The spectrum estimates for random graphs with given expected degrees *

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Abstract

We consider the random graph model $G(\mathbf{w})$ for a given expected degree sequence $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, where the probability p_{ij} of an edge between vertices v_i and v_j is defined as $w_i w_j \rho$ with $\rho = \frac{1}{\sum_{i=1}^{n} w_i}$. In this paper, we apply a matrix concentration inequality to derive an upper bound on the largest eigenvalue of the adjacency matrix of the random graph with given expected degrees. We further analyze the expectation of the largest eigenvalue of the adjacency matrix and establish its concentration properties. Additionally, by utilizing an extension of the matrix Chernoff inequality that incorporates an intrinsic dimension parameter, we investigate the expectation and tail behavior of the largest eigenvalue of the Laplacian matrix for the random graph model $G(\mathbf{w})$.

Keywords: random graphs; expected degree sequences; adjacency matrix; Laplacian matrix; largest eigenvalue

Mathematics Subject Classification: 05C50, 15A18.

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*Supported by NSFC (No. 12001421, 12131013 and 12471334) and STU Scientific Research Initiation Grant (NTF25010T) and Shaanxi Fundamental Science Research Project for Mathematics and Physics (No.22JSZ009).

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1 Introduction

The spectra of the adjacency matrix and the Laplacian matrix of graphs have many applications in graph theory. For example, the spectrum of the adjacency matrix of a graph is related to its connectivity, the number of occurrences of specific subgraphs, chromatic number, and independence number. The graph is connected if and only if the second-smallest eigenvalue of the Laplacian matrix is strictly positive. The Laplacian matrix also serves as a major tool for enumerating spanning trees and has numerous applications. For further details on the applications of the spectra of the adjacency and Laplacian matrices, we refer interested readers to two monographs [4, 9].

Also for random graphs, spectral properties of adjacency matrices and Laplacian matrices of random graphs have received considerable attention (See, e.g., [1, 6, 7, 10, 11, 14, 15]). We next present a brief overview of some results on random graphs.

For the Erdős-Rényi random graph model $G(n,p)$, let $\lambda_1(A),\ldots,\lambda_n(A)$ denote the eigenvalues of its adjacency matrix $A = A(G(n,p))$ in nonincreasing order. Füredi and Komlós [15] showed that, asymptotically almost surely, $\lambda_1(A) = (1+o(1))np$ and $\max\{\lambda_2(A), -\lambda_{n-1}(A)\} \leq (2+o(1))\sqrt{np(1-p)}$, provided $np(1-p) \gg \log^6 n$. Here, the notation $o(1)$ denotes a quantity (depending on n) that tends to 0 as n tends to infinity. These results were later extended to sparse random graphs [11, 19] and general random matrices [10, 15]. Alon, Krivelevich, and Vu [1] investigated the concentration of the s-th largest eigenvalue of a random symmetric matrix with independent random entries bounded in absolute value by one. Ding et al. [10] studied the largest eigenvalue and spectral norm of the adjacency matrix of random graphs. Friedman et al. [12, 13, 14] proved that for any $d \geq 4$, the second largest eigenvalue (in absolute value) of random d-regular graphs is almost surely $(2+o(1))\sqrt{d-1}$. Chung, Lu, and Vu [7] analyzed the spectra of adjacency matrices of random graphs with given expected degrees. Preciado et al. [23] analyzed the limiting spectral distribution of adjacency matrices of random graphs with given expected degrees. Lu and Peng [20] examined concentration of adjacency matrices of edge-independent random graphs. Oliveira [21] studied concentration properties for adjacency matrices of inhomogeneous Erdös-Rényi graphs, with subsequent improvements by Chung and Radcliffe [8]. Benaych-Georges et al. [2] established bounds on the spectral radii for a large class of sparse random matrices, which include the adjacency matrices of inhomogeneous Erdös-Rényi graphs. Hu et al. [17, 18] investigated the spectra of Hermitian adjacency matrices of general random mixed graphs, and presented a separation result between the largest eigenvalue and the remaining eigenvalues of the

Hermitian adjacency matrix of random mixed graphs.

In this paper, we consider a random graph model $G(\mathbf{w})$ for random graphs with given expected degrees and examine the spectra of both the adjacency matrix and the Laplacian matrix. Applying a matrix inequality, we derive an upper bound on the largest eigenvalue of the adjacency matrix for the random graph $G(\mathbf{w})$, which improve the results in [6, 8] with a weaken assumption. We further analyze the expectation of the largest eigenvalue of the adjacency matrix and establish its concentration properties. Additionally, we investigate the expectation and tail behavior of the largest eigenvalue of the Laplacian matrix for the random graph $G(\mathbf{w})$.

The random graph model $G(\mathbf{w})$, first introduced in [5], is a generalization of the Erdős-Rényi model. For any non-negative sequence $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, the random graph $G(\mathbf{w})$ with a given general expected degree distribution is defined on the vertex set $\{v_1, v_2, \ldots, v_n\}$, in which edges are independently assigned to each vertex pair (v_i, v_j) with probability $w_i w_j \rho$, where $\rho = \frac{1}{\sum_{i=1}^{n} w_i}$. The classical Erdős-Rényi random graph model $G(n,p)$ can be viewed as a special case of the $G(\mathbf{w})$ model by setting $\mathbf{w} = (pn, pn, ..., pn)$. For convenience, we say v_i and v_j are connected if (v_i, v_j) is an edge of $G(\mathbf{w})$, denoted $v_i \sim v_j$. Loops are allowed in our model. Let $d(v_i)$ denote the number of vertices (including itself in the case of loops) connected to vertex v_i, this quantity $d(v_i)$ is called the degree of v_i. To ensure $w_i w_j \rho$ is a valid probability for all i and j, we assume $w_{\max}^2 = \max_i w_i^2 < \sum_k w_k$. It is straightforward to verify that the expected degree of vertex v_i in $G(\mathbf{w})$ is w_i for each i:

$$\mathbb{E}d(v_i) = \sum_j \Pr(v_i \sim v_j) = \sum_j w_i w_j \rho = w_i.$$

We use d_i to denote the actual degree of v_i in the random graph $G(\mathbf{w})$, where the weight w_i represents its expected degree. Let m denote the maximum expected degree, i.e., $m = w_{\max}$. The second-order average degree of $G(\mathbf{w})$ is defined as $\tilde{d} = \frac{\sum_i w_i^2}{\sum_i w_i} = \sum_i w_i^2 \rho$. Thus, $G(\mathbf{w})$ is referred to as a random graph model with given expected degrees.

The *adjacency matrix* of $G(\mathbf{w})$, denoted by $A(G(\mathbf{w})) = (\xi_{ij})$ (or A_n, for brevity), satisfies the following:

- A_n is a random symmetric matrix with $\xi_{ij} = \xi_{ji}$ for $1 \leq i \leq j \leq n$;
- the upper-triangular entries ξ_{ij} $(1 \leq i < j \leq n)$ are independent random variables which take the value 1 with probability $w_i w_j \rho$ and 0 with probability $1 - w_i w_j \rho$.

Let $D(G(\mathbf{w})) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ (or D_n, for brevity) be a diagonal matrix, in which d_i is the degree of the vertex v_i in the graph $G(\mathbf{w})$. The matrix $L_n = D_n - A_n$ is then called the *Laplacian matrix* of $G(\mathbf{w})$.

The remainder of this paper is structured as follows: In Section 2, we review some existing results and prove new auxiliary results that will be used throughout the paper. In Section 3, we present results concerning the spectrum of the adjacency matrix of the random graph $G(\mathbf{w})$. Section 4 addresses the Laplacian matrix of the random graph $G(\mathbf{w})$.

2 Preliminaries and Auxiliary Results

We begin by introducing additional terminology and notation used throughout the paper.

2.1 Additional terminology and notation

We let $\mathbb{C}^{n_1\times n_2}$ denote the space of $n_1 \times n_2$ matrices with complex entries. For $A \in \mathbb{C}^{n_1\times n_2}$, $A^* \in \mathbb{C}^{n_2\times n_1}$ denotes the *conjugate transpose* of A. The space of $n \times n$ Hermitian matrices is denoted by $\mathbb{C}^{n\times n}_{\mathrm{Herm}}$, consisting of matrices $A \in \mathbb{C}^{n\times n}$ satisfying $A^* = A$. We identify $\mathbb{C}^n$ with the space $\mathbb{C}^{n\times 1}$ of column vectors, such that the inner product of $w, v \in \mathbb{C}^n$ is w^*v.

For each matrix $A \in \mathbb{C}^{n_1\times n_2}$, the *spectral norm* $\|A\|$ is defined as the largest *singular value* of A, i.e.,

$$\|A\| = \sqrt{\lambda_{\max}(A^*A)},$$

where $\lambda_{\max}(A^*A)$ denotes the largest eigenvalue of A^*A. The *Spectral Theorem* for Hermitian matrices states that every $A \in \mathbb{C}^{n\times n}_{\mathrm{Herm}}$ has n real eigenvalues (multiplicities counted), which correspond to an orthonormal set of eigenvectors. When $A \in \mathbb{C}^{n\times n}_{\mathrm{Herm}}$, let $\lambda_i(A)(1 \le i \le n)$ denote the i-th largest eigenvalue of A (multiplicities counted), we have $\|A\| = \max\{|\lambda_i(A)| : 1 \le i \le n\}$. The set $\{\lambda_i(A) : 1 \le i \le n\}$ is called the *spectrum* of A, denoted by $\mathrm{spec}(A)$. The *trace* of A, denoted $\mathrm{Tr}(A)$, is the sum of its eigenvalues.

We use the notation $A \succeq \mathbf{0}$ to indicate that A is *positive semidefinite*, i.e., $A \in \mathbb{C}^{n\times n}_{\mathrm{Herm}}$ with nonnegative eigenvalues (equivalently, $v^*Av \ge 0$ for all $v \in \mathbb{C}^n$). Similarly, $A \succ \mathbf{0}$ denotes that A is *positive definite*, i.e., $A \in \mathbb{C}^{n\times n}_{\mathrm{Herm}}$ with positive eigenvalues, where $\mathbf{0}$ is the zero matrix of the same size as A. The symbol $\preceq$ denotes the *positive semidefinite order* on Hermitian matrices: for two Hermitian matrices A and B, $A \preceq B$ (or $B \succeq A$) means $B - A \succeq \mathbf{0}$.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire analytic function with Taylor expansion $f(x) = \sum_{i=0}^{\infty} a_i x^i$ $(x \in \mathbb{C})$. If all coefficients a_i are real, the expression

$$f(A) = \sum_{i=0}^{\infty} a_i A^i \ (A \in \mathbb{C}^{n\times n}_{\mathrm{Herm}})$$

defines a mapping from $\mathbb{C}^{n\times n}_{\text{Herm}}$ to itself, with convergence defined as in [16]. The *Spectral Mapping Theorem* states that each eigenvalue of $f(A)$ is $f(\lambda)$ for $\lambda \in \text{spec}(A)$, i.e.,

$$\text{spec}(f(A)) = f(\text{spec}(A)). \tag{2.1}$$

In particular, we will often use the *matrix exponential*, defined as $\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$. From the Spectral Mapping Theorem, we know that $\exp(A)$ is always positive definite when A is Hermitian, and that $\exp(A)$ converges for all matrices A.

We frequently work with the trace of the matrix exponential:

$$\text{Tr}(\exp) : A \mapsto \text{Tr}(\exp(A)).$$

This trace-exponential function is monotone with respect to the positive semidefinite order, that is,

$$\forall A, B \in \mathbb{C}^{n\times n}_{\text{Herm}},\ A \preceq B \implies \text{Tr}(\exp(A)) \leq \text{Tr}(\exp(B)). \tag{2.2}$$

A concise proof of this result can be found, for example, in Section 2 of [22].

Moreover, we shall make brief use of the matrix logarithm. The *matrix logarithm* is defined as the functional inverse of the matrix exponential:

$$\text{for any } A \in \mathbb{C}^{n\times n}_{\text{Herm}},\ \log(e^A) := A. \tag{2.3}$$

This formula defines the logarithm of a positive definite matrix. In general, if $B = \exp(A)$, we call A the logarithm of B. Since our matrices will be Hermitian, requiring the logarithm to also be Hermitian ensures the uniqueness of this function.

The matrix logarithm is monotone with respect to the positive semidefinite order (see [3]):

$$\text{for any } A, B \in \mathbb{C}^{n\times n}_{\text{Herm}},\ \text{if } A \succ \mathbf{0}, B \succ \mathbf{0} \text{ and } A \preceq B,\ \text{then } \log(A) \preceq \log(B). \tag{2.4}$$

We denote the largest eigenvalue of an Hermitian matrix A by $\lambda_{\max}(A)$. Obviously,

$$\lambda_{\max}(\alpha A) = \alpha \lambda_{\max}(A) \text{ for } \alpha \geq 0. \tag{2.5}$$

If A is a random $n \times n$ matrix, we write $\mathbb{E}[A]$ to denote the entry-wise expectation of A, so $(\mathbb{E}[A])_{ij} = \mathbb{E}(A_{ij})$.

Throughout the paper, we assume n is large whenever necessary. Asymptotic notation is used under the assumption that $n \to \infty$. Given non-negative functions $f(n)$ and $g(n)$, we write $f(n) = O(g(n))$ if there exists a positive constant C, independent of n, such that $f(n) \leq Cg(n)$ for all sufficiently large values of n. We write $f(n) = o(g(n))$ if $f(n)/g(n) \to 0$ as $n \to \infty$. Any notation related to matrices that is not mentioned here follows the conventions in [16].

2.2 Auxiliary results

To establish a bound on the expectation of the largest eigenvalue of the adjacency matrix for the random graph $G(\mathbf{w})$, we begin with a theorem concerning the expected largest eigenvalue of a sum of independent, random Hermitian matrices whose spectral norms satisfy a uniform upper bound.

Theorem 1. *Consider a finite sequence $\{X_k\}$ of independent, random Hermitian matrices with common dimension n. Assume that*

$$\|X_k\| \leq c \quad \text{for each } k.$$

Define the random matrix

$$Y = \sum_k X_k.$$

Then, for $\theta > 0$,

$$\mathbb{E}\lambda_{\max}(Y) \leq \frac{c}{\theta}\log n + \lambda_{\max}(\mathbb{E}Y) + \frac{e^\theta - 1 - \theta}{\theta c}\lambda_{\max}\left(\sum_k \mathbb{E}(X_k^2)\right). \tag{2.6}$$

For the proof, we rely on the following results:

Lemma 1 (Transfer Rule [24])**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$, and suppose there exists a subset $S \subseteq \mathbb{R}$ with $f(a) \leq g(a)$ for all $a \in S$. If A is an Hermitian matrix with all eigenvalues contained in S, then $f(A) \preceq g(A)$.*

Definition 1 (Matrix MGF and CGF [25])**.** Let X be a random Hermitian matrix. The matrix moment generating function (matrix MGF) M_X and the matrix cumulant generating function (matrix CGF) Ξ_X are given by

$$M_X(\theta) = \mathbb{E}e^{\theta X} \text{ and } \Xi_X(\theta) = \log \mathbb{E}e^{\theta X} \text{ for } \theta \in \mathbb{R}.$$

We first derive an appropriate semidefinite bound for the MGF and CGF of a random Hermitian matrix with bounded spectral norm.

Lemma 2 (MGF and CGF Bound)**.** *Suppose that X is a random Hermitian matrix satisfying*

$$\|X\| \leq c.$$

Then, for $\theta > 0$,

$$\mathbb{E}e^{\theta X} \preceq e^{\theta \mathbb{E}X + \frac{1}{2}\theta^2 f(\theta c)\mathbb{E}(X^2)}$$

and

$$\log \mathbb{E}e^{\theta X} \preceq \theta \mathbb{E}X + \frac{1}{2}\theta^2 f(\theta c)\mathbb{E}(X^2),$$

where $f(x) = \frac{2(e^x - 1 - x)}{x^2}$.

Proof. We define

$$f(x) = 2\sum_{k=2}^{\infty} \frac{x^{k-2}}{k!} = \frac{2(e^x - 1 - x)}{x^2},$$

and present the following facts about f.

- $f(0) = 1$.
- $f(x)$ is monotonically increasing for $x \geq 0$.

For $0 < x \leq c$, since $f(x)$ is monotonically increasing for $x \geq 0$, we have $f(x) \leq f(c)$. Let X be as in the hypothesis of Lemma 2. For a real constant $\theta > 0$, $\|\theta X\| \leq \theta c$. Applying Lemma 1, we derive $f(\theta X) \preceq f(\theta c)I$. Noting that $e^x = 1 + x + \frac{1}{2}x^2 f(x)$, we obtain

$$\begin{aligned} e^{\theta X} &= I + \theta X + \frac{1}{2}\theta^2 f(\theta X)X^2 \\ &\preceq I + \theta X + \frac{1}{2}\theta^2 f(\theta c)X^2. \end{aligned} \tag{2.7}$$

We now use the fact that the expectation respects the positive semidefinite order (see [24]), i.e.,

$$\text{for any } A, B \in \mathbb{C}^{n\times n}_{\text{Herm}},\ A \preceq B \text{ almost surely implies } \mathbb{E}A \preceq \mathbb{E}B. \tag{2.8}$$

Using (2.7) and (2.8), we obtain

$$\begin{aligned} \mathbb{E}(e^{\theta X}) &\preceq \mathbb{E}(I + \theta X + \frac{1}{2}\theta^2 f(\theta c)X^2) \\ &= I + \theta \mathbb{E}X + \frac{1}{2}\theta^2 f(\theta c)\mathbb{E}(X^2) \\ &\preceq e^{\theta \mathbb{E}X + \frac{1}{2}\theta^2 f(\theta c)\mathbb{E}(X^2)}. \end{aligned} \tag{2.9}$$

The second relation follows from the fact that $I + A \preceq e^A$ for any $A \in \mathbb{C}^{n\times n}_{\text{Herm}}$, which is derived by applying the Transfer Rule (i.e., Lemma 1) to the inequality $1 + a \leq e^a$ valid for all $a \in R$.

To obtain the semidefinite bound for the matrix cumulant generating function, we take the logarithm of the semidefinite bound for the matrix moment generating function. This operation preserves the semidefinite order due to property (2.4), which states that the logarithm is operator monotone.

$$\log \mathbb{E}e^{\theta X} \preceq \theta \mathbb{E}X + \frac{1}{2}\theta^2 f(\theta c)\mathbb{E}(X^2).$$

This completes the proof of Lemma 2. $\square$

Lemma 3 ([25]). *Consider a finite sequence $\{X_k\}_{k=1}^m$ of independent, random, Hermitian matrices of the same size. Then*

$$\mathbb{E}\lambda_{\max}(\sum_{k=1}^{m} X_k) \leq \inf_{\theta>0} \frac{1}{\theta} \log \operatorname{Tr} \exp\left(\sum_{k=1}^{m} \log \mathbb{E}e^{\theta X_k}\right), \tag{2.10}$$

Proof of Theorem 1. Consider a finite sequence $\{X_k\}$ of independent, random Hermitian matrices with common dimension n. Assume that

$$\|X_k\| \leq c \text{ for each } k.$$

The matrix cumulant generating function bound (Lemma 2) states that

$$\log \mathbb{E}e^{\theta X_k} \preceq \theta \mathbb{E}X_k + \frac{1}{2}\theta^2 f(\theta c)\mathbb{E}(X_k^2) \text{ where } f(\theta c) = \frac{2(e^{\theta c} - 1 - \theta c)}{(\theta c)^2} \text{ for } \theta > 0. \tag{2.11}$$

We next address the upper bound (2.10) for $\mathbb{E}\lambda_{\max}(Y)$. By the Spectral Mapping Theorem (i.e., (2.1)), for any $s \geq 0$, and any $A \in \mathbb{C}^{n\times n}_{\text{Herm}}$, the largest eigenvalue of e^{sA} is $e^{s\lambda_{\max}(A)}$, and all eigenvalues of e^{sA} are nonnegative. Additionally, using the fact that the matrix exponential of an Hermitian matrix is positive definite, we bound the trace by n times the largest eigenvalue. Hence,

$$\exp(s\lambda_{\max}(A)) = \lambda_{\max}(\exp(sA)) \leq \operatorname{Tr}(\exp(sA)) \leq n\lambda_{\max}(\exp(sA)). \tag{2.12}$$

We require two more inequalities from matrix analysis: the first is the *Golden-Thompson inequality* (see, e.g., [3]), and the second can be found, e.g., in [26].

$$\forall d \in \{1, 2, 3, \ldots\}, \text{ and any } A, B \in \mathbb{C}^{d\times d}_{\text{Herm}}, \operatorname{Tr}(e^{A+B}) \leq \operatorname{Tr}(e^A e^B). \tag{2.13}$$

and if A and B are $n \times n$ positive semidefinite Hermitian matrices, then

$$0 \leq \operatorname{Tr}(A \cdot B) \leq \operatorname{Tr}(A) \cdot \lambda_{\max}(B) \leq \operatorname{Tr}(A) \cdot \operatorname{Tr}(B). \tag{2.14}$$

Now, using the fact in (2.2) that the trace of the exponential function is monotone with respect to the semidefinite order, and applying (2.5), (2.12), (2.13) and (2.14), we substitute the cumulant generating function bounds, i.e., (2.11) into the master inequality (2.10) for the expectation of the largest eigenvalue to derive

$$\mathbb{E}\lambda_{\max}(Y) = \mathbb{E}\lambda_{\max}\left(\sum_k X_k\right)$$

$$
\begin{aligned}
&\leq \inf_{\theta>0} \frac{1}{\theta} \log \operatorname{Tr} \exp \left(\sum_k \log \mathbb{E} e^{\theta X_k} \right) && \text{(use (2.10))} \\
&\leq \inf_{\theta>0} \frac{1}{\theta} \log \operatorname{Tr} \exp \left\{ \sum_k \left[\theta \mathbb{E} X_k + \frac{1}{2} \theta^2 f(\theta c) \mathbb{E}(X_k^2) \right] \right\} && \text{(use (2.2) and (2.11))} \\
&= \inf_{\theta>0} \frac{1}{\theta} \log \operatorname{Tr} \exp \left\{ \theta \mathbb{E} Y + \frac{1}{2} \theta^2 f(\theta c) \sum_k \mathbb{E}(X_k^2) \right\} \\
&\leq \inf_{\theta>0} \frac{1}{\theta} \log \left\{ \operatorname{Tr} \left[\exp(\theta \mathbb{E} Y) \cdot \exp \left(\frac{1}{2} \theta^2 f(\theta c) \sum_k \mathbb{E}(X_k^2) \right) \right] \right\} && \text{(use (2.13))} \\
&\leq \inf_{\theta>0} \frac{1}{\theta} \log \left[\lambda_{\max}(\exp(\theta \mathbb{E} Y)) \cdot \operatorname{Tr} \exp \left(\frac{1}{2} \theta^2 f(\theta c) \sum_k \mathbb{E}(X_k^2) \right) \right] && \text{(use (2.14))} \\
&\leq \inf_{\theta>0} \frac{1}{\theta} \log \left[\lambda_{\max}(\exp(\theta \mathbb{E} Y)) \cdot n \lambda_{\max} \left(\exp \left(\frac{1}{2} \theta^2 f(\theta c) \sum_k \mathbb{E}(X_k^2) \right) \right) \right] && \text{(use (2.12))} \\
&= \inf_{\theta>0} \frac{1}{\theta} \log \left[n \exp(\lambda_{\max}(\theta \mathbb{E} Y)) \cdot \exp \left(\lambda_{\max} \left(\frac{1}{2} \theta^2 f(\theta c) \sum_k \mathbb{E}(X_k^2) \right) \right) \right] && \text{(use (2.12))} \\
&= \inf_{\theta>0} \frac{1}{\theta} \log \left[n \exp(\theta \lambda_{\max}(\mathbb{E} Y)) \cdot \exp \left(\frac{1}{2} \theta^2 f(\theta c) \lambda_{\max} \left(\sum_k \mathbb{E}(X_k^2) \right) \right) \right] && \text{(use (2.5))} \\
&= \inf_{\theta>0} \frac{1}{\theta} \log \left[n \exp \left(\theta \lambda_{\max}(\mathbb{E} Y) + \frac{1}{2} \theta^2 f(\theta c) \lambda_{\max} \left(\sum_k \mathbb{E}(X_k^2) \right) \right) \right] \\
&= \inf_{\theta>0} \left[\frac{1}{\theta} \log n + \lambda_{\max}(\mathbb{E} Y) + \frac{e^{\theta c} - 1 - \theta c}{\theta c^2} \lambda_{\max} \left(\sum_k \mathbb{E}(X_k^2) \right) \right].
\end{aligned}
$$

In the ninth line, we utilize the fact in (2.5) that the largest eigenvalue is a positive-homogeneous map, which relies on the observation that $f(\theta c) > 0$ for $\theta > 0$. Finally, by substituting $\theta \mapsto \theta/c$, we obtain the expression (2.6). This completes the proof of Theorem 1. $\square$

3 The Spectrum of the Adjacency Matrix of Random Graphs with Given Expected Degrees

In this section, we study the spectrum of the adjacency matrix of random graphs with given expected degrees. We first establish an upper bound for the largest eigenvalue of the adjacency matrix for random graph model $G(\mathbf{w})$. We then study the expectation of the largest eigenvalue of the adjacency matrix and present a result on the concentration property of the largest eigenvalue of the adjacency matrix.

Theorem 2. *For the random graph $G(\mathbf{w})$, let $\epsilon > 0$ be an arbitrarily small constant. If the maximum expected degree m satisfies $m > \frac{4}{9}\log(2n/\epsilon)$, then with probability at least*

$1-\epsilon$, *the largest eigenvalue of* A_n *satisfies*

$$\lambda_{\max}(A_n) \le \tilde{d} + \sqrt{8m\log n},$$

where $\tilde{d} = \frac{\sum_i w_i^2}{\sum_i w_i}$ *denotes the second-order average degree of* $G(\mathbf{w})$.

Applying Theorem 2 with $\epsilon = \frac{1}{n}$, we immediately obtain the following corollary.

Corollary 1. *For the random graph* $G(\mathbf{w})$*, if the maximum expected degree* m *satisfies* $m > \frac{8}{9}\log n$*, then with probability at least* $1-1/n = 1-o(1)$*, the largest eigenvalue of* A_n *satisfies*

$$\lambda_{\max}(A_n) \le \tilde{d} + \sqrt{8m\log n}. \tag{3.1}$$

The spectrum of the adjacency matrix of the $G(\mathbf{w})$ model has been studied in [6, 8]. In [6], it is proven that if $m = w_{\max}$ is the maximum expected degree, then

$$\tilde{d} - \sqrt{2m^2\rho\log n} \le \lambda_{\max}(A_n) \le \tilde{d} + \sqrt{6\sqrt{m\log n}(\tilde{d}+\log n)} + 3\sqrt{m\log n}, \tag{3.2}$$

where $\rho = \frac{1}{\sum_{i=1}^n w_i}$, and $\tilde{d} = \frac{\sum_i w_i^2}{\sum_i w_i}$ is the second-order average degree of $G(\mathbf{w})$. In [8], it is proven that if the maximum expected degree m satisfies $m > \frac{8}{9}\log(\sqrt{2}n)$, then with probability at least $1-1/n = 1-o(1)$,

$$\tilde{d} - \sqrt{8m\log(\sqrt{2}n)} \le \lambda_{\max}(A_n) \le \tilde{d} + \sqrt{8m\log(\sqrt{2}n)}. \tag{3.3}$$

Obviously, the bounds in (3.3) are a signicant improvement upon the bounds in (3.2), and in (3.1) we manage to improve the upper bound in (3.3) with a slightly weaken assumption on m.

We also here provide the expectation estimate and demonstrate that the largest eigenvalue of A_n is highly concentrated around its expectation.

Theorem 3. *For the random graph* $G(\mathbf{w})$*, if the maximum expected degree* m *satisfies* $m > \frac{8}{9}\log n$*, then for* $\theta > 0$*, we have*

$$\mathbb{E}\lambda_{\max}(A_n) \le \frac{1}{\theta}\log n + \tilde{d} + \frac{e^\theta - 1 - \theta}{\theta}m.$$

Moreover, for any $t \gg 1$*, we have*

$$\Pr(|\lambda_{\max}(A_n) - \mathbb{E}[\lambda_{\max}(A_n)]| \ge t) \le e^{-(1-o(1))t^2/32}.$$

We now proceed to prove Theorem 2. For our proof, we will rely on the following known result. In [18], Hu et al. derive the following matrix inequality, which is particularly suitable for the random graph $G(\mathbf{w})$.

Theorem 4 ([18])**.** *Let $X_1, X_2, \ldots, X_m$ be independent random $n \times n$ Hermitian matrices. Moreover, assume that $\|X_i\| \le c$ for all i. Let $X = \sum_{i=1}^m X_i$. Then*

$$\Pr(\lambda_{\max}(X) \ge a) \le n \cdot \exp\left(-\frac{(a - \|\mathbb{E}(X)\|)^2}{2\|\sum_{i=1}^m \mathbb{E}(X_i^2)\| + \frac{2c}{3}(a - \|\mathbb{E}(X)\|)} \right) \quad \textit{for } a > \|\mathbb{E}(X)\|.$$

Proof of Theorem 2. Let $G(\mathbf{w})$ and A_n be defined as before, and let $m = w_{\max}$ denote the maximum expected degree of the graph $G(\mathbf{w})$.

For indices i and j with $1 \le i, j \le n$, let E^{ij} denote the $n \times n$ matrix with a 1 in the (i,j)-th position and 0 elsewhere. Recall that ξ_{ij} takes value 1 with probability $w_i w_j \rho$ and value 0 with probability $1 - w_i w_j \rho$. Thus, $\xi_{ji} = \xi_{ij}$ by symmetry. Define $X_{ij} = \xi_{ij}(E^{ij} + E^{ji})$. Then the adjacency matrix A_n can be expressed as

$$A_n = \sum_{1 \le i \le j \le n} X_{ij}.$$

To apply Theorem 4 to A_n, we first derive a suitable upper bound c for $\|X_{ij}\|$. Note that X_{ij} $(1 \le i \le j \le n)$ are independent random $n \times n$ Hermitian matrices. Since $\xi_{ij} = \{0, 1\}$, we have

$$\|X_{ij}\| = \|\xi_{ij}(E^{ij} + E^{ji})\| = |\xi_{ij}| \cdot \|E^{ij} + E^{ji}\| \le 1.$$

Thus, we choose $c = 1$. Next, we compute upper bounds for $\|\sum_{1 \le i \le j \le n} \mathbb{E}(X_{ij}^2)\|$ and $\|\mathbb{E}A_n\|$. For any $1 \le i < j \le n$:

$$\begin{aligned} \mathbb{E}(X_{ij}^2) &= \mathbb{E}[\xi_{ij}^2 (E^{ij} + E^{ji})^2] \\ &= \mathbb{E}[\xi_{ij}^2](E^{ij} + E^{ji})^2 \\ &= w_i w_j \rho (E^{ii} + E^{jj}). \end{aligned}$$

Similarly, for $i = j$, $\mathbb{E}(X_{ii}^2) = w_i^2 \rho E^{ii}$. Therefore,

$$\begin{aligned} \left\| \sum_{1 \le i \le j \le n} \mathbb{E}(X_{ij}^2) \right\| &= \left\| \sum_{i=1}^n \left(\sum_{j=1}^n w_i w_j \rho \right) E^{ii} \right\| \\ &= \max_{i=1,\ldots,n} \sum_{j=1}^n w_i w_j \rho \\ &= \max_{i=1,\ldots,n} w_i \\ &= m. \end{aligned}$$

For the expectation of A_n, note that $(\mathbb{E}A_n)_{ij} = \mathbb{E}\xi_{ij} = w_i w_j \rho$. Thus, $\mathbb{E}A_n = \rho \mathbf{w}^T \mathbf{w}$. As a rank-1 matrix, $\mathbb{E}A_n$ has eigenvalues $\rho \sum_i w_i^2$ and 0 (with multiplicity $n - 1$). Hence,

$$\|\mathbb{E}A_n\| = \rho \sum_i w_i^2 = \tilde{d}.$$

Set $a = \|\mathbb{E}A_n\| + \sqrt{4m\log(n/\epsilon)}$. By the assumption $m > \frac{4}{9}\log(n/\epsilon)$, we have $a - \|\mathbb{E}A_n\| < 3m$. Applying Theorem 4 with $c = 1$, we obtain

$$\begin{aligned}
\Pr(\lambda_{\max}(A_n) \geq a) &\leq n \cdot \exp\left(-\frac{(a - \|\mathbb{E}A_n\|)^2}{2\|\sum_{1\leq i\leq j\leq n} \mathbb{E}(X_{ij}^2)\| + \frac{2c}{3}(a - \|\mathbb{E}A_n\|)}\right) \\
&\leq n \cdot \exp\left(-\frac{4m\log(n/\epsilon)}{4m}\right) \\
&= \epsilon.
\end{aligned}$$

Thus, with probability at least $1 - \epsilon$,

$$\lambda_{\max}(A_n) \leq a = \|\mathbb{E}A_n\| + \sqrt{4m\log(n/\epsilon)} = \tilde{d} + \sqrt{8m\log n}.$$

This completes the proof of Theorem 2. □

Before presenting the proof of Theorem 3, we recall a known result that will be used in the sequel of the paper. Alon, Krivelevich and Vu[1] showed that the eigenvalues of random matrices are highly concentrated around their median values.

Theorem 5 ([1])**.** *Let A be an $n \times n$ random symmetric matrix with independent entries $a_{ij} = a_{ji}$ for $1 \leq i \leq j \leq n$, where each entry satisfies $|a_{ij}| \leq 1$. Let its eigenvalues be $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. For any $1 \leq s \leq n$, Let μ_s denote the median value of $\lambda_s(A)$, i.e., $\mu_s = \inf_\mu\{\Pr(\lambda_s(A) \leq \mu) \geq 1/2\}$. Then for any $t \geq 0$, we have*

$$\Pr(|\lambda_s(A) - \mu_s| \geq t) \leq 4e^{-\frac{t^2}{32s^2}}.$$

Proof of Theorem 3. Let $G(\mathbf{w})$ and A_n be defined as before, and let m denote the maximum expected degree of the graph $G(\mathbf{w})$. From the proof of Theorem 2, we know $A_n = \sum_{1\leq i\leq j\leq n} X_{ij}$. To apply Theorem 1 to A_n, we derive an upper bound c on $\|X_{ij}\|$ and upper bounds for $\lambda_{\max}(\sum_{1\leq i\leq j\leq n} \mathbb{E}(X_{ij}^2))$ and $\lambda_{\max}(\mathbb{E}A_n)$.

Recall that $X_{ij}(1 \leq i \leq j \leq n)$ are independent random $n \times n$ Hermitian matrices satisfying

$$\|X_{ij}\| \leq 1 = c.$$

Moreover,

$$\lambda_{\max}\left(\sum_{1\leq i\leq j\leq n} \mathbb{E}(X_{ij}^2)\right) = m \text{ and } \lambda_{\max}(\mathbb{E}A_n) = \rho\sum_i w_i^2 = \tilde{d}.$$

Applying Theorem 1, we obtain

$$\mathbb{E}\lambda_{\max}(A_n) \leq \frac{1}{\theta}\log n + \tilde{d} + \frac{e^\theta - 1 - \theta}{\theta}m.$$

Recall Jensen's inequality, which holds for a convex function f and a random variable Z:

$$f(\mathbb{E}[Z]) \leq \mathbb{E}[f(Z)]\,.$$

Denote by μ_1 the median of the largest eigenvalue of A_n. Since the absolute value function $f(x) = |x - \mu_1|$ is convex, we substitute $f(x) = |x - \mu_1|$ and $Z = \lambda_{\max}(A_n)$, then the left-hand side becomes $f(\mathbb{E}[\lambda_{\max}(A_n)]) = |\mathbb{E}[\lambda_{\max}(A_n)] - \mu_1|$, and the right-hand side is $\mathbb{E}[f(\lambda_{\max}(A_n))] = \mathbb{E}|\lambda_{\max}(A_n) - \mu_1|$. Thus, the inequality $|\mathbb{E}[\lambda_{\max}(A_n)] - \mu_1| \leq \mathbb{E}|\lambda_{\max}(A_n) - \mu_1|$ follows from Jensen's inequality. Next, recall that for any non-negative random variable X,

$$\mathbb{E}[X] = \int_0^\infty \Pr[X \geq t]dt.$$

By Theorem 5, we obtain

$|\mathbb{E}[\lambda_{\max}(A_n)] - \mu_1| \leq \mathbb{E}|\lambda_{\max}(A_n) - \mu_1| = \int_0^\infty \Pr(|\lambda_{\max}(A_n) - \mu_1| \geq t)dt \leq \int_0^\infty 4e^{-\frac{t^2}{32}}dt = 8\sqrt{2\pi}$. Thus, the expectation of $\lambda_{\max}(A_n)$ and its median differ by $O(1)$. Therefore, for all $t \gg 1$, we get

$$\Pr(|\lambda_{\max}(A_n) - \mathbb{E}[\lambda_{\max}(A_n)]| \geq t) \leq e^{-(1-o(1))t^2/32}.$$

4 The Spectrum of the Laplacian matrix of Random Graphs with Given Expected Degrees

In this section, we study the spectrum of the Laplacian matrix of random graphs with given expected degrees. Our goal is to analyze the expectation and the tail behavior of the largest eigenvalue of the Laplacian matrix for random graph model $G(\mathbf{w})$.

Theorem 6. *For the random graph $G(\mathbf{w})$ with Laplacian matrix L_n, we have for $\theta > 0$,*

$$\mathbb{E}\lambda_{\max}(L_n) \leq \frac{e^\theta - 1}{\theta} \cdot \frac{w_{\max}^2}{\sum_i w_i} n + \frac{2}{\theta}\log(2(n-1)).$$

Moreover,

$$\Pr\left\{\lambda_{\max}(L_n) \geq (1+\epsilon)\frac{w_{\max}^2}{\sum_i w_i} n\right\} \leq 2(n-1)\left[\frac{e^\epsilon}{(1+\epsilon)^{1+\epsilon}}\right]^{\frac{w_{\max}^2}{2\sum_i w_i}n} \quad \textit{for} \quad \epsilon \geq 2/\left(\frac{w_{\max}^2}{\sum_i w_i} n\right).$$

Our proof relies on the following known results.

Definition 2 (Intrinsic Dimension [25])**.** For a positive-semidefinite matrix A, its intrinsic dimension is defined as

$$\operatorname{intdim}(A) = \frac{\operatorname{Tr} A}{\|A\|}.$$

Theorem 7 (Matrix Chernoff: Intrinsic Dimension [25]). *Consider a finite sequence $\{X_k\}$ of independent, random, Hermitian matrices of common dimension n. Assume that*

$$0 \leq \lambda_{\min}(X_k) \quad \text{and } \lambda_{\max}(X_k) \leq c \quad \text{for each } k.$$

Introduce the random matrix

$$Y = \sum_k X_k.$$

Suppose that we have a semidefinite upper bound M for the expectation $\mathbb{E}Y$:

$$M \succeq \mathbb{E}Y = \sum_k \mathbb{E}X_k.$$

Define an intrinsic dimension bound and a mean bound:

$$d = \operatorname{intdim}(M) \quad \text{and} \quad \mu_{\max} = \lambda_{\max}(M).$$

Then, for $\theta > 0$,

$$\mathbb{E}\lambda_{\max}(Y) \leq \frac{e^\theta - 1}{\theta} \cdot \mu_{\max} + \frac{c}{\theta} \log(2d).$$

Furthermore,

$$\Pr\{\lambda_{\max}(Y) \geq (1+\epsilon)\mu_{\max}\} \leq 2d \cdot \left[\frac{e^\epsilon}{(1+\epsilon)^{1+\epsilon}}\right]^{\frac{\mu_{\max}}{c}} \quad \text{for} \quad \epsilon \geq c/\mu_{\max}.$$

Lemma 4 (Courant-Fischer [16]). *Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, and let k be an integer with $1 \leq k \leq n$. Then*

$$\lambda_k = \min_{w_1, w_2, \ldots, w_{n-k} \in \mathbb{C}^n} \ \max_{\substack{x \neq \mathbf{0}, x \in \mathbb{C}^n \\ x \perp w_1, w_2, \ldots, w_{n-k}}} \frac{x^* A x}{x^* x}.$$

Proof of Theorem 6. Let $G(\mathbf{w})$ and L_n be defined as before. Recall that $m = w_{\max}$ denote the maximum expected degree. For the indices i and j with $1 \leq i, j \leq n$, let E^{ij} be the $n \times n$ matrix with a 1 in the (i,j)-th position and 0 elsewhere. Recall that ξ_{ij} takes value 1 with probability $w_i w_j \rho$ and 0 with probability $1 - w_i w_j \rho$. So, $\xi_{ji} = \xi_{ij}$.

The Laplacian matrix L_n of $G(\mathbf{w})$ can be decomposed as

$$L_n = \sum_{1 \leq i < j \leq n} L_{ij} = \sum_{1 \leq i < j \leq n} \xi_{ij}(E^{ii} + E^{jj} - E^{ij} - E^{ji}),$$

where for $i < j$, the sub-Laplacian matrix L_{ij} is defined as

$$L_{ij} = \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} & & \\ & \xi_{ij} & -\xi_{ij} \\ & & \\ & -\xi_{ij} & \xi_{ij} \\ & & \end{pmatrix} \end{array}.$$

This matrix corresponds to the edge $e_{ij} = v_i v_j$, where $\xi_{ij} = 1$ indicates v_i and v_j are adjacent in $G(\mathbf{w})$.

Using the the Courant-Fischer min-max theorem (Lemma 4), we analyze $x^* L_{ij} x$ for any nonzero complex column vector $x \in \mathbb{C}^n$,

$$\begin{aligned} x^* L_{ij} x &= (x_1^*, x_2^*, \ldots, x_n^*) \begin{pmatrix} & & \\ & \xi_{ij} & -\xi_{ij} \\ & & \\ & -\xi_{ij} & \xi_{ij} \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \xi_{ij}(|x_i|^2 + |x_j|^2 - x_i^* x_j - x_i x_j^*) \\ &= \xi_{ij}(x_i - x_j)(x_i^* - x_j^*) \\ &= \xi_{ij}|x_i - x_j|^2 \\ &\geq 0. \end{aligned}$$

This implies L_{ij} is positive semidefinite for $i < j$.

For the spectral norm of $\|L_{ij}\|$:

$$\begin{aligned} \|L_{ij}\| &= \|\xi_{ij}(E^{ii} + E^{jj} - E^{ij} - E^{ji})\| \\ &\leq |\xi_{ij}| \cdot \|E^{ii} + E^{jj} - E^{ij} - E^{ji}\| \\ &\leq 2, \end{aligned}$$

Thus,

$$0 \leq \lambda_{\min}(L_{ij}) \text{ and } \lambda_{\max}(L_{ij}) \leq 2 \text{ for } 1 \leq i < j \leq n.$$

Moreover,

$$\mathbb{E}L_{ij} = \mathbb{E}[\xi_{ij}(E^{ii} + E^{jj} - E^{ij} - E^{ji})]$$

$$
\begin{aligned}
&= \mathbb{E}[\xi_{ij}](E^{ii} + E^{jj} - E^{ij} - E^{ji}) \\
&= w_i w_j \rho(E^{ii} + E^{jj} - E^{ij} - E^{ji}),
\end{aligned}
$$

Summing over all edges, the expected Laplacian matrix satisfies:

$$
\begin{aligned}
\mathbb{E}L_n &= \sum_{1 \le i < j \le n} \mathbb{E}L_{ij} \\
&= \sum_{1 \le i < j \le n} w_i w_j \rho(E^{ii} + E^{jj} - E^{ij} - E^{ji}) \\
&\preceq w_{\max}^2 \frac{1}{\sum_i w_i} \sum_{1 \le i < j \le n} (E^{ii} + E^{jj} - E^{ij} - E^{ji}) \\
&= \frac{w_{\max}^2}{\sum_i w_i} [(n-1)I_n - (J_n - I_n)] \\
&:= M,
\end{aligned}
$$

where I_n denotes the $n \times n$ identity matrix and J_n is the all-ones matrix. The spectrum of M is $\{0, \frac{w_{\max}^2}{\sum_i w_i} n\}$, with $\frac{w_{\max}^2}{\sum_i w_i} n$ having multiplicity $n-1$. Hence, M is a positive-semidefinite matrix and

$$
\|M\| = \frac{w_{\max}^2}{\sum_i w_i} n \text{ and } \mathrm{Tr}M = (n-1)\frac{w_{\max}^2}{\sum_i w_i} n.
$$

Therefore, the intrinsic dimension of M is

$$
d = \mathrm{intdim}(M) = \frac{\mathrm{Tr}M}{\|M\|} = n - 1,
$$

and

$$
\mu_{\max} = \lambda_{\max}(M) = \frac{w_{\max}^2}{\sum_i w_i} n.
$$

By Theorem 7, for $\theta > 0$,

$$
\mathbb{E}\lambda_{\max}(L_n) \le \frac{e^\theta - 1}{\theta} \cdot \frac{w_{\max}^2}{\sum_i w_i} n + \frac{2}{\theta} \log(2(n-1)).
$$

For the tail probability, let $\epsilon \ge 2/\left(\frac{w_{\max}^2}{\sum_i w_i} n\right)$, then

$$
\Pr\left\{\lambda_{\max}(L_n) \ge (1+\epsilon)\frac{w_{\max}^2}{\sum_i w_i} n\right\} \le 2(n-1)\left[\frac{e^\epsilon}{(1+\epsilon)^{1+\epsilon}}\right]^{\frac{w_{\max}^2}{2\sum_i w_i} n}.
$$

This completes the proof of Theorem 6. □

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