



Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

MacMahon's $\Omega_{\geq}$ operator: A computational framework

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ARTICLE INFO

Article history:

Received 22 December 2025

Received in revised form 25 June 2026

Accepted 21 July 2026

Available online 24 July 2026

MSC:

primary 05A15

secondary 05A17, 11P81

Keywords:

MacMahon's partition analysis

Omega operator

Partial fraction decomposition

Elliott rational function

 k -gon partition

ABSTRACT

MacMahon introduced partition analysis in his book “Combinatory Analysis” as a computational technique for solving problems related to systems of linear Diophantine equations and inequalities. This paper aims to develop a fundamental computational framework for MacMahon's partition analysis. As applications, we present simplified computations for “Han's formula”, the “ k -gon partitions problem”, and the “two-dimensional problem”. Moreover, we apply our method to solve a challenging problem.

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1. Introduction

In enumerative combinatorics, *partition analysis* was first introduced by MacMahon in his famous book, “Combinatorial Analysis” (MacMahon, 1960), published in 1915. In this work, MacMahon primarily investigated plane partitions by introducing the omega operator $\Omega_{\geq}$.

We briefly outline MacMahon's partition analysis. Let $\mathbb{F}$ be a field. A formal Laurent series in $\lambda_1, \dots, \lambda_k$ is defined as a series that can be expressed in the form

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<https://doi.org/10.1016/j.jsc.2026.102608>

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$$\sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_k=-\infty}^{\infty} a_{i_1, \dots, i_k} \lambda_1^{i_1} \cdots \lambda_k^{i_k},$$

where each $a_{i_1, \dots, i_k}$ is an element of $\mathbb{F}$. For such formal Laurent series, the definition of the omega operator $\Omega_{\geq}$ as given in MacMahon (1960) is:

Definition 1.1. The operator $\Omega_{\geq}$ acts on a formal Laurent series in $\lambda_1, \dots, \lambda_k$ with coefficients $a_{i_1, \dots, i_k}$ in $\mathbb{F}$ by

$$\Omega_{\geq} \sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_k=-\infty}^{\infty} a_{i_1, \dots, i_k} \lambda_1^{i_1} \cdots \lambda_k^{i_k} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} a_{i_1, \dots, i_k}.$$

In particular, if $F(\lambda)$ is a rational function in λ whose series expansion contains no negative power in λ , then $\Omega_{\geq} F(\lambda) = F(1)$.

We illustrate the operator with a simple example. Let $\mathbb{N}$ denote the set of all non-negative integers. Suppose $a_1, a_2 \in \mathbb{N}$ with $a_1 \geq a_2$. Consider the generating function

$$F(x_1, x_2) = \sum_{a_1 \geq a_2 \geq 0} x_1^{a_1} x_2^{a_2}.$$

Introducing a variable λ , we obtain

$$F(x_1, x_2) = \Omega_{\geq} \sum_{a_1, a_2 \geq 0} \lambda^{a_1 - a_2} x_1^{a_1} x_2^{a_2} = \Omega_{\geq} \frac{1}{(1 - x_1 \lambda)(1 - \frac{x_2}{\lambda})}. \quad (1)$$

The expression on the right-hand side of this equation is called the *crude generating function*. To compute this, we need to eliminate λ from (1). The method for this calculation will be introduced later.

Remark 1.2. In Andrews and Paule (2007b), the coefficients $a_{i_1, \dots, i_k}$ from Definition 1.1 are defined over the field of rational functions in several complex variables over $\mathbb{C}$, with each λ_i restricted to a neighborhood of the circle $|\lambda_i| = 1$. Furthermore, these coefficients are required to ensure that all series involved are absolutely convergent within their domain of definition. However, as shown by Breuer and Zafeirakopoulos (2017), polyhedral methods can be used to evaluate the Omega operator in an entirely symbolic manner. For a detailed treatment of the convergence issues involved, we refer the reader to Breuer and Zafeirakopoulos, 2017, Section 2.4.

Note that the operator $\Omega_{\geq}$ is supposed to act only on the Greek letters λ or μ , or on the corresponding indexed versions such as λ_i and μ_i in this paper. The parameters unaffected by $\Omega_{\geq}$ will be denoted by letters from the Latin alphabet.

An Elliott rational function E takes the form

$$E = \frac{L}{(1 - M_1(X) \Lambda^{c_1})(1 - M_2(X) \Lambda^{c_2}) \cdots (1 - M_n(X) \Lambda^{c_n})}, \quad (2)$$

where each $M_i(X)$ is a monomial in the variables $X = \{x_1, \dots, x_m\}$ that does not involve $\Lambda = \{\lambda_1, \dots, \lambda_k\}$; L is a Laurent polynomial in X and Λ . The *core problem* in MacMahon's partition analysis is to compute $\Omega_{\geq} E$, which amounts to eliminating the variables $\{\lambda_1, \dots, \lambda_k\}$ to obtain a rational function in $\{x_1, \dots, x_m\}$.

MacMahon's partition analysis is particularly suited for solving problems related to counting and enumerating solutions to linear Diophantine equations and inequalities. Notable applications include enumerating lattice points in convex polytopes and computing Ehrhart quasi-polynomials.

The technique has been revisited by Andrews, Paule, and Riese in a series of papers (Andrews, 1998, 2000; Andrews et al., 2001a; Andrews and Paule, 1999; Andrews et al., 2001b; Andrews,

2001a,b,c; Andrews et al., 2001c, 2004; Andrews and Paule, 2007a,b, 2022, 2024, 2025), where computer algebra plays a central role. Their work led to the development of an algorithm and corresponding software, namely the Omega package in Mathematica (Andrews et al., 2001a). The package was later upgraded to Omega2 in 2001 (Andrews, 2001a).

In 2003, Han (Han, 2003) presented a general algorithm for evaluating the MacMahon $\Omega_{\geq}$ operator and developed a Maple package called GenOmega.

In 2017, Breuer and Zafeirakopoulos (2017) integrated MacMahon's iterative approach based on the Omega operator with explicit formulas for its evaluation and geometric tools such as Brion decompositions and Barvinok's short rational function representations. They introduced a new algorithm for solving linear Diophantine systems, implemented in the Sage package Polyhedral Omega.

In Xin (2004), an efficient algorithm for MacMahon's partition analysis was developed by combining the theory of iterated Laurent series with an algorithm for partial fraction decompositions. This algorithm is implemented in the Maple package Ell, which achieves a significantly faster practical running time compared to Omega. In Xin (2015), Xin proposed a Euclid-style algorithm, with an implementation in the Maple package CTEuclid. This algorithm performs effectively and overcomes several bottlenecks encountered in the Ell package.

In this paper, we introduce a fundamental computational framework for evaluating $\Omega_{\geq} E$. Our approach leverages key ideas from the partial fraction decomposition algorithm and the rules for series expansion of multivariate rational functions presented in Xin (2004), as well as the technique for calculating the contribution of a single factor described in Xin (2015). In the same earlier work Xin (2015), the problem of extracting constant terms from multivariate rational functions was addressed. Here, we extend certain ideas from that study to the operator $\Omega_{\geq}$, offering a parallel but distinct computational framework.

Our method enables straightforward computation of numerous examples from MacMahon's book (MacMahon, 1960) and a series of papers by Andrews and his coauthors (Andrews, 1998, 2000; Andrews et al., 2001a; Andrews and Paule, 1999; Andrews et al., 2001b; Andrews, 2001a,b,c; Andrews et al., 2001c, 2004; Andrews and Paule, 2007a,b, 2022, 2024, 2025). Previous computational methods mainly rely on the definition of $\Omega_{\geq}$ or the Omega package. As applications, we demonstrate simplified computations for problems such as the " k -gon partitions problem", the "two-dimensional problem", "a hard problem", and others.

This paper is organized as follows. In Section 2, we present some basic tools for calculating $\Omega_{\geq} E$. In Section 3, we illustrate the application of these tools through several propositions. Section 4 provides a concise procedure for MacMahon's partition analysis, along with an example to demonstrate the usage of the commands. In Section 5, we reprove two problems with a relatively simple process and solve a hard problem.

2. Basic calculation tools

We now present some basic calculation tools for the $\Omega_{\geq}$ operator. In prior work, the field of iterated Laurent series was introduced to establish the uniqueness of series expansions for rational functions (Xin, 2004). Define the working field as $\mathbb{K} = \mathbb{C}((\lambda_k))((\lambda_{k-1})) \cdots ((\lambda_1))$. We adopt terminology from (Xin, 2015).

Note that every monomial $M \neq 1$ is comparable to 1 in $\mathbb{K}$ according to the following rule: identify the "smallest" variable λ_j that appears in M , i.e., $\deg_{\lambda_i} M = 0$ for all $i < j$. If $\deg_{\lambda_j} M > 0$, then M is said to be *small*, denoted $M < 1$; otherwise, M is *large*, denoted $M > 1$. Consequently, we can determine which of the following series expansions holds in $\mathbb{K}$:

$$\frac{1}{1-M} = \begin{cases} \sum_{i \geq 0} M^i & \text{if } M < 1, \\ \frac{1}{-M(1-1/M)} = \sum_{i \geq 0} -\frac{1}{M^{i+1}} & \text{if } M > 1. \end{cases}$$

Let $\widehat{\mathbb{K}} = \mathbb{K}((x_1)) \cdots ((x_m))$. For Equation (2), when expanding E as a series in $\widehat{\mathbb{K}}$, we typically require E to be in its *proper form*, meaning that $M_i(x) \Lambda^{c_i} < 1$ in $\widehat{\mathbb{K}}$ for all i .

We now employ Xin's partial fraction decomposition algorithm (Xin, 2015) to compute $\Omega_{\geq} E$ in $\widehat{\mathbb{K}}$ with respect to λ . For convenience, we express E in the following form:

$$E(\lambda) = \frac{L(\lambda)}{\prod_{i=1}^n (1 - u_i \lambda^{a_i})} \quad (\text{not in proper form}), \quad (3)$$

where $L(\lambda)$ is a Laurent polynomial, each u_i is a monomial in $\{x_1, \dots, x_m\} \cup (\Lambda \setminus \{\lambda\})$, and all a_i are **positive integers**. Note that each u_i is free of λ .

Theorem 2.1. Assume that the partial fraction decomposition of $E(\lambda)$ is given by

$$E(\lambda) = P(\lambda) + \frac{p(\lambda)}{\lambda^s} + \sum_{i=1}^n \frac{A_i(\lambda)}{1 - u_i \lambda^{a_i}}, \quad (4)$$

where each u_i is independent of λ ; $P(\lambda)$, $p(\lambda)$, and $A_i(\lambda)$ are polynomials, $\deg p(\lambda) < s$, and $\deg A_i(\lambda) < a_i$ for all i . Then,

$$\Omega_{\geq} E(\lambda) = P(1) + \sum_{u_i \lambda^{a_i} < 1} \frac{A_i(1)}{1 - u_i}. \quad (5)$$

Proof. By the definition of $\Omega_{\geq}$ (Definition 1.1), if $u_i \lambda^{a_i} < 1$, then

$$\Omega_{\geq} \frac{A_i(\lambda)}{1 - u_i \lambda^{a_i}} = \frac{A_i(1)}{1 - u_i}.$$

If $u_i \lambda^{a_i} > 1$, we have

$$\Omega_{\geq} \frac{A_i(\lambda)}{1 - u_i \lambda^{a_i}} = \Omega_{\geq} \frac{\lambda^{-a_i} A_i(\lambda)}{-u_i \left(1 - \frac{1}{u_i \lambda^{a_i}}\right)} = 0.$$

The proof is completed by applying Definition 1.1. $\square$

Theorem 2.2. (Xin, 2015) Let $E(\lambda)$ be as in (4). Then $A_i(\lambda)$ is uniquely characterized by

$$\begin{cases} A_i(\lambda) \equiv E(\lambda) \cdot (1 - u_i \lambda^{a_i}) \pmod{\langle 1 - u_i \lambda^{a_i} \rangle}, \\ \deg_{\lambda} A_i < a_i, \end{cases} \quad (6)$$

where $\langle 1 - u \lambda^a \rangle$ denotes the ideal generated by $1 - u \lambda^a$.

The primary objective is to compute $\Omega_{\geq} E(\lambda)$ for λ as in (3) in $\widehat{\mathbb{K}}$. We aim to compute

$$\mathcal{T}_{1-u_i \lambda^{a_i}} E(\lambda) := \frac{A_i(1)}{1 - u_i},$$

where $A_i(\lambda)$ is characterized by (6). The linear operator $\mathcal{T}_{1-u_i \lambda^{a_i}}$ is an analogous of the $\mathcal{A}_{1-u_i \lambda^{a_i}}$ operator in Xin (2015). In this new notation, Theorem 2.1 can be expressed as

$$\Omega_{\geq} E(\lambda) = P(1) + \sum_i \chi(u_i \lambda^{a_i} < 1) \mathcal{T}_{1-u_i \lambda^{a_i}} E(\lambda), \quad (7)$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

If $E(\lambda)$ is a proper rational function in λ , i.e., the degree of its numerator in λ is less than the degree of its denominator, then $P(\lambda)$ is identically 0 and hence $P(1) = 0$. (Don't confuse "proper rational function" with "proper form".) However, $P(1)$ might be hard to compute when the degree $\deg P(\lambda)$ is large. Its computation can be avoided by the following result.

Theorem 2.3. Let $E(\lambda)$ be as in (3). If $E(\lambda)$ is proper rational function in λ , then

$$\Omega_{\geq} E(\lambda) = \sum_{i=1}^n \left(\chi(u_i \lambda^{a_i} < 1) \cdot \frac{A_i(1)}{1 - u_i} \right). \quad (8)$$

If $E(0) = \lim_{\lambda \rightarrow 0} E(\lambda)$ and $E(1)$ exists, then

$$\Omega_{\geq} E(\lambda) = E(1) - \sum_{i=1}^n \left(\chi(u_i \lambda^{a_i} > 1) \cdot \frac{A_i(1)}{1 - u_i} \right). \quad (9)$$

Proof. Assume the partial fraction decomposition given in (4). (i) If $E(\lambda)$ is proper in λ , then $P(\lambda) = 0$, and the first equality holds. (ii) If $E(0)$ exists, then $p(\lambda)$ must be identically zero. Setting $\lambda = 1$ and applying (7) yields

$$E(1) = P(1) + \sum_{i=1}^n \frac{A_i(1)}{1 - u_i} = \Omega_{\geq} E(\lambda) + \sum_{i=1}^n \chi(u_i \lambda^{a_i} > 1) \mathcal{T}_{1-u_i \lambda^{a_i}} E(\lambda).$$

The second equality follows by subtracting the sum on the right-hand side. $\square$

Equation (9) can be viewed as a dual form of (7). Due to these two formulas, it is convenient to refer to the denominator factor $1 - u_i \lambda^{a_i}$ as *contributing* when $u_i \lambda^{a_i}$ is small and as *dually contributing* when $u_i \lambda^{a_i}$ is large.

For convenience, we introduce the following notation.

Definition 2.4. We denote

$$\Omega_{\geq} \frac{1}{1 - u_i \lambda^{a_i}} E(\lambda) (1 - u_i \lambda^{a_i}) = \mathcal{T}_{1-u_i \lambda^{a_i}} E(\lambda) = \frac{A_i(1)}{1 - u_i}.$$

In this notation, when applying $\Omega_{\geq}$ in λ , only the single underlined denominator factor contributes.

Theorems 2.1, 2.2, and 2.3 establish the basic calculation tools for the $\Omega_{\geq}$ operator. Definition 2.4 provides a convenient notational shorthand that streamlines our presentation. To illustrate the application of these tools, we now consider the following example.

Example 2.5. (MacMahon, 1960, Page 102) We compute the identity

$$\Omega_{\geq} \frac{1}{(1 - x\lambda)(1 - y\lambda)(1 - \frac{z}{\lambda})} = \frac{1 - xyz}{(1 - x)(1 - y)(1 - xz)(1 - zy)}$$

using two distinct approaches. For clarity, we provide detailed step-by-step explanations to facilitate the reader's understanding of our notation.

First approach: We proceed as follows:

$$\begin{aligned} & \Omega_{\geq} \frac{1}{(1 - x\lambda)(1 - y\lambda)(1 - \frac{z}{\lambda})} = \Omega_{\geq} \frac{1}{\underline{(1 - x\lambda)}(1 - y\lambda)(1 - \frac{z}{\lambda})} \quad (\text{since both factors contribute}) \\ & = \Omega_{\geq} \frac{1}{\underline{(1 - x\lambda)}(1 - y\lambda)(1 - \frac{z}{\lambda})} + \Omega_{\geq} \frac{1}{(1 - x\lambda)\underline{(1 - y\lambda)}(1 - \frac{z}{\lambda})} \\ & = \Omega_{\geq} \frac{1}{\underline{(1 - x\lambda)}(1 - yx^{-1})(1 - xz)} + \Omega_{\geq} \frac{1}{(1 - xy^{-1})\underline{(1 - y\lambda)}(1 - yz)} \quad (\text{by Theorem 2.2}) \\ & = \frac{1}{(1 - x)(1 - yx^{-1})(1 - xz)} + \frac{1}{(1 - xy^{-1})(1 - y)(1 - yz)} \quad (\text{setting } \lambda = 1 \text{ via (5)}) \end{aligned}$$

$$= \frac{1 - xyz}{(1 - x)(1 - y)(1 - xz)(1 - zy)}.$$

At the third equality, we apply the operations modulo $\langle 1 - x\lambda \rangle$ and modulo $\langle 1 - y\lambda \rangle$, respectively, as justified by Theorem 2.2.

Second approach: By using the dually contributing formulas, we derive:

$$\begin{aligned} & \Omega_{\geq} \frac{1}{(1 - x\lambda)(1 - y\lambda)(1 - \frac{z}{\lambda})} \\ &= \frac{1}{(1 - x)(1 - y)(1 - z)} - \Omega_{\geq} \frac{1}{(1 - x\lambda)(1 - y\lambda)(1 - \frac{z}{\lambda})} \quad (\text{by (9)}) \\ &= \frac{1}{(1 - x)(1 - y)(1 - z)} - \Omega_{\geq} \frac{-\lambda z^{-1}}{(1 - x\lambda)(1 - y\lambda)(1 - \frac{z}{\lambda})} \quad (\text{since } a_i \text{ is a positive integer in (3)}) \\ &= \frac{1}{(1 - x)(1 - y)(1 - z)} - \Omega_{\geq} \frac{-1}{(1 - xz)(1 - yz)(1 - \frac{z}{z})} \quad (\text{by Theorem 2.2}) \\ &= \frac{1}{(1 - x)(1 - y)(1 - z)} - \frac{-1}{(1 - xz)(1 - yz)(1 - \frac{1}{z})} \quad (\text{setting } \lambda = 1 \text{ via (9)}) \\ &= \frac{1 - xyz}{(1 - x)(1 - y)(1 - xz)(1 - zy)}. \end{aligned}$$

In fact, the second approach can be simplified as follows:

$$\begin{aligned} & \Omega_{\geq} \frac{1}{(1 - x\lambda)(1 - y\lambda)(1 - \frac{z}{\lambda})} = \frac{1}{(1 - x)(1 - y)(1 - z)} - \Omega_{\geq} \frac{1}{(1 - x\lambda)(1 - y\lambda)(1 - \frac{z}{\lambda})} \\ &= \frac{1}{(1 - x)(1 - y)(1 - z)} - \Omega_{\geq} \frac{\lambda}{(1 - x\lambda)(1 - y\lambda)(\lambda - z)} \\ &= \frac{1}{(1 - x)(1 - y)(1 - z)} - \Omega_{\geq} \frac{z}{(1 - xz)(1 - yz)(\lambda - z)} \\ &= \frac{1}{(1 - x)(1 - y)(1 - z)} - \frac{z}{(1 - xz)(1 - yz)(1 - z)} \\ &= \frac{1 - xyz}{(1 - x)(1 - y)(1 - xz)(1 - zy)}. \end{aligned}$$

3. Fundamental formulas

Rational functions in this section are assumed to be written in proper form.

Proposition 3.1. Let $A \neq 1$ be independent of λ . If the series expansion of $F(\lambda)$ only contains nonnegative powers of λ , then

$$\Omega_{\geq} \frac{1}{1 - A\lambda} \cdot F(\lambda^{-1}) = \frac{1}{1 - A} \cdot F(A).$$

Proof. Method 1: For any nonnegative integer m , by Definition 1.1 and the geometric series expansion, we have

$$\Omega_{\geq} \frac{1}{1 - A\lambda} \cdot \lambda^{-m} = \Omega_{\geq} \sum_{k \geq 0} \lambda^k A^k \cdot \lambda^{-m} = \Omega_{\geq} \sum_{k \geq 0} \lambda^{k-m} A^k = \sum_{k \geq m} A^k = \frac{A^m}{1 - A}.$$

The result follows from the linearity of the $\Omega_{\geq}$ operator.

Method 2 (only applies when $F(\lambda)$ is Elliott rational): Using the underlined notation for the contributing factor, we obtain

$$\Omega \frac{1}{1 - A\lambda} \cdot F(\lambda^{-1}) = \Omega \frac{1}{\underline{1 - A\lambda}} \cdot F(\lambda^{-1}) = \frac{1}{1 - A} \cdot F(A),$$

which completes the proof. $\square$

Proposition 3.1 appeared as Xin, 2005, Lemma 3.23 without a proof. Here, we focus on the case when $F(\lambda)$ is an Elliott rational function, so we provide the second proof.

Proposition 3.2. Let $B \neq 1$ be independent of λ . If $F(\lambda)$ is a rational function in λ whose series expansion only contains nonnegative powers in λ , and $F(1)$ exists, then

$$\Omega \frac{1}{1 - \frac{B}{\lambda}} \cdot F(\lambda) = \frac{F(1) - B \cdot F(B)}{1 - B}.$$

Proof. The proof is analogous to that of Proposition 3.1. We only use the second method under the assumption that $F(\lambda)$ is an Elliott rational function.

$$\begin{aligned} \Omega \frac{1}{1 - \frac{B}{\lambda}} \cdot F(\lambda) &= \frac{1}{1 - B} \cdot F(1) - \Omega \frac{1}{1 - \frac{B}{\lambda}} \cdot F(\lambda) = \frac{1}{1 - B} \cdot F(1) - \Omega \frac{\lambda}{\lambda - B} \cdot F(\lambda) \\ &= \frac{1}{1 - B} \cdot F(1) - \Omega \frac{B}{\lambda - B} \cdot F(B) = \frac{1}{1 - B} \cdot F(1) - \frac{B}{1 - B} \cdot F(B). \end{aligned}$$

Here, the second step rewrites the denominator, and the third step applies the operator to the underlined factor. The final expression yields the desired result. $\square$

The following nine fundamental formulas are presented in the book (MacMahon, 1960, Pages 102–103) without proof. Using our method, these identities can be readily established.

Proposition 3.3. (MacMahon, 1960, Pages 102–103) and (Andrews et al., 2001a, Section 2) Let s be a positive integer. The fundamental evaluations of the Omega operator are given by:

$$\Omega \frac{1}{(1 - \lambda x)(1 - \frac{y}{\lambda^s})} = \frac{1}{(1 - x)(1 - x^s y)}, \quad (10)$$

$$\Omega \frac{1}{(1 - \lambda^s x)(1 - \frac{y}{\lambda})} = \frac{1 + xy \frac{1 - y^{s-1}}{1 - y}}{(1 - x)(1 - y^s x)}, \quad (11)$$

$$\Omega \frac{1}{(1 - \lambda x)(1 - \frac{y}{\lambda})(1 - \frac{z}{\lambda})} = \frac{1}{(1 - x)(1 - xy)(1 - xz)}, \quad (12)$$

$$\Omega \frac{1}{(1 - \lambda x)(1 - \lambda y)(1 - \frac{z}{\lambda})} = \frac{1 - xyz}{(1 - x)(1 - y)(1 - xz)(1 - yz)}, \quad (13)$$

$$\Omega \frac{1}{(1 - \lambda x)(1 - \lambda y)(1 - \frac{z}{\lambda^2})} = \frac{1 + xyz - x^2 yz - xy^2 z}{(1 - x)(1 - y)(1 - xz^2)(1 - zy^2)}, \quad (14)$$

$$\Omega \frac{1}{(1 - \lambda^2 x)(1 - \frac{y}{\lambda})(1 - \frac{z}{\lambda})} = \frac{1 + xy + xz + xyz}{(1 - x)(1 - xy^2)(1 - xz^2)}, \quad (15)$$

$$\Omega \frac{1}{(1 - \lambda^2 x)(1 - \lambda y)(1 - \frac{z}{\lambda})} = \frac{1 + xz - xyz - xyz^2}{(1 - x)(1 - y)(1 - yz)(1 - xz^2)}, \quad (16)$$

$$\begin{aligned}\Omega \frac{1}{(1-\lambda x)(1-\lambda y)(1-\lambda z)(1-\frac{w}{\lambda})} &= \frac{1-xyw-xzw-yzw+xyzw+xyzw^2}{(1-x)(1-y)(1-z)(1-wx)(1-wy)(1-wz)}, \\ \Omega \frac{1}{(1-\lambda x)(1-\lambda y)(1-\frac{z}{\lambda})(1-\frac{w}{\lambda})} &= \frac{1-xyz-xyw-xyzw+xy^2zw+x^2yzw}{(1-x)(1-y)(1-xz)(1-xw)(1-yz)(1-yw)}.\end{aligned}\tag{17}$$

$$\tag{18}$$

Proof. Equations (10) and (12) are direct consequences of Proposition 3.1. Equation (11) follows from Proposition 3.2. The identity (13) is verified in Example 2.5. The proofs of (14) and (18) are analogous to the first approach in Example 2.5. Similarly, Equations (16) and (17) can be derived using the second method described in Example 2.5.

For (15), we proceed as follows:

$$\begin{aligned}\Omega \frac{1}{(1-\lambda^2x)(1-\frac{y}{\lambda})(1-\frac{z}{\lambda})} &= \Omega \frac{1}{(1-\lambda^2x)(1-\frac{y}{\lambda})(1-\frac{z}{\lambda})} = \Omega \frac{(1+\frac{y}{\lambda})(1+\frac{z}{\lambda})}{(1-\lambda^2x)(1-\frac{y^2}{\lambda^2})(1-\frac{z^2}{\lambda^2})} \\ &= \Omega \frac{1+\frac{y}{\lambda} \cdot \lambda^2x + \frac{z}{\lambda} \cdot \lambda^2x + \frac{yz}{\lambda^2} \cdot \lambda^2x}{(1-\lambda^2x)(1-xy^2)(1-xz^2)} = \frac{1+xy+xz+xyz}{(1-x)(1-xy^2)(1-xz^2)}.\end{aligned}$$

This completes the proof. $\square$

As noted in Andrews et al. (2001b): “The proofs of many of these rules are quite elementary but in case of several λ variables, elimination can be much more cumbersome.” We now demonstrate through the following example that our method remains straightforward even in the presence of multiple λ variables.

Proposition 3.4. (Andrews and Paule, 2007a, Lemma 2.1) *The following identity holds:*

$$\begin{aligned}\Omega \frac{1-AB\lambda_1\lambda_2}{(1-A\lambda_1)(1-B\lambda_2)(1-C\lambda_1\lambda_2)(1-D\lambda_1\lambda_2)(1-\frac{E}{\lambda_1\lambda_2})} \\ = \frac{(1-AB)(1-CDE)}{(1-A)(1-B)(1-C)(1-D)(1-CE)(1-DE)}.\end{aligned}$$

Proof. The operator $\Omega_{\geq}$ acts to eliminate the variables λ_1 and λ_2 . We proceed by first eliminating λ_1 , followed by λ_2 . By accounting for the dual contribution of about λ_1 , we derive

$$\begin{aligned}\Omega \frac{1-AB\lambda_1\lambda_2}{(1-A\lambda_1)(1-B\lambda_2)(1-C\lambda_1\lambda_2)(1-D\lambda_1\lambda_2)(1-\frac{E}{\lambda_1\lambda_2})} \\ = \Omega \frac{1-AB\lambda_2}{(1-A)(1-B\lambda_2)(1-C\lambda_2)(1-D\lambda_2)(1-\frac{E}{\lambda_2})} \\ - \Omega \frac{1-AB\lambda_1\lambda_2}{(1-A\lambda_1)(1-B\lambda_2)(1-C\lambda_1\lambda_2)(1-D\lambda_1\lambda_2)(1-\frac{E}{\lambda_1\lambda_2})} \\ = \Omega \frac{1-AB\lambda_2}{(1-A)(1-B\lambda_2)(1-C\lambda_2)(1-D\lambda_2)(1-\frac{E}{\lambda_2})} \\ - \Omega \frac{(1-ABE) \cdot \frac{E}{\lambda_2}}{(1-\frac{AE}{\lambda_2})(1-B\lambda_2)(1-CE)(1-DE)(1-\frac{E}{\lambda_2})}.\end{aligned}$$

For the first term above, we use the dually contributing formulas for λ_2 , while for the second term, we use the contributing formulas for λ_2 . This yields

$$\begin{aligned}
& \geq \frac{\Omega \frac{1 - AB\lambda_1\lambda_2}{(1 - A\lambda_1)(1 - B\lambda_2)(1 - C\lambda_1\lambda_2)(1 - D\lambda_1\lambda_2)(1 - \frac{E}{\lambda_1\lambda_2})}}{1 - AB} \\
& = \frac{1 - AB}{(1 - A)(1 - B)(1 - C)(1 - D)(1 - E)} - \frac{\Omega \frac{1 - AB\lambda_2}{(1 - A)(1 - B\lambda_2)(1 - C\lambda_2)(1 - D\lambda_2)(1 - \frac{E}{\lambda_2})}}{1 - AB} \\
& \quad - \frac{\Omega \frac{(1 - ABE) \cdot \frac{E}{\lambda_2}}{(1 - \frac{AE}{\lambda_2})(1 - B\lambda_2)(1 - CE)(1 - DE)(1 - \frac{E}{\lambda_2})}}{1 - AB} \\
& = \frac{1 - AB}{(1 - A)(1 - B)(1 - C)(1 - D)(1 - E)} - \frac{(1 - ABE)E}{(1 - A)(1 - BE)(1 - CE)(1 - DE)(1 - E)} \\
& \quad - \frac{(1 - ABE)BE}{(1 - ABE)(1 - B)(1 - CE)(1 - DE)(1 - BE)} \\
& = \frac{(1 - AB)(1 - CDE)}{(1 - A)(1 - B)(1 - C)(1 - D)(1 - CE)(1 - DE)}.
\end{aligned}$$

This concludes the proof. $\square$

We have also computed numerous examples from the works of Andrews and his collaborators. For instance: (Andrews et al., 2001b, Equation (12)), (Andrews, 1998, Equation (3.1)), (Andrews et al., 2001c, Equations (3) and (5)), (Andrews et al., 2001c, Lemmas 1 and 2), (Andrews et al., 2004, Proposition 1), (Andrews and Paule, 2024, Lemma 4.1), (Andrews and Paule, 2025, Lemmas 2.2, 2.3, 2.4, and 2.5), among others. Our approach readily handles these computations.

4. A computational framework

For many counting problems related to linear Diophantine inequalities, we often need to compute the generating function of the counts. For example, many counting problems in Section 5. Therefore, before our computational framework, we first need to convert the counting problem into $\Omega_{\geq E}$ using geometric series expansion, where $\Omega_{\geq E}$ is usually referred to as the *crude generating function*. Our next task is to eliminate the variables $\lambda_1, \lambda_2, \dots$

Computational Framework 4.1.

Input: An Elliott rational function E written in proper form. (For simplicity, we assume all intermediate stages only encounter proper rational functions.)

Output: $\Omega_{\geq E}$.

- (S-1). For each variable λ_i , count the number of contributing factors c_i and the number of dually contributing factors d_i . We prioritize eliminating the variable λ_{i_0} that have the smallest $\min\{c_{i_0}, d_{i_0} + 1\}$.
- (S-2). If $c_{i_0} \leq d_{i_0} + 1$, then we eliminate λ_{i_0} using the contributing formulas, i.e., Equations (6) and (8); Otherwise we eliminate λ_{i_0} using the dually contributing formulas, i.e., Equations (6) and (9).
- (S-3). For each resulting rational function, repeat stages (S-1)–(S-2) to eliminate all the remaining λ variables, yielding a summation of rational functions.
- (S-4). Finally simplify the resulting summation.

Note that the general situation involves the computation of $P(1)$, which is addressed by Equations (6) and (7), but the situation is much more complicated.

One can compare the computation in Proposition 3.4 with each stage of the computational framework described above. Proposition 3.4 is a nice example to illustrate each of the stages (S-1)–(S-4) in the above computational framework.

The following result first appeared in (MacMahon, 1960, Page 183), and has been frequently utilized in the work of Andrews et al. (Andrews, 2001c; Andrews et al., 2004; Andrews and Paule, 2007a,b).

Proposition 4.2. (MacMahon, 1960, Page 183) *The following identity holds:*

$$\begin{aligned} & \Omega \frac{1}{(1-x_1\lambda_1\lambda_2)(1-\frac{x_2\lambda_3}{\lambda_1})(1-\frac{x_4}{\lambda_3\lambda_4})(1-\frac{x_3\lambda_4}{\lambda_2})} \\ &= \frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}. \end{aligned}$$

Proof. The operator $\Omega_{\geq}$ is applied to eliminate the variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. We proceed by eliminating them in the order $\lambda_4, \lambda_3, \lambda_2, \lambda_1$ successively. This yields the following sequence of expressions:

$$\begin{aligned} & \Omega \frac{1}{(1-x_1\lambda_1\lambda_2)(1-\frac{x_2\lambda_3}{\lambda_1})(1-\frac{x_4}{\lambda_3\lambda_4})(1-\frac{x_3\lambda_4}{\lambda_2})} \\ &= \Omega \frac{1}{(1-x_1\lambda_1\lambda_2)(1-\frac{x_2\lambda_3}{\lambda_1})(1-\frac{x_3x_4}{\lambda_2\lambda_3})(1-\frac{x_3}{\lambda_2})} \\ &= \Omega \frac{1}{(1-x_1\lambda_1\lambda_2)(1-\frac{x_2}{\lambda_1})(1-\frac{x_2x_3x_4}{\lambda_1\lambda_2})(1-\frac{x_3}{\lambda_2})} \\ &= \Omega \frac{1}{(1-x_1\lambda_1)(1-\frac{x_2}{\lambda_1})(1-x_1x_2x_3x_4)(1-x_1x_3\lambda_1)} \\ &= \frac{1}{(1-x_1)(1-x_1x_2)(1-x_3)(1-x_1x_2x_3x_4)} \\ & \quad + \frac{1}{(1-x_3^{-1})(1-x_1x_2x_3)(1-x_1x_3)(1-x_1x_2x_3x_4)} \\ &= \frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}. \end{aligned}$$

This completes the proof. $\square$

For an Elliott rational function, we need to extract the numerator and denominator parts and then analyze the degree of each variable. In particular, for each monomial M , we need to determine whether $M < 1$ or $M > 1$. Thus it is more convenient to use a list to represent an Elliott rational function. For the reader's convenience, we have implemented several computational steps in Maple (Maple, 2025), which is freely available at <https://github.com/TygerLiu/TygerLiu.github.io/tree/main/Procedure/Omega-Fundamental>. The program features five main commands: R2List, L2Rational, ClassifyNCDE, ContributedDispel, and DualContributedDispel.

To illustrate their usage, we now apply these commands to rederive Proposition 4.2.

Step 1: In the Maple sheet, we start with

$$> \text{temp} := \frac{1}{(1-x_1\lambda_1\lambda_2)(1-\frac{x_2\lambda_3}{\lambda_1})(1-\frac{x_3\lambda_4}{\lambda_2})(1-\frac{x_4}{\lambda_3\lambda_4})}$$

We first convert the rational function “temp” into an equivalent list representation “LL” and turn it into proper form. This is achieved by the command R2List and Lflip:

$$\begin{aligned} & > \text{R2List}(\text{temp}, [x_1, x_2, x_3, x_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4]); \\ & > \left[-\frac{1}{x_1x_2x_3\lambda_3\lambda_4}, \frac{1}{x_1\lambda_1\lambda_2}, \frac{\lambda_1}{x_2\lambda_3}, \frac{\lambda_2}{x_3\lambda_4}, \frac{x_4}{\lambda_3\lambda_4} \right] \\ & > \text{LL} := \text{Lflip}(\%, (M) \rightarrow \text{not IsSmall}(M, [x_1, x_2, x_3, x_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4])); \\ & > \text{LL} := \left[1, x_1\lambda_1\lambda_2, \frac{x_2\lambda_3}{\lambda_1}, \frac{x_3\lambda_4}{\lambda_2}, \frac{x_4}{\lambda_3\lambda_4} \right] \end{aligned}$$

In the list “LL”, the first element corresponds to the numerator and the other elements (which are small in the proper form) corresponds to the denominator factors. Such a list representation facilitates computation in Maple.

Step 2: Count c_i and d_i for each λ_i . In the proper form “LL”, c_i (resp. d_i) is just the number of elements with positive (resp. negative) power of λ_i (the first element does not come into the count). The command `ClassifyNCIDI` can be used to generate a table, which displays the “contributing factors” and “dually contributing factors” for each variable λ_i . The column “C-Num” indicates the number c_i of contributing factors, while “DC-Num” indicates the number d_i of dually contributing factors.

> `ClassifyNCIDI(temp);`

	Numerator	Contributing	C-Num	Dually contributing	DC-Num
lambda[1]	1	$1 - x_1 \lambda_1 \lambda_2$	1	$1 - \frac{x_2 \lambda_3}{\lambda_1}$	1
> lambda[2]	1	$1 - x_1 \lambda_1 \lambda_2$	1	$1 - \frac{x_3 \lambda_4}{\lambda_2}$	1
lambda[3]	1	$1 - \frac{x_2 \lambda_3}{\lambda_1}$	1	$1 - \frac{x_4}{\lambda_3 \lambda_4}$	1
lambda[4]	1	$1 - \frac{x_3 \lambda_4}{\lambda_2}$	1	$1 - \frac{x_4}{\lambda_3 \lambda_4}$	1

In this simple example, we can choose any λ_i to eliminate, but only using the contributing formulas (Equations (6) and (8)).

Step 3: We choose to eliminate the variable λ_4 first using the command `ContributeDispel`, which corresponds to the underlined operation.

> `T1 := ContributeDispel(LL,  $\lambda_4$ , 3);`

$$> T1 := \left[1, -\frac{x_3 x_4}{\lambda_2 \lambda_3}, x_1 \lambda_1 \lambda_2, \frac{x_2 \lambda_3}{\lambda_1}, \frac{x_3}{\lambda_2} \right]$$

The command `ContributeDispel(LL,  $\lambda_4$ ,  $k$ )` specifies that the $(k + 1)$ -th term of λ_4 in “LL” is contributing.

Similarly, we successively eliminate λ_3 and λ_2 to obtain:

> `T2 := ContributeDispel(T1,  $\lambda_3$ , 3);`

$$> T2 := \left[1, -\frac{x_2 x_3 x_4}{\lambda_1 \lambda_2}, x_1 \lambda_1 \lambda_2, \frac{x_3}{\lambda_2}, \frac{x_2}{\lambda_1} \right]$$

> `T3 := ContributeDispel(T2,  $\lambda_2$ , 2); F3 := L2Rational(T3);`

$$> T3 := \left[1, x_1 x_2 x_3 x_4, x_1 x_3 \lambda_1, \frac{x_2}{\lambda_1}, x_1 \lambda_1 \right]$$

$$F3 := \frac{\lambda_1}{(1 - x_1 x_2 x_3 x_4)(1 - x_1 x_3 \lambda_1)(\lambda_1 - x_2)(1 - x_1 \lambda_1)}$$

Here, the command `L2Rational` converts the list back into a rational function.

Now the situation is a little different: $c_1 = 2$ and $d_1 = 1$. Since $c_1 = d_1 + 1$, we give two approaches to eliminate λ_1 . Applying the contributing formulas gives

> `T41 := ContributeDispel(T3,  $\lambda_1$ , 2); F41 := L2Rational(T41);`

$$> T41 := \left[1, x_1 x_2 x_3, \frac{1}{x_3}, x_1 x_2 x_3 x_4, x_1 x_3 \right]$$

$$F41 := \frac{x_3}{(1 - x_1 x_2 x_3)(1 - x_1 x_3)(1 - x_3)(1 - x_1 x_2 x_3 x_4)}$$

> `T42 := ContributeDispel(T3,  $\lambda_1$ , 4); F42 := L2Rational(T42);`

$$\begin{aligned}
&> T42 := [1, x_3, x_1x_2, x_1x_2x_3x_4, x_1] \quad F42 := \frac{1}{(1-x_3)(1-x_1x_2)(1-x_1)(1-x_1x_2x_3x_4)} \\
&> F := \text{normal}(F41 + F42); \\
&> F := \frac{1 - x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3x_4)(1-x_1x_2x_3)}
\end{aligned}$$

Applying the dually contributing formulas using the command `DualContributeDispel` gives

$$\begin{aligned}
&> T5 := \text{DualContributeDispel}(T3, \lambda_1, 3); \quad F5 := \text{L2Rational}(T5); \\
&> T5 := [x_2, x_1x_2x_3, x_1x_2, x_1x_2x_3x_4, x_2] \\
&\quad F5 := \frac{x_2}{(1-x_1x_2x_3)(1-x_1x_2)(1-x_2)(1-x_1x_2x_3x_4)} \\
&> F := \text{normal}(\text{subs}(\lambda_1 = 1, F3) - F5); \\
&> F := \frac{1 - x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3x_4)(1-x_1x_2x_3)}
\end{aligned}$$

Note that one must check the conditions when applying the contributing and dually contributing formulas.

Next, we give a complicated example to illustrate that the Command `ClassifyNCID` can help us make choices manually. Consider the following randomly constructed crude generating function:

$$\begin{aligned}
\Omega &= \frac{x_2\lambda_2}{(1-x_1\lambda_1\lambda_2)(1-\frac{x_2\lambda_3}{\lambda_1})(1-\frac{x_3\lambda_4}{\lambda_2})(1-\frac{x_4}{\lambda_3\lambda_4})(1-\frac{x_1\lambda_2}{\lambda_1})(1-\frac{x_2\lambda_4}{\lambda_2})(1-\frac{x_3}{\lambda_3\lambda_4})^2} \\
&\quad \cdot \frac{1}{(1-\frac{x_1}{\lambda_2})^2(1-\frac{x_2\lambda_2}{\lambda_1^3})^2(1-x_3\lambda_1^2)^2(1-x_1\lambda_2)(1-x_1x_3\lambda_2)(1-x_1^5\lambda_2^7)}.
\end{aligned}$$

By using `ClassifyNCID`, we obtain

	Numerator	Contributing	C-Num	Dually contributing	DC-Num
lambda[1]	$x_2\lambda_2$	$(1-x_1\lambda_1\lambda_2)(1-x_3\lambda_1^2)^2$	3	$(1-\frac{x_2\lambda_3}{\lambda_1})(1-\frac{x_1\lambda_2}{\lambda_1})(1-\frac{x_2\lambda_2}{\lambda_1^3})^2$	4
lambda[2]	$x_2\lambda_2$	M	7	$(1-\frac{x_3\lambda_4}{\lambda_2})(1-\frac{x_2\lambda_4}{\lambda_2})(1-\frac{x_1}{\lambda_2^2})^2$	4
lambda[3]	$x_2\lambda_2$	$1-\frac{x_2\lambda_3}{\lambda_1}$	1	$(1-\frac{x_4}{\lambda_3\lambda_4})(1-\frac{x_3}{\lambda_3\lambda_4})^2$	3
lambda[4]	$x_2\lambda_2$	$(1-\frac{x_3\lambda_4}{\lambda_2})(1-\frac{x_2\lambda_4}{\lambda_2})$	2	$(1-\frac{x_4}{\lambda_3\lambda_4})(1-\frac{x_3}{\lambda_3\lambda_4})^2$	3

In the above table, M denotes

$$(1-x_1\lambda_1\lambda_2) \left(1-\frac{x_1\lambda_2}{\lambda_1}\right) \left(1-\frac{x_2\lambda_2}{\lambda_1^3}\right)^2 (1-x_1\lambda_2)(1-x_1x_3\lambda_2)(1-x_1^5\lambda_2^7).$$

Therefore, according to our computational framework, we should prioritize using the contributing formulas to eliminate the variable λ_3 .

5. Applications

5.1. Han's formula

Han (2003) developed a general algorithm for computing the MacMahon $\Omega_{\geq}$ operator and implemented it in the Maple package `GenOmega`. Equation (19) below is a key result of this algorithm, and its proof occupies a significant portion (Han, 2003, Theorem 1.4). We now need only one step to prove this formula.

Let

$$A(t) = \prod_{i=1}^n (1 - x_i t), \quad B(t) = \prod_{i=1}^m (1 - y_i t)$$

be two polynomials.

Theorem 5.1. (Han, 2003, Theorem 1.4) *Let $U(\lambda)$ be a Laurent polynomial with $x_i \neq 1$ and $x_i y_j \neq 1$ for all i, j . If $\deg U(\lambda) \leq n - 1$, then*

$$\Omega \frac{U(\lambda)}{A(\lambda)B(1/\lambda)} = \sum_{i=1}^n \frac{x_i^{n-1} U(1/x_i)}{(1 - x_i)B(x_i) \prod_{j \neq i} (x_i - x_j)}. \quad (19)$$

The formula holds even when the y_i 's are not all distinct.

Proof. We compute as follows:

$$\begin{aligned} \Omega \frac{U(\lambda)}{A(\lambda)B(1/\lambda)} &= \Omega \frac{U(\lambda)}{(1 - x_1 \lambda)(1 - x_2 \lambda) \cdots (1 - x_n \lambda)B(1/\lambda)} \\ &= \sum_{i=1}^n \frac{U(1/x_i)}{(1 - x_i)B(x_i) \prod_{j \neq i} (1 - x_j/x_i)}. \end{aligned}$$

This completes the proof. $\square$

5.2. k -gon partitions

For simplicity, we adopt the notation from Andrews et al. (2001c). Let $\mathbb{P}$ denote the set of all positive integers. The set of *non-degenerate k -gon partitions* into positive parts is defined by

$$\tau_k = \left\{ (a_1, a_2, \dots, a_k) \in \mathbb{P}^k \mid 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k, a_1 + \cdots + a_{k-1} > a_k \right\}.$$

The set of *non-degenerate k -gon partitions of a positive integer n* is given by

$$\tau_k(n) = \{ (a_1, a_2, \dots, a_k) \in \tau_k \mid a_1 + a_2 + \cdots + a_k = n \},$$

and its cardinality is denoted by

$$t_k(n) = |\tau_k(n)|.$$

For any integer $k \geq 3$, we define the generating functions

$$T_k(q) = \sum_{n \geq k} t_k(n) q^n,$$

and

$$S_k(x_1, \dots, x_k) = \sum_{(a_1, \dots, a_k) \in \tau_k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}.$$

In Andrews et al. (2001a), Andrews, Paule, and Riese derived explicit expressions for $T_3(q)$, $T_4(q)$, $T_5(q)$, and $T_6(q)$ using the Omega package. They also posed the following open problem (Andrews et al., 2001a, Problem 4.2): Is it possible to identify a unified pattern for all k ? This question was later resolved in Andrews et al. (2001c). In what follows, we present a simplified proof of this result.

Theorem 5.2. (Andrews et al., 2001c, Theorem 1) Let $k \geq 3$. Then

$$S_k(x_1, \dots, x_k) = \frac{x_1 x_2 \cdots x_k}{(1 - x_1 x_2 \cdots x_k)(1 - x_2 \cdots x_k) \cdots (1 - x_k)} \\ - \frac{x_1 x_2 \cdots x_{k-1} x_k^{k-1}}{(1 - x_1 \cdots x_{k-1} x_k^{k-1})(1 - x_2 \cdots x_{k-1} x_k^{k-2}) \cdots (1 - x_{k-2} x_{k-1} x_k^2)(1 - x_{k-1} x_k)(1 - x_k)}.$$

Proof. By the definition of $\Omega_{\geq}$ and the geometric series summation, we have

$$S_k(x_1, \dots, x_k) = \Omega_{\geq} \sum_{a_1 \geq 1, a_2, \dots, a_k \geq 0} x_1^{a_1} \cdots x_k^{a_k} \lambda_1^{a_2 - a_1} \cdots \lambda_{k-1}^{a_k - a_{k-1}} \lambda_k^{a_1 + \cdots + a_{k-1} - a_k - 1} \\ = \Omega_{\geq} \frac{x_1 \lambda_1^{-1}}{(1 - \frac{x_1 \lambda_k}{\lambda_1})(1 - \frac{x_2 \lambda_1 \lambda_k}{\lambda_2})(1 - \frac{x_3 \lambda_2 \lambda_k}{\lambda_3}) \cdots (1 - \frac{x_{k-1} \lambda_{k-2} \lambda_k}{\lambda_{k-1}})(1 - \frac{x_k \lambda_{k-1}}{\lambda_k})}.$$

The $\Omega_{\geq}$ operator eliminates the variables $\lambda_1, \lambda_2, \dots, \lambda_k$. We proceed by first eliminating $\lambda_{k-1}, \lambda_{k-2}, \dots, \lambda_1$ in succession, and then λ_k . The computation unfolds as follows:

$$S_k(x_1, \dots, x_k) = \Omega_{\geq} \frac{x_1 \lambda_1^{-1}}{(1 - \frac{x_1 \lambda_k}{\lambda_1})(1 - \frac{x_2 \lambda_1 \lambda_k}{\lambda_2})(1 - \frac{x_3 \lambda_2 \lambda_k}{\lambda_3}) \cdots (1 - \frac{x_{k-1} \lambda_{k-2} \lambda_k}{\lambda_{k-1}})(1 - \frac{x_k \lambda_{k-1}}{\lambda_k})} \\ = \Omega_{\geq} \frac{x_1 \lambda_1^{-1}}{(1 - \frac{x_1 \lambda_k}{\lambda_1})(1 - \frac{x_2 \lambda_1 \lambda_k}{\lambda_2}) \cdots (1 - \frac{x_{k-2} \lambda_{k-3} \lambda_k}{\lambda_{k-2}})(1 - x_{k-1} x_k \lambda_{k-2})(1 - \frac{x_k}{\lambda_k})} \\ = \Omega_{\geq} \frac{x_1 \lambda_1^{-1}}{(1 - \frac{x_1 \lambda_k}{\lambda_1})(1 - \frac{x_2 \lambda_1 \lambda_k}{\lambda_2}) \cdots (1 - \frac{x_{k-3} \lambda_{k-4} \lambda_k}{\lambda_{k-3}})(1 - x_{k-2} x_{k-1} x_k \lambda_{k-3} \lambda_k)(1 - x_{k-1} x_k)(1 - \frac{x_k}{\lambda_k})} \\ = \Omega_{\geq} \frac{x_1 \lambda_1^{-1}}{(1 - \frac{x_1 \lambda_k}{\lambda_1})(1 - \frac{x_2 \lambda_1 \lambda_k}{\lambda_2}) \cdots (1 - x_{k-3} \cdots x_k \lambda_{k-4} \lambda_k^2)(1 - x_{k-2} x_{k-1} x_k \lambda_k)(1 - x_{k-1} x_k)(1 - \frac{x_k}{\lambda_k})} \\ = \cdots \\ = \Omega_{\geq} \frac{x_1 \lambda_1^{-1}}{(1 - \frac{x_1 \lambda_k}{\lambda_1})(1 - x_2 \cdots x_k \lambda_1 \lambda_k^{k-3})(1 - x_3 \cdots x_k \lambda_k^{k-4}) \cdots (1 - x_{k-2} x_{k-1} x_k \lambda_k)(1 - x_{k-1} x_k)(1 - \frac{x_k}{\lambda_k})} \\ = \Omega_{\geq} \frac{x_1 x_2 \cdots x_k \lambda_k^{k-3}}{(1 - x_1 \cdots x_k \lambda_k^{k-2})(1 - x_2 \cdots x_k \lambda_k^{k-3}) \cdots (1 - x_{k-2} x_{k-1} x_k \lambda_k)(1 - x_{k-1} x_k)(1 - \frac{x_k}{\lambda_k})}.$$

Now, by considering the dual contribution, we obtain

$$S_k(x_1, \dots, x_k) = \frac{x_1 x_2 \cdots x_k}{(1 - x_1 x_2 \cdots x_k)(1 - x_2 \cdots x_k) \cdots (1 - x_k)} \\ - \Omega_{\geq} \frac{x_1 x_2 \cdots x_k \lambda_k^{k-3}}{(1 - x_1 \cdots x_k \lambda_k^{k-2})(1 - x_2 \cdots x_k \lambda_k^{k-3}) \cdots (1 - x_{k-2} x_{k-1} x_k \lambda_k)(1 - x_{k-1} x_k)(1 - \frac{x_k}{\lambda_k})}.$$

An application of (9) completes the proof. $\square$

Let $x_i = q$ for all i . We obtain the following result.

Corollary 5.3. (Andrews et al., 2001c, Corollary 1) Let $k \geq 3$. Then

$$T_k(q) = \frac{q^k}{(1 - q)(1 - q^2) \cdots (1 - q^k)} - \frac{q^{2k-2}}{1 - q} \cdot \frac{1}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2k-2})}.$$

5.3. Two dimensional problem

Inspired by the work of Klosinski et al. (2000), Andrews, Paule, and Riese investigated the generating function for a two-dimensional problem:

$$S_{K,L}(x, y) = \sum_{\substack{m, n \geq 0; \\ Km \geq n; \\ Ln \geq m}} x^m y^n,$$

where $K, L \geq 2$. We now illustrate the application of our method to this problem.

Theorem 5.4. (Andrews, 2001b, Section 2) Let $K, L \geq 2$. Then

$$S_{K,L}(x, y) = \frac{1 + xy \frac{(1-x^L)(1-y^K)}{(1-x)(1-y)} - xy^K - x^L y}{(1 - xy^K)(1 - x^L y)}.$$

Proof. By the definition of the $\Omega_{\geq}$ operator and the geometric series summation, we have

$$S_{K,L}(x, y) = \Omega_{\geq} \frac{1}{(1 - \frac{x\lambda_1^K}{\lambda_2})(1 - \frac{y\lambda_2^L}{\lambda_1})}.$$

To apply the $\Omega_{\geq}$ operator, we eliminate λ_1 first, followed by λ_2 . Considering the dual contribution in λ_1 , we obtain

$$\begin{aligned} S_{K,L}(x, y) &= \Omega_{\geq} \frac{1}{(1 - \frac{x}{\lambda_2})(1 - y\lambda_2^L)} - \Omega_{\geq} \frac{1}{(1 - \frac{x\lambda_1^K}{\lambda_2})(1 - \frac{y\lambda_2^L}{\lambda_1})} \\ &= \Omega_{\geq} \frac{1}{(1 - \frac{x}{\lambda_2})(1 - y\lambda_2^L)} - \Omega_{\geq} \frac{y\lambda_2^L}{(1 - xy^K\lambda_2^{LK-1})(1 - y\lambda_2^L)}. \end{aligned}$$

The second term can be evaluated directly by setting $\lambda_2 = 1$. For the first term, we consider the dual contribution in λ_2 , yielding

$$\begin{aligned} S_{K,L}(x, y) &= \frac{1}{(1-x)(1-y)} - \Omega_{\geq} \frac{1}{(1 - \frac{x}{\lambda_2})(1 - y\lambda_2^L)} - \frac{y}{(1 - xy^K)(1-y)} \\ &= \frac{1}{(1-x)(1-y)} - \frac{x}{(1-x)(1-yx^L)} - \frac{y}{(1 - xy^K)(1-y)} \\ &= \frac{1 + xy \frac{(1-x^L)(1-y^K)}{(1-x)(1-y)} - xy^K - x^L y}{(1 - xy^K)(1 - x^L y)}. \end{aligned}$$

This completes the proof. $\square$

Our method remains applicable to several other results, including (Andrews, 2000, Theorems 3.1, 4.1, and 5.1), (Andrews and Paule, 2022, Lemma 3.1), and (MacMahon, 1960, Page 139 (No. 386–394)), among others. For brevity, we do not enumerate them all here.

5.4. A hard problem

We begin by recalling the definition of the q -shifted factorial. For a complex number a and a positive integer n , it is defined as

$$(a; q)_0 = 1, \quad (a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}).$$

In Li et al. (2024), the computation of the following generating function is a difficult problem and has not been solved:

$$F(x, y, q) = \sum_{\substack{i \geq b_0 \geq b_1 \geq \dots \geq b_k \geq 0; \\ b_0 \geq j \geq c_1 \geq c_2 \geq \dots \geq c_r \geq 0}} x^i y^j q^{b_0 + b_1 + \dots + b_k + c_1 + \dots + c_r}.$$

In this subsection, we employ MacMahon's partition analysis to derive an explicit formula for $F(x, y, q)$.

Theorem 5.5. *With the notation as above, we have*

$$F(x, y, q) = \frac{1}{(y; q)_{r+1}(x; q)_{k+2}} + \sum_{i=0}^r \frac{(-1)^{i+1} y q^{i(i+3)/2}}{(1-x)(1-yq^i)(q; q)_i(q; q)_{r-i}(xyq^{i+1}; q)_{k+1}}.$$

Proof. To establish this result, we calculate the following generating function:

$$G(x, y, t, z, q) = \sum_{\substack{i \geq b_0 \geq b_1 \geq \dots \geq b_k \geq 0; \\ b_0 \geq j \geq c_1 \geq c_2 \geq \dots \geq c_r \geq 0}} x^i y^j t^{b_0} z^{b_1} q^{b_2 + \dots + b_k + c_1 + \dots + c_r}.$$

Note that $F(x, y, q) = G(x, y, q, q, q)$. Let

$$G_1(x, y, t, q) = \sum_{i \geq b_0 \geq j \geq c_1 \geq c_2 \geq \dots \geq c_r \geq 0} x^i y^j t^{b_0} q^{c_1 + \dots + c_r} \quad \text{and} \\ G_2(z, q) = \sum_{b_1 \geq \dots \geq b_k \geq 0} z^{b_1} q^{b_2 + \dots + b_k}.$$

It is straightforward to verify that

$$G_1(x, y, t, q) = \frac{1}{(1-x)(1-xt)(1-xyt)(1-xytq) \dots (1-xytq^r)}, \\ G_2(z, q) = \frac{1}{(1-z)(1-zq) \dots (1-zq^{k-1})}.$$

By Definition 1.1 of the operator $\Omega_{\geq}$, the generating function $G(x, y, t, z, q)$ can be expressed as

$$G(x, y, t, z, q) = \Omega_{\geq} G_1(x, y, t\lambda, q) \cdot G_2\left(\frac{z}{\lambda}, q\right).$$

Applying the operator step by step, we eliminate the variable λ and obtain:

$$\begin{aligned} & G(x, y, t, z, q) \\ &= \Omega_{\geq} \frac{1}{(1-x)(1-xt\lambda)(1-xyt\lambda)(1-xytq\lambda) \dots (1-xytq^r\lambda)(1-\frac{z}{\lambda})(1-\frac{zq}{\lambda}) \dots (1-\frac{zq^{k-1}}{\lambda})} \\ &= \frac{1}{(1-x)(1-xt)(1-y)(1-yq) \dots (1-yq^r)(1-xzt)(1-xztq) \dots (1-xztq^{k-1})} \\ &+ \frac{1}{(1-x)(1-\frac{1}{y})(1-xyt)(1-q) \dots (1-q^r)(1-xyzt)(1-xyztq) \dots (1-xyztq^{k-1})} \\ &+ \frac{1}{(1-x)(1-\frac{1}{yq})(1-\frac{1}{q})(1-xytq)(1-q) \dots (1-q^{r-1})(1-xyztq)(1-xyztq^2) \dots (1-xyztq^k)} \\ &+ \frac{1}{(1-x)(1-\frac{1}{yq^2})(1-\frac{1}{q^2})(1-\frac{1}{q})(1-xytq^2)(1-q) \dots (1-q^{r-2})(1-xyztq^2) \dots (1-xyztq^{k+1})} \end{aligned}$$

$$\begin{aligned}
 & + \cdots \\
 & + \frac{1}{(1-x)(1-\frac{1}{yq^r})(1-\frac{1}{q^r})(1-\frac{1}{q^{r-1}}) \cdots (1-\frac{1}{q})(1-xyztq^r)(1-xyztq^{r+1}) \cdots (1-xyztq^{k+r-1})} \\
 & = \frac{1}{(1-x)(1-xt)(y; q)_{r+1}(xzt; q)_k} + \sum_{i=0}^r \frac{(-1)^{i+1} y q^{\frac{i(i+3)}{2}}}{(1-x)(1-yq^i)(1-xytq^i)(q; q)_i(q; q)_{r-i}(xyztq^i; q)_k}.
 \end{aligned}$$

Substituting $t = q$, $z = q$, and recalling that $F(x, y, q) = G(x, y, q, q, q)$, we arrive at the desired expression. This completes the proof. $\square$

5.5. Two generating functions

Let $\mathbb{H}$ be a ring. Consider the bivariate generating function

$$F(x, y) = \sum_{i \geq j \geq 0} a_{ij} x^i y^j, \quad a_{ij} \in \mathbb{H}.$$

Define the generating functions $G_1(x, y, z)$ and $G_2(x, z)$ by

$$G_1(x, y, z) := \sum_{m \geq i \geq \ell \geq j \geq 0} a_{ij} x^m y^\ell z^{i+j}, \quad G_2(x, z) := \sum_{i \geq \ell \geq j \geq 0} a_{ij} x^\ell z^{i+j}.$$

As highlighted in Li et al. (2024), these functions are of particular interest. We now derive their expressions via MacMahon's partition analysis.

Theorem 5.6. *Following the above notation, we have*

$$G_1(x, y, z) = \frac{F(xz, yz) - yF(xyz, z)}{(1-x)(1-y)}, \quad G_2(x, z) = \frac{F(z, xz) - xF(xz, z)}{1-x}.$$

Proof. By the definition of $\Omega_{\geq}$ and the geometric series summation, we obtain

$$\begin{aligned}
 G_1(x, y, z) &= \Omega_{\geq} \sum_{m, i, \ell, j \geq 0} a_{ij} \lambda_1^{m-i} \lambda_2^{i-\ell} \lambda_3^{\ell-j} x^m y^\ell z^{i+j} \\
 &= \Omega_{\geq} \sum_{i, j \geq 0} a_{ij} \left(\frac{z\lambda_2}{\lambda_1} \right)^i \left(\frac{z}{\lambda_3} \right)^j \frac{1}{(1-x\lambda_1)(1-\frac{y\lambda_3}{\lambda_2})} \\
 &= \Omega_{\geq} F\left(\frac{z\lambda_2}{\lambda_1}, \frac{z}{\lambda_3} \right) \frac{1}{(1-x\lambda_1)(1-\frac{y\lambda_3}{\lambda_2})}.
 \end{aligned}$$

The operator $\Omega_{\geq}$ eliminates the variables λ_1 , λ_2 , and λ_3 . We first eliminate λ_1 and λ_3 , then λ_2 . Applying this procedure:

$$\begin{aligned}
 G_1(x, y, z) &= \Omega_{\geq} F\left(\frac{z\lambda_2}{\lambda_1}, \frac{z}{\lambda_3} \right) \frac{1}{(1-x\lambda_1)(1-\frac{y\lambda_3}{\lambda_2})} = \Omega_{\geq} F\left(xz\lambda_2, \frac{z}{\lambda_3} \right) \frac{1}{(1-x)(1-\frac{y\lambda_3}{\lambda_2})} \\
 &= \Omega_{\geq} F\left(xz\lambda_2, \frac{yz}{\lambda_2} \right) \frac{1}{(1-x)(1-\frac{y}{\lambda_2})}.
 \end{aligned}$$

Note that the power series expansion of $F(xz\lambda_2, \frac{yz}{\lambda_2})$ contains only non-negative powers of λ_2 . Considering the dual contribution in λ_2 , we derive

$$\begin{aligned}
G_1(x, y, z) &= \sum_{\geq} F\left(xz\lambda_2, \frac{yz}{\lambda_2}\right) \frac{1}{(1-x)(1-\frac{y}{\lambda_2})} = \frac{F(xz, yz)}{(1-x)(1-y)} - \frac{\lambda_2 \cdot F(xz\lambda_2, \frac{yz}{\lambda_2})}{(1-x)(\lambda_2 - y)} \\
&= \frac{F(xz, yz) - yF(xyz, z)}{(1-x)(1-y)}.
\end{aligned}$$

Similarly, for $G_2(x, z)$, we obtain

$$\begin{aligned}
G_2(x, z) &= \sum_{i, \ell, j \geq 0} a_{ij} \lambda_1^{i-\ell} \lambda_2^{\ell-j} x^\ell z^{i+j} = \sum_{i, j \geq 0} a_{ij} (z\lambda_1)^i \left(\frac{z}{\lambda_2}\right)^j \frac{1}{(1-\frac{x\lambda_2}{\lambda_1})} \\
&= \sum_{\geq} F\left(z\lambda_1, \frac{z}{\lambda_2}\right) \frac{1}{(1-\frac{x\lambda_2}{\lambda_1})} = \sum_{\geq} F\left(z\lambda_1, \frac{xz}{\lambda_1}\right) \frac{1}{(1-\frac{x}{\lambda_1})} \\
&= \frac{F(z, xz) - xF(xz, z)}{1-x}.
\end{aligned}$$

This completes the proof. $\square$

CRedit authorship contribution statement

Feihu Liu: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Formal analysis, Data curation, Conceptualization. **Guoce Xin:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We are grateful to the two anonymous referees for their useful comments and suggestions. Guoce Xin is partially supported by the National Natural Science Foundation of China [12571355]. Feihu Liu was supported by the Postdoctoral Fellowship Program and China Postdoctoral Science Foundation.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.jsc.2026.102608>.

Data availability

No data was used for the research described in the article.

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