Maximal independent sets in graphs with a given matching number

Yongtang Shi[1], Jianhua Tu[2,*], Ziyuan Wang[2]

[1]Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China
[2]School of Mathematics and Statistics, Beijing Technology and Business University, Beijing 100048, China

Abstract

A maximal independent set in a graph G is an independent set that cannot be extended to a larger independent set by adding any vertex from G. This paper investigates the problem of determining the maximum number of maximal independent sets in terms of the matching number of a graph. We establish the maximum number of maximal independent sets for general graphs, connected graphs, triangle-free graphs, and connected triangle-free graphs with a given matching number, and characterize the extremal graphs achieving these maxima.

Keywords: Counting; Maximal independent sets; Matching number; Extremal combinatorics
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1 Introduction

An *independent set* in a graph is a set of vertices that induces a subgraph without any edges. A *maximal independent set* (MIS, for short) is one that cannot be a proper subset of any larger independent set. For a graph G, define $\text{MIS}(G)$ as the set of all MISs in G, and let $\text{mis}(G) = |\text{MIS}(G)|$.

Let K_n, C_n, P_n, and $K_{1,n-1}$ denote the *complete graph*, the *cycle*, the *path*, and the *star* on n vertices, respectively. Around 1960, Erdős and Moser raised the problem of determining the maximum value of $\text{mis}(G)$ in terms of the order of G, and the extremal graphs. The well-known result due to Miller and Muller [10] and Moon and Moser [11], which answers this problem, shows that for any graph G of order $n \geq 2$,

$$\text{mis}(G) \leq \begin{cases} 3^{n/3}, & \text{if } n \equiv 0 \pmod 3; \\ 4 \cdot 3^{(n-4)/3}, & \text{if } n \equiv 1 \pmod 3; \\ 2 \cdot 3^{(n-2)/3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

For two graphs G and H, we define $G \cup H$ to be their *disjoint union* and write kG for the disjoint union of k copies of G. Then the extremal graphs achieving the maximum value are $\frac{n}{3}K_3$, $\frac{n-4}{3}K_3 \cup K_4$ or $\frac{n-4}{3}K_3 \cup 2K_2$, and $\frac{n-2}{3}K_3 \cup K_2$ in the three respective cases.

Since then, there has been interest in further exploring the maximum number of MISs in specific families of graphs and in identifying the extremal graphs achieving these maxima. Researchers have determined the maximum value of $\text{mis}(G)$ for various graph families of a given order, such as trees, forests, connected graphs, bipartite graphs, unicyclic graphs, and graphs with at most r cycles. For these results, we refer to [3, 4, 6–8, 12–18].

*Corresponding author.
E-mail: tujh81@163.com (J. Tu)

A *matching* of a graph G is a set of edges with no shared endpoints. A *maximum matching* is one that has the largest possible number of edges among all matchings in the graph. The *matching number* of G, denoted by $\mu(G)$, is the number of edges in a maximum matching in G.

We observe that for many graph families, such as general graphs, connected graphs, and triangle-free graphs, the extremal graphs that attain the maximum number of MISs among all graphs of order n always have large matching numbers. Motivated by this observation, we investigate the maximum number of MISs in terms of the matching number. We determine the maximum number of MISs in terms of the matching number for four graph families: general graphs, connected graphs, triangle-free graphs, and connected triangle-free graphs. Specifically, we show that

(1) for a general graph G with matching number t,

$$\text{mis}(G) \leq 3^t,$$

(2) for a connected graph G with matching number t,

$$\text{mis}(G) \leq \begin{cases} 3 & \text{for } t = 1, \\ 3^{t-1} + 2^{t-1} & \text{for } t \geq 2, \end{cases}$$

(3) for a triangle-free graph G with matching number t,

$$\text{mis}(G) \leq \begin{cases} 5^{\frac{t}{2}} & \text{for } t \geq 2 \text{ even}, \\ 2 \cdot 5^{\frac{t-1}{2}} & \text{for } t \geq 1 \text{ odd}, \end{cases}$$

(4) for a connected triangle-free graph G with matching number t,

$$\text{mis}(G) \leq \begin{cases} 5 & \text{for } t = 2, \\ 2 \cdot (5^{\frac{t-2}{2}} + 3^{\frac{t-2}{2}}) & \text{for } t \geq 4 \text{ even}, \\ 5^{\frac{t-1}{2}} + 3^{\frac{t-1}{2}} & \text{for } t \geq 1 \text{ odd}. \end{cases}$$

For the families of graphs under consideration, we also characterize the corresponding extremal graphs that possess the maximum number of MISs.

It should be noted that while Hoang and Trung [5] have, as detailed in their Proposition 2.3, established the maximum number of MISs in general graphs with a given matching number, our approach, distinct from theirs, offers a different perspective.

The structure of the rest of this paper is as follows: In Section 2, we provide preliminary results, including key lemmas essential for proving this paper's main results. In Section 3, we establish the maximum number of MISs in general graphs with a given matching number and identify the extremal graphs that achieve this maximum, as detailed in Theorem 1. In Sections 4, 5, and 6, we successively address connected graphs, triangle-free graphs, and connected triangle-free graphs, with the main results detailed in Theorems 2, 3, and 4, respectively.

2 Preliminaries

Let G be a graph. For a vertex $v \in V(G)$, we define the *open neighborhood* of v as $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and *closed neighborhood* of v as $N_G[v] = N_G(v) \cup \{v\}$.

The *degree* of v, $d_G(v)$, is the cardinality of $N_G(v)$. If $d_G(v) = 1$, v is referred to as a *leaf* of G. A vertex that is not a leaf and is adjacent to a leaf is called a *support vertex*. To simplify notation without causing confusion, we may abbreviate $N_G(v)$, $N_G[v]$, and $d_G(v)$ to $N(v)$, $N[v]$, and $d(v)$ respectively. For a subset of vertices $S \subseteq V(G)$, the *induced subgraph* $G[S]$ consists of the vertices in S and all edges between them in G. The graph $G - S$ is the result of removing all vertices in S from G. When S consists of a single vertex v, we write $G - \{v\}$ as $G - v$ for brevity.

The friendship graph F_n is constructed by joining n copies of the triangle C_3 at a common vertex, making this vertex adjacent to all other vertices. The friendship graph F_2 is shown in Figure 1.

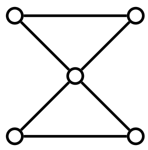

Fig. 1. F_2.

Let M be a matching in a graph G. The vertices incident to an edge of M are said to be *saturated* by M, while the others are *unsaturated*. An *M-alternating path* is a path that alternates between edges that are in M and edges that are not. An *M-augmenting path* is an M-alternating path that starts and ends with unsaturated vertices.

The following theorem is a fundamental theorem in matching theory.

Theorem A. *[1] A matching M in a graph G is a maximum matching if and only if G has no M-augmenting paths.*

The proof of the following lemma is straightforward and thus omitted.

Lemma 1. *Let G be a graph with matching number t, and let v be a vertex in G. Then,*
(1) if v is saturated by all maximum matchings of G, $\mu(G - v) = t - 1$,
(2) otherwise, $\mu(G - v) = t$.

Building on Lemma 1, we obtain the following result, whose proof is also straightforward and thus omitted.

Lemma 2. *Let G be a graph, and let v be a vertex in G that is saturated by all maximum matchings of G. Let G' be the graph obtained from G by adding $k \geq 1$ new vertices and joining these vertices to v. Then, $\mu(G') = \mu(G)$ and $\mu(G' - v) = \mu(G) - 1$.*

A *perfect matching* in a graph is a matching that saturates all vertices of the graph. Let G be a graph. If removing any vertex from G results in a graph with a perfect matching, then G is called *factor-critical*. Define $D(G)$ as the set of vertices that are unsaturated by at least one maximum matching of G.

Theorem B. *(The Gallai-Edmonds Structure Theorem [9]). For a graph G, let $D(G)$ be defined as above. Then each component of $G[D(G)]$ is factor-critical.*

Corollary 1. *If there is no vertex in a graph G that is saturated by all maximum matchings, then each component of G is factor-critical.*

Proof. This corollary can be directly derived from Theorem B. ∎

Definition 1. If $n < 6$, let $c(n) = n$; if $n \geq 6$, let

$$c(n) = \begin{cases} 2 \cdot 3^{\frac{n-3}{3}} + 2^{\frac{n-3}{3}}, & \text{if } n \equiv 0 \pmod 3, \\ 3^{\frac{n-1}{3}} + 2^{\frac{n-4}{3}}, & \text{if } n \equiv 1 \pmod 3, \\ 4 \cdot 3^{\frac{n-5}{3}} + 3 \cdot 2^{\frac{n-8}{3}}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Theorem C. *[4] For any connected graph G of order n,*

$$\text{mis}(G) \leq c(n).$$

Corollary 2. *For any connected graph G of order $n \geq 11$,*

$$\text{mis}(G) \leq 3^{\frac{n-1}{3}} + 2^{\frac{n-4}{3}}.$$

Proof. This corollary can be directly derived from Theorem C. ∎

Below are three lemmas that will be frequently used later when counting the number of MISs in graphs.

Lemma 3. *Let G be a graph. For any induced subgraph H of G,*

$$\text{mis}(H) \leq \text{mis}(G).$$

Proof. For any induced subgraph H of G, every MIS I' of H can be extended in G to form at least one MIS I such that $I \cap V(H) = I'$. This leads to the inequality

$$\text{mis}(H) \leq \text{mis}(G),$$

which completes the proof. ∎

Lemma 4. *Let G be a graph and v be a support vertex of G. Then*

$$\text{mis}(G) = \text{mis}(G - N[v]) + \text{mis}(G - v).$$

Proof. Let U be the set of leaves adjacent to v in G. The MISs of G partition into two types: those containing v, and those containing all vertices in U (but not v). Thus,

$$\text{mis}(G) = \text{mis}(G - N[v]) + \text{mis}(G - v - U) = \text{mis}(G - N[v]) + \text{mis}(G - v),$$

which completes the proof. ∎

The proof of the following lemma is straightforward and thus omitted.

Lemma 5. *Let G be a disconnected graph with $k \geq 2$ components $G_1, G_2, \cdots, G_k$. Then*

$$\text{mis}(G) = \prod_{i=1}^{k} \text{mis}(G_i).$$

The following lemma will be used in the proof of Theorem 4.

Lemma 6. *For any odd integers t_1, t_2 such that $t_1 \geq 3$ and $t_2 \geq 1$, the following inequality holds:*

$$5^{\frac{t_1+t_2}{2}} + 3^{\frac{t_1+t_2}{2}} > 2 \cdot \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right)\left(5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}\right).$$

Proof. Let $t_1 = 2a + 1$ and $t_2 = 2b + 1$ with $a \geq 1$ and $b \geq 0$. The inequality becomes:

$$5^{a+b+1} + 3^{a+b+1} > 2 \cdot (5^a + 3^a)(5^b + 3^b) = 2 \cdot (5^{a+b} + 5^a 3^b + 3^a 5^b + 3^{a+b}).$$

Thus, it suffices to show:

$$3 \cdot 5^{a+b} + 3^{a+b} > 2 \cdot 5^a 3^b + 2 \cdot 3^a 5^b. \tag{1}$$

To prove Inequality (1), we divide both sides by 5^{a+b}. Therefore, we only need to show:

$$3 + (3/5)^{a+b} > 2 \cdot \left((3/5)^a + (3/5)^b\right). \tag{2}$$

Inequality (2) is equivalent to

$$[2 - (3/5)^a] \cdot [2 - (3/5)^b] > 1. \tag{3}$$

Since $2 - (3/5)^a > 1$ and $2 - (3/5)^b \geq 1$ for $a \geq 1$ and $b \geq 0$, Inequality (3) clearly holds. Hence, Inequality (1) holds.

The proof is complete. ∎

3 Maximal independent sets in general graphs with a given matching number

For a graph G, define $\mathrm{I}(G)$ as the set of all *independent sets* in G, and let $i(G) = |\mathrm{I}(G)|$. It is important to note that the empty set is also considered an independent set.

Theorem 1. *Let G be a graph with matching number t. Then*

$$\mathrm{mis}(G) \leq 3^t,$$

with equality if and only if $G \cong tK_3 \cup rK_1$ for some $r \geq 0$.

Proof. First, we show that

$$\mathrm{mis}(G) \leq 3^t.$$

Let M be a maximum matching in G, and let $V(M)$ be the set of vertices of G that are saturated by M. The subgraph induced by $V(M)$ in G is denoted by G_M. Let $W = V(G) \setminus V(M)$. Since M is a maximum matching in G, W is an independent set of G.

It is easy to see that

$$i(G_M) \leq 3^t$$

with equality if and only if no two edges of M are joined by an edge of G (that is, M is an induced matching of G).

We prove that $\mathrm{mis}(G) \leq i(G_M)$ by constructing an injective mapping $f : \mathrm{MIS}(G) \to \mathrm{I}(G_M)$. For any MIS I of G, define $f(I) = I \cap V(M)$.

Let I' be an independent set in G_M that lies in the image of f (i.e., $I' = f(I)$ for some MIS I of G). Then I' can be uniquely extended to an MIS I of G by:

$$I = I' \cup \{v \in W \mid N(v) \cap I' = \emptyset\}.$$

This extension is forced because for any vertex $v \in W$, it must be included in I if and only if it has no neighbors in I', to maintain both independence and maximality. Thus, for any $I_1, I_2 \in \mathrm{MIS}(G)$ with $f(I_1) = f(I_2) = I'$, we must have

$$I_1 = I_2 = I' \cup \{v \in W \mid N(v) \cap I' = \emptyset\}.$$

This proves f is injective, giving

$$\mathrm{mis}(G) \leq i(G_M) \leq 3^t. \tag{4}$$

Moreover, from the preceding analysis, we have: If I' is an independent set in G_M, then there exists at most one MIS I in G such that $I \cap V(M) = I'$.

Next, we prove that if G is connected and $t \geq 2$, then $\mathrm{mis}(G) < 3^t$. We consider two cases:

Case 1. $i(G_M) < 3^t$ or there is no MIS I in G such that $I \cap V(M) = \emptyset$.

If $i(G_M) < 3^t$, then by Inequality (4) we have

$$\mathrm{mis}(G) \leq i(G_M) < 3^t.$$

If there is no MIS I in G with $I \cap V(M) = \emptyset$, then from the definition and injectivity of f as well as Inequality (4), it follows that

$$\mathrm{mis}(G) < i(G_M) \leq 3^t.$$

In either subcase, we conclude that $\mathrm{mis}(G) < 3^t$.

Case 2. $i(G_M) = 3^t$ and there exists an MIS I in G such that $I \cap V(M) = \emptyset$.

In this case, the MIS I is actually the set W, and moreover, M is an induced matching of G. Let

$$M := \{u_1v_1, u_2v_2, \cdots, u_tv_t\},\ W := \{w_1, \cdots, w_k\}.$$

We will prove that for each edge u_iv_i (where $1 \leq i \leq t$) of M, there exists a unique vertex w in W such that w is adjacent to both u_i and v_i, and w is the only neighbor of u_i in W and the only neighbor of v_i in W.

For example, consider the edge u_1v_1. Since G is connected, without loss of generality, assume that u_1 is adjacent to at least one vertex in W, say w_1. That is, assume $u_1w_1 \in E(G)$. Firstly, v_1 is adjacent to at least one vertex in W, otherwise $W \cup \{v_1\}$ would also be an independent set in G, contradicting the maximality of $I = W$. Secondly, we claim that w_1 is the unique vertex in W adjacent to v_1. To see this, suppose a different vertex in W were adjacent to v_1. This would imply the existence of an M-augmenting path, a contradiction of Theorem A. Finally, apart from w_1, neither u_1 nor v_1 has any other adjacent vertex in W, as otherwise an M-augmenting path would exist in G, contradicting Theorem A.

Since G is connected, W consists of a single vertex, implying $G \cong F_t$. Direct calculation gives $\mathrm{mis}(G) = 2^t + 1$. For $t \geq 2$, we have

$$\mathrm{mis}(G) = 2^t + 1 < 3^t.$$

Thus, we have shown that if G is connected and $t \geq 2$, then $\mathrm{mis}(G) < 3^t$.

Finally, we show that $\mathrm{mis}(G) = 3^t$ if and only if $G \cong tK_3 \cup rK_1$ for some $r \geq 0$. Assume that G has a total of k non-trivial components $G_1, \cdots, G_k$ with $\mu(G_1) = t_1, \cdots, \mu(G_k) = t_k$, where a non-trivial component is a component with at least two vertices. Clearly, $t_1 + t_2 + \cdots + t_k = t$. Then, by Lemma 5,

$$\mathrm{mis}(G) = \prod_{i=1}^{k} \mathrm{mis}(G_i) \leq \prod_{i=1}^{k} 3^{t_i} = 3^t.$$

Furthermore, if $\mathrm{mis}(G) = 3^t$, then $t_1 = t_2 = \cdots = t_k = 1$, which implies that any non-trivial component of G is a star or a K_3. For a star S, $\mathrm{mis}(S) = 2$. Thus, $\mathrm{mis}(G) = 3^t$ if and only if $G \cong tK_3 \cup rK_1$ for some $r \geq 0$.

∎

4 Maximal independent sets in connected graphs with a given matching number

For any integer $t \geq 2$, let $\mathcal{E}_t$ denote the set of connected graphs with matching number t, which are constructed by connecting a central vertex to exactly one vertex in each of $t-1$ disjoint triangles, and additionally connecting this central vertex to ℓ isolated vertices, where $\ell \geq 1$. An example of a graph in $\mathcal{E}_t$ is illustrated in Figure 2.

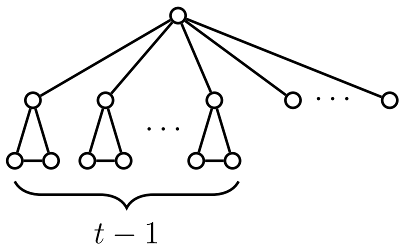

Fig. 2. An example of a graph in $\mathcal{E}_t$.

Let

$$\mathcal{H}_t := \begin{cases} \{K_3\}, & \text{if } t = 1, \\ \{F_2, C_5, K_5\} \cup \mathcal{E}_2, & \text{if } t = 2, \\ \mathcal{E}_t, & \text{if } t \geq 3. \end{cases}$$

Theorem 2. *For any connected graph G with matching number t,*

$$\mathrm{mis}(G) \leq h(t) := \begin{cases} 3 & \text{for } t = 1, \\ 3^{t-1} + 2^{t-1} & \text{for } t \geq 2, \end{cases}$$

with equality if and only if G is isomorphic to a graph in $\mathcal{H}_t$.

Proof. Let $g(t)$ be the maximum number of MISs among all connected graphs with matching number t.

Direct calculations show that for any $t \geq 1$ and any $G \in \mathcal{H}_t$,

$$\mathrm{mis}(G) = h(t),$$

which implies that $g(t) \geq h(t)$.

We prove by induction on t that $g(t) = h(t)$, and that any connected graph with matching number t and $h(t)$ MISs is isomorphic to a graph in $\mathcal{H}_t$.

For $t = 1$, a connected graph with matching number 1 is either a triangle or a star. A triangle has 3 MISs, while a star has 2 MISs. Therefore, $g(1) = 3 = h(1)$, and any connected graph with matching number 1 and $h(1) = 3$ MISs is isomorphic to K_3.

Suppose that $t \geq 2$, and for any positive integer $t' < t$, $g(t') = h(t')$, and any connected graph with matching number t' and $h(t')$ MISs is isomorphic to a graph in $\mathcal{H}_{t'}$.

Let L_t be a connected graph with $3(t-1)+1$ vertices and matching number t constructed as follows: a central vertex is connected to one vertex of each of $t-1$ disjoint triangles. One can easily verify that

$$\text{mis}(L_t) = 3^{t-1} < 3^{t-1} + 2^{t-1} = h(t).$$

Let G be a connected graph with matching number t and $g(t)$ MISs. We will show that $g(t) = h(t)$ and that G is isomorphic to a graph in $\mathcal{H}_t$. To establish this, we proceed by analyzing the cases $t = 2$ and $t \geq 3$ in turn.

Case 1. $t = 2$.

In this case, we have $\text{mis}(G) = g(2) \geq h(2) = 5$.

First, suppose no vertex in G is saturated by every maximum matching. Then, by Corollary 1, G is factor-critical, which implies $|V(G)| = 2t + 1 = 5$. By Theorem C, $\text{mis}(G) \leq c(5) = 5 = h(2)$. A direct verification shows that a factor-critical graph on 5 vertices with exactly 5 MISs must be isomorphic to one of $\{F_2, K_5, C_5\}$. Consequently, $\text{mis}(G) = g(2) = h(2)$ and G is isomorphic to a graph in $\{F_2, K_5, C_5\}$.

Next, suppose there exists a vertex v in G that is saturated by all maximum matchings. We construct a new graph G' from G by adding a new vertex and connecting it to v. By Lemmas 2 and 3, we have $\mu(G') = 2$, $\mu(G' - v) = 1$, and $\text{mis}(G') \geq \text{mis}(G) = g(2)$. This implies $\text{mis}(G') = g(2) = 5$.

Now, by Theorem 1, we have $\text{mis}(G' - v) \leq 3$. Furthermore, if $\text{mis}(G' - v) = 3$, then $G' - v \cong K_3 \cup sK_1$ for some $s \geq 1$. Since G' is connected, it follows that $G' - N_{G'}[v]$ has at most one nontrivial component, and that component has at most 2 vertices. Hence, $\text{mis}(G' - N_{G'}[v]) \leq 2$. Applying Lemma 4, we obtain:

$$g(2) = \text{mis}(G') = \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]) \leq 3 + 2 = h(2).$$

Therefore, equality holds: $g(2) = h(2) = 5$. Moreover, v must be adjacent to exactly one vertex of the triangle in $G' - v$, which implies that G' is isomorphic to a graph in $\mathcal{E}_2$. Since $\text{mis}(L_2) < h(2)$ and $\text{mis}(G) = h(2)$, it follows that G itself is isomorphic to a graph in $\mathcal{E}_2$.

Therefore, for $t = 2$, $g(2) = h(2)$ and G is isomorphic to a graph in $\mathcal{H}_2 = \{F_2, K_5, C_5\} \cup \mathcal{E}_2$.

Case 2. $t \geq 3$.

In this case, we begin by establishing the following claim.

Claim 2.1. There exists a vertex v in G that is saturated by all maximum matchings.

Proof of Claim 2.1. Suppose, for contradiction, that no vertex in G is saturated by all maximum matchings. Then, by Corollary 1 and the fact that G is connected, G is factor-critical. So $|V(G)| = 2t + 1$.

If $t = 3$, Theorem C gives

$$g(3) = \text{mis}(G) \leq c(7) = 11 < 3^2 + 2^2 = h(3),$$

contradicting $g(3) \geq h(3)$.

If $t = 4$, Theorem C gives

$$g(3) = \text{mis}(G) \leq c(9) = 22 < 3^3 + 2^3 = h(4),$$

contradicting $g(4) \geq h(4)$.

If $t \geq 5$, Corollary 2 implies

$$g(t) = \text{mis}(G) \leq 3^{\frac{2t}{3}} + 2^{\frac{2t-3}{3}} < 3^{t-1} + 2^{t-1} = h(t),$$

contradicting $g(t) \geq h(t)$.

This completes the proof of Claim 2.1. ∎

By Claim 2.1, there exists a vertex v in G saturated by all maximum matchings. We construct G' from G by adding a new vertex and connecting it to v. By Lemmas 2 and 3, we have $\mu(G') = t$, $\mu(G' - v) = t - 1$, and $\text{mis}(G') \geq \text{mis}(G) = g(t)$. This implies $\text{mis}(G') = g(t)$.

Let H be a component of $G' - v$ that has maximum matching number among all components. Define

$$\alpha := \text{mis}(H) \text{ and } \beta := \text{mis}(H - (N_{G'}(v) \cap V(H))).$$

Let

$$G_1 := G' - v - V(H) \text{ and } G_2 := G' - [N_{G'}[v] \cup V(H)].$$

Clearly, $\mu(H) \geq 1$ and $|V(G_1)| \geq 1$.

Claim 2.2. $\mu(H) = 1$.

Proof of Claim 2.2. Suppose, for contradiction, that $\mu(H) \geq 2$. Since $\mu(H) \leq \mu(G' - v) = t - 1$, the induction hypothesis and Lemma 3 imply

$$\beta \leq \alpha \leq 3^{\mu(H)-1} + 2^{\mu(H)-1}. \tag{5}$$

Construct a new graph G'' from G' by removing all vertices of H, adding $\mu(H)$ disjoint copies of K_3, and connecting v to exactly one vertex in each new copy. It is easy to see that G'' is a connected graph with matching number t. Moreover, by Lemmas 4 and 5,

$$\operatorname{mis}(G'') = \operatorname{mis}(G'' - v) + \operatorname{mis}(G'' - N_{G''}[v]) = 3^{\mu(H)} \cdot \operatorname{mis}(G_1) + 2^{\mu(H)} \cdot \operatorname{mis}(G_2). \quad (6)$$

Using Lemmas 4 and 5 and inequality (5), we obtain:

$$\begin{aligned}
\operatorname{mis}(G') &= \operatorname{mis}(G' - v) + \operatorname{mis}(G' - N_{G'}[v]) \\
&= \alpha \cdot \operatorname{mis}(G_1) + \beta \cdot \operatorname{mis}(G_2) \\
&\leq \left(3^{\mu(H)-1} + 2^{\mu(H)-1}\right) \cdot \operatorname{mis}(G_1) + \left(3^{\mu(H)-1} + 2^{\mu(H)-1}\right) \cdot \operatorname{mis}(G_2) \\
&= 3^{\mu(H)} \cdot \operatorname{mis}(G_1) + 2^{\mu(H)} \cdot \operatorname{mis}(G_2) \\
&\quad - \left[(2 \cdot 3^{\mu(H)-1} - 2^{\mu(H)-1}) \cdot \operatorname{mis}(G_1) - (3^{\mu(H)-1} - 2^{\mu(H)-1}) \cdot \operatorname{mis}(G_2)\right]. \quad (7)
\end{aligned}$$

Since $\mu(H) \geq 2$, we have $2 \cdot 3^{\mu(H)-1} - 2^{\mu(H)-1} > 3^{\mu(H)-1} - 2^{\mu(H)-1} \geq 1$. As $|V(G_1)| \geq 1$ implies $\operatorname{mis}(G_1) \geq 1$, and Lemma 3 gives $\operatorname{mis}(G_1) \geq \operatorname{mis}(G_2)$, it follows that

$$(2 \cdot 3^{\mu(H)-1} - 2^{\mu(H)-1}) \cdot \operatorname{mis}(G_1) - (3^{\mu(H)-1} - 2^{\mu(H)-1}) \cdot \operatorname{mis}(G_2) > 0.$$

Therefore, from (6) and (7),

$$g(t) = \operatorname{mis}(G') < 3^{\mu(H)} \cdot \operatorname{mis}(G_1) + 2^{\mu(H)} \cdot \operatorname{mis}(G_2) = \operatorname{mis}(G''),$$

which contradicts the assumption that $g(t)$ is the maximum number of MISs in connected graphs with matching number t. This completes the proof of Claim 2.2. ∎

By Claim 2.2, $\mu(H) = 1$. Since $\mu(H) = 1$ is the maximum matching number among all components of $G' - v$, each component of $G' - v$ is either a triangle, a star, or an isolated vertex. Thus, by Lemmas 4 and 5,

$$\begin{aligned}
\operatorname{mis}(G') &= \operatorname{mis}(G' - v) + \operatorname{mis}(G' - N_{G'}[v]) \\
&= \alpha \cdot \operatorname{mis}(G_1) + \beta \cdot \operatorname{mis}(G_2) \\
&\leq 3 \cdot \operatorname{mis}(G_1) + 2 \cdot \operatorname{mis}(G_2).
\end{aligned}$$

Furthermore, the last inequality is an equality if and only if H is a triangle and has exactly one vertex adjacent to v. Since each component with matching number 1 in $G' - v$ exhibits symmetry, $\operatorname{mis}(G')$ achieves its maximum value only when each component with matching number 1 is a triangle and has exactly one vertex adjacent to v. In other words, $\operatorname{mis}(G')$ is maximized only when G' is isomorphic to a graph in $\mathcal{E}_t$. Hence,

$$g(t) = \operatorname{mis}(G') \leq h(t).$$

Therefore, $g(t) = h(t)$, and G' is isomorphic to a graph in $\mathcal{E}_t$.

Since $\operatorname{mis}(L_t) < h(t)$ and $\operatorname{mis}(G) = h(t)$, we can infer that G' being isomorphic to a graph in $\mathcal{E}_t$ implies that G itself is also isomorphic to a graph in $\mathcal{E}_t$.

Therefore, for $t \geq 3$, $g(t) = h(t)$ and G is isomorphic to a graph in $\mathcal{E}_t$.

The proof of Theorem 2 is complete.

∎

5 Maximal independent sets in triangle-free graphs with a given matching number

When $n \geq 4$, let

$$A_n := \begin{cases} \frac{n}{2}K_2 & \text{for } n \text{ even,} \\ C_5 \cup \frac{n-5}{2}K_2 & \text{for } n \text{ odd.} \end{cases}$$

Theorem D. *[6] For any triangle-free graph G of order $n \geq 4$,*

$$\text{mis}(G) \leq \begin{cases} 2^{n/2} & \text{for } n \text{ even,} \\ 5 \cdot 2^{(n-5)/2} & \text{for } n \text{ odd,} \end{cases}$$

with equality if and only if $G \cong A_n$.

Let

$$\mathcal{M}_t := \begin{cases} \{\frac{t}{2}C_5 \cup rK_1 : r \geq 0\} & \text{for } t \geq 2 \text{ even,} \\ \{K_{1,\ell} \cup \frac{t-1}{2}C_5 \cup sK_1 : \ \ell \geq 1 \text{ and } s \geq 0\} & \text{for } t \geq 1 \text{ odd.} \end{cases}$$

Theorem 3. *Let G be a triangle-free graph with matching number t. Then*

$$\text{mis}(G) \leq m(t) := \begin{cases} 5^{t/2} & \text{for } t \geq 2 \text{ even,} \\ 2 \cdot 5^{(t-1)/2} & \text{for } t \geq 1 \text{ odd,} \end{cases}$$

with equality if and only if G is isomorphic to a graph in $\mathcal{M}_t$.

Remark 1. The definition of the function $m(t)$, together with the inequality

$$5^{t/2} > 2 \cdot 5^{(t-1)/2} \quad \text{for all integers } t \geq 1,$$

implies

$$2 \cdot 5^{(t-1)/2} \leq m(t) \leq 5^{t/2} \quad \text{for all integers } t \geq 1.$$

Moreover, it can be verified that for any integers $t_1 \geq 1$ and $t_2 \geq 1$,

$$m(t_1) \cdot m(t_2) \leq m(t_1 + t_2).$$

Proof. Prove by induction on t.

When $t = 1$, G has only one non-trivial component, and this component can only be a star. Thus

$$\text{mis}(G) = 2 = m(1),$$

and G is isomorphic to a graph in $\{K_{1,\ell} \cup sK_1 : \ \ell \geq 1 \text{ and } s \geq 0\}$.

Consider the case when $t = 2$. If there is a vertex v in G that is saturated by all maximum matchings, we construct a new graph G' from G by adding a new vertex and joining the new vertex to v. By Lemmas 2, 3, and 4, along with the inductive hypothesis, we have $\mu(G' - v) = 1$, $\mu(G' - N_{G'}[v]) \leq 1$, and

$$\mathrm{mis}(G) \leq \mathrm{mis}(G') = \mathrm{mis}(G' - v) + \mathrm{mis}(G' - N_{G'}[v]) \leq 2 + 2 < 5 = m(2).$$

If there is no vertex in G that is saturated by all maximum matchings, then by Corollary 1, each component of G is factor-critical. Since G is triangle-free and $t = 2$, G has only one non-trivial component, and this component is isomorphic to C_5.

In conclusion, when $t = 2$, $\mathrm{mis}(G) \leq m(2)$, with equality if and only if G is isomorphic to a graph in $\{C_5 \cup rK_1 : r \geq 0\}$.

Assume that $t \geq 3$, and for any integer $t' < t$, if H is a triangle-free graph with matching number t', then $\mathrm{mis}(H) \leq m(t')$, with equality if and only if H is isomorphic to a graph in $\mathcal{M}_{t'}$.

Now, let $G_1, \cdots, G_k$ be non-trivial components of G, and let $\mu(G_1) = t_1, \cdots, \mu(G_k) = t_k$. Clearly, $t_1 + \cdots + t_k = t$.

We will prove that if $k = 1$, then

$$\mathrm{mis}(G) = \mathrm{mis}(G_1) < m(t).$$

If there is a vertex v in G_1 that is saturated by all maximum matchings, we construct a new graph G'_1 from G_1 by adding a new vertex and joining the new vertex to v. By Lemma 2, $\mu(G'_1 - v) = t - 1$ and $\mu(G'_1 - N_{G'_1}[v]) \leq t - 1$.

If t is even, then by Lemmas 3 and 4, together with the inductive hypothesis,

$$\begin{aligned}
\mathrm{mis}(G_1) &\leq \mathrm{mis}(G'_1) \\
&= \mathrm{mis}(G'_1 - v) + \mathrm{mis}(G'_1 - N_{G'_1}[v]) \\
&\leq 2 \cdot \mathrm{mis}(G'_1 - v) \\
&\leq 2 \cdot 2 \cdot 5^{(t-2)/2} \\
&< 5^{t/2} \\
&= m(t).
\end{aligned}$$

If t is odd, then by Lemmas 3 and 4, together with the inductive hypothesis,

$$\begin{aligned}
\mathrm{mis}(G_1) &\leq \mathrm{mis}(G'_1) \\
&= \mathrm{mis}(G'_1 - v) + \mathrm{mis}(G'_1 - N_{G'_1}[v]) \\
&\leq 2 \cdot \mathrm{mis}(G'_1 - v) \\
&\leq 2 \cdot 5^{(t-1)/2} \\
&= m(t).
\end{aligned}$$

Moreover, if $\text{mis}(G_1) = m(t)$, then by the inductive hypothesis, $G_1' - v \cong \frac{t-1}{2}C_5 \cup r_1K_1$ for some $r_1 \geq 1$ and $G_1' - N_{G_1'}[v] \cong \frac{t-1}{2}C_5 \cup r_2K_1$ for some $r_2 \geq 0$. This forces $N_{G_1'}(v)$ to be contained in the set of the r_1 isolated vertices of $G_1' - v$. Consequently, G_1' must have at least two components: one containing v and another being a C_5, which contradicts the fact that G_1' is connected. Thus,

$$\text{mis}(G_1) < m(t).$$

If there is no vertex in G_1 that is saturated by all maximum matchings, then by Corollary 1, G_1 is factor-critical, which implies that $|V(G_1)| = 2t+1 \geq 7$. It follows from Theorem D and the fact that G_1 is a connected graph, which is clearly not isomorphic to A_{2t+1}, that

$$\text{mis}(G_1) < 5 \cdot 2^{t-2} \leq 2 \cdot 5^{(t-1)/2} \leq m(t).$$

We have proved that if $k = 1$, then $\text{mis}(G) = \text{mis}(G_1) < m(t)$. We have actually also proven that for any connected triangle-free graph F with matching number $t \geq 3$, $\text{mis}(F) < m(t)$.

Now, suppose that $k \geq 2$. If there are at least two components of G with matching number 1, then by Lemma 5, Remark 1, and the inductive hypothesis,

$$\text{mis}(G) \leq 2 \cdot 2 \cdot m(t-2) \leq 2 \cdot 2 \cdot 5^{(t-2)/2} < 2 \cdot 5^{(t-1)/2} \leq m(t).$$

From Lemma 5, Remark 1, and the inductive hypothesis, we have

$$\text{mis}(G) = \prod_{i=1}^{k} \text{mis}(G_i) \leq \prod_{i=1}^{k} m(t_i) \leq m(t).$$

Moreover, if $\text{mis}(G) = m(t)$, then for each $i \in 1, \ldots, k$, we have $t_i \leq 2$, and at most one non-trivial component of G is isomorphic to a star, while the remaining non-trivial components are isomorphic to C_5. Therefore, if $\text{mis}(G) = m(t)$, then G is isomorphic to a graph in $\mathcal{M}_t$.

The proof of Theorem 3 is complete.

∎

6 Maximal independent sets in connected triangle-free graphs with a given matching number

When n is odd and $n \geq 7$, let

$$\mathcal{D}_n := \{H_7 \cup \frac{n-7}{2}K_2\} \cup \{T_{2r+1} \cup \frac{n-2r-1}{2}K_2 \mid 0 \leq r \leq \frac{n-1}{2}\},$$

where H_7 and T_{2r+1} are shown in Figure 3.

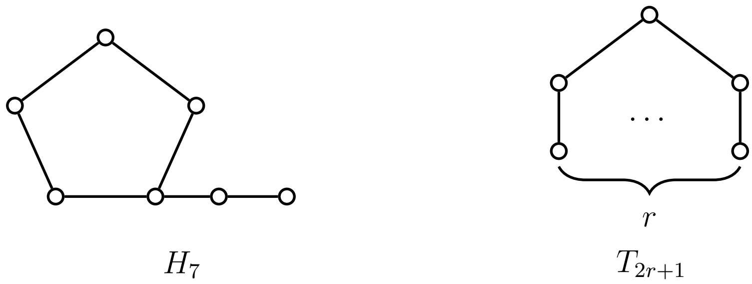

Fig. 3. Graphs H_7 and T_{2r+1}.

Theorem E. [2] *Let G be a triangle-free graph G of order $n \geq 7$. If n is odd and $G \ncong A_n$, then*

$$\mathrm{mis}(G) \leq q(n) := 2^{\frac{n-1}{2}},$$

with equality if and only if G is isomorphic to a graph in $\mathcal{D}_n$.

For a given positive integer t, we introduce a set of graphs $\mathcal{G}_t$. When t is even, the graphs in $\mathcal{G}_t$ can be constructed through the following steps:

(1) Construct three nonempty sets of vertices L_1, L_2, and L_3 with $|L_1| = \ell_1$, $|L_2| = \ell_2$, and $|L_3| = \ell_3$.

(2) Add two vertices u and v, and connect each vertex in L_1 to u, each vertex in L_2 to v, and each vertex in L_3 to both u and v.

(The class of graphs obtained at this step is referred to as $\mathcal{P}$.)

(3) Add $\frac{t-2}{2}$ vertex-disjoint C_5's, and connect exactly one vertex of each C_5 to u.

When t is odd, the graphs in $\mathcal{G}_t$ can be constructed through the following steps:

(1) Form a star $K_{1,r}$, where $r \geq 1$, and denote the central vertex of the star as w.

(2) Add $\frac{t-1}{2}$ vertex-disjoint C_5's, and connect exactly one vertex of each C_5 to w.

Two examples of graphs in $\mathcal{G}_t$ are shown in Figure 4.

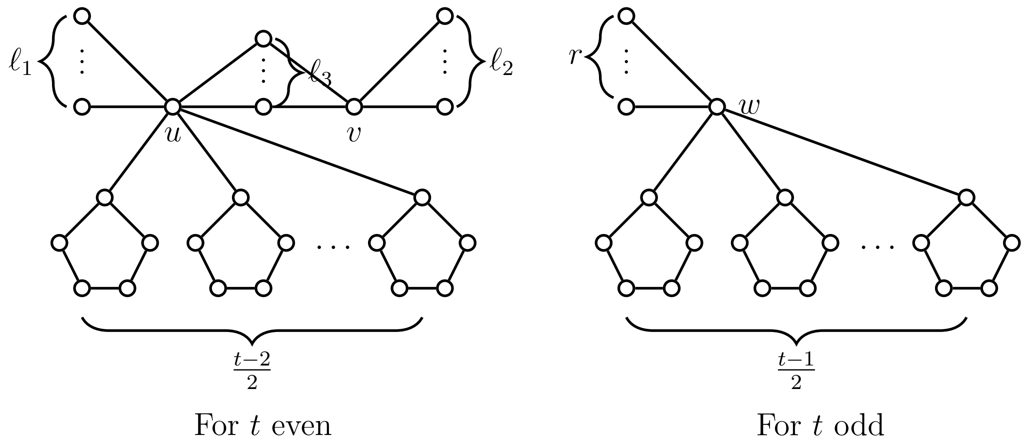

Fig. 4. Two examples of graphs in $\mathcal{G}_t$.

Next, we introduce a set of graphs $\mathcal{Q}_3$. The graphs in $\mathcal{Q}_3$ are the connected graphs which can be constructed through the following steps:

(1) Choose a graph H from $\mathcal{P} \cup \{K_{1,r_1} \cup K_{1,r_2} :\ r_1, r_2 \geq 2\}$.

(2) Construct a star $K_{1,s}$, where $s \geq 1$, and denote the central vertex of the star as w.

(3) Connect w to some neighbors of the support vertices in H, ensuring that these support vertices are still support vertices in the new graph.

Two examples of graphs in $\mathcal{Q}_3$ are shown in Figure 5.

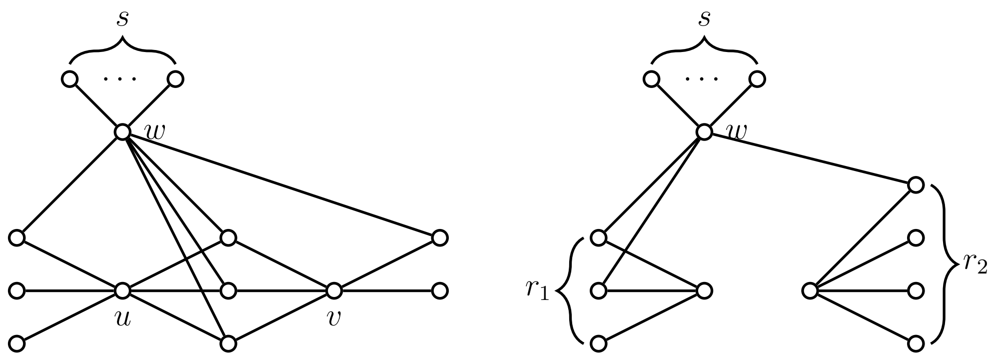

Fig. 5. Two examples of graphs in $\mathcal{Q}_3$.

Finally, we introduce a set of graphs $\mathcal{Q}_4$. The graphs in $\mathcal{Q}_4$ are the connected graphs which can be constructed through the following steps:

(1) Choose a graph H from

$$\mathcal{Q}_3 \cup \{F \cup K_{1,r} :\ F \in \mathcal{P},\ r \geq 2\} \cup \{K_{1,r_1} \cup K_{1,r_2} \cup K_{1,r_3} :\ r_1, r_2, r_3 \geq 2\}.$$

(2) Construct a star $K_{1,q}$, where $q \geq 1$, and denote the central vertex of the star as w,

(3) Connect w to some neighbors of the support vertices in H, ensuring that these support vertices are still support vertices in the new graph.

Let

$$\mathcal{F}_t := \begin{cases} \{K_{1,r} : r \geq 1\} & \text{for } t = 1, \\ \{C_5\} & \text{for } t = 2, \\ \mathcal{G}_3 \cup \mathcal{Q}_3 & \text{for } t = 3, \\ \mathcal{G}_4 \cup \mathcal{Q}_4 & \text{for } t = 4, \\ \mathcal{G}_t & \text{for } t \geq 5. \end{cases}$$

Theorem 4. *For any connected triangle-free graph G with matching number t,*

$$\operatorname{mis}(G) \leq f(t) := \begin{cases} 5 & \text{for } t = 2, \\ 2 \cdot (5^{\frac{t-2}{2}} + 3^{\frac{t-2}{2}}) & \text{for } t \geq 4 \text{ even}, \\ 5^{\frac{t-1}{2}} + 3^{\frac{t-1}{2}} & \text{for } t \geq 1 \text{ odd}, \end{cases}$$

with equality if and only if G is isomorphic to a graph in $\mathcal{F}_t$.

Proof. Let $\phi(t)$ be the maximum number of MISs among all connected triangle-free graphs with matching number t.

Direct calculations show that for any $t \geq 1$ and any $G \in \mathcal{F}_t$,

$$\text{mis}(G) = f(t),$$

which implies $\phi(t) \geq f(t)$.

We prove by induction on t that $\phi(t) = f(t)$, and that any connected triangle-free graph with matching number t and $f(t)$ MISs is isomorphic to a graph in $\mathcal{F}_t$.

For $t = 1$, it is easy to see that any connected triangle-free graph with matching number 1 is a star and has $\phi(1) = f(1) = 2$ MISs.

Suppose that $t \geq 2$, and for any positive integer $t' < t$, $\phi(t') = f(t')$, and any connected triangle-free graph with matching number t' and $f(t')$ MISs is isomorphic to a graph in $\mathcal{F}_{t'}$.

Let G be a connected triangle-free graph with matching number t and $\phi(t)$ MISs. We will show that $\phi(t) = f(t)$ and that G is isomorphic to a graph in $\mathcal{F}_t$. To establish this, we proceed by analyzing the cases $t = 2$, $t = 3$, $t = 4$, and $t \geq 5$ in turn.

Case 1. $t = 2$.

In this case, $\text{mis}(G) = \phi(2) \geq f(2) = 5$.

We first show that in this case, no vertex in G is saturated by every maximum matching. Suppose, for contradiction, that there exists a vertex v in G that is saturated by all maximum matchings. We construct a new graph G' from G by adding a new vertex and connecting it to v. By Lemmas 2 and 3, we have $\mu(G') = 2$, $\mu(G' - v) = 1$, and $\text{mis}(G') \geq \text{mis}(G) = \phi(2)$. This implies that $\text{mis}(G') = \phi(2)$. Moreover, by Lemma 4,

$$\text{mis}(G') = \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]).$$

From Lemma 3 and Theorem 3, it follows that $\text{mis}(G' - N_{G'}[v]) \leq \text{mis}(G' - v) \leq m(1) = 2$, which implies that

$$\phi(2) = \text{mis}(G') \leq 4 < 5 = f(2).$$

This contradicts the fact that $\phi(2) \geq f(2)$.

Therefore, no vertex in G is saturated by all maximum matchings. By Corollary 1 and the connectivity of G, it follows that G is factor-critical, which implies $|V(G)| = 2t+1 = 5$. One can verify that $G \cong C_5$ and

$$\phi(2) = \text{mis}(C_5) = 5 = f(2).$$

Therefore, for $t = 2$, $\phi(2) = f(2)$ and G is isomorphic to C_5, that is, to the graph in $\mathcal{F}_2$.

We present the following three claims that will be required in the subsequent proof.

Claim A. If H is a connected triangle-free graph with matching number 2 and $\mathrm{mis}(H) = 4$, then H is isomorphic to a graph in $\mathcal{P}$.

Proof of Claim A. First, H cannot be factor-critical; otherwise, the earlier analysis would imply $\mathrm{mis}(H) = 5$. Thus, there exists a vertex v in H that is saturated by all maximum matchings. Construct a graph H' from H by adding a new vertex and connecting it to v. By Lemmas 2 and 3, we have $\mu(H') = 2$, $\mu(H' - v) = 1$, and $\mathrm{mis}(H') \geq \mathrm{mis}(H) = 4$. Moreover, by Lemma 3 and Theorem 3,

$$\mathrm{mis}(H') = \mathrm{mis}(H' - v) + \mathrm{mis}(H' - N_{H'}[v]) \leq 2 \cdot \mathrm{mis}(H' - v) \leq 2 \cdot m(1) = 4,$$

which forces

$$\mathrm{mis}(H') = 4 \text{ and } \mathrm{mis}(H' - N_{H'}[v]) = \mathrm{mis}(H' - v) = m(1) = 2.$$

By Theorem 3, it follows that $H' - v \cong K_{1,\ell_1} \cup s_1K_1$ for some $\ell_1 \geq 1$, $s_1 \geq 1$. Since H' is connected and $\mathrm{mis}(H' - N_{H'}[v]) = 2$, we have $\ell_1 \geq 2$ and $H' - N_{H'}[v] \cong K_{1,\ell_2}$ for some $1 \leq \ell_2 < \ell_1$. Thus, H' is isomorphic to a graph in $\mathcal{P}$.

Finally, since $\mathrm{mis}(H) = \mathrm{mis}(H') = 4$, a direct calculation shows that H is also isomorphic to a graph in $\mathcal{P}$. ∎

Claim B. Let H be a connected triangle-free graph with matching number 2, and let L be a proper induced subgraph of H, meaning L has at least one fewer vertex than H. Define $\alpha := \mathrm{mis}(H)$ and $\beta := \mathrm{mis}(L)$. The best possible pairs for (α, β) are $(5, 3)$ and $(4, 4)$. A pair (α, β) is considered "best" if it is impossible to increase one of the values without decreasing the other. More specifically, in our context, "best" means that for any other pair (α, β), we must have $\alpha \leq 5$, and when $\alpha = 5$, $\beta < 3$; when $\alpha = 4$, $\beta < 4$. In other words, all other pairs satisfy $\alpha + \beta < 8$.

Proof of Claim B. By the earlier analysis and Claim A, if H is factor-critical, then $H \cong C_5$. In this case, $\alpha = 5$ and $\beta \leq 3$. Moreover, $\beta = 3$ if and only if $L \cong P_4$. Otherwise, $\beta \leq \alpha \leq 4$. Furthermore, $\beta = \alpha = 4$ if and only if H is isomorphic to a graph in $\mathcal{P}$ and L is isomorphic to a graph in $\mathcal{P} \cup \{K_{1,r_1} \cup K_{1,r_2} : \ r_1, r_2 \geq 1\}$. ∎

Claim C. If $t \geq 3$, then there exists a vertex v in G that is saturated by all maximum matchings.

Proof of Claim C. Suppose, for contradiction, that no vertex in G is saturated by every maximum matching. By Corollary 1 and the connectivity of G, it follows that G is factor-critical. Hence, $|V(G)| = 2t + 1 \geq 7$. Since G is connected and has at least 7 vertices, $G \ncong A_{2t+1}$. Moreover, as $\mathcal{D}_{2t+1}$ contains no factor-critical graphs, G cannot be isomorphic to any graph in $\mathcal{D}_{2t+1}$. By Theorem E,

$$\phi(t) = \mathrm{mis}(G) < q(2t + 1) = 2^t \leq f(t),$$

contradicting the fact that $\phi(t) \geq f(t)$. ∎

For $t \geq 3$, we introduce a family of graphs denoted by $\mathcal{L}_t$. When t is odd, the graphs in $\mathcal{L}_t$ are obtained from those in $\mathcal{G}_t$ by deleting all leaves. When t is even, the graphs in $\mathcal{L}_t$ are obtained from those in $\mathcal{G}_t$ by deleting all leaves adjacent to one of the support vertices. Direct calculations show that for any $t \geq 3$ and any $L \in \mathcal{L}_t$,

$$\mathrm{mis}(L) = \begin{cases} 5^{(t-1)/2} & \text{for odd } t \geq 3, \\ 2 \cdot 5^{(t-2)/2} + 3^{(t-2)/2} & \text{for even } t \geq 4. \end{cases}$$

Therefore, $\mathrm{mis}(L) < f(t)$ holds for all $t \geq 3$ and any $L \in \mathcal{L}_t$.

By Claim C, if $t \geq 3$, then there exists a vertex v in G that is saturated by all maximum matchings. We construct a graph G' from G by adding a new vertex and joining it to v. By Lemmas 2 and 3, we have $\mu(G') = t$, $\mu(G' - v) = t - 1$, and $\mathrm{mis}(G') \geq \mathrm{mis}(G) = \phi(t)$. This implies that $\mathrm{mis}(G') = \phi(t)$.

Case 2. $t = 3$.

In this case, $\mathrm{mis}(G') = \phi(3) \geq f(3) = 8$. By Lemma 4,

$$\mathrm{mis}(G') = \mathrm{mis}(G' - v) + \mathrm{mis}(G' - N_{G'}[v]).$$

Moreover, as $\mu(G' - v) = 2$, it follows from Lemma 3 and Theorem 3 that

$$\mathrm{mis}(G' - N_{G'}[v]) \leq \mathrm{mis}(G' - v) \leq m(2) = 5.$$

If $\mathrm{mis}(G' - v) = 5$, then by Theorem 3, we have $G' - v \cong C_5 \cup rK_1$ for some $r \geq 1$. Since G' is connected, it follows that $\mathrm{mis}(G' - N_{G'}[v]) \leq 3$, with equality if and only if $G' - N_{G'}[v] \cong P_4$. This implies that if $\mathrm{mis}(G' - v) = 5$, then $\phi(3) = \mathrm{mis}(G') = f(3) = 8$, and G' is isomorphic to a graph in $\mathcal{G}_3$.

If $\mathrm{mis}(G' - v) = 4$, then $\mathrm{mis}(G' - N_{G'}[v]) \leq \mathrm{mis}(G' - v) = 4$. Hence, $\phi(3) = \mathrm{mis}(G') = f(3) = 8$ and $\mathrm{mis}(G' - v) = \mathrm{mis}(G' - N_{G'}[v]) = 4$. Let G_1 and G_2 be the graphs obtained from $G' - v$ and $G' - N_{G'}[v]$, respectively, by deleting all isolated vertices. We analyze the structures of G_1 and G_2 as follows:

- If G_1 has only one component, then by Claim A, that component is isomorphic to a graph in $\mathcal{P}$ and G_2 is isomorphic to a graph in

$$\mathcal{P} \cup \{K_{1,r_1} \cup K_{1,r_2} :\ r_1, r_2 \geq 1\}.$$

- Otherwise, G_1 has two components, each of which is a star with at least two leaves, and G_2 is isomorphic to a graph in

$$\{K_{1,r_1} \cup K_{1,r_2} :\ r_1, r_2 \geq 1\}.$$

Based on the construction of graphs in $\mathcal{Q}_3$, it can be verified that G' is isomorphic to a graph in $\mathcal{Q}_3$.

We cannot have $\mathrm{mis}(G'-v) < 4$, as this would imply $\mathrm{mis}(G') < f(3) = 8$, which contradicts the fact that $\mathrm{mis}(G') \geq f(3) = 8$.

Therefore, $\phi(3) = \mathrm{mis}(G') = f(3) = 8$ and G' is isomorphic to a graph in $\mathcal{F}_3 = \mathcal{G}_3 \cup \mathcal{Q}_3$.

We introduce a family of graphs denoted by $\mathcal{R}_3$. The graphs in $\mathcal{R}_3$ are obtained from those in $\mathcal{Q}_3$ by deleting all leaves adjacent to one of the support vertices. For any $R \in \mathcal{R}_3$, direct computation yields:

$$\mathrm{mis}(R) \leq 7 < f(3) = 8.$$

We have that $\mathrm{mis}(H) < f(3)$ for all $H \in \mathcal{L}_3 \cup \mathcal{R}_3$. Therefore, from $\mathrm{mis}(G) = f(3)$, we can infer that G' being isomorphic to a graph in $\mathcal{F}_3 = \mathcal{G}_3 \cup \mathcal{Q}_3$ implies that G itself is isomorphic to a graph in $\mathcal{F}_3 = \mathcal{G}_3 \cup \mathcal{Q}_3$.

Therefore, for $t = 3$, $\phi(3) = f(3)$ and G is isomorphic to a graph in $\mathcal{F}_3$.

Case 3. $t = 4$.

In this case, $\mathrm{mis}(G') = \phi(4) \geq f(4) = 16$. By Lemma 4,

$$\mathrm{mis}(G') = \mathrm{mis}(G'-v) + \mathrm{mis}(G' - N_{G'}[v]).$$

Moreover, as $\mu(G'-v) = 3$, it follows from Lemma 3 and Theorem 3 that

$$\mathrm{mis}(G' - N_{G'}[v]) \leq \mathrm{mis}(G'-v) \leq m(3) = 10.$$

If $\mathrm{mis}(G'-v) = 10$, then by Theorem 3, we have $G'-v \cong K_{1,\ell} \cup C_5 \cup sK_1$ for some $\ell \geq 1$ and $s \geq 1$. Since G' is connected, it follows that $\mathrm{mis}(G' - N_{G'}[v]) \leq 6$, with equality if and only if $\ell \geq 2$ and $G' - N_{G'}[v] \cong P_4 \cup K_{1,\ell'}$ for some $1 \leq \ell' < \ell$. This implies that if $\mathrm{mis}(G'-v) = 10$, then $\phi(4) = \mathrm{mis}(G') = f(4) = 16$ and G' is isomorphic to a graph in $\mathcal{G}_4$.

If $\mathrm{mis}(G'-v) = 9$, then $G'-v$ must contain at least two non-trivial components; otherwise, by the inductive hypothesis (since Theorem 4 holds for $t = 3$), we would have $\mathrm{mis}(G'-v) \leq f(3) = 8$, which leads to a contradiction. Consequently, $G'-v$ contains exactly two non-trivial components, each contributing 3 MISs. However, since $\mu(G'-v) = 3$ and $f(1) = 2$, the number of MISs in $G'-v$ cannot be 9, resulting in a contradiction. Therefore, this situation is impossible.

If $\mathrm{mis}(G'-v) = 8$, then $\mathrm{mis}(G' - N_{G'}[v]) \leq \mathrm{mis}(G'-v) = 8$. Hence, $\phi(4) = \mathrm{mis}(G') = f(4) = 16$ and $\mathrm{mis}(G'-v) = \mathrm{mis}(G' - N_{G'}[v]) = 8$. Let G_1 and G_2 be the graphs obtained from $G'-v$ and $G' - N_{G'}[v]$, respectively, by deleting all isolated vertices. We analyze the structures of G_1 and G_2 as follows:

- If G_1 has only one component, then by the inductive hypothesis, that component is isomorphic to a graph in $\mathcal{G}_3 \cup \mathcal{Q}_3$ and G_2 is isomorphic to a graph in

$$\mathcal{G}_3 \cup \mathcal{Q}_3 \cup \{F \cup K_{1,r} :\ F \in \mathcal{P},\ r \geq 1\} \cup \{K_{1,r_1} \cup K_{1,r_2} \cup K_{1,r_3} :\ r_1, r_2, r_3 \geq 1\}.$$

- If G_1 has two components, then by Lemma 5, one has matching number 2 with 4 MISs, and the other has matching number 1 with 2 MISs. By Claim A, the former is isomorphic to a graph in $\mathcal{P}$, and the latter is isomorphic to a star with at least two leaves. Moreover, G_2 is isomorphic to a graph in

$$\{F \cup K_{1,r} : F \in \mathcal{P},\ r \geq 1\} \cup \{K_{1,r_1} \cup K_{1,r_2} \cup K_{1,r_3} :\ r_1, r_2, r_3 \geq 1\}.$$

- Otherwise, G_1 has three components, each of which is a star with at least two leaves. Moreover, G_2 is isomorphic to a graph in

$$\{K_{1,r_1} \cup K_{1,r_2} \cup K_{1,r_3} :\ r_1, r_2, r_3 \geq 1\}.$$

Based on the constructions of the graphs in $\mathcal{G}_4$ and $\mathcal{Q}_4$, it can be verified that G' is isomorphic to a graph in $\mathcal{G}_4 \cup \mathcal{Q}_4$.

We cannot have $\text{mis}(G' - v) < 8$, as this would imply $\text{mis}(G') < f(4) = 16$, which contradicts the fact that $\text{mis}(G') \geq f(4) = 16$.

Therefore, $\phi(4) = \text{mis}(G') = f(4) = 16$ and G' is isomorphic to a graph in $\mathcal{F}_4 = \mathcal{G}_4 \cup \mathcal{Q}_4$.

We introduce a family of graphs denoted by $\mathcal{R}_4$. The graphs in $\mathcal{R}_4$ are obtained from those in $\mathcal{Q}_4$ by deleting all leaves adjacent to one of the support vertices. For any $R \in \mathcal{R}_4$, direct computation yields:

$$\text{mis}(R) \leq 15 < f(4) = 16.$$

We have that $\text{mis}(H) < f(4)$ for all $H \in \mathcal{L}_4 \cup \mathcal{R}_4$. Therefore, from $\text{mis}(G) = f(4)$, we can infer that G' being isomorphic to a graph in $\mathcal{F}_4 = \mathcal{G}_4 \cup \mathcal{Q}_4$ implies that G itself is isomorphic to a graph in $\mathcal{F}_4 = \mathcal{G}_4 \cup \mathcal{Q}_4$.

Therefore, for $t = 4$, $\phi(4) = f(4)$ and G is isomorphic to a graph in $\mathcal{F}_4$.

Case 4. $t \geq 5$.

In this case, we present the following six claims to prove that for $t \geq 5$, $\phi(t) = f(t)$ and G is isomorphic to a graph in $\mathcal{F}_t$.

Claim 4.1. In $G' - v$, there is no component H such that $\mu(H)$ is even and $\mu(H) \geq 4$.

Proof of Claim 4.1. Suppose, for contradiction, that there exists a component H of $G' - v$ with even $\mu(H) = k \geq 4$. Let $\alpha := \text{mis}(H)$ and $\beta := \text{mis}(H - (N_{G'}(v) \cap V(H)))$. By Lemma 3 and the inductive hypothesis,

$$\beta \leq \alpha \leq f(k) = 2 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right). \tag{8}$$

Define $G_1 := G' - v - V(H)$ and $G_2 := G' - [N_{G'}[v] \cup V(H)]$. Then, by Lemmas 4 and 5, we have

$$\text{mis}(G') = \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]) = \alpha \cdot \text{mis}(G_1) + \beta \cdot \text{mis}(G_2). \tag{9}$$

Remark 2. There obviously exists an MIS containing v in G', and the number of such MISs is given by $\text{mis}(G' - N_{G'}[v]) = \beta \cdot \text{mis}(G_2)$. Therefore, if G_2 is an empty graph without vertices, we define $\text{mis}(G_2) = 1$. This convention is adopted throughout the paper.

By Lemma 3 and Remark 2, we have $\text{mis}(G_1) \geq \text{mis}(G_2) \geq 1$.

Now, construct a new graph G'' from G' by first removing all vertices of H, then adding $k/2$ disjoint copies of C_5, and finally connecting v to exactly one vertex in each new copy. Clearly, G'' is a connected triangle-free graph with matching number t. Moreover, by Lemmas 4 and 5 and the construction of G'', we have

$$\text{mis}(G'') = \text{mis}(G'' - v) + \text{mis}(G'' - N_{G''}[v]) = 5^{\frac{k}{2}} \cdot \text{mis}(G_1) + 3^{\frac{k}{2}} \cdot \text{mis}(G_2). \tag{10}$$

Since for all $k \geq 4$, we have both

$$5^{\frac{k}{2}} + 3^{\frac{k}{2}} > 4 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right) \text{ and } 5^{\frac{k}{2}} > 2 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right),$$

combining these with Inequality (8) and Equalities (9) and (10), we deduce that:

$$\begin{aligned}
\phi(t) &= \text{mis}(G') \\
&= \alpha \cdot \text{mis}(G_1) + \beta \cdot \text{mis}(G_2) \\
&\leq 2 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right) \cdot \text{mis}(G_1) + 2 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right) \cdot \text{mis}(G_2) \\
&< 2 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right) \cdot \text{mis}(G_1) + \left[5^{\frac{k}{2}} - 2 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right)\right] \cdot \text{mis}(G_2) + 3^{\frac{k}{2}} \cdot \text{mis}(G_2) \\
&\leq 2 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right) \cdot \text{mis}(G_1) + \left[5^{\frac{k}{2}} - 2 \cdot \left(5^{\frac{k-2}{2}} + 3^{\frac{k-2}{2}}\right)\right] \cdot \text{mis}(G_1) + 3^{\frac{k}{2}} \cdot \text{mis}(G_2) \\
&= 5^{\frac{k}{2}} \cdot \text{mis}(G_1) + 3^{\frac{k}{2}} \cdot \text{mis}(G_2) \\
&= \text{mis}(G'').
\end{aligned}$$

Thus, $\text{mis}(G'') > \phi(t)$, contradicting the assumption that $\phi(t)$ is the maximum number of MISs in connected triangle-free graph with matching number t.

The proof of Claim 4.1 is complete. ∎

Claim 4.2. If there exist at least two components in $G' - v$ with matching number 2, then each such component is isomorphic to C_5, and exactly one vertex in each is adjacent to v.

Proof of Claim 4.2. Suppose H_1 and H_2 are two components of $G' - v$ with matching number 2. Define

$$\alpha_1 := \text{mis}(H_1), \quad \beta_1 := \text{mis}(H_1 - (N_{G'}(v) \cap V(H_1))),$$

$$\alpha_2 := \text{mis}(H_2), \quad \beta_2 := \text{mis}(H_2 - (N_{G'}(v) \cap V(H_2))).$$

Let $G_3 := G' - v - V(H_1) - V(H_2)$ and $G_4 := G' - [N_{G'}[v] \cup V(H_1) \cup V(H_2)]$. By Lemmas 4 and 5,

$$\text{mis}(G') = \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]) = \alpha_1\alpha_2 \cdot \text{mis}(G_3) + \beta_1\beta_2 \cdot \text{mis}(G_4).$$

By Claim B, the best possible values for each pair (α_i, β_i) for $i = 1, 2$ are $(5, 3)$ and $(4, 4)$. We consider the following cases: If $(\alpha_1, \beta_1) = (\alpha_2, \beta_2) = (5, 3)$, define

$$I_1 = \alpha_1\alpha_2 \cdot \text{mis}(G_3) + \beta_1\beta_2 \cdot \text{mis}(G_4) = 25 \cdot \text{mis}(G_3) + 9 \cdot \text{mis}(G_4).$$

If $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} = \{(5, 3), (4, 4)\}$, define

$$I_2 = \alpha_1\alpha_2 \cdot \text{mis}(G_3) + \beta_1\beta_2 \cdot \text{mis}(G_4) = 20 \cdot \text{mis}(G_3) + 12 \cdot \text{mis}(G_4).$$

If $(\alpha_1, \beta_1) = (\alpha_2, \beta_2) = (4, 4)$, define

$$I_3 = \alpha_1\alpha_2 \cdot \text{mis}(G_3) + \beta_1\beta_2 \cdot \text{mis}(G_4) = 16 \cdot \text{mis}(G_3) + 16 \cdot \text{mis}(G_4).$$

By Lemma 3 and Remark 2, we have $\text{mis}(G_3) \geq \text{mis}(G_4) \geq 1$, which implies

$$I_1 > I_2 \geq I_3.$$

Hence, $\text{mis}(G')$ is maximized exactly when $(\alpha_1, \beta_1) = (\alpha_2, \beta_2) = (5, 3)$. It follows that both H_1 and H_2 are isomorphic to C_5, and in each, exactly one vertex is adjacent to v. ∎

Claim 4.3. If there exists a component H_1 in $G' - v$ such that $\mu(H_1)$ is odd and $\mu(H_1) \geq 3$, then H_1 is the only component in $G' - v$ with an odd matching number.

Proof of Claim 4.3. Suppose, for contradiction, that there exists another component H_2 in $G' - v$ such that $\mu(H_2)$ is odd. Let $\mu(H_1) = t_1 \geq 3$ and $\mu(H_2) = t_2 \geq 1$.

Define

$$\alpha_1 := \text{mis}(H_1), \quad \beta_1 := \text{mis}(H_1 - (N_{G'}(v) \cap V(H_1))),$$
$$\alpha_2 := \text{mis}(H_2), \quad \beta_2 := \text{mis}(H_2 - (N_{G'}(v) \cap V(H_2))).$$

By Lemma 3 and the inductive hypothesis,

$$\beta_1 \leq \alpha_1 \leq f(t_1) = 5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}, \tag{11}$$

and

$$\beta_2 \leq \alpha_2 \leq f(t_2) = 5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}. \tag{12}$$

Let

$$G_3 := G' - v - V(H_1) - V(H_2), \quad G_4 := G' - [N_{G'}[v] \cup V(H_1) \cup V(H_2)].$$

By Lemmas 4 and 5,

$$\text{mis}(G') = \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]) = \alpha_1\alpha_2 \cdot \text{mis}(G_3) + \beta_1\beta_2 \cdot \text{mis}(G_4). \tag{13}$$

Furthermore, by Lemma 3 and Remark 2, we have $\text{mis}(G_3) \geq \text{mis}(G_4) \geq 1$.

Now construct a new graph G'' from G' by first removing all vertices in H_1 and H_2 from G', then adding $(t_1+t_2)/2$ disjoint copies of C_5, and finally connecting v to exactly one vertex in each new copy. Then G'' is a connected triangle-free graph with matching number t. Moreover, by Lemmas 4 and 5 and the construction of G'', we obtain

$$\text{mis}(G'') = \text{mis}(G''-v) + \text{mis}(G''-N_{G''}[v]) = 5^{\frac{t_1+t_2}{2}} \cdot \text{mis}(G_3) + 3^{\frac{t_1+t_2}{2}} \cdot \text{mis}(G_4). \quad (14)$$

By Lemma 6,

$$5^{\frac{t_1+t_2}{2}} + 3^{\frac{t_1+t_2}{2}} > 2 \cdot \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right)\left(5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}\right). \quad (15)$$

Therefore, using Equalities (13) and (14), and Inequalities (11), (12), and (15), we conclude:

$$\begin{aligned}
\phi(t) &= \text{mis}(G') \\
&= \alpha_1\alpha_2 \cdot \text{mis}(G_3) + \beta_1\beta_2 \cdot \text{mis}(G_4) \\
&\leq \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right)\left(5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}\right) \cdot \text{mis}(G_3) \\
&\quad + \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right)\left(5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}\right) \cdot \text{mis}(G_4) \\
&< \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right)\left(5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}\right) \cdot \text{mis}(G_3) \\
&\quad + \left[5^{\frac{t_1+t_2}{2}} - \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right)\left(5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}\right)\right] \cdot \text{mis}(G_4) + 3^{\frac{t_1+t_2}{2}} \cdot \text{mis}(G_4) \\
&\leq \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right)\left(5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}\right) \cdot \text{mis}(G_3) \\
&\quad + \left[5^{\frac{t_1+t_2}{2}} - \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right)\left(5^{\frac{t_2-1}{2}} + 3^{\frac{t_2-1}{2}}\right)\right] \cdot \text{mis}(G_3) + 3^{\frac{t_1+t_2}{2}} \cdot \text{mis}(G_4) \\
&= 5^{\frac{t_1+t_2}{2}} \text{mis}(G_3) + 3^{\frac{t_1+t_2}{2}} \text{mis}(G_4) \\
&= \text{mis}(G'').
\end{aligned}$$

Thus, $\text{mis}(G'') > \phi(t)$, contradicting the assumption that $\phi(t)$ is the maximum number of MISs in connected triangle-free graph with matching number t.

The proof of Claim 4.3 is complete. ∎

Let $R_1, \cdots, R_k$ be non-trivial components in $G'-v$. For each $1 \leq i \leq k$, let $\mu(R_i) = t_i$, and assume without loss of generality that $t_1 \geq \cdots \geq t_k$.

Claim 4.4. $t_1 \geq 2$.

Proof of Claim 4.4. Suppose for contradiction that $t_1 = 1$. Since $t \geq 5$ and $\mu(G'-v) = t-1 \geq 4$, there are at least four components in $G'-v$ with matching number 1. Let $H = R_1 \cup R_2 \cup R_3 \cup R_4$.

Let $\alpha := \text{mis}(H)$ and $\beta := \text{mis}(H - (N_{G'}(v) \cap V(H)))$. By Lemma 3 and the inductive hypothesis,

$$\beta \leq \alpha = 2^4. \quad (16)$$

Define $G_1 := G' - v - V(H)$ and $G_2 := G' - [N_{G'}[v] \cup V(H)]$. Then, by Lemma 3 and Remark 2, we have $\text{mis}(G_1) \geq \text{mis}(G_2) \geq 1$.

Construct a new graph G'' as follows: first remove all vertices of H from G', then add two disjoint copies of C_5, and finally connect the vertex v to exactly one vertex in each new copy. Clearly, G'' remains a connected triangle-free graph with matching number t. By Lemmas 4 and 5,

$$\text{mis}(G'') = \text{mis}(G'' - v) + \text{mis}(G'' - N_{G''}[v]) = 5^2 \cdot \text{mis}(G_1) + 3^2 \cdot \text{mis}(G_2). \tag{17}$$

Therefore, applying Lemmas 4, along with Inequality (16) and Equality (17), we obtain:

$$\begin{aligned}
\phi(t) &= \text{mis}(G') \\
&= \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]) \\
&= \alpha \cdot \text{mis}(G_1) + \beta \cdot \text{mis}(G_2) \\
&\leq 2^4 \cdot \text{mis}(G_1) + 2^4 \cdot \text{mis}(G_2) \\
&< 5^2 \cdot \text{mis}(G_1) + 3^2 \cdot \text{mis}(G_2) \\
&= \text{mis}(G'').
\end{aligned}$$

Thus, $\text{mis}(G'') > \phi(t)$, contradicting the assumption that $\phi(t)$ is the maximum number of MISs in connected triangle-free graph with matching number t.

The proof of Claim 4.4 is complete. ∎

Claim 4.5. If $t_1 \geq 3$, then $G' - v \cong R_1 \cup sK_1$ for some $s \geq 1$, and moreover, G' is isomorphic to a graph in $\mathcal{G}_t$.

Proof of Claim 4.5. By Claims 4.1 and 4.3, t_1 is odd, and aside from R_1, no other component in $G' - v$ has an odd matching number.

Note that k is the number of non-trivial components in $G' - v$. We now show that $k = 1$. Suppose, for contradiction, that $k \geq 2$. Then by Claim 4.1, we have $t_2 = 2$. Let

$$\alpha := \text{mis}(R_1) \text{ and } \beta := \text{mis}(R_1 - (N_{G'}(v) \cap V(R_1))).$$

By Lemma 3 and the inductive hypothesis,

$$\beta \leq \alpha \leq f(t_1) = 5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}. \tag{18}$$

Let $G_1 := G' - v - V(R_1)$ and $G_2 := G' - [N_{G'}[v] \cup V(R_1)]$. By Lemma 3 and Remark 2, we have $\text{mis}(G_1) \geq \text{mis}(G_2) \geq 1$.

We construct a new graph G'' as follows: first remove all vertices of R_1 from G', then add $\frac{t_1-1}{2}$ new disjoint copies of C_5 and a new copy of a star $K_{1,\ell}$ for some $\ell \geq 2$, and finally connect v to exactly one vertex in each new C_5 and to a leaf of the star. It is easy to see that G'' is a connected triangle-free graph with matching number t. By Lemmas 4 and 5,

$$\mathrm{mis}(G'') = \mathrm{mis}(G'' - v) + \mathrm{mis}(G'' - N_{G''}[v]) = 2 \cdot 5^{\frac{t_1-1}{2}} \cdot \mathrm{mis}(G_1) + 2 \cdot 3^{\frac{t_1-1}{2}} \cdot \mathrm{mis}(G_2). \quad (19)$$

Therefore, using Lemmas 4 and 5, Inequality (18), and Equality (19), we obtain:

$$\begin{aligned}
\phi(t) &= \mathrm{mis}(G') \\
&= \mathrm{mis}(G' - v) + \mathrm{mis}(G' - N_{G'}[v]) \\
&= \alpha \cdot \mathrm{mis}(G_1) + \beta \cdot \mathrm{mis}(G_2) \\
&\le \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right) \cdot \mathrm{mis}(G_1) + \left(5^{\frac{t_1-1}{2}} + 3^{\frac{t_1-1}{2}}\right) \cdot \mathrm{mis}(G_2) \\
&\le 2 \cdot 5^{\frac{t_1-1}{2}} \cdot \mathrm{mis}(G_1) + 2 \cdot 3^{\frac{t_1-1}{2}} \cdot \mathrm{mis}(G_2) \\
&= \mathrm{mis}(G'').
\end{aligned}$$

Since G'' is also a connected triangle-free graph with matching number t, we have $\mathrm{mis}(G'') = \phi(t)$. From the above inequality and the fact that $5^{\frac{t_1-1}{2}} > 3^{\frac{t_1-1}{2}}$ when $t_1 \ge 3$, it follows that $\mathrm{mis}(G_1) = \mathrm{mis}(G_2)$.

However, since $t_2 = 2$, there exist at least two components in $G'' - v$ with matching number 2. By Claim 4.2 and the fact that $\mathrm{mis}(G'') = \phi(t)$, we have $R_2 \cong C_5$. Furthermore, $\mathrm{mis}(R_2) = 5$ and $\mathrm{mis}(R_2 - [N_{G'}[v] \cap R_2]) = 3$. Hence, $\mathrm{mis}(G_1)$ cannot be equal to $\mathrm{mis}(G_2)$, a contradiction. Therefore, $k = 1$, which implies $G' - v \cong R_1 \cup sK_1$ for some $s \ge 1$.

Now, $t = t_1 + 1$, so t is even and $t \ge 6$. By Lemma 3 and the inductive hypothesis,

$$\begin{aligned}
\phi(t) &= \mathrm{mis}(G') \\
&= \mathrm{mis}(G' - v) + \mathrm{mis}(G' - N_{G'}[v]) \\
&\le 2 \cdot \mathrm{mis}(G' - v) \\
&= 2 \cdot \mathrm{mis}(R_1) \\
&\le 2 \cdot f(t-1) \\
&= 2 \cdot \left(5^{\frac{t-2}{2}} + 3^{\frac{t-2}{2}}\right) \\
&= f(t).
\end{aligned}$$

Thus, $\phi(t) = f(t)$. This implies that

$$\mathrm{mis}(G' - N_{G'}[v]) = \mathrm{mis}(G' - v) = \mathrm{mis}(R_1) = f(t-1).$$

By the inductive hypothesis, R_1 is isomorphic to a graph in $\mathcal{G}_{t-1}$. Also, the neighbors of v in R_1 can only be leaves of R_1, and v is not adjacent to all of its leaves. We therefore conclude that G' is isomorphic to a graph in $\mathcal{G}_t$.

The proof of Claim 4.5 is complete. ∎

Claim 4.6. If $t_1 = 2$, then there exists at most one component with matching number 1 in $G' - v$, and moreover, G' is isomorphic to a graph in $\mathcal{G}_t$.

Proof of Claim 4.6. Suppose, for contradiction, that both R_{k-1} and R_k have matching number 1.

Let $H := R_{k-1} \cup R_k$, and define $\alpha := \text{mis}(H)$ and $\beta := \text{mis}(H - (N_{G'}(v) \cap V(H)))$. By Lemma 3 and the inductive hypothesis,

$$\beta \leq \alpha = 2^2 = 4. \tag{20}$$

Let $G_1 := G' - v - V(H)$ and $G_2 := G' - [N_{G'}[v] \cup V(H)]$. By Lemma 3 and Remark 2, we have $\text{mis}(G_1) \geq \text{mis}(G_2) \geq 1$.

We construct a new graph G'' as follows: first remove all vertices of H from G', then add a new copy of C_5, and finally connect v to exactly one vertex of the new copy. Clearly, G'' is a connected triangle-free graph with matching number t. By Lemmas 4 and 5,

$$\text{mis}(G'') = \text{mis}(G'' - v) + \text{mis}(G'' - N_{G''}[v]) = 5 \cdot \text{mis}(G_1) + 3 \cdot \text{mis}(G_2). \tag{21}$$

Therefore, using Lemmas 4 and 5, Inequality (20), and Equality (21), we obtain:

$$\begin{aligned}
\phi(t) &= \text{mis}(G') \\
&= \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]) \\
&= \alpha \cdot \text{mis}(G_1) + \beta \cdot \text{mis}(G_2) \\
&\leq 4 \cdot \text{mis}(G_1) + 4 \cdot \text{mis}(G_2) \\
&\leq 5 \cdot \text{mis}(G_1) + 3 \cdot \text{mis}(G_2) \\
&= \text{mis}(G'').
\end{aligned}$$

Since G'' is also a connected triangle-free graph with matching number t, it follows that $\text{mis}(G'') = \phi(t)$. The above inequality then implies $\text{mis}(G_1) = \text{mis}(G_2)$.

However, since $t_1 = 2$, there exist at least two components in $G'' - v$ with matching number 2. By Claim 4.2 and the fact that $\text{mis}(G'') = \phi(t)$, we have $R_1 \cong C_5$. Furthermore, $\text{mis}(R_1) = 5$ and $\text{mis}(R_1 - [N_{G'}[v] \cap R_1]) = 3$. Hence, $\text{mis}(G_1)$ cannot be equal to $\text{mis}(G_2)$, a contradiction.

Hence, there is at most one component in $G' - v$ with matching number 1.

Now, if t is odd, then $t-1$ is even and there are no components in $G'-v$ with matching number 1. By Claim 4.2, every non-trivial component of $G' - v$ is isomorphic to C_5, and exactly one vertex in each is adjacent to v. Thus,

$$\phi(t) = \text{mis}(G') = \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]) = 5^{\frac{t-1}{2}} + 3^{\frac{t-1}{2}} = f(t).$$

and G' is isomorphic to a graph in $\mathcal{G}_t$.

If t is even, then $t - 1 \geq 5$ is odd. This implies that in $G' - v$, there is exactly one component with matching number 1, and at least two components with matching number

2. By Claim 4.2, every component in $G' - v$ with matching number 2 is isomorphic to C_5, and in each such component, exactly one vertex is adjacent to v. The unique component with matching number 1 is a star. Therefore,

$$\phi(t) = \text{mis}(G') = \text{mis}(G' - v) + \text{mis}(G' - N_{G'}[v]) \leq 2 \cdot 5^{\frac{t-2}{2}} + 2 \cdot 3^{\frac{t-2}{2}} = f(t),$$

which implies that $\phi(t) = f(t)$. Moreover, in the star component, the neighbors of v can only be its leaves, and v is not adjacent to all of the leaves. Hence, we conclude that G' is isomorphic to a graph in $\mathcal{G}_t$.

The proof of Claim 4.6 is complete. ∎

By Claims 4.4–4.6, we conclude that for $t \geq 5$, $\phi(t) = f(t)$ and G' is isomorphic to a graph in $\mathcal{G}_t$.

We have that $\text{mis}(L) < f(t)$ for all $t \geq 5$ and any $L \in \mathcal{L}_t$. Therefore, from $\text{mis}(G) = f(t)$, we can infer that G' being isomorphic to a graph in $\mathcal{F}_t = \mathcal{G}_t$ implies that G itself is isomorphic to a graph in $\mathcal{F}_t = \mathcal{G}_t$.

Therefore, for $t \geq 5$, $\phi(t) = f(t)$ and G is isomorphic to a graph in $\mathcal{F}_t$.

The proof of Theorem 4 is complete.

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Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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