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A combinatorial proof for the positivity of the normalized Jacobi triple product tails



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ABSTRACT

For $k \geq 1$, we prove that

$$[q^n z^s] J_k(z,q) \geq 0, \qquad (n \geq 0,\ s \in \mathbb{Z})$$

for the normalized Jacobi triple product tails

$$J_k(z,q) = \frac{\sum_{j=k}^{\infty} (-1)^{j-k} q^{\binom{j+1}{2}} (z^{-j} + \cdots + z^j)}{(zq, q/z; q)_\infty}.$$

This result not only implies Merca's stronger nonnegativity conjecture on truncated Jacobi triple product series in full generality, but also yields infinite families of linear inequalities for two-colored partitions and partitions with parts in the residue classes $\pm S \pmod R$. We present a combinatorial proof wherein a sign-reversing involution reduces the normalized Jacobi triple product tails to the invariant subsets according to the generalized minimal–excludant of partitions. Furthermore, by combining an invertible lift operator on Frobenius arms with Konan's size- and length-preserving bijection, an injection is constructed between the consecutive

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invariant subsets, which implies the coefficientwise nonnegativity of the normalized Jacobi triple product tails.
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1. Introduction and main results

Euler's pentagonal number theorem

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{j\in\mathbb{Z}}(-1)^j q^{j(3j-1)/2},$$

stands as the foundational cornerstone of q-series and combinatorics. Multiplying it by $(q;q)_\infty^{-1}=\sum_{n\geq 0}p(n)q^n$ yields the well-known alternating recurrence relation for the partition function $p(n)$. Shanks [29] obtained a finite form of the pentagonal number series in 1951. Researchers such as Warnaar [30] and Chapman [10] have extensively explored other related finite forms and partial-sum identities. The systematic study of coefficientwise positivity for such truncations began with the following identity due to Andrews and Merca [4]

$$\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1})=1+(-1)^{k-1}\sum_{n\geq 1}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q.$$

For $n\geq 1$, they interpreted the coefficients of q^n as follows,

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\big(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1)\big)=M_k(n), \tag{1.1}$$

where $M_k(n)$ is the number of partitions of n in which k is the least positive integer that does not occur as a part and the number of parts larger than k is greater than the number of parts smaller than k. This immediately yields the corresponding nonnegativity of the truncated pentagonal number series. We refer to [20,33,37] for the subsequent bijective and analytic work clarified this interpretation which also produced companion inequalities.

Guo and Zeng [15] then established truncations of Gauss' identities on triangular numbers and square numbers, which lead to inequalities for overpartitions and partitions with odd parts distinct. Andrews and Merca [5] showed that both truncations given by Euler and Gauss admit a unified treatment through the Rogers–Fine identity. Xia, Yee, and Zhao [34] later constructed broader families of truncated forms for these three classical theta identities. These developments made it natural to seek a truncation theory at the level of the Jacobi triple product series, which reduces to Euler and Gauss' identities as special cases.

Andrews–Merca and Guo–Zeng independently conjectured that appropriately normalized finite truncations of the Jacobi triple product series have nonnegative coefficients. More precisely, for $1 \leq S < R/2$ they asked whether every positive-degree coefficient of

$$(-1)^{k-1} \frac{\sum_{j=0}^{k-1} (-1)^j q^{Rj(j+1)/2-Sj} \left(1 - q^{(2j+1)S}\right)}{\left(q^S, q^{R-S}, q^R; q^R\right)_\infty}, \tag{1.2}$$

is nonnegative. Note that the complete series over $j \geq 0$ on the numerator is just the Jacobi triple product series which equals to $(q^S, q^{R-S}, q^R; q^R)_\infty$. It has been certified analytically by Mao [24] and combinatorially by Yee in 2015 [36]. Wang and Yee [31] subsequently obtained an explicit multiple-sum form based on the Bailey pair methods and a q-series expansion of Liu [22]. The bilateral variants, averaged truncations, Hecke–Rogers analogues, and various combinatorial interpretations have since been developed in [16,19,28,32,35].

Merca observed that the preceding finite-truncation statement (1.2) can be strengthened substantially. In [26,27], Merca proposed a nonnegativity conjecture after removing the factor $(q^R; q^R)_\infty$ from the denominator of (1.2) and extending the parameter range to $1 \leq S < R$.

Conjecture 1.1. *For positive integers k, R, S with $1 \leq S < R$, the coefficient of q^n with $n \geq 1$ in*

$$\mathcal{T}_{R,S,k}(q) = \frac{\sum_{j=k}^{\infty} (-1)^{j-k} q^{Rj(j+1)/2-Sj} \left(1 - q^{(2j+1)S}\right)}{\left(q^S, q^{R-S}; q^R\right)_\infty} \tag{1.3}$$

is nonnegative.

Ballantine and Feigon [8] proved the first three truncation levels $k = 1, 2, 3$ and obtained several related identities. Ding and Sun [12] proved the specialization $R = 3S$ by applying Bailey lattice. Mao [23] used Wright's circle method to obtain asymptotic formulas for the relevant Jacobi triple product and quintuple product tails, thereby proving the corresponding conjectures for coefficients of sufficiently large exponents. In a separate analytic treatment, Ding and Sun [13] further established eventual nonnegativity with an effective procedure for a threshold $N(R, S, k)$ and also proved the nonnegativity of the coefficients of q^n when $n \geq N(R, S, k)$. Zhou [37] obtained further tail-positivity results and specializations in a systematic study of pentagonal tails. Thereby this conjecture remains open in full generality until now.

The tails of theta series also carry a natural physical interpretation. In two-dimensional conformal field theory and solvable lattice models, q-series represent graded partition functions in which q tracks energy and z acts as a charge fugacity [1,11]. Truncating these series corresponds to a finite-volume restriction or an ultraviolet cutoff [3,25]. Consequently, the tail is the high-energy remainder of the spectrum.

In this paper, we consider the normalized Jacobi triple product tails as follows

$$J_k(z,q) = \frac{\sum_{j=k}^{\infty}(-1)^{j-k}q^{T_j}\chi_j(z)}{(zq,q/z;q)_\infty}, \tag{1.4}$$

where $k \geq 1$, $T_j = \binom{j+1}{2}$ and $\chi_j(z) = \sum_{u=-j}^{j} z^u$. The Laurent variable z records the difference between the numbers of A-colored parts and B-colored parts in the two colored-partitions. Combinatorially, we prove the coefficientwise nonnegativity of $J_k(z,q)$.

Theorem 1.2. *For every integer $k \geq 1$,*

$$J_k(z,q) \in \mathbb{Z}_{\geq 0}[z,z^{-1}][[q]]. \tag{1.5}$$

Equivalently, $[q^n z^s]\, J_k(z,q) \geq 0$ for all $n \geq 0$ and $s \in \mathbb{Z}$.

The above theorem leads to inequalities on partition pairs and two-colored partitions immediately. Moreover, by substituting $(z,q) = (q^S, q^R)$ into the above theorem, it directly reduces to Merca's stronger Conjecture 1.1.

Corollary 1.3. *Merca's stronger Conjecture 1.1 holds.*

Our proof is based on the minimal excludant of integer partitions, which is denoted by mex in short. The statistic mex has long been used in combinatorial theory, see, for example, Fraenkel and Peled [14]. Andrews and Newman [6,7] introduced it systematically into partition theory and related its moments to classical partition functions. The parity of the mex was then connected with Dyson's crank, and Hopkins, Sellers and Stanton [17] formulated a shifted version that records the first missing part above a prescribed level, see also [6,7,9,17,18,21].

For a partition λ, let

$$\mathrm{mex}_r(\lambda) = \min\{t > r : t \text{ is not a part of } \lambda\},$$

and let $\mathrm{Top}(\lambda)$ denote the set of the elements contained in the top row of its Frobenius symbol [2]. For $r \geq 0$, define

$$\mathcal{M}_r = \{\lambda : \mathrm{mex}_r(\lambda) - r \text{ is odd}\}, \qquad \mathcal{F}_r = \{\lambda : r \notin \mathrm{Top}(\lambda)\}.$$

Konan [21] established an explicit bijection for these generalized mex classes.

Theorem 1.4 *(Konan [21]). For every $r \geq 0$ there is a bijection*

$$\mathsf{K}_r : \mathcal{M}_r \longrightarrow \mathcal{F}_r$$

that preserves both size and length

$$|\mathsf{K}_r(\lambda)| = |\lambda|, \qquad \ell(\mathsf{K}_r(\lambda)) = \ell(\lambda).$$

To give our combinatorial proof, first we construct a bidegree-preserving, sign-reversing involution on a set of quadruples. Then the unpaired elements are organized into invariant subsets according to the mex $\mathcal{C}_r$ with the corresponding generating function $C_r(z,q)$. Based on Konan's bijection, we derive an injection Θ_r on this subset which makes each negative block in the invariant subset inject into a preceding positive block. The argument therefore proves coefficientwise nonnegativity of $J_k(z,q)$.

This paper is organized as follows. Section 2 introduces the basic definitions and notations for partitions and formal power series. Section 3 derives the invariant subset according to the mex decomposition combinatorially and records its direct q-series counterpart. Section 4 constructs the Frobenius-arm lift and an injection between consecutive invariant subsets, thereby proving Theorem 1.2. Section 5 contains some concluding remarks. In Appendix A, we recall the detailed procedure of Konan's bijection.

2. Partitions, minimal excludants, and colored indices

In this paper, we assume that $|q| < 1$ and all the formal power series are interpreted in $\mathscr{R} = \mathbb{Z}[z, z^{-1}][[q]]$, where the coefficient of each q^n is a Laurent polynomial in z. Thereby, for a fixed bidegree (n, s) only finitely many summands can contribute to the coefficient of $q^n z^s$.

Let

$$(a;q)_n = \prod_{u=0}^{n-1}(1-aq^u), \qquad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

and $(a_1, a_2, \ldots, a_t; q)_\infty = \prod_{i=1}^{t}(a_i;q)_\infty$.

We write $F \succeq G$, or equivalently $G \preceq F$, if every coefficient of $F - G$ is nonnegative.

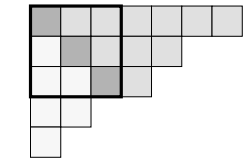
A partition [2] of a nonnegative integer n is a finite weakly decreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$, which can be denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. The set of all partitions, including the empty partition, is denoted by $\mathcal{P}$. For $\lambda \in \mathcal{P}$, let $|\lambda|$, $\ell(\lambda)$, and $m_t(\lambda)$ denote its size, length, and multiplicity of the part t. Multiset union is written as $\lambda \sqcup \mu$. Denote the staircase in the set of partitions by

$$\delta_r = (r, r-1, \ldots, 1), \qquad \delta_0 = \varnothing,$$

so that $|\delta_r| = T_r$, $\ell(\delta_r) = r$ and we call r the level of this staircase.

If d is the size of the Durfee square of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, that is, $d = \max\{i \geq 1 \colon \lambda_i \geq i\}$, and λ' is its conjugate, then the Frobenius symbol of λ is given by [2]

$$\lambda \longleftrightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_d \\ b_1 & b_2 & \cdots & b_d \end{pmatrix},$$



$\lambda = (7, 5, 4, 2, 1), \qquad d = 3$

$$a_i = \lambda_i - i, \qquad b_i = \lambda'_i - i,$$

$$\lambda \longleftrightarrow \begin{pmatrix} 6 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}.$$

The cells enclosed by the black border form the Durfee square. The lightly shaded cells give the Frobenius arms 6, 3, 1; the cells below the diagonal give the legs 4, 2, 0.

Fig. 1. The Young diagram of a partition and its Frobenius symbol.

where $a_i = \lambda_i - i$ and $b_i = \lambda'_i - i$ for $1 \le i \le d$. The two rows are strictly decreasing sequences of nonnegative integers, which represent the Frobenius arms and legs, respectively, such that

$$|\lambda| = d + \sum_{i=1}^{d} (a_i + b_i), \qquad \ell(\lambda) = b_1 + 1 \quad (\lambda \neq \varnothing). \tag{2.1}$$

Denote $\mathrm{Top}(\lambda) = \{a_1, \ldots, a_d\}$ and $\mathrm{Top}(\varnothing) = \varnothing$. See Fig. 1 as an example.

For $\lambda \in \mathcal{P}$, define the ordinary and shifted minimal excludants by

$$\mathrm{mex}(\lambda) = \min\{t \ge 1 : m_t(\lambda) = 0\}, \qquad \mathrm{mex}_r(\lambda) = \min\{t > r : m_t(\lambda) = 0\}.$$

For example, if $\lambda = (7, 5, 4, 2, 1)$, then $\mathrm{mex}_1(\lambda) = \mathrm{mex}_2(\lambda) = 3$.

Recall that

$$\mathcal{M}_r = \{\lambda \in \mathcal{P} : \mathrm{mex}_r(\lambda) - r \text{ is odd}\}, \qquad \mathcal{F}_r = \{\lambda \in \mathcal{P} : r \notin \mathrm{Top}(\lambda)\}.$$

The condition defining $\mathcal{F}_r$ concerns a Frobenius arm, not the presence of r as an ordinary part. For example, $\lambda = (7, 5, 4, 2, 1)$, then $\mathrm{Top}(7, 5, 4, 2, 1) = (6, 3, 1)$, hence it lies in $\mathcal{F}_r$ precisely when $r \notin \{1, 3, 6\}$.

Assume that for a partition with two colors A and B, an A-colored part of size i, denoted by i^A, has weight $q^i z$, whereas a B-colored part of size i, denoted by i^B, has weight $q^i z^{-1}$. Note that the superscripts A and B are just color labels, not powers. Thus a pair $(\alpha, \beta) \in \mathcal{P}^2$, where α records the A-colored parts and β records the B-colored parts, has weight

$$\mathrm{wt}(\alpha, \beta) = q^{|\alpha|+|\beta|} z^{\ell(\alpha)-\ell(\beta)}.$$

Consequently,

$$\sum_{\alpha,\beta \in \mathcal{P}} \mathrm{wt}(\alpha, \beta) = \frac{1}{(zq, q/z; q)_\infty}. \tag{2.2}$$

We also set the weight of an uncolored atom U to be $zq^0 = z$. For example, for a two colored partition pair $(\alpha = (5, 2, 2), \beta = (4, 3, 1))$, we have $\mathrm{wt}(\alpha, \beta) = q^{17} z^{3-3} = q^{17}$.

3. Preliminary cancellations and the invariant subset decomposition

We first interpret $J_k(z,q)$, both combinatorially and analytically. Since

$$\chi_j(z) = z^{-j}(1+z+\cdots+z^{2j}),$$

equation (1.4) can be rewritten as

$$J_k(z,q) = \sum_{j=k}^{\infty}(-1)^{j-k}\frac{q^{T_j}z^{-j}(1+z+\cdots+z^{2j})}{(zq,q/z;q)_\infty},$$

where the factor $q^{T_j}z^{-j}$ can be seen as the weight of a B-colored staircase

$$\delta_j = (j, j-1, \ldots, 1).$$

Let $\mathcal{X}_k$ be the set of quadruples

$$\mathcal{X}_k = \{y = (j,m,\alpha,\beta)\colon \alpha,\beta\in\mathcal{P},\ j\geq k,\ 0\leq m\leq 2j,\ \mathrm{mex}(\beta) > j\},$$

where α,β are colored by A and B, respectively. Define

$$\mathrm{sgn}(y) = (-1)^{j-k}, \qquad \mathrm{wt}(y) = q^{|\alpha|+|\beta|}z^{m+\ell(\alpha)-\ell(\beta)}.$$

Here m records the choice of the monomial z^m in $1+z+\cdots+z^{2j}$, or equivalently m copies of the atom U, which denotes a single cell in the Young diagram. Combining with (2.2), it is easy to see that

$$J_k(z,q) = \sum_{y\in\mathcal{X}_k}\mathrm{sgn}(y)\,\mathrm{wt}(y).$$

For $\beta\in\mathcal{P}$, let

$$h(\beta) := \max\{t\geq 0 : \delta_t\subseteq\beta\} = \mathrm{mex}(\beta)-1,$$

where the staircase containment is understood with multiplicity. For $m\geq 0$, set

$$L(m) := \max\left\{k, \left\lceil\frac{m}{2}\right\rceil\right\}.$$

Once m,α,β are given, the admissible values of j are precisely the integers in the interval

$$L(m)\leq j\leq h(\beta).$$

We now define a map

$$\iota : \mathcal{X}_k \longrightarrow \mathcal{X}_k$$

by

$$\iota(j,m,\alpha,\beta)=\begin{cases}(j+1,m,\alpha,\beta), & j-L \text{ is even and } j<h,\\ (j-1,m,\alpha,\beta), & j-L \text{ is odd},\\ (j,m,\alpha,\beta), & j=h \text{ and } h-L \text{ is even},\end{cases}$$

where $y=(j,m,\alpha,\beta)\in\mathcal{X}_k$, and $L=L(m), h=h(\beta)$.

Proposition 3.1. *The map ι is an involution, and the invariant subset of this involution is*

$$\mathcal{I}_\iota=\{(j,m,\alpha,\beta)\in\mathcal{X}_k : j=h(\beta),\ \operatorname{mex}(\beta)=L(m)+2d+1 \text{ for some } d\geq 0\}.$$

Proof. In the first case, $j<h$ implies

$$\delta_{j+1}\subseteq\beta.$$

In the second case, $j-L$ is odd, so $j\geq L+1$. Hence

$$j-1\geq L\geq k, \qquad \delta_{j-1}\subseteq\beta \qquad \text{and} \qquad m\leq 2L\leq 2(j-1).$$

For any element in the first case, the staircase under ι is mapped to of level $j+1$ which has odd difference from L, so applying ι again simply maps it back to itself at level j. Conversely, for any element in the second case, applying ι twice clearly maps it back to itself at level j. The elements in the third case consist precisely of the fixed points of ι. Moreover, since ι changes only j, it preserves $\operatorname{wt}(y)$. On every nonfixed orbit, j is changed by one, and therefore

$$\operatorname{sgn}(\iota(y))=-\operatorname{sgn}(y).$$

Thus ι is a weight-preserving sign-reversing involution.

Note that an element $y=(j,m,\alpha,\beta)$ is fixed under ι if and only if

$$j=h(\beta), \qquad h(\beta)-L(m)\equiv 0 \pmod 2,$$

and $\operatorname{sgn}(y)=(-1)^{L(m)-k}$. Equivalently, there exists $d\geq 0$ such that

$$\operatorname{mex}(\beta)=L(m)+2d+1,$$

which completes the proof. □

For $r \geq 1$, define the invariant subset according to the mex

$$\mathcal{C}_r = \left\{(\alpha, \beta) \in \mathcal{P}^2 : \operatorname{mex}(\beta) = r + 2d + 1 \text{ for some } d \geq 0\right\}, \tag{3.1}$$

and let

$$C_r(z, q) = \sum_{(\alpha,\beta)\in\mathcal{C}_r} q^{|\alpha|+|\beta|} z^{\ell(\alpha)-\ell(\beta)}.$$

By applying the involution ι, we obtain the following decomposition of $J_k(z, q)$ according to the invariant subset of ι. Here we present both combinatorial proof and analytic proof of such a decomposition.

Lemma 3.2 *(Invariant subset decomposition of $J_k(z, q)$). For every $k \geq 1$,*

$$\begin{aligned} J_k(z, q) = (1 + z + \cdots + z^{2k}) C_k(z, q) \\ + \sum_{r=k+1}^{\infty} (-1)^{r-k} z^{2r-1} (1 + z) C_r(z, q). \end{aligned} \tag{3.2}$$

Combinatorial proof. Under the involution ι, only its fixed points remain. Set $r = L(m)$.

If $r = k$, then $0 \leq m \leq 2k$. It follows that every surviving element has positive sign, and the corresponding (α, β) belongs to $\mathcal{C}_k$. Summing over the possible values of m gives

$$(1 + z + \cdots + z^{2k}) C_k(z, q).$$

If $r > k$, then $L(m) = \lceil m/2 \rceil = r$, and hence

$$m \in \{2r - 1, 2r\}.$$

The sign of the fixed point is $(-1)^{r-k}$, while the two possible values of m contribute to

$$z^{2r-1} + z^{2r} = z^{2r-1}(1 + z).$$

Summing over all $r \geq k + 1$ gives (3.2). Conversely, every $(\alpha, \beta) \in \mathcal{C}_r$, together with an admissible value of m described above, determines the unique fixed point $j = h(\beta)$. □

Analytic proof. Formula (3.2) also follows from (1.4). Let

$$F_j(z, q) = \frac{q^{T_j} z^{-j}}{(zq, q/z; q)_\infty},$$

and let

$$A_j(z) = 1 + z + \cdots + z^{2j}, \qquad D_r(z) = z^{2r-1}(1 + z).$$

Then

$$J_k(z,q)=\sum_{j=k}^{\infty}(-1)^{j-k}A_j(z)F_j(z,q),\qquad A_j(z)=A_k(z)+\sum_{r=k+1}^{j}D_r(z).$$

Collecting equal blocks in the region $k+1\le r\le j$ gives

$$\begin{aligned}J_k(z,q)&=A_k(z)\sum_{j=k}^{\infty}(-1)^{j-k}F_j(z,q)+\sum_{r=k+1}^{\infty}D_r(z)\sum_{j=r}^{\infty}(-1)^{j-k}F_j(z,q)\\&=A_k(z)\widehat{C}_k(z,q)+\sum_{r=k+1}^{\infty}(-1)^{r-k}D_r(z)\widehat{C}_r(z,q),\end{aligned}$$

where

$$\widehat{C}_r(z,q)=\sum_{j=r}^{\infty}(-1)^{j-r}F_j(z,q).$$

Pairing up the consecutive summands, we have

$$\widehat{C}_r(z,q)=\sum_{d\ge 0}\big(F_{r+2d}(z,q)-F_{r+2d+1}(z,q)\big).$$

Noting that $T_{j+1}-T_j=j+1$, it implies that

$$\begin{aligned}F_j(z,q)-F_{j+1}(z,q)&=\frac{q^{T_j}z^{-j}(1-q^{j+1}z^{-1})}{(zq,q/z;q)_\infty}\\&=\frac{1}{(zq;q)_\infty}\left(\prod_{i=1}^{j}\frac{q^iz^{-1}}{1-q^iz^{-1}}\right)\left(\prod_{i=j+2}^{\infty}\frac{1}{1-q^iz^{-1}}\right).\end{aligned}$$

The first factor $1/(zq;q)_\infty$ permits the generating function of arbitrary A-colored partitions. According to the following two brackets, the B-colored partitions must have parts $1,2,\dots,j$ occurring at least once, $j+1$ being absent, and all larger parts being unrestricted. Thus $F_j(z,q)-F_{j+1}(z,q)$ is the generating function for pairs (α,β) with $\mathrm{mex}(\beta)=j+1$, that is to say, $j+1$ is the first missing B-colored part. Taking $j=r+2d$ shows that $\widehat{C}_r(z,q)=C_r(z,q)$, and this proves (3.2). □

4. Frobenius operation and the combinatorial proof

In this section, we will construct a Frobenius-arm lift operator as well as an injection which can be combined with Konan's bijection to build an injection from the invariant subset $\mathcal{C}_r$ to $\mathcal{C}_{r-1}$.

Taking $(\alpha, \beta) \in \mathcal{C}_r$, since $\mathrm{mex}(\beta) > r$, by removing one copy of each part among $1, 2, \ldots, r$, it yields a unique decomposition for β:

$$\beta = \delta_r \sqcup \gamma,$$

such that $\mathrm{mex}_r(\gamma) = \mathrm{mex}(\beta) = r + 2d + 1$, and hence $\gamma \in \mathcal{M}_r$. Conversely, adjoining δ_r to any $\gamma \in \mathcal{M}_r$ produces a B-colored partition in the invariant subset $\mathcal{C}_r$.

To give the combinatorial proof of our main result, we need to introduce a Frobenius operation associated with the following sets

$$D_r := \{\lambda \in \mathcal{F}_r : r - 1 \in \mathrm{Top}(\lambda)\}, \qquad E_r := \{\mu \in \mathcal{F}_{r-1} : r \in \mathrm{Top}(\mu)\}.$$

Lemma 4.1. *For a given $r \geq 2$, there exists a bijection*

$$\mathsf{U}_r : D_r \longrightarrow E_r,$$

where for $\lambda \in D_r$, $\mathsf{U}_r(\lambda)$ is obtained by replacing the unique top-row entry $r - 1$ with r and leaving the bottom row unchanged in the Frobenius symbol of λ.

Proof. For a partition $\lambda \in D_r$, suppose its Frobenius symbol is as follows

$$\lambda = \begin{pmatrix} a_1 & \cdots & a_d \\ b_1 & \cdots & b_d \end{pmatrix},$$

and let $a_t = r - 1$. Since $\lambda \in \mathcal{F}_r$, the integer r does not occur in the top row. Hence

$$a_{t-1} \geq r + 1 \quad (t > 1), \qquad a_{t+1} \leq r - 2 \quad (t < d).$$

Thus replacing $a_t = r - 1$ with r preserves the strictly decreasing order of the top row, and thereby $\mathsf{U}_r(\lambda) \in E_r$.

Conversely, if $\mu \in E_r$, then its top row contains a unique entry r and contains no entry $r - 1$. Replacing that entry r with $r - 1$ again preserves the strictly decreasing order and produces a partition in D_r. This lowering operation is the inverse of U_r. Thus U_r is a bijection. □

We call U_r the Frobenius-arm lift operator since it extends one of the arms of λ by 1. From the properties of the Frobenius symbol (2.1), it is clear that

$$|\mathsf{U}_r(\lambda)| = |\lambda| + 1, \qquad \ell(\mathsf{U}_r(\lambda)) = \ell(\lambda).$$

Fig. 2 illustrates this operation.

Let $\widehat{r}$ denote a marked A-colored part of size r. The next lemma establishes an injection.

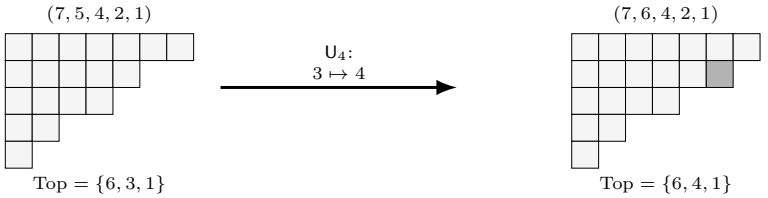


Fig. 2. The Frobenius-arm lift U_4. One cell is added to the arm corresponding to the top-row entry 3; the bottom row and the partition length are unchanged.

Lemma 4.2. *The map*

$$\mathsf{B}_r : \mathcal{P} \times \{\widehat{r}\} \times \mathcal{F}_r \hookrightarrow \mathcal{P} \times \mathcal{F}_{r-1}$$

with

$$\mathsf{B}_r(\alpha, \widehat{r}, \lambda) = \begin{cases} (\alpha \sqcup (r), \lambda), & \text{if } r-1 \notin \mathrm{Top}(\lambda), \\ (\alpha \sqcup (r-1), \mathsf{U}_r(\lambda)), & \text{if } r-1 \in \mathrm{Top}(\lambda) \end{cases}$$

is an injection for $r \geq 2$. More precisely, it is a bijection onto the disjoint union $\mathcal{I}_r = \mathcal{I}_r^{(0)} \sqcup \mathcal{I}_r^{(1)} \subset \mathcal{P} \times \mathcal{F}_{r-1}$, where

$$\mathcal{I}_r^{(0)} = \{(\alpha', \mu) : r \notin \mathrm{Top}(\mu),\ m_r(\alpha') \geq 1\},$$
$$\mathcal{I}_r^{(1)} = \{(\alpha', \mu) : r \in \mathrm{Top}(\mu),\ m_{r-1}(\alpha') \geq 1\},$$

and in both branches $|\alpha'| + |\mu| = |\alpha| + r + |\lambda|$, $\ell(\alpha') = \ell(\alpha) + 1$, and $\ell(\mu) = \ell(\lambda)$.

Proof. It is obvious that $\mathcal{I}_r^{(0)} \cap \mathcal{I}_r^{(1)} = \emptyset$. The first branch of $\mathsf{B}_r(\alpha, \widehat{r}, \lambda)$ lands in $\mathcal{I}_r^{(0)}$: the output top row still omits r, and the newly inserted part r ensures that $m_r(\alpha') \geq 1$. The second branch lands in $\mathcal{I}_r^{(1)}$: the arm lift operator puts r into the output top row and α' contains the newly inserted part $r-1$.

Conversely, for $(\alpha', \mu) \in \mathcal{I}_r$, set

$$(\alpha, \widehat{r}, \lambda) = \begin{cases} (\alpha' \setminus (r), \widehat{r}, \mu), & r \notin \mathrm{Top}(\mu), \\ (\alpha' \setminus (r-1), \widehat{r}, \mathsf{U}_r^{-1}(\mu)), & r \in \mathrm{Top}(\mu). \end{cases}$$

Here $\alpha' \setminus (t)$ denotes the partition obtained by removing one copy of the part t. In the first case, $(\alpha', \mu) \in \mathcal{I}_r^{(0)}$ thus $\mu \in \mathcal{F}_{r-1}$ and its top row also omits r. So we have $\lambda = \mu \in \mathcal{F}_r$ and thereby $(\alpha, \widehat{r}, \lambda) \in \mathcal{P} \times \{\widehat{r}\} \times \mathcal{F}_r$. In the second case, it is also true since $\lambda = \mathsf{U}_r^{-1}(\mu)$ is again an element of $\mathcal{F}_r$. Therefore, B_r is a bijection onto the set $\mathcal{I}_r$.

We can see that the size and length statements are obvious from the definition of U_r. Thus, the proof is complete. □

To show the map B_r is not onto $\mathcal{P} \times \mathcal{F}_{r-1}$, we consider $(\varnothing, \varnothing)$ which lies outside its image.

Employing the maps U_r, B_r, and Konan's bijection, we can finish the proof of our main result (1.5).

Theorem 4.3 *(Invariant subset injection). For every $r \geq 2$, there is an injection*

$$\begin{aligned} \Theta_r : &\{(U,U)\} \times \mathcal{C}_r \hookrightarrow \mathcal{C}_{r-1} \\ &(U, U, \alpha, \beta) \mapsto (\alpha', \beta') \end{aligned}$$

where $(\alpha, \beta) \in \mathcal{C}_r$, $\beta = \delta_r \sqcup \gamma$, with $\gamma \in \mathcal{M}_r$, and $\lambda = \mathsf{K}_r(\gamma) \in \mathcal{F}_r$ is obtained by using Konan's bijection as given in Theorem 1.4, then α' is determined by

$$\mathsf{B}_r(\alpha, \widehat{r}, \lambda) = (\alpha', \mu), \qquad \mu \in \mathcal{F}_{r-1},$$

β' is given by

$$\beta' = \delta_{r-1} \sqcup \gamma'.$$

with $\gamma' = \mathsf{K}_{r-1}^{-1}(\mu) \in \mathcal{M}_{r-1}$, such that

$$|\alpha'| + |\beta'| = |\alpha| + |\beta|, \qquad \ell(\alpha') - \ell(\beta') = \ell(\alpha) - \ell(\beta) + 2.$$

Consequently,

$$z^2 C_r(z,q) \preceq C_{r-1}(z,q). \tag{4.1}$$

Proof. By replacing the staircase δ_r with δ_{r-1}, the removed part r is marked by color B of weight $q^r z^{-1}$. Together with the two atoms (U,U), it has weight

$$(q^r z^{-1}) z^2 = q^r z,$$

which can be seen as the weight of a marked A-colored part $\widehat{r}$. Since $\gamma' \in \mathcal{M}_{r-1}$, there exists $d' \geq 0$ such that $\operatorname{mex}(\beta') = \operatorname{mex}_{r-1}(\gamma') = (r-1) + 2d' + 1$. Combine it with the fact that α' is an arbitrary A-colored partition containing part r or $r-1$ and check the definition of $\mathcal{C}_r$ in (3.1), we have $(\alpha', \beta') \in \mathcal{C}_{r-1}$.

To prove the injectivity, we construct an explicit left inverse of Θ_r on its image. Let

$$(\alpha', \beta') \in \operatorname{Im}(\Theta_r).$$

Since $(\alpha', \beta') \in \mathcal{C}_{r-1}$, the partition β' contains at least one copy of each of the parts $1, 2, \ldots, r-1$. Hence the separation operation

$$\gamma' := \beta' \setminus \delta_{r-1}$$

is uniquely determined. Moreover,

$$\mathrm{mex}_{r-1}(\gamma') = \mathrm{mex}(\beta') = r + 2d$$

for some $d \geq 0$, and therefore $\gamma' \in \mathcal{M}_{r-1}$. We may thus recover uniquely

$$\mu := \mathsf{K}_{r-1}(\gamma') \in \mathcal{F}_{r-1}.$$

Because (α', β') lies in the image of Θ_r, the pair (α', μ) lies in the image of the marked-part injection B_r. Lemma 4.2 shows that the branch of B_r used in the construction is determined uniquely by the Frobenius top row of μ.

If $r \notin \mathrm{Top}(\mu)$, then $(\alpha', \mu) \in \mathcal{I}_r^{(0)}$ which admits the first branch of B_r. In particular, α' contains a part r, and the unique predecessor pair is

$$\alpha = \alpha' \setminus (r), \qquad \lambda = \mu.$$

If $r \in \mathrm{Top}(\mu)$, then $(\alpha', \mu) \in \mathcal{I}_r^{(1)}$ which admits the second branch of B_r. Since $\mu \in \mathcal{F}_{r-1}$, we know its Frobenius top row contains no entry $r-1$. Further note that α' contains a part $r-1$, and in this case the unique predecessor pair is

$$\alpha = \alpha' \setminus (r-1), \qquad \lambda = \mathsf{U}_r^{-1}(\mu).$$

Thus in either case the triple $(\alpha, \widehat{r}, \lambda)$ is recovered uniquely from (α', μ) and $\lambda \in \mathcal{F}_r$.

Finally, applying the inverse map of Konan's bijection at level r, we obtain

$$\gamma := \mathsf{K}_r^{-1}(\lambda) \in \mathcal{M}_r, \qquad \beta := \delta_r \sqcup \gamma,$$

and therefore a unique element $(U, U; \alpha, \beta) \in \{(U, U)\} \times \mathcal{C}_r$ is derived. Denoting this reverse map by Ψ_r, together with Θ_r gives

$$\Psi_r(\Theta_r(U, U; \alpha, \beta)) = (U, U; \alpha, \beta).$$

Hence Ψ_r is a left inverse of Θ_r, and thereby Θ_r is injective.

Konan's map preserves size and length, while Lemma 4.2 gives

$$|\alpha'| + |\gamma'| = |\alpha| + r + |\gamma|, \qquad \ell(\alpha') = \ell(\alpha) + 1, \qquad \ell(\gamma') = \ell(\gamma).$$

Then by the relations $|\delta_r| - |\delta_{r-1}| = r$ and $\ell(\delta_r) - \ell(\delta_{r-1}) = 1$, it yields the two conditions

$$|\alpha'| + |\beta'| = |\alpha| + |\beta|, \qquad \ell(\alpha') - \ell(\beta') = \ell(\alpha) - \ell(\beta) + 2.$$

After the two atoms (U, U) are included, the initial element $(U, U; \alpha, \beta)$ and its image $\Theta_r(U, U; \alpha, \beta)$ contribute to the terms of the same bidegree in the generating function, and thus the injection gives (4.1). □

We now use an example to illustrate each step of the injection Θ_r. Let

$$r = 2, \qquad \alpha = \varnothing, \qquad \beta = (4, 4, 2, 1),$$

and take the two atoms (U, U) into consideration. Since $\mathrm{mex}(\beta) = 3$, we have $(\alpha, \beta) \in \mathcal{C}_2$.

(1) *Remove the staircase δ_r from β.* The staircase at level 2 is $\delta_2 = (2, 1)$, and hence

$$\beta = \delta_2 \sqcup \gamma, \qquad \gamma = (4, 4).$$

Moreover, $\mathrm{mex}_2(\gamma) = 3$, so $\gamma \in \mathcal{M}_2$.

(2) *Encode by Konan's bijection on γ.* Following the detailed procedures as given in Appendix A, it gives

$$\lambda = \mathsf{K}_2(\gamma) = \mathsf{K}_2(4, 4) = (5, 3).$$

By the Frobenius symbol, we have

$$(4, 4) \longleftrightarrow \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}, \qquad (5, 3) \longleftrightarrow \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $\mathrm{Top}(\lambda) = \{4, 1\}$ avoids 2, as required for $\lambda \in \mathcal{F}_2$. The encoding of Konan's map preserves both size and length: $|\lambda| = |\gamma| = 8$ and $\ell(\lambda) = \ell(\gamma) = 2$.

(3) *Convert the removable B-colored part and the two atoms into a marked A-colored part $\widehat{r}$.* Replace δ_2 with $\delta_1 = (1)$ and remove the B-colored part 2^B. Together with the two atoms U, this part has total weight

$$(q^2 z^{-1}) z^2 = q^2 z,$$

which can be seen as the weight of a marked A-colored part of size 2.

(4) *Apply B_r to $(\alpha, \widehat{r}, \lambda) = (\varnothing, \widehat{2}, (5, 3))$.* Since $r - 1 = 1$ occurs in $\mathrm{Top}(\lambda) = \{4, 1\}$, we apply the second branch of B_2 which raises the top-row entry 1 to 2, and leaves the bottom row unchanged, which leads to

$$\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix}.$$

The latter Frobenius symbol corresponds to $\mu = (5, 4)$. Consequently,

$$\mathsf{B}_2(\varnothing, \widehat{2}, (5,3)) = ((1),(5,4)),$$

so $\alpha' = (1)$ and $\mu = (5,4)$. Notice that $\mathrm{Top}(\mu) = \{4,2\}$ omits 1, and hence $\mu \in \mathcal{F}_1$.

(5) *Decode at the new level* $\mathsf{K}_{r-1}^{-1}(\mu)$. At level 1, $\mathrm{mex}_1(5,4) = 2$, so $(5,4) \in \mathcal{M}_1$. Therefore

$$\gamma' = \mathsf{K}_1^{-1}(\mu) = \mathsf{K}_1^{-1}(5,4) = (5,4).$$

(6) *Attach* δ_{r-1} *to derive* $\beta' = \delta_{r-1} \sqcup \gamma'$. Adjoining $\delta_1 = (1)$ gives

$$\beta' = \delta_1 \sqcup \gamma' = (5,4,1).$$

Since $\mathrm{mex}(\beta') = 2 = 1 + 2 \cdot 0 + 1$, the output (α', β') belongs to $\mathcal{C}_1$. Altogether, we have

$$\Theta_2(U, U; \varnothing, (4,4,2,1)) = ((1),(5,4,1)).$$

Note that the above process can be reversed step by step to recover the original elements. Under the injection Θ_r, the bidegrees contributing to the generating functions can be checked directly. As in the above example, the initial element in the invariant subset $\mathcal{C}_2$ has weight

$$q^{|\beta|} z^{-\ell(\beta)} = q^{11} z^{-4},$$

and multiplied by the weight of the two atoms gives $q^{11} z^{-2}$. The target element in $\mathcal{C}_1$ has weight

$$q^{|\alpha'|+|\beta'|} z^{\ell(\alpha')-\ell(\beta')} = q^{1+10} z^{1-3} = q^{11} z^{-2}.$$

Thus the total weight is preserved.

Now, we are ready to prove our main result.

Proof of Theorem 1.2. It is easy to see that

$$1 + z + \cdots + z^{2k} = \sum_{m=0}^{2k-2} z^m + z^{2k-1}(1+z).$$

For $a \geq 0$, denote $r_a = k + 2a + 1$. In (3.2), the negative blocks occur exactly at $r = r_a$ for some a and we pair them with the preceding positive blocks at $r_a - 1$ for $a > 0$. When $a = 0$, the needed positive block comes from $z^{2k-1}(1+z)$ multiplied by $C_k(z,q)$. Hence

$$J_k(z,q) = \sum_{m=0}^{2k-2} z^m \cdot C_k(z,q) + \sum_{a=0}^{\infty} z^{2r_a-3}(1+z)\Big(C_{r_a-1}(z,q) - z^2 C_{r_a}(z,q)\Big).$$

For $k \geq 1$, each $C_k(z,q)$ is a generating function with nonnegative coefficients, and each difference in the parentheses is nonnegative by using (4.1). Thus $J_k(z,q)$ is coefficientwise nonnegative. □

From Theorem 1.2, Merca's stronger conjecture on truncated Jacobi triple product tails can be verified directly.

Proof of Corollary 1.3. Let $1 \leq S < R$. A monomial $q^n z^s$ contributed by the jth summand in (1.4) arises from the product of a monomial $q^{n-T_j} z^v$ in the expansion of $1/(zq, q/z; q)_\infty$, with $|v| \leq n - T_j$, and a monomial z^u in $\chi_j(z)$, with $|u| \leq j$. Note that $j \geq k \geq 1$ and $T_j \geq j$, we then have

$$|s| \leq j + n - T_j \leq n.$$

By setting $(z,q) = (q^S, q^R)$, the monomial $q^n z^s$ becomes q^{Rn+Ss}, where

$$Rn + Ss \geq (R-S)n.$$

For each fixed exponent $N \geq 0$, only pairs (n,s) satisfying $0 \leq n \leq N/(R-S)$ and $|s| \leq n$ can contribute to the coefficient of q^N, and there are only finitely many such pairs. Moreover,

$$\chi_j(q^S) = q^{-Sj}\frac{1-q^{(2j+1)S}}{1-q^S}, \qquad (q^{R+S}, q^{R-S}; q^R)_\infty = \frac{(q^S, q^{R-S}; q^R)_\infty}{1-q^S}.$$

Substituting these relations into (1.4) gives

$$J_k(q^S, q^R) = \frac{\displaystyle\sum_{j=k}^{\infty} (-1)^{j-k} q^{RT_j - Sj}\left(1 - q^{(2j+1)S}\right)}{(q^S, q^{R-S}; q^R)_\infty} = \mathcal{T}_{R,S,k}(q).$$

Therefore, Theorem 1.2 implies that every coefficient of $\mathcal{T}_{R,S,k}(q)$ is nonnegative. □

Based on Theorem 1.2 and Corollary 1.3, we conclude this section with some partition inequalities. Let $p_2(n,s)$ be the number of ordered pairs of partitions (α, β) satisfying $|\alpha| + |\beta| = n$ and $\ell(\alpha) - \ell(\beta) = s$. For $1 \leq S < R$, we let $p_{R,S}(N)$ count two-colored partitions of N with A-colored parts being congruent to $S \pmod R$ and B-colored parts being congruent to $-S \pmod R$. Set $p_2(n,s) = p_{R,S}(n) = 0$ for $n < 0$. By extracting the coefficient $[q^n z^s]$ of $J_k(z,q)$ in Theorem 1.2 and the coefficient $[q^N]$ of $\mathcal{T}_{R,S,k}(q)$ in Corollary 1.3, respectively, we obtain the following inequalities.

Corollary 4.4 *(Truncated partition inequalities). For $k \geq 1$, $n \geq 0$, and $s \in \mathbb{Z}$,*

$$\sum_{j=k}^{\infty}(-1)^{j-k}\sum_{u=-j}^{j} p_2(n-T_j, s-u) \geq 0.$$

For $1 \leq S < R$ and $N \geq 0$,

$$\sum_{j=k}^{\infty}(-1)^{j-k}\Big(p_{R,S}(N-RT_j+Sj) - p_{R,S}(N-RT_j-(j+1)S)\Big) \geq 0.$$

5. Concluding remarks

Our proof divides the coefficientwise nonnegativity problem into two combinatorial stages. First, the involution on quadruples transforms our target into invariant subsets according to the mex. Second, the injection Θ_r maps each negative invariant subset associated to $z^2C_r(z,q)$ into the preceding positive one associated to $C_{r-1}(z,q)$ which preserves the bidegree of (q,z) in the generating functions. A natural next question is whether the invariant subset inequality

$$z^2C_r(z,q) \preceq C_{r-1}(z,q)$$

admits a direct map on the colored pairs (α,β) which gives an explicit partition-theoretic interpretation of the coefficientwise nonnegativity of $C_{r-1}(z,q) - z^2C_r(z,q)$. Moreover, it is also interesting to consider which kind of truncated theta series admits an analogous generalized-mex–Frobenius injection.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Konan's bijection

This appendix is included to introduce the detailed procedures of Konan's bijection $\mathsf{K}_r : \mathcal{M}_r \longrightarrow \mathcal{F}_r$ on the following two sets

$$\mathcal{M}_r = \{\lambda : \mathrm{mex}_r(\lambda) - r \text{ is odd}\}, \qquad \mathcal{F}_r = \{\lambda : r \notin \mathrm{Top}(\lambda)\},$$

which can be found in [21, Sections 2–4].

A.1. The forward and inverse maps

We adopt the notation of the staircase as given in [21, Section 2]:

$$\Delta_{r,t} = (r+t, r+t-1, \ldots, r+1), \qquad \Delta_{r,0} = \varnothing.$$

It is also compatible with the terminal state used in the bijection.

For a partition λ, denote $\lambda_0 = \infty$ and

$$d_s(\lambda) = \max\{i \in \{0, 1, \ldots, \ell(\lambda)\} : \lambda_i - i \geq s\}.$$

Then $s \in \mathrm{Top}(\lambda)$ exactly when $d_s(\lambda) \geq 1$ and $\lambda_{d_s(\lambda)} - d_s(\lambda) = s$.

Let $\gamma \in \mathcal{M}_r$ and write

$$\mathrm{mex}_r(\gamma) = r + 2t + 1, \qquad \gamma = \Delta_{r,2t} \sqcup \lambda,$$

where the part $r + 2t + 1$ is absent from λ. For $(\Delta_{r,2t}, \lambda)$, set $s = r + 2t$ and $d = d_s(\lambda)$. Konan's forward map is iterated according to the following two moves.

Move K1: $s \in \mathrm{Top}(\lambda)$**.** Let $\lambda_d = d + s$. Delete this critical part, add one to the first $d-1$ parts, and insert a part $s+1$:

$$\begin{aligned}&(\Delta_{r,2t}; \lambda_1, \ldots, \lambda_{d-1}, d+s, \lambda_{d+1}, \ldots)\\ &\quad\longmapsto (\Delta_{r,2t}; \lambda_1 + 1, \ldots, \lambda_{d-1} + 1, s+1, \lambda_{d+1}, \ldots).\end{aligned} \tag{K1}$$

The list on the right-hand side of (K1) specifies a multiset of residual parts, which must be reordered into weakly decreasing order before the next step. In particular, the inserted part $s+1$ need not occupy the dth position; its ordered placement is made explicit in [21, Proposition 15]. For example, if $r = 2$ and $\gamma = (5, 4, 4)$, then $t = 0$, $s = 2$, and $d = 2$. The first K1 move gives the list $(6, 3, 4)$, so the next residual partition is $(6, 4, 3)$. At each subsequent step, $d = d_s(\lambda)$ is evaluated using the current reordered residual partition.

Under this move, the staircase keeps unchanged, and the residual partition corresponding to λ keeps both the size and the length.

Move K2: $s \notin \mathrm{Top}(\lambda)$ **and** $t \geq 1$**.** Remove the two largest parts $s, s-1$ from the staircase, subtract one from the first d residual parts, and insert $d+s$ and $s-1$:

$$\begin{aligned}&(\Delta_{r,2t}; \lambda_1, \ldots, \lambda_d, \lambda_{d+1}, \ldots)\\ &\quad\longmapsto (\Delta_{r,2t-2}; \lambda_1 - 1, \ldots, \lambda_d - 1, d+s, s-1, \lambda_{d+1}, \ldots),\end{aligned} \tag{K2}$$

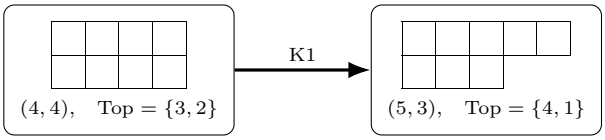


Fig. 3. A one-step instance of Konan's K1 move. The operation preserves size and length while removing the forbidden Frobenius arm 2.

with the residual parts reordered. Under this move, the size of λ increases by $s+(s-1)$ and its length increases by two, exactly offsetting the removed parts of the staircase.

Iteration stops at a state $(\Delta_{r,0};\nu)$ with $r \notin \mathrm{Top}(\nu)$. Konan sets $\mathsf{K}_r(\gamma)=\nu$ and proves that this procedure terminates and has an inverse. The inverse algorithm for K_r^{-1} is given in [21, Section 3.2].

A.2. Examples

We first show the example corresponding to the one after Theorem 4.3. Let $r=2$ and $\gamma=(4,4)$. Since $\mathrm{mex}_2((4,4))=3$, we have $t=0$ and the shifted staircase $\Delta_{r,2t}$ is empty. The top row of $(4,4)$ is $(3,2)$, so K1 should be applied with $s=2$ and $d=2$. We delete the second part 4, increase the first part to 5 and insert 3. Thus

$$\mathsf{K}_2(4,4)=(5,3).$$

Both partitions have size 8 and length 2, and $\mathrm{Top}(5,3)=\{4,1\}$ avoiding 2. See Fig. 3.

We further consider a more involved example. Take

$$r=2, \qquad \gamma=(6,5,4,3).$$

Since

$$\mathrm{mex}_2(\gamma)=7=2+2\cdot 2+1,$$

we have $t=2$, and the initial decomposition is

$$\gamma=\Delta_{2,4}\sqcup\varnothing, \qquad \Delta_{2,4}=(6,5,4,3).$$

At the initial state, $s=r+2t=6$ and $d_6(\varnothing)=0$. Since $6\notin\mathrm{Top}(\varnothing)$, K2 removes the two largest parts 6 and 5 of the staircase, and inserts these two parts into the residual partition. Hence

$$(\Delta_{2,4};\varnothing)\xrightarrow[s=6,\ d=0]{\mathrm{K2}}(\Delta_{2,2};(6,5)).$$

At the second state,

$$s=4, \qquad d_4(6,5)=1, \qquad \mathrm{Top}(6,5)=\{5,3\}.$$

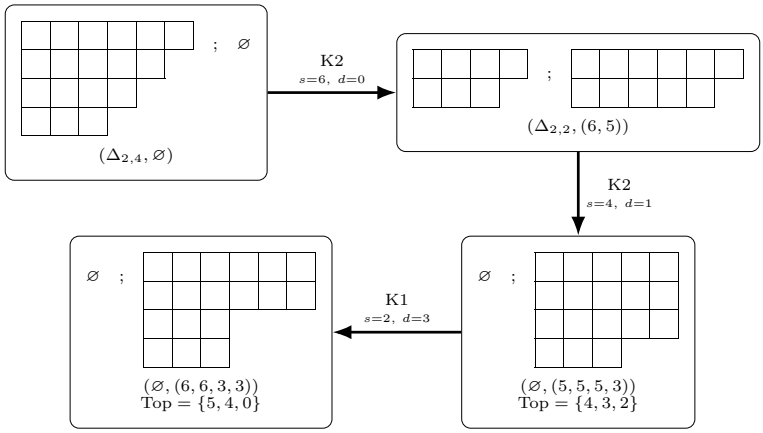


Fig. 4. The complete orbit of Konan's forward map for $r = 2$ and $\gamma = (6, 5, 4, 3)$.

Thus $4 \notin \mathrm{Top}(6, 5)$, and we apply K2 once more. This deletes the staircase parts 4 and 3, decreases the first residual part 6 to 5, and inserts

$$d + s = 5, \qquad s - 1 = 3.$$

After reordering the residual parts, this gives

$$(\Delta_{2,2}; (6, 5)) \xrightarrow[s=4,\ d=1]{\text{K2}} (\varnothing; (5, 5, 5, 3)).$$

The shifted staircase is now empty, but the algorithm has not yet terminated. Note that

$$\mathrm{Top}(5, 5, 5, 3) = \{4, 3, 2\},$$

so the forbidden Frobenius arm $r = 2$ is still present. Here

$$s = 2, \qquad d_2(5, 5, 5, 3) = 3, \qquad \lambda_3 = 5 = d + s.$$

Using move K1, delete the third part 5, increase the first two parts from 5 to 6, and insert $s + 1 = 3$. Consequently, we have

$$(\varnothing; (5, 5, 5, 3)) \xrightarrow[s=2,\ d=3]{\text{K1}} (\varnothing; (6, 6, 3, 3)).$$

Finally, $\mathrm{Top}(6, 6, 3, 3) = (5, 4, 0)$, which does not contain 2. Thus the resulting partition belongs to $\mathcal{F}_2$, and

$$\mathsf{K}_2(6, 5, 4, 3) = (6, 6, 3, 3).$$

The preservation of size and length can also be observed directly from the complete execution of Konan's forward map for $r = 2$ and $\gamma = (6, 5, 4, 3)$ as illustrated in Fig. 4.

Data availability

No data was used for the research described in the article.

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