Schubert polynomials and patterns in permutations

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Abstract

This paper investigates the size of the support of a Schubert polynomial $\mathfrak{S}_w(x)$ indexed by a permutation w. This number also equals the number of lattice points in the Newton polytope of $\mathfrak{S}_w(x)$. We establish a lower bound for this number in terms of the occurrences of patterns in w. The analysis is carried out in the general framework of dual characters of flagged Weyl modules. Our result considerably improves the bounds for principal specializations of Schubert polynomials or dual flagged Weyl characters previously obtained by Weigandt, Gao, and Mészáros–St. Dizier–Tanjaya. Some problems and conjectures are discussed.

Keywords: Schubert polynomial, key polynomial, flagged Weyl module, dual character, support, principal specialization

AMS Classifications: 05E10, 05E14, 05A19, 14N15

1 Introduction

As usual, let S_n be the symmetric group of permutations of $[n] := \{1, 2, \ldots, n\}$. Given a permutation $w \in S_n$, let $\mathfrak{S}_w(x)$ denote the associated Schubert polynomial. They were introduced by Lascoux and Schützenberger [15] to represent Schubert classes in the cohomology ring of the flag manifold. Schubert polynomials can be defined in a recursive procedure. For the longest permutation $w_0 = n \cdots 21$, set $\mathfrak{S}_{w_0}(x) = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$. For $w \neq w_0$, locate a position $1 \leq i < n$ such that $w(i) < w(i+1)$, and set $\mathfrak{S}_w(x) = \partial_i \mathfrak{S}_{ws_i}(x)$, where ws_i is obtained from w by swapping $w(i)$ and $w(i+1)$, and ∂_i is the divided difference operator acting on a polynomial $f(x)$ by

$$\partial_i f(x) = \frac{f(x) - f(x)|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}.$$

For a (weak) composition $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then one can express

$$\mathfrak{S}_w(x) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} a_\alpha \, x^\alpha.$$

It is famously known that $a_\alpha \in \mathbb{Z}_{\geq 0}$ [16]. The support of $\mathfrak{S}_w(x)$ is defined as

$$\mathrm{supp}(\mathfrak{S}_w(x)) = \{\alpha \colon a_\alpha \neq 0\}.$$

By the work of Fink, Mészáros and St. Dizier [7] (first conjectured by Monical, Tokcan and Yong [21]), the elements in $\mathrm{supp}(\mathfrak{S}_w(x))$ are in one-to-one correspondence with the lattice points in the Newton polytope of $\mathfrak{S}_w(x)$. Recall that the Newton polytope of a polynomial f in $x_1, \ldots, x_n$ is the convex hull in $\mathbb{R}^n$ generated by the vectors in the support of f.

We use θ_w to stand for the size of $\mathrm{supp}(\mathfrak{S}_w(x))$, or equivalently, the number of lattice points in the Newton polytope of $\mathfrak{S}_w(x)$. Given $u = u(1) \cdots u(m) \in S_m$ with $m \leq n$, we say that a subsequence $w(i_1) \cdots w(i_m)$ of w is a u pattern if $w(i_1) \cdots w(i_m)$ has the same relative order as u. Let $p_u(w)$ denote the number of appearances of u patterns in w. For example, we have $p_{132}(1432) = 3$.

Our first main result is a lower bound for θ_w in terms of the numbers $p_u(w)$.

Theorem 1.1. *For $w \in S_n$, we have*

$$\begin{aligned}\theta_w \geq 1 &+ p_{132}(w) + p_{1432}(w) + p_{13254}(w) + 3p_{14253}(w) \\ &+ p_{14352}(w) + 4p_{15243}(w) + p_{15324}(w) + 2p_{15342}(w) \\ &+ p_{15432}(w) + p_{24153}(w) + 2p_{25143}(w) + p_{35142}(w).\end{aligned} \tag{1.1}$$

As comparison, the principal specialization $\nu_w := \mathfrak{S}_w(x)|_{x_i=1}$ of $\mathfrak{S}_w(x)$ has received much attention in recent years. By the nonnegativity of coefficients, we have

$$\theta_w \leq \nu_w.$$

The equality holds if and only if $\mathfrak{S}_w(x)$ is zero-one, that is, each coefficient a_α is equal to either 0 or 1. A criterion for zero-one Schubert polynomials was first given by Fink, Mészáros and St. Dizier [8], see [11] for an alternative proof.

A classical formula due to Macdonald [16], see also [9, 12], states that

$$\nu_w = \frac{1}{\ell!} \sum_{(a_1,\ldots,a_\ell)\in \mathrm{Red}(w)} a_1 \cdots a_\ell,$$

where ℓ is the length of w, and the sum runs over reduced words of w. It is well known that $\nu_w = 1$ if and only if w is dominant, that is, w has no 132 pattern. Stanley [23] conjectured that $\nu_w = 2$ if and only if w has exactly one 132 pattern. This conjecture was confirmed by Weigandt [24] by proving a lower bound

$$\nu_w \geq 1 + p_{132}(w). \tag{1.2}$$

This bound was later strengthened by Gao [10, Theorem 2.1], where it was shown that

$$\nu_w \geq 1 + p_{132}(w) + p_{1432}(w). \tag{1.3}$$

Both proofs in [10, 24] make use of the pipe dream model of Schubert polynomials.

Because of $\theta_w \leq \nu_w$, Theorem 1.1 immediately yields a lower bound for ν_w which largely improves the bound in (1.3).

Corollary 1.2. *For $w \in S_n$, ν_w is bounded below by the right-hand side of* (1.1).

We deal with Theorem 1.1 in the general setting of dual characters of flagged Weyl modules associated to diagrams in the square grid $[n] \times [n]$. A diagram D means a subset of boxes in $[n] \times [n]$. The associated flagged Weyl module $\mathcal{M}_D$ is a representation of the Borel group B of invertible upper-triangular complex matrices [13, 14, 17]. Let $\chi_D(x) = \chi_D(x_1, \ldots, x_n)$ denote the dual character of $\mathcal{M}_D$. As will be explained in Section 2, $\chi_D(x)$ specializes to a Schubert polynomial (resp., key polynomial) when D is the Rothe diagram of a permutation (resp., skyline diagram of a composition).

We use θ_D to represent the size of the support of $\chi_D(x)$, which also equals the number of lattice points in the Newton polytope, called Schubitope, of $\chi_D(x)$ [7]. We deduce a lower bound for θ_D by using the appearances of certain subdiagrams of D. Precisely, consider the configurations in Figure 1.1, where a blank/shaded box means the absence/presence. Let (i, j) denote the box of $[n] \times [n]$ in row i and column j in

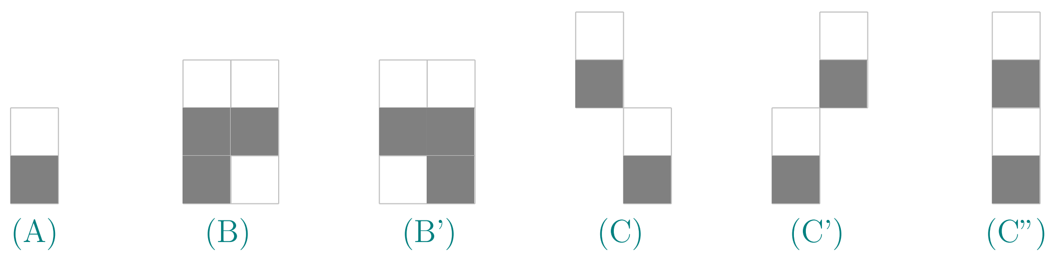

Figure 1.1. Configurations for Theorem 1.3.

matrix coordinate. Define

- $r_1(D)$: the number of subdiagrams of D which are equal to the configuration (A) in Figure 1.1;
- $r_2(D)$: the number of subdiagrams of D which are equal to the configuration (B), or (B') in Figure 1.1;
- $r_3(D)$: the number of subdiagrams of D which are equal to the configuration (C), or (C'), or (C") in Figure 1.1.

To avoid confusion, we explain the above notions in more detail. For example, a subdiagram of D which is equal to the configuration (C) in Figure 1.1 means a subset

$$\{(i_1, j_1), (i_2, j_1), (i_3, j_2), (i_4, j_2)\}$$

of boxes in $[n] \times [n]$ such that (1) $i_1 < i_2 < i_3 < i_4$ and $j_1 < j_2$, and (2) exactly two of the boxes, (i_2, j_1) and (i_4, j_2), belong to D.

The statistic $r_1(D)$ has been investigated by Mészáros, St. Dizier and Tanjaya [19], where it is called the rank of D and is denoted rank(D). Let $\chi_D(1, \ldots, 1) := \chi_D(x)|_{x_i=1}$ be the principal specialization of $\chi_D(x)$. Since $\chi_D(x)$ has nonnegative coefficients, it follows that

$$\chi_D(1, \ldots, 1) \geq \theta_D.$$

A criterion for the equality was conjectured in [19] and proved by Guo, Lin and Peng [11]. As shown in [19, Theorem 2], $\chi_D(1, \ldots, 1)$ is bounded below by $1 + r_1(D)$, which recovers the bound in (1.2) when D is the Rothe diagram of a permutation.

We prove the following lower bound for θ_D.

Theorem 1.3. *For any diagram D, we have*

$$\theta_D \geq 1 + r_1(D) + r_2(D) + r_3(D). \tag{1.4}$$

Theorem 1.3 leads to a strengthen of the above mentioned bound for $\chi_D(1, \ldots, 1)$ by Mészáros, St. Dizier and Tanjaya [19, Theorem 2].

Corollary 1.4. *For any diagram D, $\chi_D(1, \ldots, 1)$ is bounded below by the right-hand side of* (1.4).

When restricting to the skyline diagram $D(\alpha)$ of a composition $\alpha \in \mathbb{Z}^n_{\geq 0}$, Theorem 1.3 yields a lower bound for the supports of the key polynomial $\kappa_\alpha(x)$. Key polynomials, also called the Demazure characters, are the characters of a Demazure modules for the general linear group [4, 5]. They can also be defined in a recursive manner. If $\alpha = (\alpha_1 \geq \cdots \geq \alpha_n)$ is weakly decreasing, then set $\kappa_\alpha(x) = x^\alpha$. Otherwise, choose $1 \leq i < n$ such that $\alpha_i < \alpha_{i+1}$, and set $\kappa_\alpha(x) = \partial_i \, x_i \, \kappa_{s_i\alpha}(x)$, where $s_i\alpha$ is obtained from α by swapping the parts α_i and α_{i+1}.

Theorem 1.5. *For any (weak) composition α, we have*

$$\kappa_\alpha(1, \ldots, 1) \geq \theta_{D(\alpha)} \geq 1 + \sum_{\mathrm{inv}_1(\alpha)} (\alpha_{i_2} - \alpha_{i_1}) + \sum_{\mathrm{inv}_2(\alpha)} (\alpha_{i_2} - \alpha_{i_3}) \cdot (\alpha_{i_3} - \alpha_{i_1})$$
$$+ \sum_{\mathrm{inv}_3(\alpha)} (\alpha_{i_2} - \alpha_{i_1}) \cdot (\alpha_{i_4} - \alpha_{i_3}),$$

where $\mathrm{inv}_1(\alpha) = \{(i_1, i_2) \colon i_1 < i_2,\ \alpha_{i_1} < \alpha_{i_2}\}$, $\mathrm{inv}_2(\alpha) = \{(i_1, i_2, i_3) \colon i_1 < i_2 < i_3,\ \alpha_{i_1} < \alpha_{i_3} < \alpha_{i_2}\}$, *and* $\mathrm{inv}_3(\alpha) = \{(i_1, i_2, i_3, i_4) \colon i_1 < i_2 < i_3 < i_4,\ \alpha_{i_1} < \alpha_{i_2},\ \alpha_{i_3} < \alpha_{i_4}\}$.

Taking only the first summation, Theorem 1.5 reduces to the following lower bound by Mészáros, St. Dizier and Tanjaya [19, Corollary 20]:

$$\kappa_\alpha(1, \ldots, 1) \geq 1 + \sum_{\mathrm{inv}_1(\alpha)} (\alpha_{i_2} - \alpha_{i_1}).$$

This paper is organized as follows. Section 2 lays out basic information that we need about the dual characters of flagged Weyl modules. Section 3 is devoted to a proof of Theorem 1.3, based on which we complete the proofs of Theorems 1.1 and 1.5 in Section 4. We conclude in Section 5 with some problems and conjectures. We put some tables that are needed in the proof of Theorem 1.1 in the appendix section.

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2 Dual characters of flagged Weyl modules

In this section, we review some necessary background on flagged Weyl modules, and explain how their dual characters specialize to Schubert and key polynomials.

Recall that a diagram D is a subset of boxes in $[n]\times[n]$. Write $D=(D_1,D_2,\ldots,D_n)$, where, for $1\leq j\leq n$, D_j denotes the j-th column of D. We also represent D_j by a subset of $[n]$, that is, $i\in D_j$ if and only if the box (i,j) belongs to D. For example, the diagram in Figure 2.2 can be expressed as $D=(\{2,3,4\},\emptyset,\{1,2\},\{3\})$.

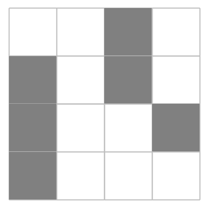


Figure 2.2. A diagram in $[4]\times[4]$.

Let $R=\{r_1<\cdots<r_k\}$ and $S=\{s_1<\cdots<s_k\}$ be two k-element subsets of $[n]$. We say $R\leq S$ if $r_i\leq s_i$ for $1\leq i\leq k$. This defines a partial order on all k-element subsets of $[n]$, which is usually called the Gale order. For two diagrams $C=(C_1,\ldots,C_n)$ and $D=(D_1,\ldots,D_n)$, denote $C\leq D$ if $C_j\leq D_j$ for each $1\leq j\leq n$.

Let $\mathrm{GL}(n,\mathbb{C})$ be the general linear group of $n\times n$ invertible complex matrices, and B the Borel subgroup of $\mathrm{GL}(n,\mathbb{C})$ which consists of upper-triangular matrices. We use Y to denote the upper-triangular matrix of variables y_{ij} with $1\leq i\leq j\leq n$:

$$Y=\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n}\\ 0 & y_{22} & \cdots & y_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & y_{nn}\end{bmatrix}.$$

Let $\mathbb{C}[Y]$ be the linear space of polynomials in $\{y_{ij}\}_{i\leq j}$ over $\mathbb{C}$. Define the (right) action of B on $\mathbb{C}[Y]$ by $f(Y)\cdot b=f(b^{-1}\cdot Y)$, where $b\in B$ and $f\in\mathbb{C}[Y]$. For two subsets R and S of $[n]$ with the same cardinality, let Y_S^R be the submatrix of Y with rows indexed by R and columns indexed by S. It is easily checked that $\det\left(Y_S^R\right)\neq 0$ if and only if $R\leq S$. For $C=(C_1,\ldots,C_n)$ and $D=(D_1,\ldots,D_n)$ with $C\leq D$, denote

$$\det\left(Y_D^C\right)=\prod_{j=1}^{n}\det\left(Y_{D_j}^{C_j}\right).$$

The flagged Weyl module associated to D is the subspace

$$\mathcal{M}_D=\mathrm{Span}_{\mathbb{C}}\left\{\det\left(Y_D^C\right):C\leq D\right\},$$

which is a B-module with the action inherited from the action of B on $\mathbb{C}[Y]$.

Let $X = \mathrm{diag}(x_1, \ldots, x_n)$ be the diagonal matrix. The character of $\mathcal{M}_D$ is defined as

$$\mathrm{char}(\mathcal{M}_D)(x_1, \ldots, x_n) = \mathrm{tr}(X \colon \mathcal{M}_D \to \mathcal{M}_D).$$

It is readily verified that $\det\left(Y_D^C\right)$ for $C \leq D$ is an eigenvector of X with eigenvalue

$$\prod_{j=1}^{n} \prod_{i \in C_j} x_i^{-1}.$$

The dual character is defined to be $\chi_D(x) := \mathrm{char}(\mathcal{M}_D)(x_1^{-1}, \ldots, x_n^{-1})$. The weight vector $\mathrm{wt}(C) = (\alpha_1, \ldots, \alpha_n)$ of a diagram $C = (C_1, \ldots, C_n)$ is defined by letting α_i be the number of appearances of i in $C_1, \ldots, C_n$. Geometrically, α_i is the number of boxes lying in row i. The diagram in Figure 2.2 has weight vector $(1, 2, 2, 1)$. By the above arguments, we have the following combinatorial characterization on the support of $\chi_D(x)$, see Adve, Robichaux and Yong [1] for discussions about the computational complexity for deciding the support.

Proposition 2.1. *The support of* $\chi_D(x)$ *is* $\{\mathrm{wt}(C) \colon C \leq D\}$.

The Rothe diagram $D(w)$ of a permutation $w \in S_n$ can be constructed as follows. For $1 \leq i \leq n$, place a dot in row i and column $w(i)$. Then $D(w)$ is obtained by ignoring all boxes to the right of a dot in the same row, and all boxes below a dot in the same column. The skyline diagram $D(\alpha)$ of a composition $\alpha = (\alpha_1, \ldots, \alpha_n)$ consists of the leftmost α_i boxes in row i for each $1 \leq i \leq n$. Figure 2.3 displays the Rothe diagram of $w = 1432$ and the skyline diagram of $\alpha = (1, 3, 0, 2)$. As specializations, it is well known

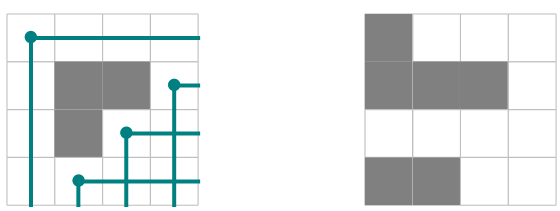


Figure 2.3. The Rothe diagram of $w = 1432$ and the skyline diagram of $\alpha = (1, 3, 0, 2)$.

that $\chi_{D(w)}(x) = \mathfrak{S}_w(x)$ [13, 14] and $\chi_{D(\alpha)}(x) = \kappa_\alpha(x)$ [5].

3 Proof of Theorem 1.3

Let D be a given diagram in $[n] \times [n]$. For simplicity, we denote

$$r_i = r_i(D) \quad \text{for } i = 1, 2, 3.$$

To finish the proof of Theorem 1.3, by Proposition 2.1, it suffices to construct $r_1 + r_2 + r_3$ diagrams less than D which have distinct weight vectors. To accomplish this, we shall design three algorithms to produce such diagrams.

3.1 The first algorithm

The first algorithm produces a chain $C^1 < C^2 < \cdots < C^{r_1} < D$, including r_1 diagrams less than D. It essentially obeys a similar idea to that in the proof of [19, Lemma 17].

For $(i,j) \in D$, denote by $r_1(D;i,j)$ the number of blank boxes above (i,j) in the same column, namely,

$$r_1(D;i,j) = \#\left\{i' \colon i' < i, (i',j) \notin D\right\}.$$

By definition, it follows that

$$r_1 = \sum_{(i,j)\in D} r_1(D;i,j).$$

Algorithm 1. First, we construct C^{r_1}. Among the boxes (i,j) of D such that $r_1(D;i,j) > 0$, choose the top-left most one, say (i_0,j_0). Note that the box (i_0-1,j_0) right above it must be blank. Set $C^{r_1} = D \setminus \{(i_0,j_0)\} \cup \{(i_0-1,j_0)\}$. It is clear that $C^{r_1} < D$ and $r_1(C^{r_1}) = r_1(D)-1$. Replacing D by C^{r_1} and doing the same operation, we are given $C^{r_1-1} < C^{r_1}$ with $r_1(C^{r_1-1}) = r_1(C^{r_1})-1$. Repeating this procedure yields the desired chain $C^1 < C^2 < \cdots < C^{r_1} < D$. An illustration for this algorithm is depicted in Figure 3.4, where the position marked with a crossing means the box which has been moved up.

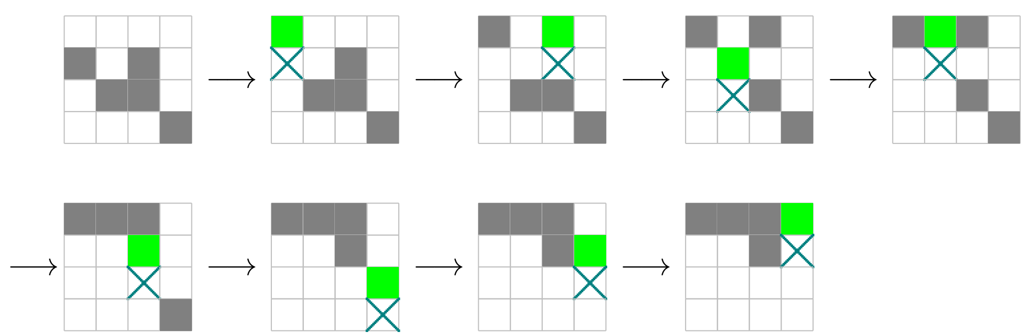


Figure 3.4. An illustration for performing Algorithm 1.

Remark 3.1. *The diagrams produced by Algorithm 1 are generally different from the diagrams constructed in [19, Lemma 17]. The reason that we adopt Algorithm 1 is that we need to produce diagrams whose weight vectors are distinct from those generated by Algorithm 2 and Algorithm 3 in the next two subsections.*

Proposition 3.2. *The weight vectors of the r_1 diagrams generated in Algorithm 1 are distinct.*

Proof. As explained in [19, Lemma 18], if $C < D$, then C and D have distinct weight vectors. This allows us to conclude the proof. ∎

3.2 The second algorithm

The second algorithm will give rise to r_2 diagrams less than D with distinct weight vectors. Moreover, these weight vectors are distinct from the weight vectors of diagrams generated by Algorithm 1.

Given a pair of row indices $1 < i_1 < i_2 \leq n$ and a pair of column indices $1 \leq j_1 < j_2 \leq n$, let $r_2(D; i_1, i_2; j_1, j_2)$ denote the number of row indices i with $i < i_1$ such that the subdiagram of D, which includes the six boxes restricted to rows $\{i, i_1, i_2\}$ and columns $\{j_1, j_2\}$, is either the configuration (B) or (B') in Figure 1.1. Clearly, we have

$$r_2 = \sum_{\substack{1<i_1<i_2\leq n \\ 1\leq j_1<j_2\leq n}} r_2(D; i_1, i_2; j_1, j_2).$$

Algorithm 2. The algorithm will be performed for any given pair of row indices $1 < i_1 < i_2 \leq n$. We shall take the diagram in Figure 3.5 as a running example.

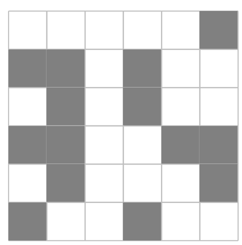
$D =$  .

Figure 3.5. A diagram for illustrating Algorithm 2.

Now, let us fix $1 < i_1 < i_2 \leq n$. A box $(i_1, j) \in D$ in row i_1 is called a pivot if

(1) the two boxes (i_1, j) and (i_2, j) form the configuration ;

(2) there exists (at least) one column index j' such that $r_2(D; i_1, i_2; j, j') > 0$ if $j < j'$, and $r_2(D; i_1, i_2; j', j) > 0$ if $j' < j$.

For example, if we take $(i_1, i_2) = (4, 5)$, then the diagram in Figure 3.5 has two pivots in the fourth row: the boxes $(4, 1)$ and $(4, 5)$.

For each pivot (i_1, j) in row i_1, we will produce a family of diagrams according to the following two steps.

Step 0. Move all boxes of D in rows $1, 2, \ldots, i_1 - 1$ to the topmost positions, and also move any further left pivot (i_1, j') with $j' < j$ up to $(i_1 - 1, j')$.

Step 1. For $m = 1, 2, \ldots$, locate all the column indices j' such that $r_2(D; i_1, i_2; j, j') \geq m$ if $j < j'$, and $r_2(D; i_1, i_2; j', j) \geq m$ if $j' < j$. Suppose that there are k such column indices, say $j_1 < \cdots < j_k$. For $1 \leq t \leq k$, let C^t be the diagram obtained by moving the boxes $(i_2, j_1), \ldots, (i_2, j_t)$ in row i_2 up to the positions $(i_1 - m, j_1), \ldots, (i_1 - m, j_t)$. Evidently, this procedure will terminate since m cannot increase arbitrarily.

Let $S_{i_1,i_2,j}(D)$ be the set of all diagrams generated in Step 1. Running over all the pivots (i_1, j) in row i_1 yields the set, denoted $S_{i_1,i_2}(D)$, of (disjoint) union of all $S_{i_1,i_2,j}(D)$. Now we see that

$$r_2 = \sum_{1<i_1<i_2\le n} \#S_{i_1,i_2}(D).$$

Let us illustrate how to use Algorithm 2 to generate $S_{4,5}(D)$ for the diagram D in Figure 3.5. We first consider the pivot $(4, 1)$. In Step 0, all the boxes above row 4 are moved up to the topmost positions, see Figure 3.6. In Step 1, the algorithm generates

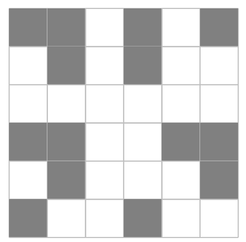


Figure 3.6. The diagram generated in Step 0 for the pivot $(4, 1)$.

two diagrams when $m = 1$, see Figure 3.7. It is easily checked that the algorithm will

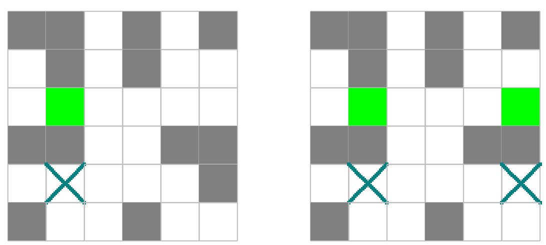


Figure 3.7. Diagrams generated in Step 1 for the pivot $(4, 1)$ and for $m = 1$.

terminate when $m > 1$. We next consider the pivot $(4, 5)$. The diagram in Step 0 is given in Figure 3.8, which is obtained by first moving all the boxes above row 4 to the topmost positions and then moving the pivot $(4, 1)$ up to the position immediately above (which is indicated by a crossing). Based on this diagram, we then implement the operation in

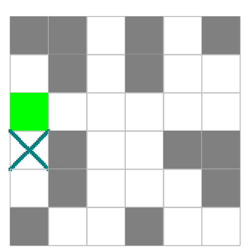


Figure 3.8. The diagram in Step 0 for the pivot (4,5).

Step 1. When $m = 1$, Step 1 produces two diagrams, see Figure 3.9. When $m = 2$, we further obtain a single diagram, as given in Figure 3.10. Hence the set $S_{4,5}(D)$ has a total of five diagrams.

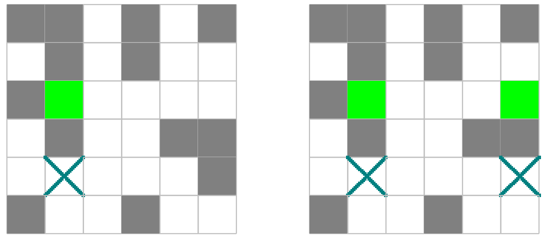


Figure 3.9. Diagrams generated in Step 1 for the pivot $(4, 5)$ and for $m = 1$.

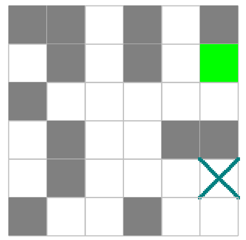


Figure 3.10. The diagram generated in Step 1 for the pivot $(4, 5)$ and for $m = 2$.

We next show that after applying Algorithm 2 to all pairs $1 < i_1 < i_2 \leq n$, the resulting r_2 diagrams have distinct weights. Moreover, these diagrams have different weights from the diagrams generated by Algorithm 1.

Proposition 3.3. *Let $1 < i_1 < i_2 \leq n$. Then the diagrams in $S_{i_1,i_2}(D)$ have distinct weight vectors. Moreover, for $1 < i_1' < i_2' \leq n$ with $(i_1', i_2') \neq (i_1, i_2)$, the diagrams in $S_{i_1,i_2}(D)$ and $S_{i_1',i_2'}(D)$ have distinct weight vectors.*

Proof. Let $C, C' \in S_{i_1,i_2}(D)$ with $C' \neq C$, and let $C'' \in S_{i_1',i_2'}(D)$. We first check that C and C' have different weights. The arguments are divided into two cases.

Case 1. C and C' correspond to the same pivot. By the construction in Step 1 of Algorithm 2, C and C' are obtained from the same diagram (generated in Step 0) by moving some boxes in row i_2 up to the same row (which happens for the same m) or two different rows (which happens for distinct m). If the boxes are moved to the same row, then C and C' have different numbers of boxes in row i_2. If the boxes are moved to two different rows, then C and C' have different numbers of boxes in these two rows. In both situations, C and C' have distinct weights.

Case 2. C and C' correspond to distinct pivots. In this case, notice that C and C' have different numbers of boxes in row i_1. So C and C' have distinct weights.

We next check that C and C'' have distinct weights. We also have two cases.

Case 1': $i_1 \neq i_1'$. Assume $i_1 < i_1'$ without loss of generality. Let

$$k = \#\{1 \leq j \leq n \colon r_1(D; i_1, j) > 0\}.$$

Concerning C'', Step 0 will move all boxes of D above row i_1' to the topmost positions, and so exactly k boxes in row i_1 are moved up to higher rows. However, applying Algorithm 2 to the pair (i_1, i_2), it is not hard to observe that there are at most $k - 1$

boxes of D (lying in row i_1 or i_2) that are moved up to rows higher than row i_1 (because the last pivot in row i_1 will not be moved up). So the weights of C and C'' cannot be the same.

Case 2': $i_1 = i_1'$, but $i_2 \neq i_2'$. Suppose that $i_2 < i_2'$. Then C'' has less boxes than C in row i_2'. So C and C'' have distinct weights. This completes the proof. ∎

Proposition 3.4. *For $1 < i_1 < i_2 \leq n$, the diagrams in $S_{i_1,i_2}(D)$ and the diagrams generated by Algorithm 1 have distinct weight vectors.*

Proof. Let $C \in S_{i_1,i_2}(D)$, and C' be any diagram generated by Algorithm 1. Suppose to the contrary that C and C' have the same weight vector. Note that C has less boxes than D in row i_2. Thus some boxes of D in row i_2 must be moved up to form C'. This meanwhile tells that in the construction of C', the boxes of D lying above row i_2 are moved up to the topmost positions. Particularly, there are k boxes of D in row i_1 that are moved up to higher rows, where, as in Case 1' in Proposition 3.3,

$$k = \#\{1 \leq j \leq n \colon r_1(D; i_1, j) \neq 0\}.$$

However, as explained in Case 1' in Proposition 3.3, there are at most $k-1$ boxes of D, that are moved up to the places above row i_1, to form C. This implies that C' has more boxes above row i_1 than C, leading to a contradiction. ∎

3.3 The third algorithm

We lastly describe the third algorithm which will produce r_3 diagrams whose weights are different from those generated by Algorithm 1 as well as Algorithm 2.

Denote by $D_{>i}$ the subdiagram of D which includes the boxes of D below row i. A box (i, j) of D is called a PIVOT if $r_1(D; i, j) > 0$ (that is, there is at least one blank box above (i, j) in the same column). Here we use capital letters to distinguish with the pivots defined in Subsection 3.2.

Lemma 3.5. *We have*

$$r_3 = \sum_{(i,j)} r_1(D; i, j) \times r_1(D_{>i}),$$

where the sum ranges over all PIVOTs of D, and $r_1(D_{>i})$ is the number of subdiagrams of $D_{>i}$ which are equal to the configuration (A) in Figure 1.1.

Proof. Consider the subdiagrams of D which are equal to the configuration (C), or (C'), or (C") in Figure 1.1. Note that the second box in each subdiagram is a PIVOT. Given a PIVOT box (i, j), the subdiagrams, whose second box is (i, j), are counted by $r_1(D; i, j) \times r_1(D_{>i})$, and so the proof is complete. ∎

Algorithm 3. Fix a PIVOT box $(i, j) \in D$. This algorithm will produce $r_1(D; i, j) \times r_1(D_{>i})$ diagrams less than D.

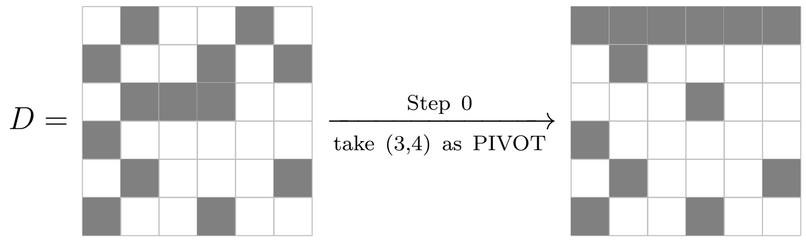


Figure 3.11. An illustration for Step 0 in Algorithm 3.

Step 0. Move the boxes of D above row i, along with the boxes of D in row i to the left of (i, j), up to the topmost positions. See Figure 3.11 for an illustration. Notice that in the resulting diagram, there are $r_1(D; i, j)$ blank boxes right above (i, j).

Step 1. Apply Algorithm 1 to $D_{>i}$. This gives rise to a chain of $r_1(D_{>i})$ diagrams less than D, say $C^1 < C^2 < \cdots < C^k$, where $k = r_1(D_{>i})$. For $1 \le p \le k$ and $1 \le q \le r_1(D; i, j)$, let $C^{p,q}$ be the diagram obtained from C^p by moving the PIVOT (i, j) up to the position $(i-q+1, j)$. This is well defined since there are $r_1(D; i, j)$ blank boxes right above (i, j). Note that $C^{p,1} = C^p$. Set

$$T_{i,j}(D) = \{C^{p,q} \colon 1 \le p \le k,\ 1 \le q \le r_1(D; i, j)\}.$$

Proposition 3.6. *Running over all PIVOTs of D, the diagrams generated by Algorithm 3 have distinct weight vectors.*

Proof. Assume that $C \in T_{i,j}(D)$ and $C' \in T_{i',j'}(D)$. We show that C and C' have distinct weights. We consider the following two cases separately.

Case 1: $(i, j) = (i', j')$. This can be seen as follows. For $1 \le q \le r_1(D; i, j)$, notice that

$$C^{1,q} < C^{2,q} < \cdots < C^{k,q},$$

where $k = r_1(D_{>i})$ as in Step 1. As explained in Proposition 3.2, one can conclude that $C^{1,q}, \ldots, C^{k,q}$ have distinct weights. Moreover, by the construction in Step 1, the weights of $C^{p,q}$ and $C^{p',q'}$ for $q \neq q'$ are obviously distinct.

Case 2: $(i, j) \neq (i', j')$. Without loss of generality, assume that $i < i'$, or $i = i'$ and $j < j'$. We assert that in row $i - r_1(D; i, j)$, C' has more boxes than C. To see this, a key observation is that when Step 0 in Algorithm 3 is applied to the PIVOT (i', j'), the box (i, j) is moved up to the position $(i - r_1(D; i, j), j)$. However, when Algorithm 3 is applied to the PIVOT (i, j), Step 1 contributes no box to row $i - r_1(D; i, j)$ since the box (i, j) can be moved up at most to the position $(i - r_1(D; i, j) + 1, j)$ (that is, $(i - r_1(D; i, j), j)$ is a blank box in C). This implies that C' has at least one more box than C in row $i - r_1(D; i, j)$, and so C and C' have distinct weights. ∎

Proposition 3.7. *For any PIVOT (i, j), the diagrams in $T_{i,j}(D)$ and the diagrams generated by Algorithm 1 have distinct weight vectors.*

Proof. Let $C \in T_{i,j}(D)$. Write

$$C^1 < C^2 < \cdots < C^{r_1} < D$$

for the chain of diagrams produced by Algorithm 1. Let $1 \le \ell \le r_1$ be the index such that C^ℓ is the diagram obtained from its preceding diagram $C^{\ell+1}$ by moving (i,j) up to the position $(i-1,j)$ (here we set $C^{r_1+1} = D$). Notice that

- $C^{\ell+1}$ is exactly the resulting diagram after applying Step 0 in Algorithm 3 to D;
- for $1 \le s < r_1(D;i,j)$, the diagram $C^{\ell+1-s}$ is obtained from $C^{\ell+1}$ by moving (i,j) up to the position $(i-s,j)$.

We show that the weights of C and C^t are distinct in two cases:

Case 1. $\ell+1-r_1(D;i,j) < t \le r_1$. In this case, note that $(C^t)_{>i} = D_{>i}$. However, $C_{>i}$ and $D_{>i}$ have different weights, and so the weights of C and C^t are distinct.

Case 2. $1 \le t \le \ell+1-r_1(D;i,j)$. In this case, during the construction of C^t, the box (i,j) of D is moved up to the position $(i-r_1(D;i,j),j)$. For the analogous reason to the proof of the $(i,j) \ne (i',j')$ case in Proposition 3.6, C^t has more boxes than C in row $i-r_1(D;i,j)$. This completes the proof. ∎

Proposition 3.8. *For any PIVOT (i,j) and any row indices $i_1 < i_2$, the diagrams in $T_{i,j}(D)$ and the diagrams in $S_{i_1,i_2}(D)$ generated by Algorithm 2 have distinct weight vectors.*

Proof. Let $C \in T_{i,j}(D)$ and $C' \in S_{i_1,i_2}(D)$. We have two cases.

Case 1: $i \ne i_1$. If $i < i_1$, then we can show that C' has at least one more box than C in row $i-r_1(D;i,j)$. The arguments are completely similar to Case 2 in the proof of Proposition 3.6, and so is omitted. If $i > i_1$, then, as in Case 1' in the proof of Proposition 3.3, let

$$k = \#\{1 \le j \le n\colon r_1(D;i_1,j) \ne 0\}.$$

In the construction of C, there are k boxes of D in row i_1 that are moved up to the area higher than row i_1. While, in the construction of C', there are at most $k-1$ boxes (from row i_1 or i_2) of D that are moved up to the area higher than row i_1. So we see that C has more boxes than C' in the area above row i_1, and thus C and C' have distinct weights.

Case 2: $i = i_1$. In this case, note that $C_{>i}$ has the same number of boxes as $D_{>i}$, while $(C')_{>i}$ has less boxes than $D_{>i}$. This implies that C and C' must have distinct weights. So the proof is complete. ∎

We can finally give a proof of Theorem 1.3.

Proof of Theorem 1.3. We see that Algorithms 1, 2 and 3 together produce a total of $r_1 + r_2 + r_3$ diagrams that are less than D. It follows from the propositions proved in Subsections 3.1, 3.2 and 3.3 that these diagrams have distinct weight vectors. This finishes the proof. ∎

4 Proofs of Theorems 1.1 and 1.5

In this section, we specialize D in Theorem 1.3 to a Rothe diagram or a skyline diagram, thereby completing the proofs of Theorems 1.1 and 1.5.

4.1 Proof of Theorem 1.1

Let $D = D(w)$ be the Rothe diagram of a permutation $w \in S_n$. It suffices to prove the following.

Theorem 4.1. *For $w \in S_n$, we have*

$$\begin{aligned} r_1(D(w)) + r_2(D(w)) + r_3(D(w)) & \\ \geq p_{132}(w) + p_{1432}(w) + p_{13254}(w) + 3p_{14253}(w) & \\ + p_{14352}(w) + 4p_{15243}(w) + p_{15324}(w) + 2p_{15342}(w) & \\ + p_{15432}(w) + p_{24153}(w) + 2p_{25143}(w) + p_{35142}(w). & \end{aligned} \tag{4.1}$$

Proof. For each permutation u appearing on the right-hand side of (4.1), let $P_u(w)$ denote the set of u patterns in w, and a_u be the coefficient of $p_u(w)$. For every u pattern in $P_u(w)$, we shall construct a_u subdiagams of $D(w)$, each of which is equal to one of the configurations in Figure 1.1. Then we conclude the proof by explaining that such constructed subdiagrams are different from each other.

We first look at the construction of subdiagrams of $D(w)$ for $u = 132$ or 1432. Let $w(i_1)w(i_2)w(i_3) \in P_{132}(w)$. The subdiagram of $D(w)$ generated by $w(i_1)w(i_2)w(i_3)$ is defined as $\{(i_1, w(i_2)), (i_2, w(i_2))\}$, which is the configuration (A) in Figure 1.1. The construction is displayed in the first line of Table 1, where the boxes forming the subdiagram are marked with ✓. This correspondence has appeared in the proof of [19, Corollary 19], which is in fact a bijection between $P_{132}(w)$ and the set of subdiagrams of $D(w)$ which are equal to the configuration (A) in Figure 1.1.

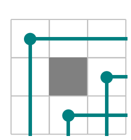
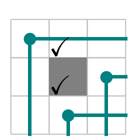
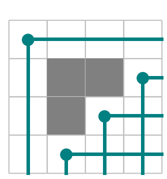
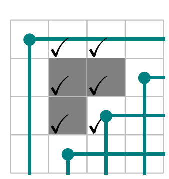
patterns	subdiagrams
 132 pattern	 (A)
 1432 pattern	 (B)

Table 1. Subdiagrms generated by 132 or 1432 patterns.

We next turn to the pattern $w(i_1)w(i_2)w(i_3)w(i_4) \in P_{1432}(w)$. The subdiagram of $D(w)$ generated by this pattern is defined as

$$\{(i_1, w(i_4)), (i_1, w(i_3)), (i_2, w(i_4)), (i_2, w(i_3)), (i_3, w(i_4)), (i_3, w(i_3))\},$$

which forms the configuration (B) in Figure 1.1. See the second line of Table 1 for an illustration of the construction.

There are 10 permutations u in S_5 appearing on the right-hand side of (4.1). We put the constructions of subdiagrams of $D(w)$ for these u patterns in Tables 4 and 5 in the appendix.

Let $\mathrm{Sub}(D(w))$ denote the collection (as multiset) of all subdiagrams of $D(w)$ which could be produced by the u patterns of w as displayed in Tables 1, 4 and 5. The remaining work is to check that the subdiagrams in $\mathrm{Sub}(D(w))$ are different. To do this, a crucial feature that we can observe from Tables 1, 4 and 5 is that for any given subdiagram, say D_{sub}, in $\mathrm{Sub}(D(w))$, we are able to recover the (unique) pattern in w from which D_{sub} is generated. To be specific, we have the following explanations.

- D_{sub} is the configuration (A) in Figure 1.1. In this case, D_{sub} has two boxes. Assume that the boxes lie in rows $\{i_1 < i_2\}$ and column j. Then the corresponding 132 pattern of w includes the entries of w at the positions $\{i_1, i_2, w^{-1}(j)\}$, where w^{-1} is the inverse of w.

- D_{sub} is the configuration (B), or (B') in Figure 1.1. Assume that the six boxes in D_{sub} lie in rows $\{i_1 < i_2 < i_3\}$ and columns $\{j_1 < j_2\}$. Then the corresponding pattern of w includes the entries of w at the positions $\{i_1, i_2, i_3, w^{-1}(j_1), w^{-1}(j_2)\}$. It should be noted that there may happen that $i_3 = w^{-1}(j_2)$, and in this case the pattern is a 1432 pattern.

- D_{sub} is the configuration (C), or (C'), or (C") in Figure 1.1. Assume that the four boxes in D_{sub} lie in rows $\{i_1 < i_2 < i_3 < i_4\}$, and among the four boxes, the lowest two lie in column j. Then the corresponding pattern of w includes the entries of w at the positions $\{i_1, i_2, i_3, i_4, w^{-1}(j)\}$.

In view of the above observations, give two subdiagrams, say D^1_{sub} and D^2_{sub}, in $\mathrm{Sub}(D(w))$, we can verify $D^1_{\mathrm{sub}} \neq D^2_{\mathrm{sub}}$ by contradiction. Suppose otherwise that $D^1_{\mathrm{sub}} = D^2_{\mathrm{sub}}$. Then they are generated by the same u pattern in w. This only possibly happens in the case of the $u = 15342$ pattern in Table 5. However, the two subdigrams generated by a $u = 15342$ pattern are obviously distinct. This arrives at a contradiction. So the proof is complete. ∎

Remark 4.2. *We remark that there may possibly exist instances of subdiagrams of $D(w)$, which are equal to the configurations in Figure 1.1, but cannot be produced by the patterns of w listed in Tables 1, 4, or 5. For example, consider the Rothe diagram $D(w)$ for $w =$ 162435. Take the subdiagram consisting of the boxes with check marks. This subdiagram cannot be generated by any $u \in S_5$ pattern of w. If it were generated by a u pattern, then the u pattern of w would be* 16243 *and so $u =$ 15243. *However, in view of the construction in Table 4, the four subdiagrams of $D($162435$)$ generated by the pattern* 16243 *do not include the one we are considering.*

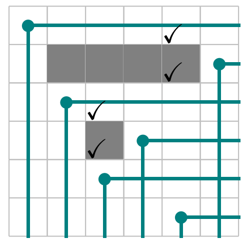


Figure 4.12. A subdiagram of $D(162435)$.

4.2 Proof of Theorem 1.5

Now we let $D = D(\alpha)$ in Theorem 1.3 be the skyline diagram of a weak composition α. We finish the proof of Theorem 1.5 by verifying the following equality.

Theorem 4.3. *For any (weak) composition α, we have*

$$r_1(D(\alpha)) + r_2(D(\alpha)) + r_3(D(\alpha)) = \sum_{\mathrm{inv}_1(\alpha)} (\alpha_{i_2} - \alpha_{i_1}) + \sum_{\mathrm{inv}_2(\alpha)} (\alpha_{i_2} - \alpha_{i_3}) \cdot (\alpha_{i_3} - \alpha_{i_1}) + \sum_{\mathrm{inv}_3(\alpha)} (\alpha_{i_2} - \alpha_{i_1}) \cdot (\alpha_{i_4} - \alpha_{i_3}). \tag{4.2}$$

Proof. It is obvious that

$$r_1(D(w)) = \sum_{(i_1,i_2)\in \mathrm{inv}_1(\alpha)} (\alpha_{i_2} - \alpha_{i_1}).$$

We next check that

$$r_2(D(w)) = \sum_{\substack{(i_1,i_2,i_3)\\ \in \mathrm{inv}_2(\alpha)}} (\alpha_{i_2} - \alpha_{i_3}) \cdot (\alpha_{i_3} - \alpha_{i_1}). \tag{4.3}$$

This can be seen as follows. First, since $D(\alpha)$ is left-justified, there is no subdiagram of $D(\alpha)$ which is the configuration (B'). We are now left with the enumeration of subdigrams of $D(\alpha)$ which are equal to the configuration (B). Compute such subdiagrams which lie in rows $\{i_1 < i_2 < i_3\}$. Clearly, we have $\alpha_{i_1} < \alpha_{i_3} < \alpha_{i_2}$. Observe that for each such subdiagram, the column index of its left three boxes is greater than α_{i_1} but less than or equal to α_{i_3}, while the column index of its right three boxes is greater than α_{i_3} but less than or equal to α_{i_2}. So there are a total of $(\alpha_{i_2} - \alpha_{i_3}) \cdot (\alpha_{i_3} - \alpha_{i_1})$ subdiagrams, which are the configuration (B), lying in rows $\{i_1 < i_2 < i_3\}$. This justifies (4.3).

Using similar analysis, one can readily verify that

$$r_3(D(\alpha)) = \sum_{\substack{(i_1,i_2,i_3,i_4)\\ \in \mathrm{inv}_3(\alpha)}} (\alpha_{i_2} - \alpha_{i_1}) \cdot (\alpha_{i_4} - \alpha_{i_3}).$$

This completes the proof of (4.2). ∎

5 Concluding remarks

In this section, we investigate some problems and conjectures concerning θ_w.

5.1 The maximum value of θ_w

Let α_n be the largest principal specialization for Schubert polynomials:

$$\alpha_n = \max\{\nu_w : w \in S_n\}$$

Merzon and Smirnov [18] predicted that the maximum value α_n is achieved on layered permutations. For positive integers $b_1, \ldots, b_k$ summing up to n, the associated layered permutation $w(b_1, \ldots, b_k)$ in S_n is defined as the concatenation $w^1 \cdots w^k$ of k words, where w^i is obtained by permuting the entries in the interval $[b_1+\cdots+b_{i-1}+1, b_1+\cdots+b_i]$ decreasingly. Here we set $b_0 = 0$. For example, $w(2, 3, 2, 1) = 21543768$.

Conjecture 5.1 (Merzon–Smirnov [18])**.** *For $n \geq 1$, the permutations in S_n attaining α_n are layered permutations.*

Remark 5.2. *Very recently, the Merzon–Smirnov conjecture was disproved by Anderson, Panova and Petrov [2] by finding a counterexample at $n = 17$.*

Consider the largest value of θ_w with $w \in S_n$:

$$\beta_n = \max\{\theta_w : w \in S_n\}$$

In Table 2, we list the values of α_n and β_n for $n \leq 9$, together with the permutations achieving these maximum values. When $n \leq 9$, the permutations attaining β_n are layered

n	α_n	permutations attaining α_n	β_n	permutations attaining β_n
1	1	1	1	1
2	1	12, 21	1	12, 21
3	2	132	2	132
4	5	1432	5	1432
5	14	12543, 15432, 21543	14	15432
6	84	126543, 216543	65	126543, 216543
7	660	1327654	347	1276543, 2176543
8	9438	13287654	2151	13287654
9	163592	132987654	17319	132987654

Table 2. The values of α_n and β_n for $n \leq 9$.

permutations. We do not know if this is still true for general n.

The asymptotic behavior of α_n was first sought by Stanley [23].

Conjecture 5.3 (Stanley [23]). *There exists a limit*

$$\lim_{n\to\infty} \frac{\log \alpha_n}{n^2}.$$

Morales, Pak and Panova [22] showed that there is a limit when restricted to layered permutations. That is, letting

$$\gamma_n = \max\{\nu_w \colon w \text{ are layered permutations in } S_n\},$$

there exists a limit

$$\lim_{n\to\infty} \frac{\log_2 \gamma_n}{n^2} \approx 0.2932362762.$$

Zhang [25] recently considered the asymptotic property for the largest value of $\mathfrak{S}_w(x)\big|_{x_i=q^{i-1}}$ with $q = -1$ for multi-layered permutations.

Problem 5.4. *Does there exist an asymptotic behavior for β_n similar to Conjecture 5.3.*

Remark 5.5. *Chou and Setiabrata [3] recently gave an answer to Problem 5.4 by establishing the following asymptotic for β_n:*

$$\lim_{n\to\infty} \frac{\ln \beta_n}{n \ln n} = 1.$$

5.2 A positivity conjecture

For $w \in S_n$, we may write

$$\nu_w = 1 + \sum_{\substack{u\in S_m\\ m\leq n}} c_u\, p_u(w),$$

where the coefficients c_u for $u \in S_m$ are determined recursively by

$$c_u = \nu_u - 1 - \sum_{\substack{\sigma\in S_\ell\\ \ell<m}} c_\sigma\, p_\sigma(u).$$

As observed by Gao [10, Lemma 3.1], the coefficients c_u own the stability property, that is, $c_u = 0$ for $u \in S_m$ with $u(m) = m$. The following appealing conjecture appears as [10, Conjecture 3.2].

Conjecture 5.6 (Gao [10]). *For any permutation u, we have $c_u \in \mathbb{Z}_{\geq 0}$.*

The above conjecture has been confirmed for permutations avoiding both 1432 and 1423 patterns by Mészáros and Tanjaya [20], and for permutations avoiding 1243 patterns by Dennin [6].

Similarly, for $w \in S_n$, one may express

$$\theta_w = 1 + \sum_{u} d_u\, p_u(w).$$

The coefficients d_u can be similarly computed in a recursive procedure:

$$d_u = \theta_u - 1 - \sum_{\substack{\sigma \in S_\ell \\ \ell < m}} d_\sigma \, p_\sigma(u).$$

Imitating the arguments in the proof of [10, Lemma 3.1], we can show that $d_u = 0$ for $u \in S_m$ with $u(m) = m$.

Conjecture 5.7. *For any permutation u, we have $d_u \in \mathbb{Z}_{\geq 0}$.*

Conjecture 5.7 has been verified for n up to 8. The data imply that

- when $1 \leq n \leq 5$, we have $0 \leq d_u \leq c_u$, and $d_u > 0$ if and only if $c_u > 0$;
- when $n = 6, 7$, we still have $0 \leq d_u \leq c_u$. But there exist two permutations in S_6, $u^1 = 136245$ and $u^2 = 146235$, such that $d_{u^1} = d_{u^2} = 0$ but $c_{u^1} = c_{u^2} = 1$;
- when $n = 8$, we no longer have $0 \leq d_u \leq c_u$. The only exception is $u = 13452786$ for which $c_u = 3$ and $d_u = 4$.

In Table 3, we list the values of $c_u > 0$ (or $d_u > 0$) for permutations $u \in S_m$ with $m \leq 5$, where the permutations appearing in the lower bound in Theorem 1.1 are underlined.

permutation	c_u	d_u	permutation	c_u	d_u
<u>132</u>	1	1	<u>1432</u>	1	1
12453	1	1	<u>15342</u>	2	2
12534	1	1	15423	1	1
12543	5	4	<u>15432</u>	3	3
<u>13254</u>	3	2	21453	1	1
13524	3	2	21534	1	1
13542	4	3	21543	5	4
<u>14253</u>	3	3	<u>24153</u>	1	1
<u>14352</u>	1	1	<u>25143</u>	2	2
14523	1	1	31524	1	1
14532	1	1	31542	2	2
<u>15243</u>	4	4	<u>35142</u>	1	1
<u>15324</u>	1	1			

Table 3. Permutations in S_m for $m \leq 5$ with nonzero values of c_u and d_u.

From Table 3, we see that there are 13 permutations in S_5 that do not appear in the lower bound in Theorem 1.1. Among the 12 permutations appearing in the lower bound, the coefficients for $p_{13254}(w)$ and $p_{15432}(w)$ are not optimal if assuming Conjecture 5.7. New algorithms or tools are needed to explore for further improving the lower bound established in Theorem 1.1.

Problem 5.8. *Strengthen the lower bound in Theorem 1.1.*

As an attempt to enhance the bound, it would be interesting to establish a bound for θ_w encompassing all $p_u(w)$ for u being permutations listed in Table 3.

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6 Appendix

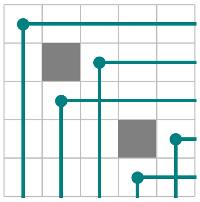
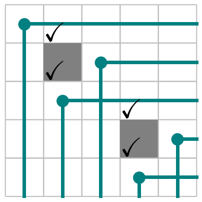
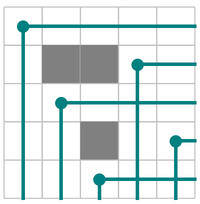
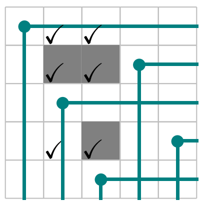
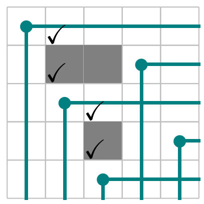
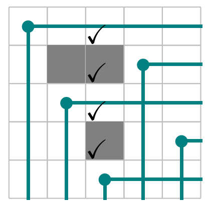
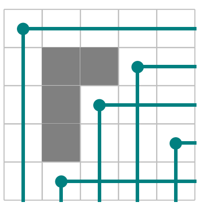
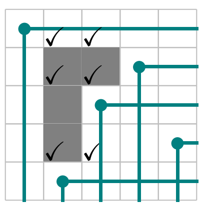
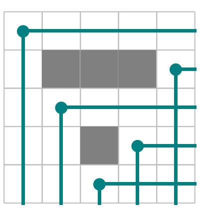
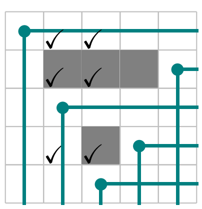
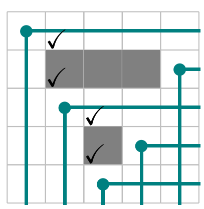
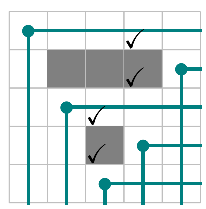
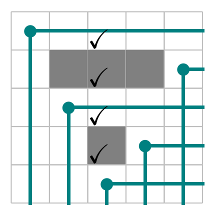
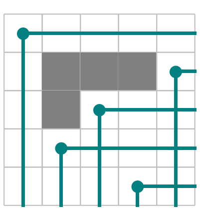
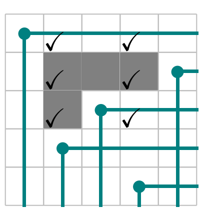
patterns	subdiagrams			
 13254 pattern	 (C)			
 14253 pattern	 (B’)	 (C)	 (C”)	
 14352 pattern	 (B)			
 15243 pattern	 (B’)	 (C)	 (C’)	 (C”)
 15324 pattern	 (B)			

Table 4. Subdiagrams in the proof of Theorem 1.1 generated by u patterns with $u \in S_5$.

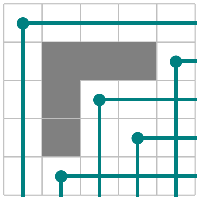
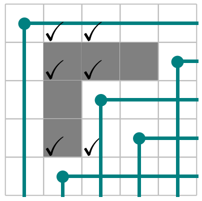
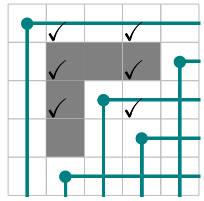
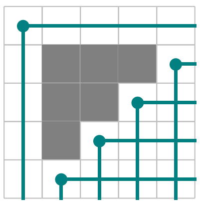
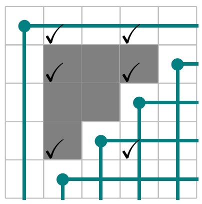
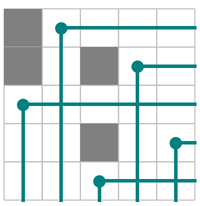
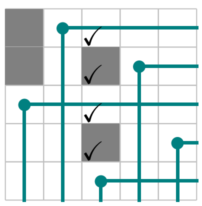
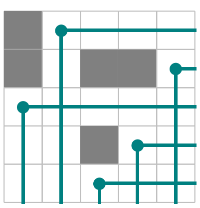
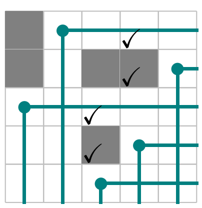
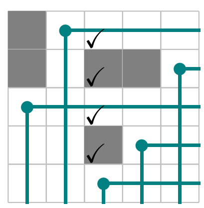
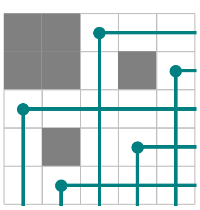
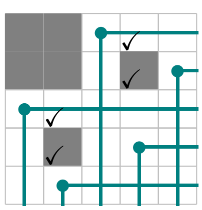
patterns	subdiagrams	
 15342 pattern	 (B)	 (B)
 15432 pattern	 (B)	
 24153 pattern	 (C”)	
 25143 pattern	 (C’)	 (C”)
 35142 pattern	 (C’)	

Table 5. Subdiagrams in the proof of Theorem 1.1 generated by u patterns with $u \in S_5$ (continued).

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