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A refinement of the fixed–pixed points equidistribution on restricted permutations

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ABSTRACT

Motivated by a recent conjecture of Bsila, Cox, Hugo, Styron and Zhuang concerning fixed points and pixed points on pattern-avoiding permutations, we prove a bivariate refinement involving descent statistics. Given a set of permutations Π, let $\mathfrak{S}_n(\Pi)$ denote the set of permutations in the symmetric group $\mathfrak{S}_n$ that avoid every element of Π in the sense of pattern avoidance. For each set Π appearing in their conjecture, we show that the pairs of statistics (des, fix) and (ides, pix) are equidistributed over $\mathfrak{S}_n(\Pi)$. Our proof is based on explicit ordinary generating functions for the corresponding pattern-avoiding classes.

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1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of the set $[n] = \{1, 2, \ldots, n\}$. We call $i \in [n]$ a *fixed point* of $\sigma \in \mathfrak{S}_n$ if $\sigma(i) = i$. Let $\mathrm{fix}(\sigma)$ denote the number of fixed points of σ. A permutation $\sigma \in \mathfrak{S}_n$ is called a *derangement* if $\mathrm{fix}(\sigma) = 0$. Researches on fixed points and derangements can be found in [1–6]. We call i an *ascent* of σ if $\sigma(i) < \sigma(i+1)$ for $i \in [n-1]$ or $i = n$. In the 1980s, Désarménien introduced another family of permutations, now called *desarrangements*, and proved that they are equinumerous with derangements [7]. A desarrangement is a permutation whose first ascent is even. Note that a decreasing permutation $nn-1\cdots1$ is a desarrangement if and only if n is even. The empty word is also regarded as a desarrangement.

Example 1. For $1 \le n \le 4$, the desarrangements in $\mathfrak{S}_n$ are, respectively,

$$\varnothing, \quad \{21\}, \quad \{213, 312\},$$

and

$$\{2134, 2143, 3124, 3142, 3241, 4123, 4132, 4231, 4321\}.$$

The intrinsic relationship between derangements and desarrangements has been further explored on the basis of the definition of pixed points. As a combinatorial statistic on permutations, pixed points were originally proposed and thoroughly studied by Foata and Han in the setting of hyperoctahedral groups. Foata and Han [8] introduced the pixed factorization of a permutation, that is, every permutation $\sigma \in \mathfrak{S}_n$ admits a unique factorization $\sigma = \iota\delta$, where ι is an increasing prefix and δ is a desarrangement. The letters of ι are the *pixed points* of σ, and we denote by $\mathrm{pix}(\sigma)$ the number

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of pixed points of σ. In particular, the identity permutation has n pixed points. Under this terminology, derangements are the permutations with no fixed points, while desarrangements are the permutations with no pixed points.

Example 2. Let $\sigma = 135764829 \in \mathfrak{S}_9$. Then the pixed factorization of σ is

$$\sigma = \underbrace{1357}_{\iota}\,\underbrace{64829}_{\delta}.$$

Here $\iota = 1357$ is the longest increasing prefix of σ and the first ascent of $\delta = 64829$ occurs at position 2. Thus δ is a desarrangement and $\mathrm{pix}(\sigma) = 4$.

Pattern avoidance is a central topic in the modern study of permutations. Given two permutations $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$, we say that σ contains the pattern τ if there exist indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that the subsequence $\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)$ has the same relative order as τ. Otherwise, we say that σ *avoids* τ. For example, the permutation $\sigma = 31524$ contains the pattern 132, since the subsequence 354 has the same relative order as 132. On the other hand, $\sigma = 31524$ avoids pattern 321, since it has no decreasing subsequence of length 3. More generally, for a set of patterns Π, we write

$$\mathfrak{S}_n(\Pi) = \{\sigma \in \mathfrak{S}_n : \sigma \text{ avoids every pattern in } \Pi\}. \tag{1.1}$$

When the patterns in Π are listed explicitly, we often omit the enclosing braces. For example, $\mathfrak{S}_n(123, 132)$ stands for $\mathfrak{S}_n(\{123, 132\})$.

The study of pattern-avoiding permutations goes back to the work of Knuth on stack-sortable permutations, where the permutations sortable by a single stack were shown to be precisely the 231-avoiding permutations [9]. These permutations are counted by the Catalan numbers, and this observation initiated a rich interaction between permutation patterns and classical Catalan objects. Later, Simion and Schmidt [10] carried out a systematic study of permutations avoiding patterns of length three, proving in particular that all single patterns of length three are Catalan-enumerated. Over the past few decades, pattern avoidance has become a broad framework for studying refined enumeration, bijective constructions, permutation statistics, and structural decompositions of permutation classes; see [11,12] for an overview.

Recently, Bsila, Cox, Hugo, Styron and Zhuang revisited desarrangements from the point of view of permutation statistics and pattern avoidance [1]. They obtained several generating functions for desarrangements with respect to some permutation statistics, including descents, peaks, valleys, double ascents and double descents, and they also gave a complete enumeration of desarrangements avoiding prescribed sets of patterns of length three. Their work led to new interpretations of several classical integer sequences, including Catalan, Fine, Jacobsthal and Fibonacci numbers, in terms of pattern-avoiding desarrangements.

Let

$$\mathcal{P} = \left\{ \begin{array}{ccc} \{132, 312\}, & \{132, 321\}, & \{213, 231\}, \\ \{123, 132, 312\}, & \{123, 213, 231\}, & \{123, 312, 321\}, \\ \{132, 312, 321\}, & \{213, 231, 312\}, & \{213, 231, 321\} \end{array} \right\}. \tag{1.2}$$

At the end of their paper, Bsila et al. [1] posed the following conjecture.[1]

Conjecture 1.1. *For all $n \ge 0$ and for every pattern set $\Pi \in \mathcal{P}$, the permutation statistics* fix *and* pix *are equidistributed over $\mathfrak{S}_n(\Pi)$. Equivalently,*

$$\sum_{\sigma \in \mathfrak{S}_n(\Pi)} x^{\mathrm{fix}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} x^{\mathrm{pix}(\sigma)}. \tag{1.3}$$

The purpose of this paper is to prove a refinement of this conjecture. Instead of considering only the one-variable distributions of fix and pix, we include descent statistics. We call $i \in [n-1]$ a *descent* of $\sigma \in \mathfrak{S}_n$ if $\sigma(i) > \sigma(i+1)$. We denote by $\mathrm{Des}(\sigma)$ the set of descents of σ, and denote its cardinality by $\mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)|$. We call $i \in [n-1]$ an *inverse descent* of $\sigma \in \mathfrak{S}_n$ if $\sigma^{-1}(i) > \sigma^{-1}(i+1)$, and we denote by $\mathrm{ides}(\sigma)$ the number of inverse descents of σ. Equivalently, i is an inverse descent of σ if $i+1$ appears to the left of i in σ. Note that $\mathrm{ides}(\sigma)$ is simply the number of $i \in [n-1]$ such that $i+1$ appears somewhere to the left of i in σ.

Example 3. Let $\sigma = 314265 \in \mathfrak{S}_6$. The descents of σ are 1,3 and 5, since $3 > 1$, $4 > 2$, and $6 > 5$. Hence $\mathrm{des}(\sigma) = 3$. Moreover, $\sigma^{-1} = 241365$, whose descents are 2 and 5. Equivalently, the inverse descents of σ are obtained by comparing the positions of i and $i+1$: here 3 appears to the left of 2, and 6 appears to the left of 5. Therefore, the inverse descents of σ are 2 and 5, and $\mathrm{ides}(\sigma) = 2$.

[1] After completing our work, we learned that Henke, Stephens, and Zhuang had independently obtained a bijective proof of the original conjecture [13].

For each pattern set $\Pi \in \mathcal{P}$, we define the following two enumerative polynomials

$$F_{\Pi}(t;x,y) := \sum_{n\geq 0}\left(\sum_{\pi\in\mathfrak{S}_n(\Pi)} x^{\mathrm{des}(\pi)}y^{\mathrm{fix}(\pi)}\right)t^n \tag{1.4}$$

and

$$P_{\Pi}(t;x,y) := \sum_{n\geq 0}\left(\sum_{\pi\in\mathfrak{S}_n(\Pi)} x^{\mathrm{ides}(\pi)}y^{\mathrm{pix}(\pi)}\right)t^n. \tag{1.5}$$

The following theorem is our main result.

Theorem 1.2. *For every $\Pi \in \mathcal{P}$, we have*

$$F_{\Pi}(t;x,y) = P_{\Pi}(t;x,y). \tag{1.6}$$

Equivalently, for $n \geq 0$, the two pairs (des, fix) *and* (ides, pix) *are equidistributed over $\mathfrak{S}_n(\Pi)$.*

Remark 1.1. For a pattern set $\Pi \in \mathcal{P}$, let

$$d_n(\Pi) := |\{\sigma \in \mathfrak{S}_n(\Pi) : \mathrm{fix}(\sigma) = 0\}| \text{ and } \widetilde{d}_n(\Pi) := |\{\sigma \in \mathfrak{S}_n(\Pi) : \mathrm{pix}(\sigma) = 0\}|. \tag{1.7}$$

Thus $d_n(\Pi)$ (resp. $\widetilde{d}_n(\Pi)$) is the number of derangements (resp. desarrangements) in $\mathfrak{S}_n(\Pi)$. It follows from setting $x = 1$ and $y = 0$ in Theorem 1.2 that

$$d_n(\Pi) = \widetilde{d}_n(\Pi), \tag{1.8}$$

for all $n \geq 0$ and all $\Pi \in \mathcal{P}$. The numbers $d_n(\Pi)$ can be obtained from the refined enumeration of pattern-avoiding permutations by the number of fixed points, due to Robertson–Saracino–Zeilberger [14] and Mansour–Robertson [12]. For the nine pattern classes in $\mathcal{P}$, Theorem 1.2 refines the enumeration of pattern-avoiding desarrangements in Theorem 3.28 of Bsila et al. [1] by incorporating the descent statistics. Note that the additional case $\Pi = \{132\}$ appearing in their theorem is not covered by this refinement, since it does not satisfy the full equidistribution of fix and pix.

Remark 1.2. Stump proved that $\mathrm{des}(\sigma) = \mathrm{ides}(\sigma)$ for every permutation σ avoiding a single pattern in $\{132, 213, 231, 312\}$ in [15, Corollary 3.2]. Since every $\Pi \in \mathcal{P}$ contains at least one of these four patterns, it follows that $\mathrm{des}(\sigma) = \mathrm{ides}(\sigma)$ holds for every $\sigma \in \mathfrak{S}_n(\Pi)$ with $\Pi \in \mathcal{P}$. Hence the equidistribution of (des, fix) and (ides, pix) established in this paper is equivalent to that of (des, fix) and (des, pix), which was recently proved bijectively by Henke et al. [13].

We shall prove Theorem 1.2 by generating function techniques. The two-pattern and three-pattern cases are treated in Sections 2.1 and 2.2, respectively.

2. Proof of Theorem 1.2

Before proving Theorem 1.2, we recall two basic operators on permutations. For any $\sigma \in \mathfrak{S}_n$, its *reversal* σ^r is given by $\sigma^r(i) = \sigma(n+1-i)$; its *complement* σ^c is given by $\sigma^c(i) = n+1-\sigma(i)$. The *reversal-complement* of σ is defined by

$$\sigma^{rc} := (\sigma^r)^c = (\sigma^c)^r. \tag{2.1}$$

Equivalently,

$$\sigma^{rc}(i) = n+1-\sigma(n+1-i) \text{ for } 1 \leq i \leq n. \tag{2.2}$$

In what follows, we shall use the following elementary fact repeatedly.

Lemma 2.1. *The reversal-complement map $\sigma \mapsto \sigma^{rc}$ preserves the statistics* des *and* fix*:*

$$\mathrm{des}(\sigma^{rc}) = \mathrm{des}(\sigma) \text{ and } \mathrm{fix}(\sigma^{rc}) = \mathrm{fix}(\sigma). \tag{2.3}$$

Proof. For $1 \leq i \leq n-1$, by the definition of σ^{rc}, we have

$$\begin{aligned} i \in \mathrm{Des}(\sigma^{rc}) &\Longleftrightarrow \sigma^{rc}(i) > \sigma^{rc}(i+1) \\ &\Longleftrightarrow n+1-\sigma(n+1-i) > n+1-\sigma(n-i) \\ &\Longleftrightarrow \sigma(n+1-i) < \sigma(n-i) \\ &\Longleftrightarrow n-i \in \mathrm{Des}(\sigma), \end{aligned}$$

and for $1 \le i \le n$,

$$\begin{aligned}\sigma^{rc}(i) = i &\Longleftrightarrow n+1-\sigma(n+1-i) = i\\ &\Longleftrightarrow \sigma(n+1-i) = n+1-i.\end{aligned}$$

It follows that $\sigma \mapsto \sigma^{rc}$ preserves both des and fix. □

2.1. Double-avoidance classes in $\mathcal{P}$

In this section, we prove Theorem 1.2 for three double-avoiding classes Π, namely $\{132, 312\}$, $\{213, 231\}$ and $\{132, 321\}$, by computing their ordinary generating functions $F_\Pi(t; x, y)$ and $P_\Pi(t; x, y)$, respectively.

2.1.1. The class $\mathfrak{S}_n(132, 312)$

Given a permutation $\sigma \in \mathfrak{S}_n$, we call $i \in \{2, 3, \ldots, n\}$ an *ascent top position* (resp. a *descent bottom position*) if $\sigma(i-1) < \sigma(i)$ (resp. $\sigma(i-1) > \sigma(i)$). Let $\mathrm{Asct}(\sigma)$ be the set of all ascent tops of σ. A well-established bijection ξ between the restricted permutation class $\mathfrak{S}_n(132, 312)$ and the power set $2^{\{2,3,\ldots,n\}}$ originally introduced by Simion and Schmidt [10], is defined by

$$\sigma \longmapsto \xi(\sigma) := \mathrm{Asct}(\sigma). \tag{2.4}$$

Inversely, for any given subset $A = \{a_1 < a_2 < \cdots < a_m\} \in 2^{\{2,3,\ldots,n\}}$, the corresponding permutation σ is uniquely constructed by setting

$$\sigma(a_i) = n - m + i \text{ for } 1 \le i \le m, \tag{2.5}$$

while the remaining positions are filled, from left to right, by $n-m, n-m-1, \ldots, 1$ in strictly decreasing order. The following lemma shows how this bijection provides a natural way to evaluate the permutation statistics des and ides on $\mathfrak{S}_n(132, 312)$.

Lemma 2.2. *For $\sigma \in \mathfrak{S}_n(132, 312)$ and $B = [n] \setminus \mathrm{Asct}(\sigma)$, we have*

$$\mathrm{des}(\sigma) = \mathrm{ides}(\sigma) = |B| - 1. \tag{2.6}$$

Proof. Every index $i \in \{2, \ldots, n\}$ is either an ascent top position or a descent bottom position of σ; hence the descent bottoms are exactly $\{2, \ldots, n\} \setminus \mathrm{Asct}(\sigma)$, and $\mathrm{des}(\sigma) = |B| - 1$.

For $\mathrm{ides}(\sigma)$. Set $m = |\mathrm{Asct}(\sigma)|$. If $m = 0$, then $\sigma = n(n-1)\cdots 1$, and $\mathrm{ides}(\sigma) = n - 1 = |B| - 1$. Assume now $m \ge 1$. By (2.5), the values $n-m+1, \ldots, n$ occupy the positions in $\mathrm{Asct}(\sigma)$ in increasing order, while values $1, \ldots, n-m$ occupy the positions in B in decreasing order. Recall that v is an inverse descent if and only if $v+1$ lies to the left of v. For $v \in \{n-m+1, \ldots, n-1\}$, $v+1$ lies to the right of v, so none of these is an inverse descent. Each $v \in \{1, \ldots, n-m-1\}$ is an inverse descent, since $v+1$ precedes v; hence, there are exactly $n-m-1$ inverse descents. Finally, $\sigma(1) = n-m$ while $n-m+1$ is at position $a_1 \ge 2$, so $n-m$ is not an inverse descent. Therefore, $\mathrm{ides}(\sigma) = n-m-1 = |B| - 1$. □

Example 4. Let $\sigma = 6\,7\,5\,4\,8\,9\,3\,10\,2\,1 \in \mathfrak{S}_{10}$. On one hand, we have $\mathrm{Asct}(\sigma) = \{2, 5, 6, 8\}$ and $B = [10] \setminus \mathrm{Asct}(\sigma) = \{1, 3, 4, 7, 9, 10\}$. On the other hand, we have $\mathrm{des}(\sigma) = |\{2, 3, 6, 8, 9\}|$ and $\mathrm{ides}(\sigma) = |\{1, 2, 3, 4, 5\}|$. Thus $\mathrm{des}(\sigma) = \mathrm{ides}(\sigma) = |B| - 1$.

Theorem 2.3. *Let $T_1 = \{132, 312\}$. We have*

$$F_{T_1}(t; x, y) = 1 + \frac{yt}{1-yt} + \frac{xt^2(1+xyt)(1-xt)}{(1-yt)(1-x^2t^2)(1-(1+x)t)}. \tag{2.7}$$

Proof. Let $\mathbb{P}$ be the set of positive integers. Suppose that $B = \{b_1 < b_2 < \cdots < b_N\}$. In the reverse construction of the Simion–Schmidt bijection ξ (see (2.5)), we have $\sigma(b_j) = N + 1 - j$ for $1 \le j \le N$. It follows that b_j is a fixed point if and only if $b_j + j = N + 1$. Since the sequence $b_j + j$ is strictly increasing, this can happen for at most one index j. Define

$$\epsilon(B) = \begin{cases} 1, & \text{if there exists } j \text{ such that } b_j + j = N+1;\\ 0, & \text{otherwise.}\end{cases} \tag{2.8}$$

The positions $b_N + 1, b_N + 2, \ldots, n$ are forced to be fixed points in the reverse construction of ξ, and hence

$$\mathrm{fix}(\sigma) = n - b_N + \epsilon(B). \tag{2.9}$$

Let $\max B$ denote the maximum element of the set B. By combining the bijection ξ with Lemma 2.2 and (2.9), we can rewrite the polynomials $F_{\Pi}(t;x,y)$ for $\Pi = T_1 = \{132, 312\}$, defined in (1.4) as

$$\begin{aligned} F_{T_1}(t;x,y) &= 1 + \sum_{n\geq 1} \sum_{\substack{B\subseteq[n]\\ 1\in B}} x^{|B|-1} y^{n-\max B+\epsilon(B)} t^n \\ &= 1 + \sum_{\substack{B\subseteq\mathbb{P}\\ 1\in B}} x^{|B|-1} y^{\epsilon(B)} t^{\max B} \sum_{n\geq \max B} (yt)^{n-\max B} \\ &= 1 + \frac{1}{1-yt} H(t;x,y), \end{aligned} \tag{2.10}$$

where

$$H(t;x,y) := \sum_{\substack{B\subseteq\mathbb{P}\\ 1\in B}} x^{|B|-1} y^{\epsilon(B)} t^{\max B}. \tag{2.11}$$

Since y only records whether the possible extra fixed point inside B exists, we have

$$H(t;x,y) = H_0(t;x) + (y-1)H_1(t;x), \tag{2.12}$$

where $H_0(t;x)$ is the sum over all B and $H_1(t;x)$ is the sum over those B with $\epsilon(B) = 1$.

The set $B = \{1\}$ contributes t to both sums. If $\max B = L \geq 2$, then $1, L \in B$, while the elements of $\{2, 3, \ldots, L-1\}$ may be chosen freely, so

$$H_0(t;x) = t + \sum_{L\geq 2} x(1+x)^{L-2} t^L = t + \frac{xt^2}{1-(1+x)t}. \tag{2.13}$$

For $H_1(t;x)$. Suppose $\epsilon(B) = 1$ with $N \geq 2$, and let $p = b_j$ be the unique element of B satisfying $b_j + j = N + 1$. Then B consists of 1, p, and an arbitrary subset $A \subseteq \{2, 3, \ldots, p-1\}$, and exactly $p-1$ elements larger than p, one of which is $L = \max B$. Moreover, $|B| - 1 = p + |A|$. Conversely, every such choice produces a set B with $\epsilon(B) = 1$. For fixed p and L, the elements after p are formed by L together with $p-2$ elements chosen from $\{p+1, p+2, \ldots, L-1\}$. Summing over A and over L (using the binomial series), we obtain

$$H_1(t;x) = t + \sum_{p\geq 2} x^p (1+x)^{p-2} \sum_{L\geq 2p-1} \binom{L-p-1}{p-2} t^L = t + \sum_{p\geq 2} x^p (1+x)^{p-2} \frac{t^{2p-1}}{(1-t)^{p-1}}. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), and using $1 - t - x(1+x)t^2 = (1+xt)(1-(1+x)t)$, we obtain

$$H(t;x,y) = yt + \frac{xt^2(1+xyt)(1-xt)}{(1-x^2t^2)(1-(1+x)t)}. \tag{2.15}$$

Combining this with (2.10) yields (2.7). □

Theorem 2.4. *Let $T_1 = \{132, 312\}$. We have*

$$P_{T_1}(t;x,y) = 1 + \frac{yt}{1-yt} + \frac{xt^2(1+xyt)(1-xt)}{(1-yt)(1-x^2t^2)(1-(1+x)t)}. \tag{2.16}$$

Proof. If $\mathrm{Asct}(\sigma) = \{2, 3, \ldots, n\}$, then $\sigma = 12\cdots n$ and $B = [n] \setminus \mathrm{Asct}(\sigma) = \{1\}$, and the pixed factorization gives $\mathrm{pix}(\sigma) = n$, while Lemma 2.2 gives $\mathrm{ides}(\sigma) = 0$. Hence this case, together with the empty permutation, contributes

$$1 + \sum_{n\geq 1} y^n t^n = 1 + \frac{yt}{1-yt}. \tag{2.17}$$

Assume now $|B| \geq 2$ and write $B = \{b_1 < b_2 < \cdots < b_N\}$, so $b_1 = 1$. Since the positions $2, 3, \ldots, b_2 - 1$ belong to $\mathrm{Asct}(\sigma)$, the maximal increasing prefix of σ has length $b_2 - 1$. Let $r \geq 1$ be the length of the maximal run of consecutive elements of B starting at b_2, i.e., $b_2, b_2+1, \ldots, b_2+r-1 \in B$ and $b_2 + r \notin B$, and let $\eta(B)$ be 1 if r is odd and 0 if r is even. The pixed factorization then shows that $\mathrm{pix}(\sigma)$ is determined by the parity of r, namely

$$\mathrm{pix}(\sigma) = b_2 - 1 - \eta(B). \tag{2.18}$$

Combining the bijection ξ with Lemma 2.2, (2.17) and (2.18), and summing over all $n \geq \max B$, as in the proof of Theorem 2.3, we obtain

$$P_{T_1}(t;x,y) = 1 + \frac{yt}{1-yt} + \frac{1}{1-t} K(t;x,y), \tag{2.19}$$

where

$$K(t;x,y)=\sum_{\substack{B\subseteq\mathbb{P}\\ 1\in B,\,|B|\geq 2}} x^{|B|-1}y^{b_2-1-\eta(B)}t^{\max B}. \tag{2.20}$$

To evaluate $K(t;x,y)$, set $\ell := b_2-1\geq 1$, so that $B=\{1\}\cup\{\ell+1,\ell+2,\dots,\ell+r\}\cup C$ with $C\subseteq\{\ell+r+2,\ell+r+3,\dots\}$. (note that $\ell+r+1\notin B$ by the maximality of r). The initial run contributes $x^r y^{\ell-\eta(B)}$. If $C=\emptyset$, then $\max B=\ell+r$, contributing $t^{\ell+r}$. If $C\neq\emptyset$, then $L\in C$ while the elements $C\cap\{\ell+r+2,\ell+r+3,\dots,L-1\}$ may be chosen freely, which gives $\sum_{L\geq\ell+r+2}x(1+x)^{L-(\ell+r+2)}t^L=\frac{xt^{\ell+r+2}}{1-(1+x)t}$. Hence, for fixed ℓ and r, the tail contributes

$$t^{\ell+r}+\frac{xt^{\ell+r+2}}{1-(1+x)t}=t^{\ell+r}\frac{(1-t)(1-xt)}{1-(1+x)t}. \tag{2.21}$$

It follows that

$$K(t;x,y)=\frac{(1-t)(1-xt)}{1-(1+x)t}\sum_{\ell\geq 1}\sum_{r\geq 1}x^r t^{\ell+r}y^{\ell-\chi(2\nmid r)}, \tag{2.22}$$

where $\chi(P)$ is equal to 1 if the proposition P is true, and 0 otherwise. Separating r according to its parity, we get

$$\begin{aligned}K(t;x,y)&=\frac{(1-t)(1-xt)}{1-(1+x)t}\sum_{\ell\geq 1}t^\ell\left(y^\ell\sum_{\substack{r\geq 1\\ r\ \text{even}}}(xt)^r+y^{\ell-1}\sum_{\substack{r\geq 1\\ r\ \text{odd}}}(xt)^r\right)\\&=\frac{(1-t)(1-xt)}{1-(1+x)t}\sum_{\ell\geq 1}t^\ell\left(y^\ell\frac{x^2t^2}{1-x^2t^2}+y^{\ell-1}\frac{xt}{1-x^2t^2}\right)\\&=\frac{xt^2(1+xyt)(1-t)(1-xt)}{(1-yt)(1-x^2t^2)(1-(1+x)t)}.\end{aligned} \tag{2.23}$$

Substituting (2.23) into (2.19), we obtain (2.16). □

2.1.2. The class $\mathfrak{S}_n(213,231)$

We first describe the structure of permutations in $\mathfrak{S}_n(213,231)$. Let $\sigma\in\mathfrak{S}_n(213,231)$, and write $\sigma=L\,n\,R$, where L and R are the words to the left and right of the letter n, respectively. Since σ avoids 213, the word L must be increasing; since σ avoids 231, every letter in L must be smaller than every letter in R. Hence $L=12\cdots a$ for some $0\leq a\leq n-1$, and every $\sigma\in\mathfrak{S}_n(213,231)$ has a unique decomposition

$$\sigma=12\cdots a\,n\,\tau^{+a}, \tag{2.24}$$

where τ^{+a} is obtained by adding a to each letter of $\tau\in\mathfrak{S}_{n-a-1}(213,231)$.

Example 5. Let $\sigma=1237465\in\mathfrak{S}_7(213,231)$. Then the letter 7 separates σ as $\sigma=L\,7\,R$, where $L=123$ and $R=\tau^{+3}=465$. Thus $a=3$. Subtracting $a=3$ from each letter of R, we obtain $\tau=132\in\mathfrak{S}_3(213,231)$.

Theorem 2.5. *Let $T_2=\{213,231\}$. We have*

$$F_{T_2}(t;x,y)=1+\frac{yt}{1-yt}+\frac{xt^2(1+xyt)(1-xt)}{(1-yt)(1-x^2t^2)(1-(1+x)t)}. \tag{2.25}$$

Proof. By Lemma 2.1, the reversal-complement map $\sigma\mapsto\sigma^{rc}$ restricts to a bijection between $\mathfrak{S}_n(132,312)$ and $\mathfrak{S}_n(213,231)$, thus we have $F_{T_2}(t;x,y)=F_{T_1}(t;x,y)$, where $T_1=\{132,312\}$. By the formula for $F_{T_1}(t;x,y)$ given in (2.7), we obtain (2.25). □

Theorem 2.6. *Let $T_2=\{213,231\}$. We have*

$$P_{T_2}(t;x,y)=1+\frac{yt}{1-yt}+\frac{xt^2(1+xyt)(1-xt)}{(1-yt)(1-x^2t^2)(1-(1+x)t)}. \tag{2.26}$$

Proof. If $a=n-1$ in (2.24), then $\sigma=12\cdots n$, with $\mathrm{ides}(\sigma)=0$ and $\mathrm{pix}(\sigma)=n$. These identity permutations, together with the empty permutation, contribute

$$1+\sum_{n\geq 1}y^nt^n=1+\frac{yt}{1-yt}. \tag{2.27}$$

Suppose now $a \le n-2$, so that $m := n-a-1 \ge 1$ and $\tau \in \mathfrak{S}_m(213,231)$ is nonempty. Since n occurs before all letters of τ^{+a}, the pair $(n-1,n)$ always creates one new inverse descent, and the remaining inverse descents are exactly those coming from τ; hence

$$\mathrm{ides}(\sigma) = \mathrm{ides}(\tau) + 1. \tag{2.28}$$

For the pixed statistic, we distinguish whether τ is a desarrangement. If τ is a desarrangement, so is τ^{+a}, and the pixed prefix of σ is $12\cdots an$. If τ is not a desarrangement, then $n\,\tau^{+a}$ is a desarrangement and the pixed prefix is $12\cdots a$. Therefore,

$$\mathrm{pix}(\sigma) = \begin{cases} a+1, & \text{if } \tau \text{ is a desarrangement;} \\ a, & \text{otherwise.} \end{cases} \tag{2.29}$$

Define

$$A(t;x) = \sum_{n\ge 0}\left(\sum_{\sigma\in\mathfrak{S}_n(213,231)} x^{\mathrm{ides}(\sigma)}\right)t^n, \tag{2.30}$$

and

$$B(t;x) = \sum_{n\ge 1}\left(\sum_{\substack{\sigma\in\mathfrak{S}_n(213,231)\\ \sigma \text{ is a desarrangement}}} x^{\mathrm{ides}(\sigma)}\right)t^n. \tag{2.31}$$

We first compute $A(t;x)$. The empty permutation and identity permutations contribute $1/(1-t)$. For the non-identity part of (2.24), the segment $12\cdots a$ contributes $\sum_{a\ge 0} t^a = 1/(1-t)$, the letter n contributes t, and by (2.28), a factor x, and the nonempty τ contributes $A(t;x)-1$. Thus

$$A(t;x) = \frac{1}{1-t} + \frac{xt}{1-t}\big(A(t;x)-1\big). \tag{2.32}$$

Solving this equation gives

$$A(t;x) = \frac{1-xt}{1-(1+x)t}. \tag{2.33}$$

For $B(t;x)$. A desarrangement cannot have an ascent at position 1, so in (2.24) we must have $a=0$; hence every nonempty desarrangement in this class is of the form $\sigma = n\,\tau$, with nonempty $\tau \in \mathfrak{S}_{n-1}(213,231)$. By the parity observation above, $n\tau$ is a desarrangement if and only if τ is not, and the initial letter n creates one new inverse descent. Consequently

$$B(t;x) = xt\big(A(t;x) - 1 - B(t;x)\big). \tag{2.34}$$

Substituting (2.33) into the above identity, we obtain

$$B(t;x) = \frac{xt^2}{(1+xt)(1-(1+x)t)}. \tag{2.35}$$

We can now assemble $P_{T_2}(t;x,y)$. In the non-identity part of (2.24), the segment $12\cdots a$ contributes $\sum_{a\ge 0}(yt)^a = 1/(1-yt)$; the letter n contributes a factor xt; and, by (2.29), the nonempty τ contributes $A(t;x)-1+(y-1)B(t;x)$. Adding the contribution from (2.27), we obtain

$$P_{T_2}(t;x,y) = 1 + \frac{yt}{1-yt} + \frac{xt}{1-yt}\left(A(t;x) - 1 + (y-1)B(t;x)\right). \tag{2.36}$$

Substituting (2.33) and (2.35) into (2.36) yields (2.26). □

2.1.3. The class $\mathfrak{S}_n(132,321)$

By the Simion–Schmidt structural description [10, Proposition 11], every permutation $\sigma_{k,m}$ in $\mathfrak{S}_n(132,321)$ has the form

$$\sigma_{k,m} := (m-k+1)(m-k+2)\cdots m\,12\cdots(m-k)(m+1)\cdots n, \tag{2.37}$$

for some $1 \le k \le m \le n$. Note that $k=m$ gives the identity permutation.

Theorem 2.7. *Let $T_3 = \{132, 321\}$. We have*

$$F_{T_3}(t;x,y) = \frac{1}{1-yt} + \frac{xt^2}{(1-t)^2(1-yt)}. \tag{2.38}$$

Proof. The identity permutation $\sigma_{n,n}$ contributes y^n to the coefficient of t^n in $F_{T_3}(t;x,y)$. For a non-identity element $\sigma_{k,m}$, by (2.37), the three segments are increasing, and the only descent occurs between the first two, while the fixed points are precisely the entries $m+1, m+2, \ldots, n$; hence

$$\mathrm{des}(\sigma_{k,m}) = 1 \text{ and } \mathrm{fix}(\sigma_{k,m}) = n-m. \tag{2.39}$$

For fixed m, there are exactly $m-1$ non-identity permutations, and each permutation contributes xy^{n-m}. Thus,

$$\sum_{\sigma\in\mathfrak{S}_n(132,321)} x^{\mathrm{des}(\sigma)}y^{\mathrm{fix}(\sigma)} = y^n + x\sum_{m=2}^{n}(m-1)y^{n-m}. \tag{2.40}$$

Multiplying (2.40) by t^n and summing over $n\geq 1$, the first item, together with the empty permutation, gives $1/(1-yt)$, while

$$x\sum_{n\geq 2}\sum_{m=2}^{n}(m-1)y^{n-m}t^n = \frac{x}{1-yt}\sum_{m\geq 2}(m-1)t^m = \frac{xt^2}{(1-t)^2(1-yt)}, \tag{2.41}$$

which yields (2.38). □

Theorem 2.8. *Let $T_3 = \{132, 321\}$. We have*

$$P_{T_3}(t;x,y) = \frac{1}{1-yt} + \frac{xt^2}{(1-t)^2(1-yt)}. \tag{2.42}$$

Proof. For the identity permutation $\sigma_{n,n}$, we have $\mathrm{ides}(\sigma_{n,n}) = 0$ and $\mathrm{pix}(\sigma_{n,n}) = n$, it contributes y^n to the coefficients of t^n. For $1\leq k < m \leq n$, it follows from (2.37) that

$$\sigma_{k,m}^{-1} = (k+1)(k+2)\cdots m\,12\cdots k\,(m+1)\cdots n = \sigma_{m-k,m}, \tag{2.43}$$

and hence $\mathrm{ides}(\sigma_{k,m}) = \mathrm{des}(\sigma_{m-k,m}) = 1$. Including the letter m in the pixed prefix would leave a suffix beginning with an ascent, which is not a desarrangement; on the other hand, the suffix

$$m, 1, 2, \ldots, m-k, m+1, \ldots, n, \tag{2.44}$$

has its first ascent at position 2 and is therefore a desarrangement. Hence the pixed prefix of $\sigma_{k,m}$ is $(m-k+1)\cdots(m-1)$, and $\mathrm{pix}(\sigma_{k,m}) = k-1$. It follows that

$$\sum_{\sigma\in\mathfrak{S}_n(132,321)} x^{\mathrm{ides}(\sigma)}y^{\mathrm{pix}(\sigma)} = y^n + \sum_{m=2}^{n}\sum_{k=1}^{m-1}xy^{k-1} = y^n + x\sum_{j=0}^{n-2}(n-1-j)y^j. \tag{2.45}$$

Multiplying (2.45) by t^n and summing over $n\geq 1$, the first item, together with the empty permutation, gives $1/(1-yt)$, while

$$x\sum_{n\geq 2}\sum_{j=0}^{n-2}(n-1-j)y^jt^n = x\sum_{i\geq 1}it^{i+1}\sum_{n\geq i+1}(yt)^{n-1-i} = \frac{xt^2}{(1-t)^2(1-yt)}, \tag{2.46}$$

which yields (2.42). □

2.2. Triple-avoidance classes in $\mathcal{P}$

In this section, we prove Theorem 1.2 for the remaining three-avoiding classes Π, namely $\{123, 132, 312\}$, $\{123, 213, 231\}$, $\{132, 312, 321\}$, $\{213, 231, 321\}$, $\{213, 231, 312\}$, and $\{123, 312, 321\}$ by computing their generating functions $F_\Pi(t;x,y)$ and $P_\Pi(t;x,y)$, respectively.

2.2.1. The class $\mathfrak{S}_n(123, 132, 312)$

Following the Simion–Schmidt description [10, Proposition 16], every permutation in $\mathfrak{S}_n(123, 132, 312)$ has the form

$$\hat{\sigma}_{n,k} = (n-1)(n-2)\cdots k\,n\,(k-1)(k-2)\cdots 1, \quad \text{for } 1\leq k\leq n. \tag{2.47}$$

Theorem 2.9. *Let $T_4 = \{123, 132, 312\}$. We have*

$$F_{T_4}(t;x,y) = 1 + yt + t^2\left(\frac{y^2+x}{1-x^2t^2} + \frac{(1+y)x^2t^2}{(1-x^2t^2)^2}\right) + t\left(\frac{(1+y)xt^2}{(1-x^2t^2)^2} + \frac{yx^2t^2}{1-x^2t^2}\right). \tag{2.48}$$

Proof. By (2.47), the sum over $\sigma \in \mathfrak{S}_n(T_4)$ is a sum over $k = 1, \ldots, n$. If $k < n$, then the only ascent of $\hat{\sigma}_{n,k}$ occurs at position $n-k$; if $k = n$, then $\hat{\sigma}_{n,n} = n(n-1)\cdots 1$. Hence

$$\mathrm{des}(\hat{\sigma}_{n,k}) = \begin{cases} n-2, & \text{if } 1 \le k < n; \\ n-1, & \text{if } k = n. \end{cases} \tag{2.49}$$

For fixed points: the letter n occurs in position $n-k+1$, so it is a fixed point if and only if $k = 1$; for $k \le a \le n-1$, the letter a occurs in position $n-a$, so it is a fixed point if and only if n is even and $a = n/2$; for $1 \le a \le k-1$, the letter a occurs in position $n-a+1$, so it is a fixed point if and only if n is odd and $a = (n+1)/2$. Therefore

$$\mathrm{fix}(\hat{\sigma}_{n,k}) = \chi(k=1) + \chi(n \text{ is even and } k \le n/2) + \chi(n \text{ is odd and } k > (n+1)/2). \tag{2.50}$$

Equivalently, if $n = 2m+1$, then

$$\mathrm{fix}(\hat{\sigma}_{2m+1,k}) = \begin{cases} 1, & k = 1 \text{ or } m+2 \le k \le 2m+1; \\ 0, & 2 \le k \le m+1, \end{cases} \tag{2.51}$$

while if $n = 2$ m, then

$$\mathrm{fix}(\hat{\sigma}_{2m,k}) = \begin{cases} 2, & k = 1; \\ 1, & 2 \le k \le m; \\ 0, & m+1 \le k \le 2\ m. \end{cases} \tag{2.52}$$

Hence, for $n = 2m+1$, by (2.51), we have

$$\sum_{\sigma \in \mathfrak{S}_{2m+1}(T_4)} x^{\mathrm{des}(\sigma)} y^{\mathrm{fix}(\sigma)} = x^{2m} y + x^{2m-1} m(1+y). \tag{2.53}$$

For $n = 2$ m, by (2.52), we have

$$\sum_{\sigma \in \mathfrak{S}_{2m}(T_4)} x^{\mathrm{des}(\sigma)} y^{\mathrm{fix}(\sigma)} = x^{2m-1} + x^{2m-2}\big((m-1)(1+y) + y^2\big). \tag{2.54}$$

Together with the initial contributions 1 and yt corresponding to $n = 0$ and $n = 1$, respectively, the stated generating function (2.48) follows by multiplying (2.53) and (2.54) by t^{2m+1} and t^{2m}, respectively, then summing over $m \ge 1$. □

Theorem 2.10. *Let* $T_4 = \{123, 132, 312\}$. *We have*

$$P_{T_4}(t; x, y) = 1 + yt + t^2\left(\frac{y^2+x}{1-x^2t^2} + \frac{(1+y)x^2t^2}{(1-x^2t^2)^2}\right) + t\left(\frac{(1+y)xt^2}{(1-x^2t^2)^2} + \frac{yx^2t^2}{1-x^2t^2}\right). \tag{2.55}$$

Proof. By (2.47), if $k < n$, then every value except $n-1$ is an inverse descent of $\hat{\sigma}_{n,k}$ (the letter n lies to the right of $n-1$), while $\hat{\sigma}_{n,n} = n(n-1)\cdots 1$ is decreasing; hence

$$\mathrm{ides}(\hat{\sigma}_{n,k}) = \begin{cases} n-2, & \text{if } 1 \le k < n; \\ n-1, & \text{if } k = n. \end{cases} \tag{2.56}$$

For the pixed statistic. If $k < n$, then the unique ascent of $\hat{\sigma}_{n,k}$ occurs at position $n-k$, so $\hat{\sigma}_{n,k}$ itself is a desarrangement exactly when $n-k$ is even.

Suppose first that $n = 2m+1$. Then $n-k$ is even exactly when k is odd. Thus, for odd $k < 2m+1$, we have $\mathrm{pix}(\hat{\sigma}_{2m+1,k}) = 0$. For even k, deleting the first letter shifts the unique ascent to an even position, so the remaining word is a desarrangement and $\mathrm{pix}(\hat{\sigma}_{2m+1,k}) = 1$. Finally, when $k = 2m+1$, deleting the first letter leaves a decreasing word of even length, which is a desarrangement. Hence

$$\mathrm{pix}(\hat{\sigma}_{2m+1,k}) = \begin{cases} 0, & k \text{ is odd and } k < 2m+1; \\ 1, & k \text{ is even}; \\ 1, & k = 2m+1. \end{cases} \tag{2.57}$$

Now suppose that $n = 2$ m. Then $n-k$ is even exactly when k is even, so $\mathrm{pix}(\hat{\sigma}_{2m,k}) = 0$ for even k. If k is odd and $k < 2m-1$, deleting the first letter shifts the unique ascent to an even position, and hence $\mathrm{pix}(\hat{\sigma}_{2m,k}) = 1$. In the remaining case $k = 2m-1$, the first two letters form the increasing word $(2m-1, 2m)$, and deleting them leaves a decreasing word of even length, which is a desarrangement. Therefore

$$\mathrm{pix}(\hat{\sigma}_{2m,k}) = \begin{cases} 0, & k \text{ is even}; \\ 1, & k \text{ is odd and } k < 2m-1; \\ 2, & k = 2m-1. \end{cases} \tag{2.58}$$

Therefore, using (2.56) and (2.57), for $n = 2m + 1$,

$$\sum_{\sigma\in\mathfrak{S}_{2m+1}(T_4)} x^{\mathrm{ides}(\sigma)}y^{\mathrm{pix}(\sigma)} = x^{2m}y + x^{2m-1}m(1+y), \tag{2.59}$$

and using (2.56) and (2.58), for $n = 2$ m,

$$\sum_{\sigma\in\mathfrak{S}_{2m}(T_4)} x^{\mathrm{ides}(\sigma)}y^{\mathrm{pix}(\sigma)} = x^{2m-1} + x^{2m-2}\big((m-1)(1+y)+y^2\big). \tag{2.60}$$

The right hand sides of (2.59) and (2.60) coincide with (2.53) and (2.54), respectively, so the final summation is the same as in the proof of Theorem 2.9 and yields (2.55). □

2.2.2. The class $\mathfrak{S}_n(123, 213, 231)$

Since the reversal-complement map $\sigma \mapsto \sigma^{rc}$ is a bijection between $\mathfrak{S}_n(123, 132, 312)$ and $\mathfrak{S}_n(123, 213, 231)$, applying it to (2.47) shows that every permutation in $\mathfrak{S}_n(123, 213, 231)$ has the form

$$\tilde{\sigma}_{n,k} = n(n-1)\cdots(n-k+2)\,1\,(n-k+1)(n-k)\cdots 2, \quad \text{for } 1 \le k \le n. \tag{2.61}$$

Theorem 2.11. *Let* $T_5 = \{123, 213, 231\}$. *We have*

$$F_{T_5}(t; x, y) = 1 + yt + t^2\left(\frac{y^2+x}{1-x^2t^2} + \frac{(1+y)x^2t^2}{(1-x^2t^2)^2}\right) + t\left(\frac{(1+y)xt^2}{(1-x^2t^2)^2} + \frac{yx^2t^2}{1-x^2t^2}\right). \tag{2.62}$$

Proof. By Lemma 2.1, the reversal-complement map $\sigma \mapsto \sigma^{rc}$ restricts to a bijection between $\mathfrak{S}_n(123, 213, 231)$ and $\mathfrak{S}_n(123, 132, 312)$, which preserves the numbers of descents and fixed points, thus $F_{T_5}(t; x, y) = F_{T_4}(t; x, y)$, for $T_4 = \{123, 132, 312\}$. The stated formula then follows from Theorem 2.9. □

Theorem 2.12. *Let* $T_5 = \{123, 213, 231\}$. *We have*

$$P_{T_5}(t; x, y) = 1 + yt + t^2\left(\frac{y^2+x}{1-x^2t^2} + \frac{(1+y)x^2t^2}{(1-x^2t^2)^2}\right) + t\left(\frac{(1+y)xt^2}{(1-x^2t^2)^2} + \frac{yx^2t^2}{1-x^2t^2}\right). \tag{2.63}$$

Proof. For $\tilde{\sigma}_{n,k}$ with $k < n$, the only value that fails to be an inverse descent is 1, whose successor 2 lies at the end of the word; for $k = n$ the word is decreasing. Hence

$$\mathrm{ides}(\tilde{\sigma}_{n,k}) = \begin{cases} n-2, & 1 \le k < n; \\ n-1, & k = n. \end{cases} \tag{2.64}$$

For $\mathrm{pix}(\tilde{\sigma}_{n,k})$. If $k < n$, the unique ascent of $\tilde{\sigma}_{n,k}$ occurs at position k (the adjacent pair $1 < n-k+1$). Thus $\tilde{\sigma}_{n,k}$ is itself a desarrangement exactly when k is even. If k is odd and $k < n$, deleting the first letter shifts this unique ascent to position $k-1$, which is even.

Now suppose $n = 2m + 1$. If k is even, then $\tilde{\sigma}_{2m+1,k}$ is already a desarrangement, so its pix is 0. If k is odd and $k < 2m + 1$, deleting the first letter gives a desarrangement, so the pix is 1. Finally, when $k = 2m + 1$, the word $\tilde{\sigma}_{2m+1,2m+1} = (2m+1)(2m)\cdots 1$ is decreasing of odd length; deleting the first letter leaves a decreasing word of even length, which is a desarrangement. Hence

$$\mathrm{pix}(\tilde{\sigma}_{2m+1,k}) = \begin{cases} 0, & k \text{ is even}; \\ 1, & k \text{ is odd}. \end{cases} \tag{2.65}$$

Next suppose $n = 2$ m. If k is even, then $\tilde{\sigma}_{2m,k}$ is already a desarrangement, so its pix is 0. If k is odd and $3 \le k \le 2m-1$, deleting the first letter shifts the unique ascent to the even position $k-1$, so the pix is 1. The remaining case is $k = 1$. In this case, $\tilde{\sigma}_{2m,1} = 1\ 2m\ 2m-1\ \ldots\ 2$ and the initial increasing word has length 2, since deleting the first two letters leaves the decreasing word $(2m-1)\cdots 2$ of even length, which is a desarrangement. Therefore

$$\mathrm{pix}(\tilde{\sigma}_{2m,k}) = \begin{cases} 0, & k \text{ is even}; \\ 1, & k \text{ is odd and } 3 \le k \le 2m-1; \\ 2, & k = 1. \end{cases} \tag{2.66}$$

Combining (2.64) and (2.65), for $n = 2m + 1$, we get

$$\sum_{\sigma\in\mathfrak{S}_{2m+1}(T_5)} x^{\mathrm{ides}(\sigma)}y^{\mathrm{pix}(\sigma)} = x^{2m}y + mx^{2m-1}(1+y). \tag{2.67}$$

Similarly, combining (2.64) and (2.66), for $n = 2$ m, we get

$$\sum_{\sigma\in\mathfrak{S}_{2m}(T_5)} x^{\mathrm{ides}(\sigma)}y^{\mathrm{pix}(\sigma)} = x^{2m-1} + x^{2m-2}\big((m-1)(1+y)+y^2\big). \tag{2.68}$$

The right hand sides of (2.67) and (2.68) coincide with (2.53) and (2.54), respectively, so the final summation is the same as in the proof of Theorem 2.9 and yields (2.63). □

2.2.3. The class $\mathfrak{S}_n(132, 312, 321)$

We claim that every permutation in $\mathfrak{S}_n(132, 312, 321)$ has the form

$$\bar{\sigma}_{n,k} = 2\,3\,\ldots,k\,1\,(k+1)(k+2)\ldots n, \quad \text{for } 1 \le k \le n. \tag{2.69}$$

Here, the case $k = 1$ gives the identity permutation $\bar{\sigma}_{n,1} = 12\cdots n$.

Indeed, let $\sigma \in \mathfrak{S}_n(132, 312, 321)$. Since σ avoids 321, the subword to the left of 1 is increasing; since σ avoids 132, the subword to the right of 1 is also increasing. Moreover, if some letter a to the left of 1 and some letter b to the right of 1 satisfied $a > b$, then the three letters $a\,1\,b$ would form a 312-pattern. Hence every letter to the left of 1 is smaller than every letter to the right of 1, so these letters are precisely $2, 3, \ldots, k$ and $k+1, k+2, \ldots, n$ for some k, which is (2.69).

Theorem 2.13. *Let* $T_6 = \{132, 312, 321\}$. *We have*

$$F_{T_6}(t; x, y) = 1 + \frac{yt}{1-yt} + \frac{xt^2}{(1-t)(1-yt)}. \tag{2.70}$$

Proof. By (2.69), $\bar{\sigma}_{n,1} = 12\cdots n$ contributes y^n. For $2 \le k \le n$, the only descent of $\bar{\sigma}_{n,k}$ occurs between the first and second segments, and the fixed points are exactly the entries $k+1, k+2, \ldots, n$; hence $\mathrm{des}(\bar{\sigma}_{n,k}) = 1$ and $\mathrm{fix}(\bar{\sigma}_{n,k}) = n-k$. It follows that

$$\sum_{\sigma\in\mathfrak{S}_n(T_6)} x^{\mathrm{des}(\sigma)} y^{\mathrm{fix}(\sigma)} = y^n + \sum_{k=2}^{n} xy^{n-k}. \tag{2.71}$$

Consequently, after multiplying by t^n and summing over $n \ge 1$, and after adding the initial contribution 1 of the empty permutation corresponding to $n = 0$, we obtain (2.70). □

Theorem 2.14. *Let* $T_6 = \{132, 312, 321\}$. *We have*

$$P_{T_6}(t; x, y) = 1 + \frac{yt}{1-yt} + \frac{xt^2}{(1-t)(1-yt)}. \tag{2.72}$$

Proof. By (2.69), $\bar{\sigma}_{n,1} = 12\cdots n$ contributes y^n. For $2 \le k \le n$, the only inverse descent of $\bar{\sigma}_{n,k}$ is at the value 1 (the letter 2 lies to its left), so $\mathrm{ides}(\bar{\sigma}_{n,k}) = 1$. For the pixed statistic, write

$$\bar{\sigma}_{n,k} = \underbrace{2\cdots(k-1)}_{\iota}\ \underbrace{k1(k+1)\cdots n}_{\delta}.$$

The word δ has its first ascent at position 2, so δ is a desarrangement, while including k in the prefix would leave a suffix beginning with an ascent. Hence ι is the pixed prefix and $\mathrm{pix}(\bar{\sigma}_{n,k}) = k-2$. It follows that

$$\sum_{\sigma\in\mathfrak{S}_n(T_6)} x^{\mathrm{ides}(\sigma)} y^{\mathrm{pix}(\sigma)} = y^n + \sum_{k=2}^{n} xy^{k-2}. \tag{2.73}$$

This coincides with (2.71), so the rest of the computation is the same as in the proof of Theorem 2.13 and yields (2.72). □

2.2.4. The class $\mathfrak{S}_n(213, 231, 321)$

Since the reversal-complement map $\sigma \mapsto \sigma^{rc}$ is a bijection between $\mathfrak{S}_n(132, 312, 321)$ and $\mathfrak{S}_n(213, 231, 321)$, applying it to (2.69) shows that every permutation in $\mathfrak{S}_n(213, 231, 321)$ has the form

$${}^{\star}\sigma_{n,k} := 1\,2\,\ldots\,(k-1)\,n\,k\,(k+1)\,\ldots\,(n-1), \quad \text{for } 1 \le k \le n. \tag{2.74}$$

Theorem 2.15. *Let* $T_7 = \{213, 231, 321\}$. *We have*

$$F_{T_7}(t; x, y) = 1 + \frac{yt}{1-yt} + \frac{xt^2}{(1-t)(1-yt)}. \tag{2.75}$$

Proof. By Lemma 2.1, the reversal-complement map $\sigma \mapsto \sigma^{rc}$ restricts to a bijection between $\mathfrak{S}_n(213, 231, 321)$ and $\mathfrak{S}_n(132, 312, 321)$; so $F_{T_7}(t; x, y) = F_{T_6}(t; x, y)$, for $T_6 = \{132, 312, 321\}$. The formula (2.75) then follows from (2.70). □

Theorem 2.16. *Let* $T_7 = \{213, 231, 321\}$. *We have*

$$P_{T_7}(t; x, y) = 1 + \frac{yt}{1-yt} + \frac{xt^2}{(1-t)(1-yt)}. \tag{2.76}$$

Proof. By (2.74), ${}^{\star}\sigma_{n,n} = 12\cdots n$ contributes y^n. For $1 \le k \le n-1$, the only inverse descent of ${}^{\star}\sigma_{n,k}$ is at the value $n-1$ (the letter n lies to its left), so $\mathrm{ides}({}^{\star}\sigma_{n,k}) = 1$. For the pixed statistic, write

$$ {}^{\star}\sigma_{n,k} = \underbrace{12\cdots(k-1)}_{\iota}\,\underbrace{nk(k+1)\cdots(n-1)}_{\delta}. $$

The word δ has its first ascent at position 2, so δ is a desarrangement, while including n in the prefix would leave a suffix beginning with an ascent. Hence ι is the pixed prefix and $\mathrm{pix}({}^{\star}\sigma_{n,k}) = k-1$. It follows that, for $n \ge 1$,

$$ \sum_{\sigma\in\mathfrak{S}_n(T_7)} x^{\mathrm{ides}(\sigma)} y^{\mathrm{pix}(\sigma)} = y^n + \sum_{k=1}^{n-1} x y^{k-1}. \tag{2.77} $$

This coincides with (2.71), so the rest of the computation is the same as in the proof of Theorem 2.13 and yields (2.76). □

2.2.5. The class $\mathfrak{S}_n(213, 231, 312)$

We claim that every permutation in $\mathfrak{S}_n(213, 231, 312)$ is of the form

$$ \mathring{\sigma}_{n,k} = 12\cdots k\, n(n-1)\cdots(k+1), \quad \text{for } 0 \le k \le n-1. \tag{2.78} $$

Indeed, let $\sigma \in \mathfrak{S}_n(213, 231, 312)$. The letter 1 must occur at one of the two ends of σ: if some letter a were to its left and some letter b to its right, then $a1b$ would form a 213-pattern if $a < b$ and a 312-pattern if $a > b$. If 1 is the last letter, the subword preceding it must be decreasing (any increasing pair $a < b$ occurring before 1 would give a 231-pattern), so $\sigma = n(n-1)\cdots 21$, which corresponds to $k = 0$.

Otherwise, 1 is the first letter; deleting it and standardizing the remaining word yields an element of $\mathfrak{S}_{n-1}(213, 231, 312)$, and induction on n gives the claimed form.

Theorem 2.17. *Let $T_8 = \{213, 231, 312\}$. We have*

$$ F_{T_8}(t; x, y) = 1 + \frac{yt + xt^2}{(1-yt)(1-x^2t^2)}. \tag{2.79} $$

Proof. From (2.78), the initial segment $12\cdots k$ is increasing, and the final segment $n(n-1)\cdots(k+1)$ is decreasing, so the descents occur precisely inside this final decreasing segment: $\mathrm{des}(\mathring{\sigma}_{n,k}) = n-k-1$. For fixed points: The letters $1, 2, \ldots, k$ are fixed points and hence contribute k fixed points. In the final decreasing segment, for $1 \le j \le n-k$, the letter at position $k+j$ is $n-j+1$; it is a fixed point if and only if $2j = n-k+1$. Hence the segment contributes one additional fixed point when $n-k$ is odd, and

$$ \mathrm{fix}(\mathring{\sigma}_{n,k}) = k + \chi(n-k \text{ is odd}). \tag{2.80} $$

Thus, for $n \ge 1$,

$$ \sum_{\sigma\in\mathfrak{S}_n(T_8)} x^{\mathrm{des}(\sigma)} y^{\mathrm{fix}(\sigma)} = \sum_{k=0}^{n-1} x^{n-k-1} y^{k+\chi(n-k \text{ is odd})}. \tag{2.81} $$

Multiplying by t^n, summing over $n \ge 1$, adding the contribution of the empty permutation, and setting $\ell = n-k$, we obtain

$$ \begin{aligned} F_{T_8}(t; x, y) &= 1 + \left(\sum_{k\ge0}(yt)^k\right)\left(\sum_{\ell\ge1} t^\ell x^{\ell-1} y^{\chi(\ell \text{ is odd})}\right) \\ &= 1 + \frac{1}{1-yt}\left(\sum_{m\ge0} t^{2m+1}x^{2m}y + \sum_{m\ge1} t^{2m}x^{2m-1}\right), \end{aligned} \tag{2.82} $$

which is (2.79). □

Theorem 2.18. *Let $T_8 = \{213, 231, 312\}$. We have*

$$ P_{T_8}(t; x, y) = 1 + \frac{yt + xt^2}{(1-yt)(1-x^2t^2)}. \tag{2.83} $$

Proof. By (2.78), each letter v is an inverse descent of $\mathring{\sigma}_{n,k}$ for $k+1 \le v \le n-1$; hence

$$ \mathrm{ides}(\mathring{\sigma}_{n,k}) = n-k-1. \tag{2.84} $$

For the pixed points, the initial increasing part of $\mathring{\sigma}_{n,k}$ is $12\cdots k$, and the remaining suffix is the decreasing word $n(n-1)\cdots(k+1)$, of length $n-k$. We distinguish two cases according to the parity of $n-k$. If $n-k$ is even, then the decreasing suffix $n(n-1)\cdots(k+1)$ is a desarrangement. Thus the pixed factorization is

$$\mathring{\sigma}_{n,k} = \underbrace{12\cdots k}_{\iota}\ \underbrace{n(n-1)\cdots(k+1)}_{\delta},$$

and consequently $\mathrm{pix}(\mathring{\sigma}_{n,k}) = k$.

If $n-k$ is odd, then the decreasing suffix $n(n-1)\cdots(k+1)$ is not a desarrangement. In this case, the first letter n of the final segment must be included in the increasing prefix. The remaining suffix $(n-1)(n-2)\cdots(k+1)$ has even length and is therefore a desarrangement. Hence the pixed factorization is

$$\mathring{\sigma}_{n,k} = \underbrace{12\cdots k\,n}_{\iota}\ \underbrace{(n-1)(n-2)\cdots(k+1)}_{\delta},$$

and so $\mathrm{pix}(\mathring{\sigma}_{n,k}) = k+1$. Thus, in all cases, we have

$$\mathrm{pix}(\mathring{\sigma}_{n,k}) = k + \chi(n-k \text{ is odd}). \tag{2.85}$$

Combining (2.84) and (2.85), we obtain, for fixed $n \geq 1$,

$$\sum_{\sigma\in\mathfrak{S}_n(T_8)} x^{\mathrm{ides}(\sigma)} y^{\mathrm{pix}(\sigma)} = \sum_{k=0}^{n-1} x^{n-k-1} y^{k+\chi(n-k \text{ is odd})}. \tag{2.86}$$

Since the right-hand side of (2.86) is identical to that of (2.81), so the same summation as in the proof of Theorem 2.17 yields (2.83). □

2.2.6. The class $\mathfrak{S}_n(123, 312, 321)$

Let $T_9 = \{123, 312, 321\}$. By the Erdős–Szekeres theorem [16, p. 160], $\mathfrak{S}_n(123, 321) = \varnothing$ for $n \geq 5$, and therefore $\mathfrak{S}_n(T_9) = \varnothing$ for $n \geq 5$. For $1 \leq n \leq 4$, direct enumeration yields $\mathfrak{S}_1(T_9) = \{1\}$, $\mathfrak{S}_2(T_9) = \{12, 21\}$, $\mathfrak{S}_3(T_9) = \{132, 213, 231\}$, and $\mathfrak{S}_4(T_9) = \{2143\}$. Consequently, we have

$$F_{T_9}(t; x, y) = P_{T_9}(t; x, y) = 1 + yt + (y^2 + x)t^2 + x(2y + 1)t^3 + x^2t^4.$$

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Data availability

No data was used for the research described in the article.

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