Richardson tableaux and noncrossing partial matchings

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Abstract

Richardson tableaux are a remarkable subfamily of standard Young tableaux introduced by Karp and Precup to index the irreducible components of Springer fibers equal to Richardson varieties. We prove that the set of insertion tableaux of noncrossing partial matchings on $\{1, 2, \ldots, n\}$ by applying the Robinson–Schensted algorithm exactly coincides with the set of Richardson tableaux of size n. This leads to a natural one-to-one correspondence between the set of Richardson tableaux of size n and the set of Motzkin paths with n steps, in response to a problem proposed by Karp and Precup. As applications, we recover some known properties and establish new ones for Richardson tableaux.

Keywords: Richardson tableau, Springer fiber, Robinson–Schensted algorithm, evacuation map, noncrossing partial matching, q-Catalan number

AMS Classifications: 05E10, 05E14, 05A19, 14N15

1 Introduction

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ be a partition of n with ℓ parts, and N be an $n \times n$ nilpotent matrix which is in Jordan canonical form with block sizes $\lambda_1, \ldots, \lambda_\ell$ weakly decreasing. The Springer fiber $\mathcal{B}_\lambda$ indexed by λ is a subvariety of the flag variety $\mathrm{Fl}_n(\mathbb{C}) = \{F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n \colon \dim(F_j) = j\}$ [24]. Precisely,

$$\mathcal{B}_\lambda = \{F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n \in \mathrm{Fl}_n(\mathbb{C}) \colon N(F_j) \subset F_{j-1} \text{ for } 1 \leq j \leq n\}.$$

The cohomology ring of a Springer fiber carries a representation of the symmetric group where the top-dimensional cohomology is irreducible [25]. By the work of Spaltenstein [22], the irreducible components, denoted $\mathcal{B}_T$, of $\mathcal{B}_\lambda$ are indexed by standard Young tableaux T of shape λ. It remains an open problem, originally posed by Springer [26], to expand the cohomology class $[\mathcal{B}_T]$ in terms of the basis of Schubert classes in the cohomology of $\mathrm{Fl}_n(\mathbb{C})$.

Some partial answers to Springer's problem are known. For example, Güemes [10] computed the Schubert expansions for Springer fiber components in the hook-shape cases, and Graham and Zierau [11] obtained formulas, in equivariant cohomology and K-theory, for smooth components arising from K-orbits. Recently, motivated by the study of totally nonnegative Springer fibers by Lusztig [17], Karp and Precup [12] showed that $\mathcal{B}_T$ is a Richardson variety if and only if T is a Richardson tableau. Later, Spink and Tewari [23] found a combinatorial formula to expand the classes of such Richardson varieties in terms of Schubert classes, which, together with [12], gives an answer to Springer's problem in the case when T is a Richardson tableau. For the interactions between tableau combinatorics and components of Springer fibers, see also [5, 6, 29].

This paper focuses on the combinatorics of Richardson tableaux. Using the generating function method, Karp and Precup [12] proved that the number of Richardson tableaux of size n is equal to the number of Motzkin paths with n steps. They [12, Problem 4.5] asked for an explicit correspondence between Richardson tableaux and Motzkin paths. Our goal is to provide such a construction which is achieved by applying the Robinson–Schensted (RS) algorithm to noncrossing partial matchings. From this perspective, some properties of Richardson tableaux are revealed in a more intuitive manner. We hope that the RS algorithm could shed some new light on the study of the geometry of Springer fibers.

Let us proceed with some terminology and notation. We always identify a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell > 0)$ of n with its Young diagram, namely, a left-justified array of boxes with λ_i boxes in row i. A semistandard Young tableau T of shape λ is a filling of positive integers into the boxes of λ such that each box receives exactly one number, the numbers in each row are weakly increasing from left to right, and the numbers along each column are strictly increasing from top to bottom. A semistandard tableau T is called standard if the integers appearing in T are exactly $1, 2, \ldots, n$. See Figure 1.1 for an illustration.

1	1	2
2	3	
3	4	

1	2	4
3	5	
6	7	

Figure 1.1. Two semistandard Young tableaux. The right tableau is standard.

A partial matching on $[n] = \{1, 2, \ldots, n\}$ is a set partition of $[n]$ such that each block contains either a single number or exactly two numbers. For a partial matching on $[n]$, the associated involution w is defined by setting $w(i) = i$ if $\{i\}$ is a singleton, and $w(i) = j$ if $\{i, j\}$ is a two-element block. For example, the involution corresponding to the partial matching $\{\{1, 5\}, \{2\}, \{3, 4\}, \{6\}, \{7, 8\}\}$ is 52431687. In the following, we shall not distinguish between a partial matching on $[n]$ and an involution of $[n]$.

We may represent a partial matching by its diagram: list n dots labeled $1, 2, \ldots, n$ from left to right, and then draw an arc between i and j if i, j lie in the same block, see Figure 1.2 for the diagram representations of two partial matchings. We use (i, j),

Figure 1.2. Two partial matchings on [8].

where $i < j$, to denote an arc connecting i and j. Let (i_1, j_1) and (i_2, j_2) $(i_1 < i_2)$ be two arcs. They form a crossing if $i_1 < i_2 < j_1 < j_2$, and a nesting if $i_1 < i_2 < j_2 < j_1$. A partial matching is called noncrossing if it has no crossing, see the left picture in Figure 1.2. Crossings and nestings were systematically studied by Chen, Deng, Du, Stanley and Yan [2], and also play an important role in the study of RNA structures [3]. Geometrically, noncrossing matchings appeared for exmaple in total nonnegativity [14] and two-row and two column components of Springer fibers [9, 13, 16, 28].

As a classical result, the RS algorithm associates with a permutation w of $[n]$ a pair (T, T') of standard tableaux of size n with the same shape. Denote $w \xrightarrow{\text{RS}} (T, T')$. Schützenberger's theorem [19] says that the inverse w^{-1} of w satisfies $w^{-1} \xrightarrow{\text{RS}} (T', T)$. This symmetry implies that w is an involution if and only if $T = T'$.

For a standard tableau T of size n and $1 \leq j \leq n$, let $T_{<j}$ be the subtableau occupied by the entries less than j. Let $r_j(T)$ be the row number in which j appears in T. We say that T is a Richardson tableau if it satisfies the following condition

> **Richardson condition**: For each $1 \leq j \leq n$ with $r_j(T) \geq 2$, the largest entry of $T_{<j}$ in row $r_j(T) - 1$ is greater than every entry of $T_{<j}$ lying in or strictly below row $r_j(T)$.

See [12, Theorem 1.5] for other equivalent characterizations for Richardson tableaux.

Our main result is stated as follows.

Theorem 1.1. *The set of insertion tableaux of noncrossing partial matchings on $[n]$ is exactly the set of Richardson tableaux of size n.*

We give applications of Theorem 1.1. The first one explains that Theorem 1.1 gives an answer to [12, Problem 4.5] raised by Karp and Precup.

Remark 1.2. *There is a bijection between noncrossing partial matchings on $[n]$ and Motzkin paths with n steps. A Motzkin path with n steps is a path from $(0, 0)$ to $(n, 0)$, using possibly up steps $(1, 1)$, down steps $(1, -1)$ and horizontal steps $(1, 0)$, which never goes strictly below the x-axis. Given a noncrossing partial matching on $[n]$, we construct a Motzkin path as follows. For $1 \leq i \leq n$, the i-th step of the path is $(1, 1)$ (resp., $(1, -1)$) if i is a left (resp., right) endpoint of an arc, and is $(1, 0)$ if $\{i\}$ is a singleton. Combined with Theorem 1.1, this yields a bijection between Motzkin paths and Richardson tableaux, answering [12, Problem 4.5].*

As shown in [12, Section 3], the set of Richardson tableaux is invariant under the evacuation map. This phenomenon can also be deduced from Theorem 1.1. Let $\mathrm{evac}(T)$ denote the evacuation tableau of T, see Section 2 for the definition.

Corollary 1.3 ([12, Corollary 3.23])**.** *The set of Richardson tableaux is closed under the evacuation map. That is, if T is a Richardson tableau, then so is* $\mathrm{evac}(T)$.

Proof. Let $w_0 = n \cdots 21$ be the longest permutation of $[n]$. It is well known that if $w \xrightarrow{\text{RS}} (T, T')$, then $w_0 w w_0 \xrightarrow{\text{RS}} (\mathrm{evac}(T), \mathrm{evac}(T'))$, see for example [27, Theorem A1.2.10].

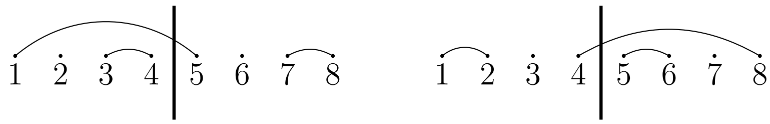

Figure 1.3. Diagrams of w and w_0ww_0 for $w = 52431687$.

Assume now that w is a noncrossing involution. Note that w_0ww_0 is also an involution. Moreover, the diagram of w_0ww_0 is obtained from that of w after a reflection along a vertical line passing through the midpoint, see Figure 1.3 for an illustration. This implies w_0ww_0 is also a noncrossing involution. So, invoking Theorem 1.1, the insertion tableau $\mathrm{evac}(T)$ of w_0ww_0 is a Richardson tableau. ∎

A prime Richardson tableau is a Richardson tableau which cannot be decomposed into a concatenation of smaller Richardson tableaux, see Section 3 for the precise definition. For $n \geq 2$, the shape of a prime Richardson tableau must be of the form $\lambda = (\lambda_1, \ldots, \lambda_{\ell-2}, 1, 1)$, that is, the last two rows have length equal to one [12, Corollary 3.15]. Theorem 1.1 leads to an alternative explanation of the following relation.

Corollary 1.4 ([12, Corollary 4.3])**.** *The number of prime Richardson tableaux of shape* $\lambda = (\lambda_1, \ldots, \lambda_{\ell-2}, 1, 1)$ *equals the number of Richardson tableaux of shape* $(\lambda_1, \ldots, \lambda_{\ell-2})$.

Proof. See Remark 3.6. ∎

Remark 1.5. *Since the proof of Theorem 1.1 uses several properties concerning Richardson tableaux from [12], we do not give entirely new proofs of Corollaries 1.3 and 1.4. However, Theorem 1.1 provides a nicer perspective and an intuitive explanation for these two corollaries.*

We say that a column of a standard tableau is odd (resp., even) if it has an odd (resp., even) number of boxes. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the n-th Catalan number.

Corollary 1.6. *Let* $0 \leq k \leq n$ *with* $k \equiv n \pmod 2$*. The number of Richardson tableaux of size* n *and with* k *odd columns is equal to*

$$\binom{n}{k} C_{\frac{n-k}{2}}. \tag{1.1}$$

Proof. A theorem due to Schützenberger [20] states that the number of fixed points of an involution equals the number of odd columns in its insertion tableau, see also [1]. So (1.1) is equivalent to showing that the number of noncrossing partial matchings on $[n]$ with k singletons is $\binom{n}{k}C_{\frac{n-k}{2}}$. To see this, there are $\binom{n}{k}$ ways to choose k elements of $[n]$ as singletons, and the remaining $n-k$ elements form noncrossing perfect matchings which are counted by the Catalan number $C_{\frac{n-k}{2}}$. ∎

We also consider q-analogues of Corollary 1.6, which turn out to be closely related to q-Catalan numbers. It is worth pointing out that q-Catalan numbers (more generally, rational q-Catalan numbers) have emerged in counting points of certain Richardson varieties over a finite field by Galashin and Lam [8]. For a permutation w of $[n]$, a position $i \in [n-1]$ is a descent (resp., ascent) if $w(i) > w(i+1)$ (resp., $w(i) < w(i+1)$). Let $\mathrm{maj}(w)$ (resp., $\mathrm{comaj}(w)$) be the sum of descents (resp., ascents) of w. For a standard tableau T of size n, an entry $i \in [n-1]$ is a descent of T if $i+1$ appears in a row strictly below i, and otherwise, we call i an ascent. Similarly, we use $\mathrm{maj}(T)$ (resp., $\mathrm{comaj}(T)$) to denote the sum of descents (resp., ascents) of T. Clearly,

$$\mathrm{maj}(w) + \mathrm{comaj}(w) = \mathrm{maj}(T) + \mathrm{comaj}(T) = \binom{n}{2}.$$

A Richardson tableau is called even if it has no odd columns. Evidently, an even Richardson tableau must have even size. For $m \geq 0$, write $[m]_q = \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1}$, and $[m]_q! = [1]_q[2]_q \cdots [m]_q$. We establish the following q-counting for even Richardson tableaux.

Corollary 1.7. *For $n \geq 1$, write $\mathrm{ERT}(2n)$ for the set of even Richardson tableaux of size $2n$. Then we have*

$$\sum_{T \in \mathrm{ERT}(2n)} q^{\mathrm{comaj}(T)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{[2n]_q!}{[n+1]_q! \cdot [n]_q!}, \tag{1.2}$$

which is the n-th q-Catalan number. For $0 \leq m \leq n-1$, let $\mathrm{ERT}(2n, m)$ denote the set of even Richardson tableaux of size $2n$ with m ascents. Then we have the following refinement via the q-Narayana number:

$$\sum_{T \in \mathrm{ERT}(2n,m)} q^{\mathrm{comaj}(T)} = \frac{1}{[n]_q} \begin{bmatrix} n \\ m \end{bmatrix}_q \begin{bmatrix} n \\ m+1 \end{bmatrix}_q q^{m^2+m}. \tag{1.3}$$

Proof. We first prove (1.2). Let T be an even Richardson tableau of size $2n$, which, by the proof of Corollary 1.6, corresponds to a noncrossing involution w of $[2n]$ without fixed points. By Remark 1.2, w corresponds to a Dyck path $D = D_1 D_2 \cdots D_{2n}$ where D_i is either an up step $(1, 1)$ or a down step $(1, -1)$. We say $i \in [2n-1]$ is a valley point if $D_i = (1, -1)$ and $D_{i+1} = (1, 1)$. Let $\mathrm{maj}(D)$ denote the sum of valley points of D. A famous q-counting due to MacMahon [15] (see also [7]) says that

$$\sum_{D} q^{\mathrm{maj}(D)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q, \tag{1.4}$$

where the sum runs over Dyck paths with $2n$ steps.

By [27, Lemma 7.23.1], it is known that i is a descent of w if and only if it is a descent of T, and so we have $\mathrm{maj}(w) = \mathrm{maj}(T)$, or equivalently, $\mathrm{comaj}(w) = \mathrm{comaj}(T)$. Moreover, we have the following crucial observation:

$$i \text{ is a valley point of } D \text{ if and only if it is an ascent of } w.$$

Therefore, we have $\mathrm{maj}(D) = \mathrm{comaj}(w)$, and hence $\mathrm{maj}(D) = \mathrm{comaj}(T)$. This, along with (1.4), leads to (1.2).

To conclude the identity in (1.3), it is enough to notice that

$$\sum_{D} q^{\mathrm{maj}(D)} = \frac{1}{[n]_q} \begin{bmatrix} n \\ m \end{bmatrix}_q \begin{bmatrix} n \\ m+1 \end{bmatrix}_q q^{m^2+m}, \tag{1.5}$$

where D is restricted to Dyck paths with $2n$ steps and m valley points. The formula in (1.5) was stated by MacMahon without proof, see [7, Section 4] for a proof. ∎

We conjecture that Corollary 1.6 generally has the following q-analogue which has been checked for n up to 14.

Conjecture 1.8. *For $0 \le k \le n$ with $k \equiv n \pmod 2$, let $\mathrm{RT}(n,k)$ denote the set of Richardson tableaux of size n and with k odd columns. Then*

$$\sum_{T \in \mathrm{RT}(n,k)} q^{\mathrm{comaj}(T)} = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q C_{\frac{n-k}{2}}(q), \tag{1.6}$$

where $C_m(q) = \frac{1}{[m+1]_q} \begin{bmatrix} 2m \\ m \end{bmatrix}_q$ denotes the m-th q-Catalan number. Equivalently,

$$\sum_{w} q^{\mathrm{comaj}(w)} = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q C_{\frac{n-k}{2}}(q),$$

where the sum is over all noncrossing involutions of $[n]$ with exactly k fixed points.

Setting $k = 0$ and replacing n by $2n$, the formula in (1.6) specializes to (1.2).

Remark 1.9. *A formula for q-counting the statistic $\mathrm{maj}(T)$ of Richardson tableaux of any fixed shape has been explored in [12, Theorem 4.8]. We do not know how to deduce this formula from the perspective of noncrossing involutions.*

By Corollary 1.3, the evacuation map acts on the set of Richardson tableaux of size n. In fact, this is an involution since $\mathrm{evac}(\mathrm{evac}(T)) = T$. Let M_n denote the n-th Motzkin number, counting Motzkin paths with n steps. As usual, we set $M_0 = 1$.

Corollary 1.10. *The number of Richardson tableaux of size n which are fixed by the evacuation map is equal to*

$$M_{\lfloor \frac{n}{2} \rfloor} + \sum_{k \ge 1} \sum_{1 \le i_1 < \cdots < i_k \le \lfloor \frac{n}{2} \rfloor} \prod_{p=1}^{k+1} M_{i_p - i_{p-1} - 1}, \tag{1.7}$$

where we set $i_0 = 0$ and $i_{k+1} = \lfloor \frac{n}{2} \rfloor + 1$.

Proof. In view of the proof of Corollary 1.3, we need to count noncrossing partial matchings P on $[n]$ which are invariant after a reflection along the vertical line (denoted L) passing through the midpoint. We first consider the case when n is even. Each arc (i,j) of P either lies on the left-hand/right-hand side of L, or intersects with L. In the latter case, we have $i \le \frac{n}{2} < j$. Since P is noncrossing and symmetric with respect to L, we must have $j = n + 1 - i$, that is, the arc (i, j) must be invariant after a reflection along L. Such an arc (i, j), where $1 \le i \le \frac{n}{2}$ and $j = n + 1 - i$, is called a big arc.

Note that the configuration of arcs lying on the left-hand side of L completely determines P. Suppose that P has k big arcs. If $k = 0$, then the dots $1, 2, \ldots, \frac{n}{2}$ form a noncrossing partial matching. Such noncrossing partial matchings are counted by $M_{\frac{n}{2}}$. We next consider $k \ge 1$. Let $1 \le i_1 < i_2 < \cdots < i_k \le \frac{n}{2}$ be the left endpoints of these big arcs. Set $i_0 = 0$ and $i_{k+1} = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then, for $1 \le p \le k+1$, the local configuration on the dots $i_{p-1}+1, \ldots, i_p - 1$ between i_{p-1} and i_p forms a noncrossing partial matching. So these noncrossing partial matchings contribute

$$\prod_{1 \le p \le k+1} M_{i_p - i_{p-1} - 1}.$$

This verifies the formula in (1.7).

In the case when n is odd, we notice that the midpoint $\frac{n+1}{2}$ must be a singleton. The remaining arguments are completely analogous to the even case. ∎

This paper is organized as follows. In Section 2, we review the necessary background on the Robinson–Schensted algorithm and the evacuation map acting on standard Young tableaux. In Section 3, we present a proof of Theorem 1.1. Some remarks are given in Section 4.

Acknowledgements

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2 RS algorithm and evacuation map

We give an overview of the Robinson–Schensted algorithm, the evacuation map as well as the definition of Richardson tableaux. We refer to [27, Chapter 7] for more detailed information.

Given a semistandard tableau T and positive integer i, the Robinson–Schensted-Knuth (RSK) algorithm produces a new semistandard tableau, denoted $T \leftarrow i$, by inserting i into T. The procedure relies on a row bumping operation. First, look at the first row R of T. If i is greater than or equal to every entry in R, then place i at the end of R and stop. Otherwise, locate the leftmost entry, say i', in R which is greater than i, and replace this entry by i. In this case, we continue to insert the bumped entry i' into the second row by applying the same bumping operation. The procedure will eventually stop, and the resulting tableau is denoted $T \leftarrow i$. It can be seen that the shape of $T \leftarrow i$

has an extra corner box compared to the shape of T. In Figure 2.4, we illustrate the procedure of inserting $i = 1$ into the left semistandard tableau in Figure 1.1.

$$
\begin{array}{|c|c|c|}\hline 1 & 1 & \mathbf{2} \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array} \leftarrow 1
\qquad
\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & \mathbf{3} \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array} \;\leftarrow 2
\qquad
\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & \mathbf{4} \\ \cline{1-2}\end{array} \;\leftarrow 3
\qquad
\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}
$$

Figure 2.4. The bumping procedure in the RSK algorithm.

Remark 2.1. *The RSK row insertion is reversible. Specifically, let T be a semistandard tableau of shape λ, B a corner box of λ, and μ the partition obtained by removing B from λ. Then we can apply the reverse procedure of the RSK algorithm to recover the (unique) semistandard tableau P of shape μ and integer i such that $T = P \leftarrow i$.*

Let $w = w(1)w(2)\cdots w(n)$ be a permutation of $[n]$. The RSK algorithm can be used to generate a pair (T, T') of standard tableaux of the same shape. When applied to words without repeated entries, the RSK algorithm is usually called the Robinson–Schensted (RS) algorithm. Let T_0 be the empty tableau, and $T_i = T_{i-1} \leftarrow w(i)$ for $1 \le i \le n$. Set $T = T_n$. To define T', let λ^i be the shape of T_i. Clearly, λ^i is obtained from λ^{i-1} by adding a box in some corner. Then T' is the standard tableau such that the entry i is assigned in the corner box of λ^i not contained in λ^{i-1}. The standard tableaux T and T' are called the insertion and recording tableaux of w, respectively. This correspondence will be denoted $w \xrightarrow{\text{RS}} (T, T')$. A remarkable theorem due to Schützenberger states that if w^{-1} is the inverse of w, then $w^{-1} \xrightarrow{\text{RS}} (T', T)$, see also [27, Section 7.13]. So, when w is an involution, we have $T = T'$. This gives a bijection between the set of involutions of $[n]$ and the set of standard tableaux of size n.

We now describe the evacuation map. Let T be a standard tableau of shape λ. The evacuation map acts on T based on the jeu de taquin slide operation. Remove the entry 1 from the upper left corner of T, and decrease the remaining entries by 1. Next, slide this empty box until it reaches an outer corner of T. The rule is described below. Among the boxes directly below or directly to the right of this empty box, locate the one filled with a smaller number, and then swap it with the empty box. Repeat the same procedure until the empty box slides to a position which is a corner on the southeast border of λ. Now the entries $1, \ldots, n-1$ form a standard tableau, denoted $\Delta(T)$, of shape μ. Put the number n into the corner box of λ which is not contained in μ. Continuing this procedure eventually yields a standard tableau of shape λ, which is denoted $\mathrm{evac}(T)$.

For example, take T as the right standard tableau in Figure 1.1. Applying the operation Δ, we obtain a sequence of standard tableaux listed in Figure 2.5. So we get $\mathrm{evac}(T)$ as given in Figure 2.6.

Remark 2.2. *In the special case when T has rectangular shape $a \times b$ with $n = ab$, its evacuation has a particularly simple description (see [18, Lemma 3.1]):* $\mathrm{evac}(T)$ *can be obtained from T by rotating by $180°$ and replacing each entry r by $n + 1 - r$.*

$$
\underset{T}{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}}
\qquad
\underset{\Delta(T)}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array}}
\qquad
\underset{\Delta^2(T)}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}}
\qquad
\underset{\Delta^3(T)}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}}
\qquad
\underset{\Delta^4(T)}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}}
\qquad
\underset{\Delta^5(T)}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}}
\qquad
\underset{\Delta^6(T)}{\begin{array}{|c|}\hline 1 \\ \hline\end{array}}
$$

Figure 2.5. Successive applications of Δ to T.

$$
\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array}
$$

Figure 2.6. The evacuation tableau of T.

Recall that $w_0 = n \cdots 21$ denotes the longest permutation. The conjugate permutation $w_0 w w_0$ maps i to $n+1-w(n+1-i)$ for $1 \leq i \leq n$. The following connection was proved by Schützenberger [19], see also [27, Theorem A1.2.10].

Theorem 2.3 ([19])**.** *If $w \xrightarrow{\text{RS}} (T, T')$, then $w_0 w w_0 \xrightarrow{\text{RS}} (\text{evac}(T), \text{evac}(T'))$.*

From the above theorem, it can be seen that the evacuation map is actually an involution on standard tableaux, meaning that $\text{evac}(\text{evac}(T)) = T$.

3 Noncrossing involutions and Richardson tableaux

In this section, we establish the relationship between the insertion tableaux of noncrossing involutions and Richardson tableaux, thus finishing the proof of Theorem 1.1.

Let u and v be permutations of $[m]$ and $[n]$, respectively. The direct sum $u \oplus v$ of u and v is the permutation of $[m+n]$ defined by

$$u(1) \cdots u(m)\, v(1) + m \cdots v(n) + m,$$

that is, concatenate u and v, and then increase each entry $v(i)$ by m. For example, for $u = 2413$ and $v = 21$, we have $u \oplus v = 241365$. A permutation w is called prime if it cannot be written as a direct sum of smaller permutations.

Remark 3.1. *Let w be a noncrossing involution of $[n]$. Then w is prime if and only if $w(1) = n$. Equivalently, the diagram of w is either a fixed point (in the case $n = 1$) , or contains the arc $(1, n)$, see Figure 3.7 for an illustration. If $w(1) < n$, then we can write w as a direct sum of smaller permutations by restricting w to $\{1, \ldots, w(1)\}$ and $\{w(1)+1, \ldots, n\}$, respectively. Therefore, w can be uniquely written as a direct sum of prime noncrossing involutions.*

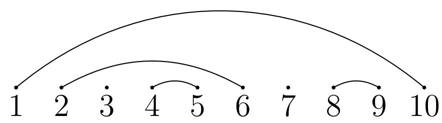

Figure 3.7. A prime noncrossing partial matching.

We now define the concatenation of standard tableaux [12, Section 3.3]. Let T be a standard tableau of size m, and T' be a standard tableau of size n. The concatenation $T \circ T'$ is a standard tableau of size $m+n$ defined by increasing each entry of T' by m, and then concatenating the corresponding rows of T and T'. This is best understood by an example as illustrated in Figure 3.8. Similarly, a standard tableau is prime if it

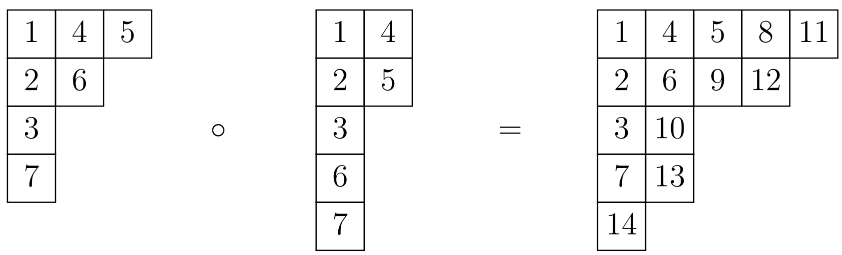

Figure 3.8. The concatenation of two standard tableaux.

cannot be decomposed into a concatenation of smaller standard tableaux. Note that each standard tableau has a unique decomposition into prime pieces. For example, the left two standard tableaux in Figure 3.8 are both prime Richardson tableaux.

Remark 3.2. *There is a geometric meaning of the concatenation [4, Proposition 7.1]:*

$$\mathcal{B}_{T\circ T'} \simeq \mathcal{B}_T \times \mathcal{B}_{T'},$$

which could reduce the study of $\mathcal{B}_T$ to the prime tableaux.

By the construction of the RS algorithm, the following fact is immediate.

Lemma 3.3. *Let T and T' be the insertion tableaux of u and v, respectively. Then the insertion tableau of $u \oplus v$ is equal to $T \circ T'$.*

Let $\mathrm{NC}(n)$ be the set of noncrossing involutions of $[n]$, and $\mathrm{RT}(n)$ be the set of Richardson tableaux of size n. The following is a restatement of Theorem 1.1.

Theorem 3.4. *For $n \geq 1$, there exists a bijection $\Phi\colon \mathrm{NC}(n) \to \mathrm{RT}(n)$ by sending $w \in \mathrm{NC}(n)$ to its insertion tableau.*

Before giving a proof of Theorem 3.4, we list the insertion tableaux of all partial matchings on [4], as illustrated in Figure 3.9. There are a total of ten partial matchings. The rightmost one in the second row is not noncrossing, whose insertion tableau is the only standard tableau of size 4 which is not Richardson. Moreover, among the nine Richardson tableaux, two are prime tableaux which exactly correspond to the two prime noncrossing partial matchings drawn in the fourth column.

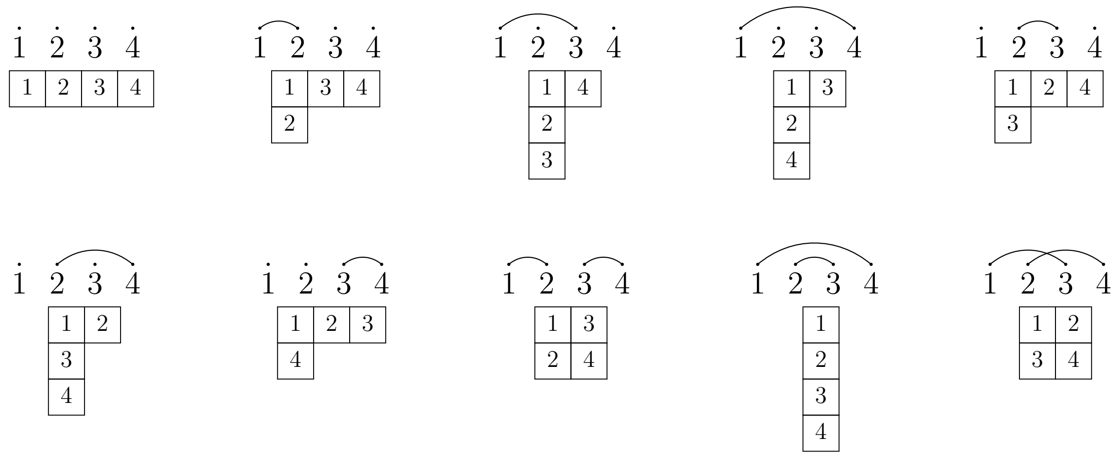

Figure 3.9. Insertion tableaux of partial matchings on [4].

Proof of Theorem 3.4. The construction is by induction on size n. For $n = 1$ or 2, this is clear as shown in Figure 3.10. Assume now that $n > 2$. We first show that Φ is an

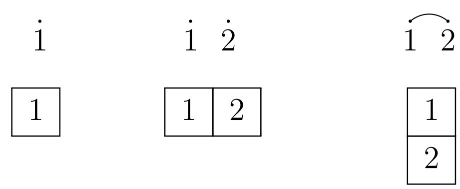

Figure 3.10. The correspondence Φ for $n = 1, 2$.

injection from $\mathrm{NC}(n)$ to $\mathrm{RT}(n)$. Let w be a noncrossing involution of $[n]$, and T be its insertion tableau. We need to verify that T is a Richardson tableau. There are two cases.

(1) The noncrossing involution w is not prime. In this case, we decompose $w = w^1 \oplus w^2 \oplus \cdots \oplus w^k$ into prime noncrossing involutions. It follows from Lemma 3.3 that $T = T_1 \circ T_2 \circ \cdots \circ T_k$, where T_i is the insertion tableau of the prime piece w^i. By induction, each T_i is a Richardson tableau. It is easily checked that the concatenation of Richardson tableaux still satisfies the Richardson condition given above Theorem 1.1, see also [12, Lemma 3.9] for a proof. This concludes that T is a Richardson tableau.

(2) The noncrossing involution w is prime. In this case, we have $w(1) = n$ and $w(n) = 1$. Let U be the insertion tableau of $w(1)w(2)\cdots w(n-1)$, and U' be the insertion tableau of $w(2)w(3)\cdots w(n-1)$. Since $w(1) = n$, it is easily seen that U is obtained from U' by placing a box filled with n as a new row on the bottom. Now T is obtained by inserting $w(n) = 1$ into U. From the construction of the RS algorithm, it follows that T is obtained from U by moving each entry in the first column down by one unit, and then placing 1 into the top-left corner box.

Let us explain the above construction by an example. In Figure 3.11, we list a prime noncrossing partial matching on [8], whose associated involution is 85342761, together with its corresponding tableaux T, U and U'.

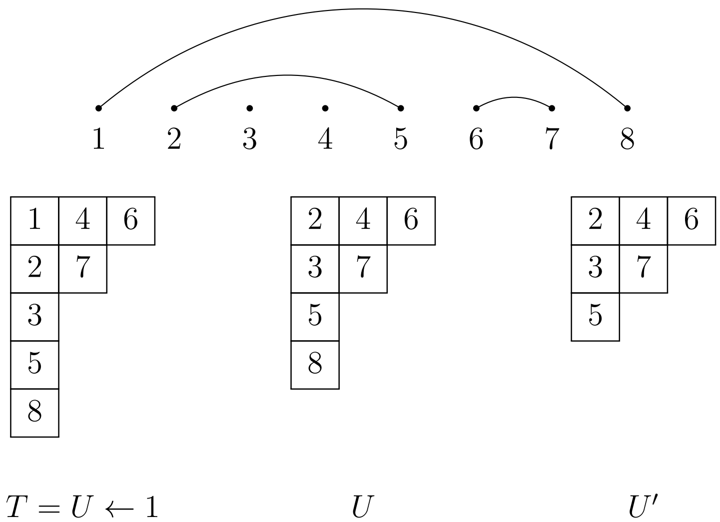

Figure 3.11. An illustration of the construction for the prime case.

Note that the tableau U' is the insertion tableau of a noncrossing involution of $\{2, 3, \ldots, n-1\}$. By induction, U' satisfies the Richardson condition. Since U is obtained by placing n on the bottom of U', it is clear that U also satisfies the Richardson condition. Furthermore, since T is obtained from U by shifting every entry in the first column down, it is obvious that T satisfies the Richardson condition, and so it is a Richardson tableau.

To complete the proof, we need to verify that for each Richardson tableau $T \in \mathrm{RT}(n)$, there exists a noncrossing involution w of $[n]$ whose insertion tableau equals T. The proof is also by induction on n. Similarly, the arguments are divided into two cases.

(1) The Richardson tableau T is not prime. Write $T = T_1 \circ T_2 \circ \cdots \circ T_k$ for the decomposition into prime tableaux. It is straightforward to check that each prime tableau in the decomposition satisfies the Richardson condition, and thus is a Richardson tableau (see also [12, Lemma 3.9]). By induction, there exists a unique noncrossing involution w^i whose insertion tableau is T_i. Set $w = w^1 \oplus w^2 \oplus \cdots \oplus w^k$. By Lemma 3.3, we see that $\Phi(w) = T$.

(2) The Richardson tableau T is prime. By [12, Corollary 3.15], the shape of T is of form $\lambda = (\lambda_1, \ldots, \lambda_{\ell-2}, 1, 1)$, and moreover it follows from [12, Corollary 3.16] that the largest number n appears in the bottom row. Let $T(i, j)$ denote the entry of T lying in row i and column j. Let U be the standard tableau on $\{1, 2, \ldots, n-1\}$ obtained from T by removing the lowest box containing n. Then U has $\ell - 1$ rows, and the last row of U has one box. The following claim is a key ingredient.

Claim. For any row index $2 \leq i \leq \ell - 1$ such that the box $(i-1, 2)$ lies in U, we have $U(i, 1) < U(i-1, 2)$.

The above claim can be proved by induction on n. By the definition of a Richardson tableau, it is clear that U is also a Richardson tableau. Using [12, Proposition 3.13], there exists a unique decomposition $U = P \circ Q$ where P is a prime Richardson tableau with exactly $\ell - 1$ rows, and Q is a Richardson tableau with at most $\ell - 2$ rows. So the box $(i, 1)$ is in P. If $(i-1, 2)$ lies in Q, the claim is clearly true since every entry in Q is greater than every entry in P. It remains to consider the case when $(i-1, 2)$ lies in P. In this case, since P is prime, the shape of P is of form $\mu = (\mu_1, \ldots, \mu_{\ell-3}, 1, 1)$. This, along with the assumption that $(i-1, 2)$ lies in P, implies that $i \leq \ell - 2$. We now apply induction to P, concluding that $P(i, 1) < P(i-1, 2)$. This verifies the claim since $U(i, 1) = P(i, 1)$ and $U(i-1, 2) = P(i-1, 2)$.

Now, we locate the lowest box $(\ell - 1, 1)$ of U. By Remark 2.1, we can perform the reverse procedure of the RS algorithm to get the tableau U' such that $U = U' \leftarrow x$ where the shape of U' is obtained from that of U by removing the box $(\ell - 1, 1)$. According to the above claim, we see that the 'reverse bumping route' is in the first column. This implies $x = 1$ where U' is simply obtained from U by deleting the entry 1 in the first column, and then moving the remaining entries in the first column up by one unit.

Since U is a Richardson tableau, by the above construction, it is obvious that U' is a tableau on $\{2, \ldots, n-1\}$ which satisfies the Richardson condition (as described above Theorem 1.1). Hence, by induction, there exists a unique noncrossing involution u of $\{2, \ldots, n-1\}$ such that the insertion tableau of u is U'. Let $w = n \cdot u \cdot 1$ be the prime noncrossing involution of $[n]$ by appending n and 1 respectively on the left and right sides of u. Then U is the insertion tableau of $u \cdot 1$. Notice that the insertion tableau of w is obtained from U by placing n as a single row on the bottom. This tells us that the insertion tableau of w is T.

Combining the above, we see that Φ is indeed a bijection, completing the proof. ∎

The construction in the proof of Theorem 3.4 yields the following byproduct.

Corollary 3.5. *If restricting Φ to the set of prime noncrossing involutions of $[n]$, then the image set is equal to the set of prime Richardson tableaux of size n.*

Remark 3.6. *There is a bijection between the set of prime noncrossing partial matchings on $[n]$ and the set of noncrossing partial matchings on $[n-2]$ by removing the arc $(1, n)$. So Corollary 3.5 induces a bijection, denoted Ψ, between the set of prime Richardson tableaux of size n and the set of Richardson tableaux of size $n-2$. Moreover, by the construction in the proof of Theorem 3.4, Ψ will send a prime Richardson tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_{\ell-2}, 1, 1)$ to a Richardson tableau of shape $(\lambda_1, \ldots, \lambda_{\ell-2})$. So Ψ yields an alternative proof of [12, Corollary 4.3] which is restated in Corollary 1.4.*

4 Concluding remarks

Following [12], for each standard tableau T, there is an associated Richardson variety R_{v_T, w_T} defined as follows. The reading word of T is obtained by reading the entries of T from bottom to top, and within each row from left to right. If we instead read the rows from top to bottom, then we get the top-down reading word of T. Then the permutations v_T and w_T are determined as follows: $w_0 w_T^{-1} w_0$ is the reading word of T, and v_T^{-1} is the top-down reading word of $\mathrm{evac}(T)$. It was shown that R_{v_T,w_T} is the unique minimal Richardson variety containing $\mathcal{B}_T$ [12, Theorem 7.9]. Now $v_T \leq w_T$ in the Bruhat order follows geometrically from the nonemptiness of R_{v_T,w_T}. It would be interesting to find a direct combinatorial proof of this Bruhat inequality from the definitions of v_T and w_T [12, Remark 6.2]. Another problem we would like to mention is to find the maximal Richardson varieties contained in $\mathcal{B}_T$ [12, Problem 7.10]. It is hoped that noncrossing matchings could play a helpful role for addressing the above two problems.

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