by transforming the terms on $V_n(x)$ and $W_n(x)$ into $U_n(x)$ and applying the orthogonality relation of q-Hermite polynomials (1.6), we obtain triple-sum identities with a free variable ℓ. When $\ell=0$, it will reduce to double-sum Rogers–Ramanujan type identities.

Theorem 5.1. *We have*

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_{2n+1}}\sum_{m=-\infty}^{\infty}\sum_{k=0}^{\ell}(-1)^m q^{\binom{2m+2k-\ell+1}{2}+\binom{m+1}{2}}\begin{bmatrix}\ell\\ k\end{bmatrix}$$
$$\cdot\left(\begin{bmatrix}2n+1\\ n-2m-2k+\ell\end{bmatrix}-q^{2m+2k-\ell+1}\begin{bmatrix}2n+1\\ n-2m-2k+\ell-1\end{bmatrix}\right)$$
$$=\frac{2}{(q)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{3\binom{2j+\ell+1}{2}+\binom{j+1}{2}}(1-q^{6j+3\ell+3})(q^{j+1})_\ell. \tag{5.1}$$

Proof. Rewriting Andrews' identity (1.8) on $V_n(x)$ by using (2.4) and (3.1) into $U_n(x)$, then expanding it by (1.7), we have that

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-qe^{i\theta},-qe^{-i\theta})_n}{(q)_{2n+1}}$$
$$=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{3\binom{n+1}{2}}(1-q^{3n+3})\ U_n(x)$$
$$=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{3\binom{n+1}{2}}(1-q^{3n+3})\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\begin{bmatrix}n-j\\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}}H_{n-2j}(x\,|\,q). \tag{5.2}$$

Multiplying both sides by

$$\frac{(q)_\infty}{2\pi}H_\ell(x\,|\,q)(e^{2i\theta},e^{-2i\theta})_\infty, \tag{5.3}$$

and integrating the identity from $-\pi$ to π, then the right side of (5.2) becomes

$$\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{3\binom{n+1}{2}}(1-q^{3n+3})\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\begin{bmatrix}n-j\\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}}$$
$$\cdot\frac{(q)_\infty}{2\pi}\int_{-\pi}^{\pi}H_{n-2j}(x\,|\,q)H_\ell(x\,|\,q)(e^{2i\theta},e^{-2i\theta})_\infty\ d\theta.$$

Note that the above integrand is an even function, it becomes

$$\frac{2}{(q)_\infty}\sum_{n=0}^{\infty}q^{3\binom{n+1}{2}}(1-q^{3n+3})\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\begin{bmatrix}n-j\\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}}$$
$$\cdot\frac{(q)_\infty}{2\pi}\int_{0}^{\pi}H_{n-2j}(x\,|\,q)H_\ell(x\,|\,q)(e^{2i\theta},e^{-2i\theta})_\infty\ d\theta.$$

By the orthogonality relation (1.6), it can be further simplified to be

$$\frac{2}{(q)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{3\binom{2j+\ell+1}{2}+\binom{j+1}{2}}(1-q^{6j+3\ell+3})\frac{(q)_{j+\ell}}{(q)_j}.$$

For the left-hand side of (5.2), it turns to be

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n+1}} \cdot \frac{(q)_\infty}{2\pi} \int_{-\pi}^{\pi} H_\ell(x \,|\, q)(-qe^{i\theta}, -qe^{-i\theta})_n (e^{2i\theta}, e^{-2i\theta})_\infty \, d\theta,$$

which directly completes the proof by using Corollary 3.2. □

When $\ell = 0$ in the above theorem, we obtain the following double-sum identity of Rogers–Ramanujan type.

Corollary 5.2. *We have*

$$\sum_{n \geq k \geq 0} \frac{q^{n(n+1)+k(k+1)}}{(q)_k (q)_{n-k} (q^{n+1})_{n+1}} = \frac{(q^3, q^{10}, q^{13}; q^{13})_\infty}{(q)_\infty}. \tag{5.4}$$

Proof. By taking $\ell = 0$ in Theorem 5.1, it becomes

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_{2n+1}} \sum_{m=-\infty}^{\infty} (-1)^m q^{\binom{2m+1}{2}+\binom{m+1}{2}} \left(\begin{bmatrix} 2n+1 \\ n-2m \end{bmatrix} - q^{2m+1} \begin{bmatrix} 2n+1 \\ n-2m-1 \end{bmatrix} \right)$$

$$= \frac{2}{(q)_\infty} \sum_{j=0}^{\infty} (-1)^j q^{3\binom{2j+1}{2}+\binom{j+1}{2}} (1 - q^{6j+3}). \tag{5.5}$$

For the right-hand side of (5.5), by taking $j \mapsto -j-1$ into the second term, and applying Jacobi's triple product identity (2.3), it implies that

$$\frac{2}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{3\binom{2j+1}{2}+\binom{j+1}{2}} = \frac{2(q^3, q^{10}, q^{13}; q^{13})_\infty}{(q)_\infty}.$$

For the left-hand side, letting $m \to -m-1$ in the second term, it turns to be

$$2 \sum_{n=0}^{\infty} \sum_{m=-\lfloor \frac{n+1}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m \, q^{n(n+1)+m(5m+3)/2}}{(q)_{n-2m} \, (q)_{n+2m+1}},$$

by using Warnaar's identity (4.2) to simplify the identity above, the proof is completed. □

Note that (5.4) gives a double-sum expression for Andrews–Gordon identity (1.3) with $k = 6$ and $i = 3$, in which it takes the form of a quintuple sum.

From the identity due to Sun [35, (1.6)]

$$\sum_{n=0}^{\infty} \frac{q^{2n^2} \prod_{j=1}^{n} (1 + 2xq^{2j-1} + q^{4j-2})}{(q^2; q^2)_{2n}} = \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} q^{3n^2} (V_n(x) + V_{n-1}(x)), \tag{5.6}$$

we obtain the following triple-sum identity with a free variable ℓ.

Theorem 5.3. *We have*

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_{2n}}\sum_{m=-\infty}^{\infty}\sum_{k=0}^{\ell}(-1)^m q^{(2m+2k-\ell)^2+\binom{m+1}{2}}\begin{bmatrix}\ell\\ k\end{bmatrix}$$
$$\cdot\left(\begin{bmatrix}2n\\ n-2m-2k+\ell\end{bmatrix}_{q^2}-q^{4(2m+2k-\ell+1)}\begin{bmatrix}2n\\ n-2m-2k+\ell-2\end{bmatrix}_{q^2}\right)$$
$$=\frac{2}{(q^2;q^2)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{\binom{j+1}{2}+3(2j+\ell)^2}(1-q^{12(2j+\ell+1)})(q^{j+1})_\ell.$$

Proof. By substituting (2.4) into identity (5.6), we obtain

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}(-qe^{i\theta},-qe^{-i\theta};q^2)_n}{(q^2;q^2)_{2n}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{3n^2}\big(U_n(x)-U_{n-2}(x)\big)$$
$$=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{3n^2}(1-q^{12n+12})U_n(x).$$

By replacing $U_n(x)$ by (1.7), it gives

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}(-qe^{i\theta},-qe^{-i\theta};q^2)_n}{(q^2;q^2)_{2n}}$$
$$=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{3n^2}(1-q^{12n+12})\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\begin{bmatrix}n-j\\ j\end{bmatrix}(-1)^j q^{\binom{j+1}{2}}H_{n-2j}(x\,|\,q).$$

Then by multiplying both hand sides by (5.3), applying the orthogonality relation (1.6) and Lemma 3.4, the proof is completed. □

When $\ell=0$ in Theorem 5.3, it reduces to the following double-sum identity of Rogers–Ramanujan type.

Corollary 5.4. *We have*

$$\sum_{n=0}^{\infty}\sum_{m=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m q^{2n^2+m(9m+1)/2}}{(q^2;q^2)_{2n}}\begin{bmatrix}2n\\ n-2m\end{bmatrix}_{q^2}=\frac{(q^{12},q^{13},q^{25};q^{25})_\infty}{(q^2;q^2)_\infty}.$$

Moreover, we refer to Chu and Wang's result [14, Corollary 38] with $l=6$, which is a quintuple sum representation for the right-hand side of the above identity.

From Andrews' identity (1.9) on $W_n(x)$, we obtain the following result with a free variable ℓ.

Theorem 5.5. *We have*

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_{2n}}\sum_{m=-\infty}^{\infty}\sum_{k=0}^{\ell}(-1)^m q^{\binom{2m+2k-\ell}{2}+\binom{m+1}{2}}\begin{bmatrix}\ell\\ k\end{bmatrix}$$
$$\cdot\left(\begin{bmatrix}2n-1\\ n-2m-2k+\ell\end{bmatrix}+q^{2m+2k-\ell}\begin{bmatrix}2n-1\\ n-2m-2k+\ell-1\end{bmatrix}-q^{2(2m+2k-\ell)+1}\right.$$
$$\left.\cdot\begin{bmatrix}2n-1\\ n-2m-2k+\ell-2\end{bmatrix}-q^{3(2m+2k-\ell+1)}\begin{bmatrix}2n-1\\ n-2m-2k+\ell-3\end{bmatrix}\right)$$
$$=\frac{2}{(q)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{\binom{j+1}{2}+(2j+\ell)(6j+3\ell+1)/2}(1-q^{4j+2\ell+1}+q^{6j+3\ell+2}$$
$$-q^{10j+5\ell+5})(q^{j+1})_\ell.$$

Proof. By using formula (2.5), identity (1.9) can be written as follows:

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-e^{i\theta},-e^{-i\theta})_n}{(q)_{2n}}=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1}+q^{3n+2}-q^{5n+5})U_n(x).$$

Expanding $U_n(x)$ in terms of $H_n(x\,|\,q)$ by using (1.7), multiplying both hand sides by (5.3), and then applying the orthogonality relation (1.6) and Corollary 3.7, the proof is completed. □

Taking $\ell=0$ in Theorem 5.5, we get the identity as follows.

Corollary 5.6. *We have*

$$1+\sum_{n=1}^{\infty}\sum_{m=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m\ q^{n(n+1)+m(5m-3)/2}\ (1+q^m)}{(q)_{2n}}\begin{bmatrix}2n-1\\ n-2m\end{bmatrix}$$
$$=\frac{1}{(q)_\infty}\big((q^5,q^8,q^{13};q^{13})_\infty)-q(q,q^{12},q^{13};q^{13})_\infty\big).\tag{5.7}$$

Proof. When $\ell=0$, the right-hand side of Theorem 5.5 turns to be

$$\frac{2}{(q)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{\binom{j+1}{2}+j(6j+1)}(1-q^{4j+1}+q^{6j+2}-q^{10j+5}).$$

Setting $j\mapsto -j-1$ into the third and fourth summands, it becomes

$$\frac{2}{(q)_\infty}\sum_{j=-\infty}^{\infty}(-1)^j q^{\binom{j+1}{2}+j(6j+1)}(1-q^{4j+1}).$$

By using Jacobi's triple product identity (2.3), it leads to the right-hand side. For the left-hand side, by applying $m\mapsto -m-1$ to the third and fourth summands and then simplifying, we complete the proof. □

Noting that by adding (4.4) and (5.7) together, it implies that

$$2+\sum_{n=1}^{\infty}\sum_{m=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m q^{n^2+m(5m-3)/2}(1+q^n)(1+q^m)}{(q)_{2n}}\begin{bmatrix}2n-1\\ n-2m\end{bmatrix}$$
$$=\frac{1}{(q)_\infty}\big((q^5,q^8,q^{13};q^{13})_\infty+(q^6,q^7,q^{13};q^{13})_\infty\big).$$

From the identity given by Yao [39, (3.55)]

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n\prod_{j=0}^{n-1}(1+2xq^{2j}+q^{4j})}{(q^2;q^2)_{2n}}$$
$$=\frac{(q^2;q^2)_\infty}{(q)_\infty(q^4;q^4)_\infty}\sum_{n=0}^{\infty}q^{2n^2+n}(1-q^{2n+1})\big(U_n(x)+U_{n-1}(x)\big),$$

we obtain the following triple-sum identity with a free variable ℓ.

Theorem 5.7. *We have*

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(q^2;q^2)_{2n}}\sum_{m=-\infty}^{\infty}\sum_{k=0}^{\ell}(-1)^m q^{2\binom{2m+2k-\ell}{2}+\binom{m+1}{2}}\begin{bmatrix}\ell\\ k\end{bmatrix}$$
$$\cdot\Bigg(\begin{bmatrix}2n-1\\ n-2m-2k+\ell\end{bmatrix}_{q^2}+q^{2(2m+2k-\ell)}\begin{bmatrix}2n-1\\ n-2m-2k+\ell-1\end{bmatrix}_{q^2}$$
$$-q^{4(2m+2k-\ell)+2}\cdot\begin{bmatrix}2n-1\\ n-2m-2k+\ell-2\end{bmatrix}_{q^2}-q^{6(2m+2k-\ell+1)}$$
$$\cdot\begin{bmatrix}2n-1\\ n-2m-2k+\ell-3\end{bmatrix}_{q^2}\Bigg)$$
$$=\frac{2(q^2;q^2)_\infty}{(q)_\infty(q^4;q^4)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{\binom{j+1}{2}+(2j+\ell)(4j+2\ell+1)}$$
$$\cdot(1-q^{2(2j+\ell)+1}+q^{4(2j+\ell)+3}-q^{6(2j+\ell)+6})(q^{j+1})_\ell.$$

Proof. The above Yao's identity can be rewritten as

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n(-e^{i\theta},-e^{-i\theta};q^2)_n}{(q^2;q^2)_{2n}}$$
$$=\frac{(q^2;q^2)_\infty}{(q)_\infty(q^4;q^4)_\infty}\sum_{n=0}^{\infty}q^{2n^2+n}(1-q^{2n+1}+q^{4n+3}-q^{6n+6})U_n(x).$$

By using the expansion formula (1.7), multiplying both sides by (5.3), then applying the orthogonality relation (1.6) and Corollary 3.8, the proof is complete. □

By taking $\ell=0$ in Theorem 5.7, it yields the following identity.

Corollary 5.8. *We have*

$$1+\sum_{n=1}^{\infty}\sum_{m=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m q^{n(n+2)+m(9m-5)/2}(1+q^m)}{(q;q^2)_n(q^4;q^4)_n}\begin{bmatrix}2n-1\\ n-2m\end{bmatrix}_{q^2}$$
$$=\frac{(q^2;q^4)_\infty}{(q)_\infty}((q^6,q^{11},q^{17};q^{17})_\infty-q(q^2,q^{15},q^{17};q^{17})_\infty).$$

Finally, we remark that if ℓ is taken to be a special value greater than 0 in the theorems obtained in this section, it will lead to a series of identities related to partial theta functions, and more details of partial theta functions can be seen in [3, 7].

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References

[1] W. A. Al-Salam and M. E. H. Ismail, q-beta integrals and q-Hermite polynomials, *Pacific J. Math.* **135**(2) (1988) 209–221.
[2] G. E. Andrews, An analytic generalization of the Rogers–Ramanujan identities for odd moduli, *Proc. Natl. Acad. Sci. USA* **71** (1974) 4082–4085.
[3] G. E. Andrews, Ramanujan's "lost" notebook, I: Partial theta functions, *Adv. Math.* **41** (1981) 137–172.
[4] G. E. Andrews, *The Theory of Partitions* (Cambridge University, Cambridge, 1984).
[5] G. E. Andrews, q-orthogonal polynomials, Rogers–Ramanujan identities, and mock theta functions, *Proc. Steklov Inst. Math.* **276**(1) (2012) 21–32.
[6] G. E. Andrews, Dyson's "favorite" identity and Chebyshev polynomials of the third and fourth kind, *Ann. Comb.* **23** (2019) 443–464.
[7] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook: Part II* (Springer, New York, 2009).
[8] G. E. Andrews and A. K. Uncu, Sequences in overpartitions, *Ramanujan J.* **61**(2) (2023) 715–727.

[9] R. Askey and J. Wilson, *Some Basic Hypergeometric Orthogonal Polynomials That Generalize Jacobi Polynomials*, Memoirs of the American Mathematical Society, Vol. 54 (American Mathematical Society, Providence, 1985), iv+55 pp.
[10] W. N. Bailey, Some identities in combinatory analysis, *Proc. London Math. Soc.* **49** (1947) 421–425.
[11] W. N. Bailey, Identities of the Rogers–Ramanujan type, *Proc. London Math. Soc.* **50**(2) (1949) 1–10.
[12] Z. N. Cao and L. Q. Wang, Multi-sum Rogers–Ramanujan type identities, *J. Math. Anal. Appl.* **522**(2) (2023) 24.
[13] D. D. Chen and Z. G. Liu, New proofs of theorems on q-orthogonal functions, *Int. J. Number Theory* **21**(2) (2025) 473–486.
[14] W. C. Chu and C. Y. Wang, Iteration process for multiple Rogers–Ramanujan identities, *Ukrainian Math. J.* **64**(1) (2012) 110–139.
[15] K. Garrett, M. E. H. Ismail and D. Stanton, Variants of the Rogers–Ramanujan identities, *Adv. Appl. Math.* **23** (1999) 274–299.
[16] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd edn. (Cambridge University Press, Cambridge, 2004).
[17] B. Gordon, A combinatorial generalization of the Rogers–Ramanujan identities, *Amer. J. Math.* **83** (1961) 393–399.
[18] M. E. H. Ismail and D. R. Masson, q-Hermite polynomials, biorthogonal rational functions, and q-beta integrals, *Trans. Amer. Math. Soc.* **346** (1994) 63–116.
[19] S. Kanade and M. C. Russell, Staircases to analytic sum-sides for many new integer partition identities of Rogers–Ramanujan type, *Electron. J. Combin.* **26** (2019) 1–33.
[20] J. M. Laughlin, A. V. Sills and P. Zimmer, Rogers–Ramanujan-Slater type identities, *Electron. J. Combin.* **15** (2008) 1–61.
[21] P. A. MacMahon, *Combinatory Analysis. Vol. 2* (Cambridge University Press, New York, 1916).
[22] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials* (Chapman and Hall, New York, 2003).
[23] W. Nahm, Conformal field theory and the dilogarithm, in *11th Int. Conf. Mathematical Physics* (International Press, Cambridge, 1994), pp. 662–667.
[24] W. Nahm, Conformal field theory, dilogarithms and three dimensional manifold, in *Proc. Int. Conf. Interface Between Physics and Mathematics* (World Scientific, Singapore, 1994), pp. 154–165.
[25] W. Nahm, Conformal field theory and torsion elements of the Bloch group, in *Frontiers in Number Theory, Physics and Geometry II* (Springer, Berlin, 2007), pp. 67–132.
[26] L. J. Rogers, On a three-fold symmetry in the elements of Heine's series, *Proc. London Math. Soc.* **24** (1893) 171–179.
[27] L. J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* **25** (1894) 318–343.
[28] L. J. Rogers, On two theorems of combinatory analysis and some allied identities, *Proc. London Math. Soc.* **12** (1917) 315–336.
[29] L. J. Rogers and S. Ramanujan, Proof of certain identities in combinatory analysis, *Proc. Cambridge Philos. Soc.* **19** (1919) 211–216.
[30] D. D. M. Sang and D. Y. H. Shi, An Andrews–Gordon type identity for overpartitions, *Ramanujan J.* **37**(3) (2015) 653–679.
[31] I. Schur, Ein beitrag zur additiven zahlentheorie und zur theorie der kettenbrüche, *Sitzungsber. Berl. Akad.* (1917) 302–321.

[32] A. V. Sills, Finite Rogers–Ramanujan type identities, *Electron. J. Combin.* **10** (2003) 1–122.
[33] L. J. Slater, A new proof of Rogers's transformations of infinite series, *Proc. London Math. Soc.* **53** (1951) 460–475.
[34] L. J. Slater, Further identities of the Rogers–Ramanujan type, *Proc. London Math. Soc.* **54** (1952) 147–167.
[35] L. H. Sun, Rogers–Ramanujan type identities and Chebyshev polynomials of the third kind, *Ramanujan J.* **60**(3) (2023) 761–794.
[36] G. Szegö, Ein beitrag zur theorie der thetafunktionen, *Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl.* **19** (1926) 242–251.
[37] L. Q. Wang, New proofs of some double sum Rogers–Ramanujan type identities, *Ramanujan J.* **62**(1) (2023) 251–272.
[38] S. O. Warnaar, The generalized Borwein conjecture. I. The Burge transform, in *q-Series with Applications to Combinatorics, Number Theory, and Physics* (American Mathematical Society, Rhode Island, 2001), pp. 243–267.
[39] O. X. M. Yao, Rogers–Ramanujan type identities and Chebyshev polynomials, *Proc. Amer. Math. Soc.* **153**(3) (2025) 1215–1229.