

# Spectral radius and rainbow $k$ -factors of graphs

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**Abstract:** Let  $\mathcal{G} = \{G_1, \dots, G_{\frac{kn}{2}}\}$  be a set of graphs on the same vertex set  $V = \{1, \dots, n\}$  where  $k \cdot n$  is even. We say  $\mathcal{G}$  admits a rainbow  $k$ -factor if there exists a  $k$ -regular graph  $F$  on the vertex set  $V$  such that all edges of  $F$  are from different members of  $\mathcal{G}$ . In this paper, we show a sufficient spectral condition for the existence of a rainbow  $k$ -factor for  $k \geq 2$ , which is that if  $\rho(G_i) \geq \rho(K_{k-1} \vee (K_1 \cup K_{n-k}))$  for each  $G_i \in \mathcal{G}$ , then  $\mathcal{G}$  admits a rainbow  $k$ -factor unless  $G_1 = G_2 = \dots = G_{\frac{kn}{2}} \cong K_{k-1} \vee (K_1 \cup K_{n-k})$ .

**Keywords:** Extremal graph theory; Spectral radius; Rainbow  $k$ -factor

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## 1 Introduction

The spanning subgraphs of graphs possessing some given properties are called *factors*. Factor theory is one of the fundamental areas of graph theory and closely relates to graph spectra. A considerable number of influential results on spectra and factors have been established, see, for example, the survey [13]. This paper primarily focuses on  $[a, b]$ -factor, which is defined as a spanning subgraph  $H$  of  $G$  such that  $d_H(v) \in [a, b]$  for each  $v \in V(G)$ . Particularly, a  $[k, k]$ -factor is also called a  $k$ -factor.

In 2005, Brouwer and Haemers [4] provided a sufficient condition for the existence of a perfect matching (1-factor) in a regular graph in terms of the third largest eigenvalue. Subsequently, this condition was improved in [8, 9, 10] and extended to  $k$ -factor in [24, 25]. In terms of spectral radius, Fiedler and Nikiforov [14] showed tight conditions on the spectral radius that guarantee the existence of a Hamiltonian path (a special  $[1, 2]$ -factor) and a Hamiltonian cycle (a special 2-factor), respectively. In 2021, O [27] gave an upper bound on the spectral radius for graphs that do not contain perfect matchings. Denote by  $\vee$  and  $\cup$  the *join* and *union* operations on graphs, respectively. Let  $H_{n,k} \cong K_{k-1} \vee (K_1 \cup K_{n-k})$ . Fan, Lin and Lu [12] showed that if  $\rho(G) > \rho(H_{n,a})$  and  $n \geq 3a + b + 1$ , then the graph  $G$  contains an  $[a, b]$ -factor, which partially confirmed the conjecture on  $[a, b]$ -factor proposed by Cho, Hyun, O and Park [7]. Further, their result also gave the spectral condition for the existence of a  $k$ -factor.

**Theorem 1.1** ([12], Corollary 1.4). *Let  $k$  and  $n \geq 4k - 1$  be two positive integers such that  $k \cdot n$  is even and let  $G$  be a graph of order  $n$ . If  $\rho(G) > \rho(H_{n,k})$ , then  $G$  contains a  $k$ -factor.*

Let  $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$  be a collection of (not necessarily distinct) graphs on the same vertex set  $V$ , and let  $H$  be a graph with  $m$  edges on the vertex set  $V(H) \subseteq V$ . If there exists a bijection  $\varphi : E(H) \rightarrow \{1, \dots, m\}$  such that  $e \in E(G_{\varphi(e)})$  for each  $e \in E(H)$ , then we say that  $H$  is a *rainbow* of  $\mathcal{G}$ , or  $\mathcal{G}$  admits a *rainbow*  $H$ . Recently, the study of rainbow structures has attracted

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extensive attention, yielding many beautiful results from a variety of perspectives, including degree [3, 6, 16, 19, 22], size [1, 17, 20, 26] and spectra [15, 18, 29]. For more results, we refer the readers to the survey [28].

In this paper, we discuss the spectral condition on the set of graphs  $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$  that guarantee the existence of a rainbow factor in  $\mathcal{G}$ . Guo, Lu, Ma, and Ma [15] identified the spectral conditions related to rainbow matchings. Later, He, Li and Feng [18] gave sufficient spectral conditions for rainbow Hamiltonian paths and rainbow linear forests. Recently, Zhang and van Dam [29] proved that  $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$  admits a rainbow Hamiltonian cycle if  $\rho(G_i) > n - 2$  for  $i = 1, 2, \dots, n$  unless  $G_1 = G_2 = \dots = G_n \cong H_{n,2}$ . Inspired by these extensive works, we consider an extension of Theorem 1.1 to the rainbow version, by presenting a spectral condition for a rainbow  $k$ -factor.

**Theorem 1.2.** *Let  $k$  and  $n \geq 4k - 1$  be two positive integers such that  $k \cdot n$  is even and let  $\mathcal{G} = \{G_1, \dots, G_{\frac{kn}{2}}\}$  be a set of graphs on the same vertex set  $V = \{1, \dots, n\}$ . If  $\rho(G_i) \geq \rho(H_{n,k})$  for  $i = 1, 2, \dots, \frac{kn}{2}$ , then  $\mathcal{G}$  admits a rainbow  $k$ -factor unless  $G_1 = G_2 = \dots = G_{\frac{kn}{2}} \cong H_{n,k}$ .*

Our paper is organized as follows: In Section 2, we introduce the so-called Kelmans operation and do some preliminary work. In Section 3, we provide the proof of our main result. In addition, we refer the readers to [2, 5] for notation and terminology not defined here.

## 2 Preliminaries

Let  $G$  be a simple graph with vertex set  $V(G) = \{1, \dots, n\}$  and edge set  $E(G)$ . The *adjacency matrix* of  $G$  is defined as  $A(G) = (a_{ij})_{n \times n}$  where  $a_{ij} = 1$  if  $ij \in E(G)$  and 0 otherwise. The *eigenvalues* of  $G$  are the eigenvalues of  $A(G)$ . The largest eigenvalue is also known as the *spectral radius* of  $G$ , denoted by  $\rho(G)$ . Two graphs  $G_1$  and  $G_2$  are said to be *isomorphic*, denoted  $G_1 \cong G_2$ , if there exists a bijection  $\phi : V(G_1) \rightarrow V(G_2)$  such that  $uv \in E(G_1)$  if and only if  $\phi(u)\phi(v) \in E(G_2)$ . Furthermore, two graphs  $G_1$  and  $G_2$  are *identical*, written  $G_1 = G_2$ , if  $V(G_1) = V(G_2)$  and  $E(G_1) = E(G_2)$ .

Given a graph  $G$  and  $u, v \in V(G)$ , we can obtain the graph  $KO_{uv}(G)$  from  $G$  via replacing the edge  $vw$  by a new edge  $uw$  for all  $w \in N_G(v) \setminus (N_G(u) \cup \{u\})$ . This operation is called the *Kelmans operation* (from  $v$  to  $u$ ) due to Kelmans [21]. In 2009, Csikvári [11] proved that the spectral radius of a graph does not decrease after the Kelmans operation.

**Lemma 2.1** ([11], Theorem 2.1). *Let  $G$  be a graph on vertex set  $V(G)$  and let  $u, v \in V(G)$ . Then*

$$\rho(KO_{uv}(G)) \geq \rho(G).$$

In 2019, Liu, Lai and Das found that the above inequality is strict when  $G$  is connected and  $G \not\cong KO_{uv}(G)$ .

**Lemma 2.2** ([23], Corollary 4.1). *Let  $G$  be a connected graph on vertex set  $V(G)$  and let  $u, v \in V(G)$ . Then  $\rho(KO_{uv}(G)) > \rho(G)$  unless  $G \cong KO_{uv}(G)$ .*

Observe that  $KO_{uv}(G) \cong KO_{vu}(G)$ . Denote by  $KO(G)$  the graph obtained from  $G$  by applying the Kelmans operation to each vertex pair  $(u, v)$  such that  $u < v$ . It implies that  $KO_{uv}(KO(G)) \cong KO(G)$  for any vertex pair  $(u, v)$ . Then, we obtain the following proposition in  $KO(G)$ , which will be useful in the subsequent proofs.

**Proposition 2.3.** *Let  $G$  be a graph on vertex set  $V = \{1, \dots, n\}$  and let  $x, y \in V$  such that  $x < y$ . If  $xy \in E(KO(G))$ , then  $ij \in E(KO(G))$  for each integer  $1 \leq i \leq x$  and  $i < j \leq y$ .*

For a given graph set  $\mathcal{G} = \{G_1, \dots, G_m\}$ , define  $KO_{uv}(\mathcal{G}) = \{KO_{uv}(G_1), \dots, KO_{uv}(G_m)\}$  and  $KO(\mathcal{G}) = \{KO(G_1), \dots, KO(G_m)\}$ .

**Lemma 2.4.** Let  $\mathcal{G} = \{G_1, \dots, G_{\frac{kn}{2}}\}$  be a set of graphs on the same vertex set  $V = \{1, \dots, n\}$ . If  $KO(\mathcal{G})$  admits a rainbow  $k$ -factor, then so does  $\mathcal{G}$ .

*Proof.* It suffices to show that for any vertices  $u, v \in V$ , if  $KO_{uv}(\mathcal{G})$  admits a rainbow  $k$ -factor, then so does  $\mathcal{G}$ . Assume that  $F$  is a rainbow  $k$ -factor of  $KO_{uv}(\mathcal{G})$  where  $E(F) = \{e_1, e_2, \dots, e_{\frac{kn}{2}}\}$  and  $e_i \in E(KO_{uv}(G_i))$  for  $i \in [1, \frac{kn}{2}]$ . If  $e_i \in E(G_i)$  for each integer  $i \in [1, \frac{kn}{2}]$ , then  $F$  is also a rainbow  $k$ -factor of  $\mathcal{G}$ . Otherwise, there exists an edge  $e_r = uv \in E(KO_{uv}(G_r))$  but  $e_r \notin E(G_r)$ . According to the definition of the Kelmans operation, we have  $vw \in E(G_r)$ . If  $vw \in E(F)$ , then without loss of generality, assume  $e_s = vw \in E(KO_{uv}(G_s))$ . This implies both  $vw \in E(G_s)$  and  $uw \in E(G_s)$ . In this case, let  $e_r = vw$  and  $e_s = uw$ . If  $vw \notin E(F)$ , there is a vertex  $w'$  satisfying  $e_t = vw' \in E(F)$  and  $uw' \notin E(F)$  since  $d_F(u) = d_F(v) = k$ . Note that  $e_t = vw' \in KO_{uv}(G_t)$ , which implies that  $vw' \in G_t$  and  $uw' \in G_t$ . Replace  $uv$  by  $vw$  and  $vw'$  by  $uw'$ , so that  $e_r = vw$  and  $e_t = uw'$ . Then we obtain  $e_i \in E(G_i)$  for any  $i \in [1, \frac{kn}{2}]$ . Hence,  $\mathcal{G}$  admits a rainbow  $k$ -factor.  $\square$

Moreover, the proofs also involve several established results.

**Lemma 2.5** ([5]). *If  $H$  is a subgraph of a connected graph  $G$ , then  $\rho(H) \leq \rho(G)$  with equality if and only if  $H \cong G$ .*

**Theorem 2.1** ([15], Theorem 3). *Let  $\mathcal{G} = \{G_1, \dots, G_{\frac{n}{2}}\}$  be a set of graphs on the same vertex set  $V = \{1, \dots, n\}$  where  $n$  is even. If  $\rho(G_i) \geq n - 2$  for  $i = 1, 2, \dots, \frac{n}{2}$ , then  $\mathcal{G}$  admits a rainbow perfect matching unless  $G_1 = G_2 = \dots = G_{\frac{n}{2}} \cong H_{n,1}$ .*

**Theorem 2.2** ([29], Theorem 4.3). *Let  $\mathcal{G} = \{G_1, \dots, G_n\}$  be a set of graphs on the same vertex set  $V = \{1, \dots, n\}$  where  $n \geq 4$ . If  $\rho(G_i) > n - 2$  for  $i = 1, 2, \dots, n$ , then  $\mathcal{G}$  admits a rainbow Hamiltonian cycle unless  $G_1 = G_2 = \dots = G_n \cong H_{n,2}$ .*

### 3 Proof of Theorem 1.2

**Lemma 3.1.** *Let  $G$  be a graph on vertex set  $V = \{1, \dots, n\}$  and let  $u, v \in V$  such that  $u < v$ . If  $\rho(G) = \rho(H_{n,k})$  and  $KO_{uv}(G) \cong H_{n,k}$ , then  $G \cong H_{n,k}$ .*

*Proof.* The condition  $KO_{uv}(G) \cong H_{n,k}$  implies that there is a vertex subset  $A \subset V$  of size  $n - 1$  such that  $KO_{uv}(G)[A] \cong K_{n-1}$ . Since  $n - 1 \leq |\{u, v\} \cup A| \leq n$ , we have  $1 \leq |\{u, v\} \cap A| \leq 2$ . Observe that  $G \cong KO_{uv}(G) \cong H_{n,k}$  when  $|\{u, v\} \cap A| = 2$ . Thus, assume  $|\{u, v\} \cap A| = 1$  in the remaining proof. If  $G$  is disconnected, then  $\rho(G) \leq \rho(K_{n-1}) = n - 2 \leq \rho(H_{n,k})$ , and the equality holds if and only if  $k = 1$ . It follows that  $G \cong H_{n,1}$ . If  $G$  is connected, then by Lemma 2.2, we have  $\rho(KO_{uv}(G)) > \rho(G)$  unless  $G \cong KO_{uv}(G) \cong H_{n,k}$ . Combining this with  $\rho(G) = \rho(H_{n,k}) = \rho(KO_{uv}(G))$  yields  $G \cong H_{n,k}$ . Hence, the proof is complete.  $\square$

From Lemma 2.1 and Lemma 3.1, we can deduce the following corollary.

**Corollary 3.1.** *Let  $G$  be a graph of order  $n$ . If  $\rho(G) = \rho(H_{n,k})$  and  $KO(G) \cong H_{n,k}$ , then  $G \cong H_{n,k}$ .*

Recall that the join and union operations are denoted by  $\vee$  and  $\cup$ , respectively. Moreover, the union of disjoint sets is denoted by  $\dot{\cup}$ . For simplicity, let  $[n] = \{1, \dots, n\}$ .

**Lemma 3.2.** *Let  $k \geq 2$  and  $n \geq 4k - 1$  be two integers such that  $k \cdot n$  is even and let  $\mathcal{G} = \{G_1, \dots, G_{\frac{kn}{2}}\}$  be a set of graphs on the same vertex set  $V = \{1, \dots, n\}$ . If  $G_i \cong H_{n,k}$  for each  $i \in [\frac{kn}{2}]$ , and there exist  $i_1, i_2 \in [\frac{kn}{2}]$  such that  $G_{i_1} \neq G_{i_2}$ , then  $\mathcal{G}$  admits a rainbow  $k$ -factor.*

*Proof.* For simplicity, let  $V(G_i) = A_i \dot{\cup} B_i \cup C_i$  such that  $G_i[A_i] \cong K_1$ ,  $G_i[B_i] \cong K_{k-1}$ ,  $G_i[C_i] \cong K_{n-k}$ , and  $G_i = G_i[B_i] \vee (G_i[A_i] \cup G_i[C_i])$  for  $1 \leq i \leq \frac{kn}{2}$ .

**Claim 1.** If  $A_1 = A_2 = \dots = A_{\frac{kn}{2}}$ , then  $\mathcal{G}$  admits a rainbow  $k$ -factor.

*Proof.* Without loss of generality, let  $A_1 = A_2 = \dots = A_{\frac{kn}{2}} = \{u\}$  and  $G_1 \neq G_2$ . Then  $|\cup_{i=1}^k B_i| \geq k$ . Thus, there exists a subset  $B' = \{v_1, \dots, v_k\} \subset V$  such that  $v_i \in B_i$  for  $1 \leq i \leq k$ . Let  $A' = \{u\}$  and  $C' = V \setminus (A' \cup B')$ . Now we construct a graph  $H$  on the vertex set  $V$  that satisfies  $H[B'] \cong K_k$ ,  $H[C'] \cong K_{n-k-1}$  and  $H = H[B'] \vee (H[A'] \cup H[C'])$ . Note that  $\rho(H) > \rho(H_{n,k})$ . By Theorem 1.1, the graph  $H$  contains a  $k$ -factor  $F$  with  $E(F) = \{e_1, e_2, \dots, e_{\frac{kn}{2}}\}$ . As  $d_H(u) = k = d_F(u)$ , we obtain  $\{uv_1, uv_2, \dots, uv_k\} \subset E(F)$ . So, assume  $e_i = uv_i$  when  $1 \leq i \leq k$ . The definition of  $B'$  implies that  $e_i \in E(G_i)$  for each  $1 \leq i \leq k$ . Furthermore, for any  $i \in \{k+1, \dots, \frac{kn}{2}\}$ , we have  $e_i \in E(G_i)$ . Therefore,  $\mathcal{G}$  admits a rainbow  $k$ -factor.  $\square$

Next, partition  $\mathcal{G}$  as  $\mathcal{G} = \mathcal{G}_1 \dot{\cup} \mathcal{G}_2 \dot{\cup} \dots \dot{\cup} \mathcal{G}_q$  satisfies

- 1) for  $i \in [1, q]$ ,  $n_i = |\mathcal{G}_i|$  and  $n_1 \geq n_2 \geq \dots \geq n_q$ ;
- 2) for any two graphs  $G_i \in \mathcal{G}_\ell$  and  $G_j \in \mathcal{G}_m$ , if  $\ell = m$ , then  $A_i = A_j$ ; otherwise  $A_i \neq A_j$ , where  $i, j \in [\frac{kn}{2}]$  and  $\ell, m \in [q]$ .

Note that  $q \geq 2$  and  $n_1 < \frac{kn}{2}$  by Claim 1. For any  $G_i \in \mathcal{G}_\ell$ , assume that  $A_i = \{u_\ell\}$  where  $1 \leq \ell \leq q$ . Then we divide the proof into two cases depending on whether  $n$  is even or odd.

**Case 1.  $n$  is even.**

Let  $\hat{k}_i = \lfloor \frac{n_i}{n/2} \rfloor$  for  $i \in [q]$ . Note that  $k > \hat{k}_1 \geq \hat{k}_2 \geq \dots \geq \hat{k}_q \geq 0$ . When  $\hat{k}_1 > 0$ , define  $p \leq q$  as the largest integer such that  $\hat{k}_p > 0$  and let  $\hat{k}_1 + \hat{k}_2 + \dots + \hat{k}_p = \hat{k} \leq k$ ; when  $\hat{k}_1 = 0$ , let  $p = 0$  and  $\hat{k} = 0$ . Set  $\mathcal{G}' = \mathcal{G}'_1 \dot{\cup} \mathcal{G}'_2 \dot{\cup} \dots \dot{\cup} \mathcal{G}'_p$  where  $\mathcal{G}'_i \subseteq \mathcal{G}_i$  and  $|\mathcal{G}'_i| = \frac{\hat{k}_i n}{2}$  for  $i \in [p]$ . Then  $|\mathcal{G}'| = \frac{\hat{k} n}{2}$  and we have the following claim.

**Claim 2.** The graph set  $\mathcal{G}'$  admits a rainbow  $\hat{k}$ -factor.

*Proof.* The result is trivial for  $p = 0$ . Assume that  $\mathcal{G}'_\ell = \{G_{\ell,1}, G_{\ell,2}, \dots, G_{\ell, \frac{\hat{k}_\ell n}{2}}\}$  for  $\ell \in [1, p]$  and  $p \geq 1$ . Similarly to the proof of Claim 1, we deduce that each  $\mathcal{G}'_\ell$  admits a rainbow  $\hat{k}_\ell$ -factor  $F_\ell$  where  $E(F_\ell) = \{e_{\ell,1}, e_{\ell,2}, \dots, e_{\ell, \frac{\hat{k}_\ell n}{2}}\}$  such that  $e_{\ell,i} \in E(G_{\ell,i})$  for  $1 \leq i \leq \frac{\hat{k}_\ell n}{2}$ . When  $p = 1$ , we are done. Now consider  $p \geq 2$ . If  $E(F_r) \cap E(F_s) = \emptyset$  for any  $1 \leq r < s \leq p$ , then  $F_1 \cup F_2 \cup \dots \cup F_p$  is a rainbow  $\hat{k}$ -factor of  $\mathcal{G}'$ . Otherwise, suppose that  $e_{r,i_1} = e_{s,i_2} = vv'$ . Note that  $\hat{k} \geq 2$  by  $p \geq 2$ . Then we could find an edge  $ww' = e_{t,i_3} \in E(F_t)$  satisfying  $w, w' \in V \setminus \{v, v'\}$  and  $vw, vw', v'w, v'w' \notin E(F)$ , since

$$\frac{\hat{k}n}{2} - 2 - 2 \cdot (\hat{k} - 2) - 2(\hat{k} - 2)(\hat{k} - 1) = \frac{\hat{k}n}{2} - 2\hat{k}(\hat{k} - 2) - 2 \geq 1.$$

Without loss of generality, let  $r \neq t$ . Notice that at least one of the vertices  $v$  and  $v'$  is not equal to  $u_r$ , and at least one is not equal to  $u_t$ . Assume  $v \neq u_r$  and  $v' \neq u_t$ , similarly  $w \neq u_t$  and  $w' \neq u_r$ . Then replace  $vv' \in E(G_{r,i_1})$  by  $vw' \in E(G_{r,i_1})$  and  $ww'$  by  $v'w$ . It implies that  $e_{r,i_1} = vw'$ ,  $e_{s,i_2} = vv'$  and  $e_{t,i_3} = v'w$  after the replacement. Repeating the above steps until  $E(F_r) \cap E(F_s) = \emptyset$  for  $1 \leq r < s \leq p$ , then we obtain a rainbow  $\hat{k}$ -factor of  $\mathcal{G}'$ .  $\square$

From Claim 2, let  $F'$  be a rainbow  $\hat{k}$ -factor of  $\mathcal{G}'$  where  $E(F') = \cup_{i=1}^p \{e_{i,1}, \dots, e_{i, \frac{\hat{k}_i n}{2}}\}$  such that  $e_{i,j} \in E(G_{i,j})$  and  $G_{i,j} \in \mathcal{G}'_i$ . If  $\hat{k} = k$ , we are done. For  $0 \leq \hat{k} \leq k - 1$ , assume that  $\mathcal{G}'' = \mathcal{G} \setminus \mathcal{G}' = \{G_{\frac{kn}{2}+1}, \dots, G_{\frac{kn}{2}}\}$ . Observe that for any  $\frac{n}{2}$  graphs in the set  $\mathcal{G}''$ , there exist at least two graphs, say  $G_a$  and  $G_b$ , such that  $A_a \neq A_b$ . Let  $G'_j = G_j[A_j] \cup (G_j[B_j] \vee G_j[C_j]) \cong H_{n,1}$  for  $j = \frac{kn}{2} + 1, \dots, \frac{kn}{2}$ . Using Theorem 2.1, we obtain that  $\{G'_{\frac{kn}{2}+1}, \dots, G'_{\frac{kn}{2}}\}$  admits  $k - \hat{k}$  rainbow perfect matchings, denoted by  $M_{\hat{k}+1}, \dots, M_k$ . Then  $M_{\hat{k}+1}, \dots, M_k$  are  $k - \hat{k}$  rainbow perfect matchings of  $\mathcal{G}''$  as well since  $G'_i$  is a spanning subgraph of  $G_i$ . Without loss of generality, assume that  $E(M_i) = \{e_{i,1}, \dots, e_{i, \frac{n}{2}}\}$  such that  $e_{i,j} \in E(G'_{\frac{(i-1)n}{2}+j}) \subset E(G_{\frac{(i-1)n}{2}+j})$  where  $i = \hat{k} + 1, \dots, k$  and  $j = 1, \dots, \frac{n}{2}$ .

If  $E(M_i) \cap E(M_j) = \emptyset$  and  $E(F') \cap E(M_j) = \emptyset$  for any integers  $\hat{k} + 1 \leq i, j \leq k$ , we are done as  $F' \cup M_{\hat{k}+1} \cup \dots \cup M_k$  is a rainbow  $k$ -factor of  $\mathcal{G}$ . Otherwise, suppose that  $e_{r,i_1} = e_{s,i_2} = vv'$  where  $\hat{k} < r < s \leq k$  or  $1 \leq r \leq p \leq \hat{k} < s \leq k$ . So we have  $e_{s,i_2} \in E(G_{\frac{(s-1)n}{2}+i_2})$ . Assume that  $G_{\frac{(s-1)n}{2}+i_2} \in \mathcal{G}_{\ell_0}$  where  $\ell_0 \in [q]$ . Notice that  $v, v' \neq u_{\ell_0}$  since  $vv' \in E(G_{\frac{(s-1)n}{2}+i_2})$ . Then we could find an edge  $ww' = e_{t,i_3} \in E(F' \cup M_{\hat{k}+1} \cup \dots \cup M_k)$  satisfying that  $w, w' \in V \setminus \{v, v', u_{\ell_0}\}$  and  $vw, vw', v'w, v'w' \notin E(F' \cup M_{\hat{k}+1} \cup \dots \cup M_k)$ , since

$$\frac{kn}{2} - 2 - 2 \cdot (k-2) - k - 2(k-2)(k-1) = \frac{kn}{2} - k(2k-3) - 2 \geq 1.$$

If  $e_{t,i_3} \in E(F')$ , then  $e_{t,i_3} \in E(G_{t,i_3})$  and  $G_{t,i_3} \in \mathcal{G}$ ; otherwise  $e_{t,i_3} \in E(M_{\hat{k}+1} \cup \dots \cup M_k)$  and  $e_{t,i_3} \in E(G_{\frac{(t-1)n}{2}+i_3})$ , without loss of generality, we could still assume that  $G_{\frac{(t-1)n}{2}+i_3} \in \mathcal{G}_t$  (it is possible that  $t = \ell_0$ ). Thus, all vertices in  $\{v, v', w, w'\}$  are distinct from  $u_{\ell_0}$  and at least one of the vertices  $w$  and  $w'$  ( $v$  and  $v'$ ) is distinct from  $u_t$ , say  $w \neq u_t$  ( $v \neq u_t$ ). Then replace  $vv' \in E(G_{\frac{(s-1)n}{2}+i_2})$  and  $ww' \in E(G_{\frac{(s-1)n}{2}+i_2})$  by  $v'w' \in E(G_{\frac{(s-1)n}{2}+i_2})$  and  $vw$ , respectively. It implies that  $e_{r,i_1} = vv'$ ,  $e_{s,i_2} = v'w'$  and  $e_{t,i_3} = vw$  after the replacement. Repeating the above steps, we finally obtain a rainbow  $k$ -factor of  $\mathcal{G}$ . Hence, we are done with Case 1.

**Case 2.**  $n$  is odd.

In this case,  $k$  is even. Let  $\tilde{k}_i = \lfloor \frac{n_i}{n} \rfloor$  for  $i \in [q]$ . Observe that  $\frac{k}{2} > \tilde{k}_1 \geq \tilde{k}_2 \geq \dots \geq \tilde{k}_q \geq 0$ . When  $\tilde{k}_1 > 0$ , define  $p \leq q$  as the largest integer such that  $\tilde{k}_p > 0$  and let  $\tilde{k}_1 + \tilde{k}_2 + \dots + \tilde{k}_p = \tilde{k} \leq \frac{k}{2}$ ; when  $\tilde{k}_1 = 0$ , let  $p = 0$  and  $\tilde{k} = 0$ . Set  $\mathcal{G}^* = \mathcal{G}_1^* \cup \mathcal{G}_2^* \cup \dots \cup \mathcal{G}_p^* = \cup_{i=1}^p \{G_{i,1}, \dots, G_{i,\tilde{k}_i}\}$  where  $\mathcal{G}_i^* \subseteq \mathcal{G}_i$  and  $|\mathcal{G}_i^*| = \tilde{k}_i n$  for  $i \in [p]$ . Similarly to the proof of Claim 2, we have the following claim.

**Claim 3.** The graph set  $\mathcal{G}^*$  admits a rainbow  $2\tilde{k}$ -factor.

According to Claim 3, it suffices to show that the result holds for  $0 \leq \tilde{k} \leq \frac{k}{2} - 1$ . Let  $F^*$  be a rainbow  $2\tilde{k}$ -factor of  $\mathcal{G}^*$  where  $E(F^*) = \cup_{i=1}^p \{e_{i,1}, \dots, e_{i,\tilde{k}_i}\}$  such that  $e_{i,j} \in E(G_{i,j})$  and  $G_{i,j} \in \mathcal{G}_i^*$ . Observe that for any  $n$  graphs in the graph set  $\mathcal{G}^{**} = \mathcal{G} \setminus \mathcal{G}^* = \{G_{\tilde{k}+1}, \dots, G_{\frac{kn}{2}}\}$ , there exist at least two graphs, say  $G_a$  and  $G_b$ , such that  $A_a \neq A_b$ . Using Theorem 2.2, we deduce that  $\mathcal{G}^{**}$  admits  $\frac{k}{2} - \tilde{k}$  rainbow Hamiltonian cycles, denoted by  $M_{\tilde{k}+1}, \dots, M_{\frac{k}{2}}$ , where  $E(M_i) = \{e_{i,1}, \dots, e_{i,n}\}$  satisfies  $e_{i,j} \in G_{(i-1)n+j}$  for  $i = \tilde{k} + 1, \dots, \frac{k}{2}$  and  $j = 1, 2, \dots, n$ .

If  $E(M_i) \cap E(M_j) = \emptyset$  and  $E(F^*) \cap E(M_j) = \emptyset$  for any integers  $\tilde{k} + 1 \leq i, j \leq \frac{k}{2}$ , we are done as  $\mathcal{G}$  admits a rainbow  $k$ -factor  $F^* \cup M_{\tilde{k}+1} \cup \dots \cup M_{\frac{k}{2}}$ . Otherwise, suppose that  $e_{r,i_1} = e_{s,i_2} = vv'$  where  $\tilde{k} < r < s \leq \frac{k}{2}$  or  $1 \leq r \leq p \leq \tilde{k} < s \leq \frac{k}{2}$ . Note that  $e_{r,i_1} \in E(G_{(r-1)n+i_1})$  (or  $e_{r,i_1} \in E(G_{r,i_1})$ ) and  $e_{s,i_2} \in E(G_{(s-1)n+i_2})$ . Without loss of generality, assume that  $G_{(r-1)n+i_1} \in \mathcal{G}_{\ell_1}$  (or  $G_{r,i_1} \in \mathcal{G}_{\ell_1}$ ) and  $G_{(s-1)n+i_2} \in \mathcal{G}_{\ell_2}$  where  $1 \leq \ell_1, \ell_2 \leq q$ . Similarly, there exists an edge  $ww' = e_{t,i_3} \in E(F^* \cup M_{\tilde{k}+1} \cup \dots \cup M_{\frac{k}{2}})$  such that  $w, w' \in V \setminus \{v, v', u_{\ell_2}\}$  and  $vw, vw', v'w, v'w' \notin E(F^* \cup M_{\tilde{k}+1} \cup \dots \cup M_{\frac{k}{2}})$ . If  $e_{t,i_3} \in E(F^*)$ , we have  $e_{t,i_3} \in E(G_{t,i_3})$  and  $G_{t,i_3} \in \mathcal{G}$ ; otherwise  $e_{t,i_3} \in E(M_{\tilde{k}+1} \cup \dots \cup M_{\frac{k}{2}})$  and  $e_{t,i_3} \in E(G_{\frac{(t-1)n}{2}+i_3})$ , without loss of generality, we could still assume that  $G_{\frac{(t-1)n}{2}+i_3} \in \mathcal{G}_t$ . If  $\ell_1 \neq t$  or  $\ell_2 \neq t$ , do similar replacements as in Claim 2. Hence, assume  $\ell_1 = \ell_2 = t$  in the remaining proof. Notice that  $w, w' \neq u_t$ . If  $v, v' \neq u_t$ , replace  $vv' \in E(G_{(s-1)n+i_2})$  by  $vw \in E(G_{(s-1)n+i_2})$  and  $ww'$  by  $v'w'$  so that  $e_{r,i_1} = vv'$ ,  $e_{s,i_2} = vw$  and  $e_{t,i_3} = v'w'$ . Otherwise, assume  $v = u_t$ . Recall that for any  $n$  graphs in the graph set  $\mathcal{G}^{**}$ , there are two graphs, say  $G_a \in \mathcal{G}_{\ell_a}$  and  $G_b \in \mathcal{G}_{\ell_b}$ , such that  $\ell_a \neq \ell_b$ . It implies that in  $M_s$ , there is an edge  $e_{s,i_4} = xx' \in E(G_{(s-1)n+i_4})$  such that  $G_{(s-1)n+i_4} \in \mathcal{G}_{\ell_4}$  and  $\ell_4 \neq \ell_2$  (so  $\ell_4 \neq t$ ). Since  $x \neq u_{\ell_4}$  or  $x' \neq u_{\ell_4}$ , we set  $x' \neq u_{\ell_4}$ .

**Subcase 2.1.**  $x \neq u_t$  and  $x' \neq u_t$ .

If  $v' \neq x$ , replace  $u_t v' \in E(G_{(s-1)n+i_2})$  by  $xv' \in E(G_{(s-1)n+i_2})$  and  $xx'$  by  $u_t x'$ , so that  $e_{r,i_1} = u_t v'$ ,  $e_{s,i_2} = xv'$ ,  $e_{s,i_4} = u_t x'$ , and  $e_{t,i_3} = ww'$ . For  $v' = x$ , replace  $u_t v' \in E(G_{(s-1)n+i_2})$  by  $ww' \in E(G_{(s-1)n+i_2})$ ,  $v' x'$  by  $u_t x'$ , and  $ww'$  by  $v' w'$ , so that  $e_{r,i_1} = u_t v'$ ,  $e_{s,i_2} = ww'$ ,  $e_{s,i_4} = u_t x'$ , and  $e_{t,i_3} = v' w'$ .

**Subcase 2.2.**  $x = u_t$  or  $x' = u_t$ .

Without loss of generality, let  $x' = u_t$  and  $w \neq u_{\ell_4}$ . Then  $x \neq u_t$ . If  $x \in B_{(s-1)n+i_2}$ , replace  $u_t v' \in E(G_{(s-1)n+i_2})$  by  $u_t x \in E(G_{(s-1)n+i_2})$ ,  $xu_t$  by  $wu_t$ , and  $ww'$  by  $v'w'$ , such that  $e_{r,i_1} = u_t v'$ ,  $e_{s,i_2} = u_t x$ ,  $e_{s,i_4} = wu_t$ , and  $e_{t,i_3} = v'w'$ . If  $x \notin B_{(s-1)n+i_2}$ , there exists a vertex  $y \in B_{(s-1)n+i_2}$  such that  $u_t y \notin E(F^* \cup M_{\tilde{k}+1} \cup \dots \cup M_{\frac{k}{2}})$ . Since  $d_{M_s}(y) = 2$ , we can find an edge  $yy' \in E(M_s)$  such that  $y' \neq v'$ . Assume that  $e_{s,i_5} = yy' \in E(G_{(s-1)n+i_5})$  and  $G_{(s-1)n+i_5} \in \mathcal{G}_{\ell_5}$ . Note that  $y, y' \notin \{u_t, v'\}$ . If  $\ell_5 \neq t$  and  $y = u_{\ell_5}$ , then  $y' \neq u_{\ell_5}$  and  $v' \neq u_{\ell_5}$ . Thus, replace  $u_t v' \in E(G_{(s-1)n+i_2})$  by  $u_t y \in E(G_{(s-1)n+i_2})$  and  $yy'$  by  $v'y'$ , so that  $e_{r,i_1} = u_t v'$ ,  $e_{s,i_2} = u_t y$ ,  $e_{s,i_4} = xu_t$ ,  $e_{s,i_5} = v'y'$ , and  $e_{t,i_3} = ww'$ . If  $\ell_5 \neq t$  and  $y \neq u_{\ell_5}$ , then replace  $u_t v' \in E(G_{(s-1)n+i_2})$  by  $v'y' \in E(G_{(s-1)n+i_2})$  and  $yy'$  by  $u_t y$ , so that  $e_{r,i_1} = u_t v'$ ,  $e_{s,i_2} = v'y'$ ,  $e_{s,i_4} = xu_t$ ,  $e_{s,i_5} = u_t y$ , and  $e_{t,i_3} = ww'$ . If  $\ell_5 = t$ ,  $u_t v' \in E(G_{(s-1)n+i_2})$  by  $u_t y \in E(G_{(s-1)n+i_2})$  and  $yy'$  by  $v'y'$ , so that  $e_{r,i_1} = u_t v'$ ,  $e_{s,i_2} = u_t y$ ,  $e_{s,i_4} = xu_t$ ,  $e_{s,i_5} = v'y'$  and  $e_{t,i_3} = ww'$ .

Notice that Subcases 2.1 and 2.2 may lead to multiple edges in the graph  $F^* \cup M_{\tilde{k}+1} \cup \dots \cup M_{\frac{k}{2}}$ . However, by repeatedly applying the above steps, we will eventually obtain  $E(M_i) \cap E(M_j) = \emptyset$  and  $E(F^*) \cap E(M_j) = \emptyset$  for any  $i, j \in \{\tilde{k}+1, \dots, \frac{k}{2}\}$ . It follows that  $\mathcal{G}$  admits a rainbow  $k$ -factor. Hence, the proof is complete.  $\square$

**Lemma 3.3.** *Let  $n, k$  and  $p$  be three positive integers. If  $k \geq 2$ ,  $n \geq 4k - 1$  and  $1 \leq p \leq \lceil \frac{n-k}{2} \rceil - 1$ , then*

$$\rho(K_{k+p-1} \vee ((p+1)K_1 \cup K_{n-k-2p})) < \rho(H_{n,k}).$$

*Proof.* For simplicity, assume that  $G = K_{k+p-1} \vee ((p+1)K_1 \cup K_{n-k-2p})$  and  $G' = H_{n,k} = K_{k-1} \vee (K_1 \cup K_{n-k})$ . Partition the vertex set of  $G$  as  $V(G) = V((p+1)K_1) \cup V(K_{k+p-1}) \cup V(K_{n-k-2p})$ , where  $V((p+1)K_1) = \{u_1, u_2, \dots, u_{p+1}\}$ ,  $V(K_{k+p-1}) = \{v_1, v_2, \dots, v_{k+p-1}\}$  and  $V(K_{n-k-2p}) = \{v_{k+p}, v_{k+p+1}, \dots, v_{n-p-1}\}$ . One can find that

$$G' = G - \sum_{i=k}^{k+p-1} u_i v_i + \sum_{i=2}^{p+1} \sum_{j=k+p}^{n-p-1} u_i v_j + \sum_{i=2}^p \sum_{j=i+1}^{p+1} u_i u_j.$$

Let  $x$  be the Perron vector of  $A(G)$  and let  $\rho = \rho(G)$ . By symmetry, the components of  $x$  corresponding to the vertices in  $V((p+1)K_1)$  (resp.  $V(K_{k+p-1})$  or  $V(K_{n-k-2p})$ ) are equal, denoted by  $x_1$  (resp.  $x_2$  or  $x_3$ ). Then from  $A(G)x = \rho x$ , we have

$$x_2 = \frac{(\rho - (n - k - 2p - 1))x_3}{k + p - 1}. \quad (1)$$

Let  $y$  be the Perron vector of  $A(G')$  and let  $\rho' = \rho(G')$ . Similarly, the components of  $y$  corresponding to the vertices in  $V(K_1)$  (resp.  $V(K_{k-1})$  or  $V(K_{n-k})$ ) are equal, denoted by  $y_1$  (resp.  $y_2$  or  $y_3$ ). Then from  $A(G')y = \rho' y$ , we have

$$\begin{aligned} \rho' y_1 &= (k-1)y_2, \\ \rho' y_3 &= (k-1)y_2 + (n-k-1)y_3. \end{aligned}$$

Since  $G'$  contains  $K_{n-1}$  as a proper subgraph, we have  $\rho' \geq n-2$ . Combining with the above two equations, we get

$$y_3 = \frac{\rho' y_1}{\rho' - (n-k-1)}. \quad (2)$$

It is clearly that neither  $G$  nor  $G'$  is a complete graph. So  $\rho < n-1$  and  $\rho' < n-1$ . Combining

with (1) and (2), we yield

$$\begin{aligned}
y^\top(\rho' - \rho)x &= \rho'y^\top x - \rho y^\top x \\
&= y^\top A(G')x - y^\top A(G)x \\
&= \sum_{i=2}^{p+1} \sum_{j=k+p}^{n-p-1} (x_{u_i}y_{v_j} + y_{u_i}x_{v_j}) + \sum_{i=2}^{p+1} \sum_{j=2, j \neq i}^{p+1} x_{u_i}y_{u_j} - \sum_{i=k}^{k+p-1} (x_{u_1}y_{v_i} + y_{u_1}x_{v_i}) \\
&= p(n-k-2p)(x_1y_3 + x_3y_3) + p(p-1)x_1y_3 - p(x_1y_3 + y_1x_2) \\
&= p[(n-k-2p+p-1-1)x_1y_3 + (n-k-2p)x_3y_3 - y_1x_2] \\
&> p[(n-k-2p)x_3y_3 - y_1x_2] \text{ (since } 1 \leq p \leq \lceil \frac{n-k}{2} \rceil - 1, n \geq 4k-1 \text{ and } k \geq 2) \\
&= px_3y_1 \left[ \frac{\rho'(n-k-2p)}{\rho' - (n-k-1)} - \frac{\rho - (n-k-2p-1)}{k+p-1} \right] \text{ (by (1) and (2))} \\
&> px_3y_1 \left( \frac{(n-1)(n-k-2p)}{k} - \frac{k+2p}{k+p-1} \right) \text{ (since } \rho < n-1 \text{ and } \rho' < n-1) \\
&= \frac{px_3y_1}{k(k+p-1)} ((n-1)(n-k-2p)(k+p-1) - k(k+2p)) \\
&> \frac{px_3y_1}{k(k+p-1)} (2k(k+p-1) - k(k+2p)) \text{ (since } n-1 > 2k \text{ and } n-k-2p \geq 1) \\
&\geq 0 \text{ (since } k \geq 2).
\end{aligned}$$

It implies  $\rho < \rho'$ . Therefore, the proof is complete.  $\square$

Now we shall give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** For  $k = 1$ , the result follows from Theorem 2.1. Considering  $k \geq 2$ , suppose to the contrary that  $\mathcal{G} = \{G_1, \dots, G_{\frac{kn}{2}}\}$  does not admit a rainbow  $k$ -factor. According to Lemma 2.4, the graph set  $KO(\mathcal{G}) = \{KO(G_1), \dots, KO(G_{\frac{kn}{2}})\}$  does not admit a rainbow  $k$ -factor. For simplicity, denote  $G'_t := KO(G_t)$  for  $t \in [\frac{kn}{2}]$  and  $\mathcal{G}' := KO(\mathcal{G})$ . With Lemma 2.1 and the spectral conditions of Theorem 1.2, we obtain that for each integer  $t \in [\frac{kn}{2}]$ ,

$$\rho(G'_t) \geq \rho(G_t) \geq \rho(H_{n,k}). \quad (3)$$

Next, we show that  $\rho(G_t) = \rho(H_{n,k})$  and  $G'_t \cong H_{n,k}$  for  $t \in [\frac{kn}{2}]$ . Denote by  $\{i, j\}$  the edge incident with the vertices  $i$  and  $j$ .

**Claim 1.** For each  $G'_t \in \mathcal{G}'$ ,  $\{i, j\} \in E(G'_t)$  where  $i \in [k-1]$  and  $j \in [n]$ .

*Proof.* First of all, assert that  $\{i, n\} \in E(G'_t)$  for  $i \in [k-1]$  and  $t \in [\frac{kn}{2}]$ . Otherwise,  $G'_t$  will be a proper subgraph of  $H_{n,k}$ . By Lemma 2.5, we have  $\rho(G'_t) < \rho(H_{n,k})$ , a contradiction. Then using Proposition 2.3, we deduce that  $\{i, j\} \in E(G'_t)$  for each  $t \in [\frac{kn}{2}]$  when  $i \in [k-1]$  and  $j \in [n]$ .  $\square$

**Claim 2.** For each  $G'_t \in \mathcal{G}'$ ,  $\{k+i, n-i\} \in E(G'_t)$  where  $i \in [\lceil \frac{n-k}{2} \rceil - 1]$ .

*Proof.* Assume that there exist integers  $p \in [\lceil \frac{n-k}{2} \rceil - 1]$  and  $t_1 \in [\frac{kn}{2}]$  satisfying  $\{k+p, n-p\} \notin E(G'_{t_1})$ . By Proposition 2.3, every edge  $\{i, j\}$  of  $G'_{t_1}$ , where  $i < j$ , satisfies  $i < k+p$  or  $j < n-p$ . Then  $G'_{t_1}$  is a subgraph of  $K_{k+p-1} \vee (K_{n-k-2p} \cup (p+1)K_1)$ . From Lemmas 2.5 and 3.3, we obtain that

$$\rho(G'_{t_1}) \leq \rho(K_{k+p-1} \vee (K_{n-k-2p} \cup (p+1)K_1)) < \rho(H_{n,k}),$$

a contradiction. Thus, we have  $\{k+i, n-i\} \in E(G'_t)$  for  $t = 1, 2, \dots, \frac{kn}{2}$  and  $i = 1, 2, \dots, \lceil \frac{n-k}{2} \rceil - 1$ .  $\square$

**Claim 3.** For any  $G'_t \in \mathcal{G}'$ ,  $\{k, n\} \notin E(G'_t)$ .

*Proof.* Suppose by way of contradiction that there exists an integer  $t_2 \in [\frac{kn}{2}]$  such that  $\{k, n\} \in E(G'_{t_2})$ . Without loss of generality, assume  $t_2 = \frac{(k-1)n}{2} + k$  for even  $n$  and  $t_2 = \frac{(k-1)(n-1)}{2} + k - 1$  for odd  $n$ . Let  $F$  be a  $k$ -regular graph with vertex set  $V(F) = V$  and edge set  $E(F)$ . Moreover, when  $n$  is even, define  $E(F) = \{e_{i,j} | 1 \leq i \leq k, 1 \leq j \leq \frac{n}{2}\}$  where

$$e_{i,j} = \begin{cases} \{j, \frac{n}{2} + i - j\}, & \text{when } 1 \leq j \leq i - 1, \\ \{j, n + i - j\}, & \text{when } i \leq j \leq \frac{n}{2}. \end{cases}$$

Claims 1 and 2 imply that  $e_{i,j} \in E(G'_t)$  for each  $G'_t \in \mathcal{G}'$  when  $i \neq k$  or  $j \neq k$ . In addition,  $e_{k,k} \in E(G'_{\frac{(k-1)n}{2}+k})$  by assumption. It implies that  $e_{i,j} \in E(G'_{\frac{(i-1)n}{2}+j})$  for each  $i \in [k]$  and  $j \in [\frac{n}{2}]$ . When  $n$  is odd, define  $E(F) = \{e_{0,j} | 1 \leq j \leq \frac{k}{2}\} \cup \{e_{i,j} | 1 \leq i \leq k, 1 \leq j \leq \frac{n-1}{2}\}$  where  $e_{0,1} = \{1, \frac{n+1}{2}\}$  and

$$e_{i,j} = \begin{cases} \{2j-2, 2j-1\}, & \text{when } i=0 \text{ and } 2 \leq j \leq \frac{k}{2}, \\ \{j, n+1-j\}, & \text{when } i=1 \text{ and } 1 \leq j \leq \frac{n-1}{2}, \\ \{j+1, n+i-1-j\}, & \text{when } i \geq 2 \text{ and } i-1 \leq j \leq \frac{n-1}{2}, \\ \{j, \frac{n+1}{2} + i - 1 - j\}, & \text{when } i \geq 3 \text{ and } 1 \leq j \leq i-2. \end{cases}$$

Similarly, we can find that  $e_{i,j} \in E(G'_{\frac{(i-1)(n-1)}{2}+j})$  for each  $i \in [k]$  and  $j \in [\frac{n-1}{2}]$  and  $e_{0,j} \in E(G'_{\frac{k(n-1)}{2}+j})$  for  $j \in [\frac{k}{2}]$ . Thus, the graph  $F$  is a rainbow  $k$ -factor of  $\mathcal{G}'$ , a contradiction.  $\square$

According to Claim 3 and Proposition 2.3, each  $G'_t \in \mathcal{G}'$  is a subgraph of  $H_{n,k}$ . By Lemma 2.5, we deduce that for  $t = 1, 2, \dots, \frac{kn}{2}$ ,  $\rho(G'_t) \leq \rho(H_{n,k})$ , with equality if and only if  $G'_t \cong H_{n,k}$ . Combining with (3), we obtain  $\rho(G'_t) = \rho(G_t) = \rho(H_{n,k})$  and  $G'_t \cong H_{n,k}$  for each  $t = 1, 2, \dots, \frac{kn}{2}$ . By Corollary 3.1 and Lemma 3.2, it follows that  $G_1 = \dots = G_{\frac{kn}{2}} \cong H_{n,k}$ . Therefore, the proof is complete.  $\square$

## References

- [1] R. Aharoni, M. DeVos, S. González Hermsillo de la Maza, A. Montejano, and R. Šámal, A rainbow version of Mantel's theorem, *Adv. Comb.*, Paper No. 2, 12pp, 2020.
- [2] J. A. Bondy and U. S. R. Murty, Graph theory with applications, American Elsevier Publishing Co., Inc., New York, 1976.
- [3] P. Bradshaw, Transversals and bipancyclicity in bipartite graph families, *Electron. J. Combin.*, 28(4): Paper No. 4.25, 20pp, 2021.
- [4] A. E. Brouwer and W. H. Haemers, Eigenvalues and perfect matchings, *Linear Algebra Appl.*, 395: 155–162, 2005.
- [5] A. E. Brouwer and W. H. Haemers, Spectra of graphs, Universitext, Springer, New York, 2012.
- [6] Y. Cheng, W. Sun, G. Wang, and L. Wei, Transversal Hamilton paths and cycles, arXiv:2406.13998.
- [7] E.-K. Cho, J. Y. Hyun, S. O and J. R. Park, Sharp conditions for the existence of an even  $[a, b]$ -factor in a graph, *Bull. Korean Math. Soc.*, 58(1): 31–46, 2021.

- [8] S. M. Cioabă, Perfect matchings, eigenvalues and expansion, *C. R. Math. Acad. Sci. Soc. R. Can.*, 27(4): 101–104, 2005.
- [9] S. M. Cioabă and D. A. Gregory, Large matchings from eigenvalues, *Linear Algebra Appl.*, 422(1): 308–317, 2007.
- [10] S. M. Cioabă, D. A. Gregory and W. H. Haemers, Matchings in regular graphs from eigenvalues, *J. Combin. Theory Ser. B*, 99(2): 287–297, 2009.
- [11] P. Csikvári, On a conjecture of V. Nikiforov, *Discrete Math.*, 309(13): 4522–4526, 2009.
- [12] D. Fan, H. Lin and H. Lu, Spectral radius and  $[a, b]$ -factors in graphs, *Discrete Math.*, 345(7): Paper No. 112892, 9pp, 2022.
- [13] D. Fan, H. Lin, H. Lu and S. O, Eigenvalues and factors: a survey, arXiv:2312.15902.
- [14] M. Fiedler and V. Nikiforov, Spectral radius and Hamiltonicity of graphs, *Linear Algebra Appl.*, 432(9): 2170–2173, 2010.
- [15] M. Guo, H. Lu, X. Ma, and X. Ma, Spectral radius and rainbow matchings of graphs, *Linear Algebra Appl.*, 679: 30–37, 2023.
- [16] P. Gupta, F. Hamann, A. Müyesser, O. Parczyk, and A. Sgueglia, A general approach to transversal versions of Dirac-type theorems, *Bull. Lond. Math. Soc.*, 55(6): 2817–2839, 2023.
- [17] Z. He, P. Frankl, E. Győri, Z. Lv, N. Salia, C. Tompkins, K. Varga, and X. Zhu, Extremal results for graphs avoiding a rainbow subgraph, *Electron. J. Combin.*, 31(1): Paper No. 1.28, 11pp, 2024.
- [18] X. He, Y. Li and L. H. Feng, Spectral radius and rainbow Hamilton paths of a graph, *Discrete Math.*, 347(10): Paper No. 114128, 11pp, 2024.
- [19] F. Joos and J. Kim, On a rainbow version of Dirac’s theorem, *Bull. Lond. Math. Soc.*, 52(3): 498–504, 2020.
- [20] P. Keevash, M. Saks, B. Sudakov, and J. Verstraëte, Multicolour Turán problems, *Adv. in Appl. Math.*, 33(2): 238–262, 2004.
- [21] A. K. Kelmans, On graphs with randomly deleted edges, *Acta Math. Acad. Sci. Hungar.*, 37(1-3): 77–88, 1981.
- [22] L. Li, P. Li, and X. Li, Rainbow structures in a collection of graphs with degree conditions, *J. Graph Theory*, 104(2): 341–359, 2023.
- [23] M. Liu, H-J. Lai and K. Ch. Das, Spectral results on Hamiltonian problem, *Discrete Math.*, 342(6): 1718–1730, 2019.
- [24] H. Lu, Regular factors of regular graphs from eigenvalues, *Electron. J. Combin.*, 17(1): Research Paper 159, 12pp, 2010.
- [25] H. Lu, Regular graphs, eigenvalues and regular factors, *J. Graph Theory*, 69(4): 349–355, 2012.
- [26] C. Magnant, Density of Gallai multigraphs, *Electron. J. Combin.*, 22(1): Paper 1.28, 6pp, 2015.
- [27] S. O, Spectral radius and matchings in graphs, *Linear Algebra Appl.*, 614: 316–324, 2021.

- [28] W. Sun, G. Wang and L. Wei, Transversal Structures in Graph Systems: A Survey, arXiv:2412.01121.
- [29] Y. Zhang and E. R. van Dam, Rainbow Hamiltonicity and the spectral radius, *Discrete Math.*, 348(11): Paper No. 114600, 7 pp, 2025.